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Strong convergence of split equality variational inequality and fixed point problem

Abstract. The main purpose of this paper is to introduce a new algorithm for finding a solution of split equality variational inequality problem for monotone and Lipschitz continuous operators and common fixed points of a finite family of quasi-nonexpansive mappings in the setting of infinite dimensional Hilbert spaces. Under suitable conditions, we prove that the sequence generated by the proposed new algorithm converges strongly to a solution of the split equality variational inequality and fixed point problem in Hilbert spaces. Our results improve and generalize some recent results in the literature.

Keywords. Split equality problem, fixed point, quasi-nonexpansive mapping, variational inequality.

Mathematics Subject Classification (2010):47J25, 47N10, 65J15, 90C25.

1 - Introduction

Let C and Q be nonempty closed convex subsets of real Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively. The split feasibility problem (SFP) is formulated as:

$$(1) \qquad to\,finding \quad x^* \in C \quad \text{such that} \quad \mathcal{A}x^* \in Q,$$

where $\mathcal{A} : \mathcal{H}_1 \to \mathcal{H}_2$ is a bounded linear operator.

In 1994, the split feasibility problem (SFP) in finite dimensional Hilbert spaces was first introduced by Censor and Elfving [**9**] for modeling inverse problems which arise from phase retrievals and in medical image reconstruction. Since then, the split feasibility problem (SFP) has received much attention

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due to its applications in signal processing, image reconstruction, with particular progress in intensity-modulated radiation therapy, approximation theory, control theory and geophysics. For examples, one can refer to [**6**–**8**] and related literature.

Let C be a nonempty closed convex subset of $\mathcal{H}$. Let $F : \mathcal{H} \to \mathcal{H}$ be a nonlinear operator. It is well known that the Variational Inequality Problem is to find $u \in C$ such that

$$\langle Fu, v-u\rangle \geq 0, \qquad \forall v \in C. \tag{2}$$

We denote by $VI(C,F)$ the solution set of (2). The theory of variational inequalities has played an important role in the study of a wide class of problems arising in pure and applied sciences including mechanics, optimization and optimal control, partial differential equation, operations research and engineering sciences. During the last decades this problem has been studied by many authors, (see [**5**, **21**, **33**, **34**]).

We recall the following definition on $F : \mathcal{H} \to \mathcal{H}$. The fixed point set F is denoted by $Fix(F) := \{x \in \mathcal{H} : F(x) = x\}$. The operator F is called

- Lipschitz continuous on $C \subset \mathcal{H}$ with constant $L > 0$ if

$$\|F(x) - F(y)\| \leq L\|x-y\|, \qquad \forall x, y \in C.$$

- Nonexpansive on C if

$$\|F(x) - F(y)\| \leq \|x-y\|, \qquad \forall x, y \in C.$$

- Quasi- nonexpansive on C if $Fix(F) \neq \emptyset$ and

$$\|F(x) - p\| \leq \|x-p\|, \qquad \forall x \in C, \quad p \in Fix(F).$$

- Monotone on C if

$$\langle F(x) - F(y), x-y\rangle \geq 0, \quad \forall x, y \in C.$$

- Inverse strongly monotone with constant $\beta > 0$, $(\beta - ism)$ if

$$\langle F(x) - F(y), x-y\rangle \geq \beta\|F(x) - F(y)\|^2, \quad \forall x, y \in C.$$

We note that every β- inverse strongly monotone operator is monotone and Lipschitz continuous. It is known that if F is β- inverse strongly monotone, and $\lambda \in (0, 2\beta)$ then $P_C(I - \lambda F)$ is nonexpansive. It is worth noting that there exists a monotone Lipschitz continuous operator F such that $P_C(I - \lambda F)$ fails to be nonexpansive [**14**].

In [**10**], Censor et al. introduced and studied the following split variational inequality problem:

Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two real Hilbert spaces. Given operators $f : \mathcal{H}_1 \to \mathcal{H}_1$ and $g : \mathcal{H}_2 \to \mathcal{H}_2$, a bounded linear operator $\mathcal{A} : \mathcal{H}_1 \to \mathcal{H}_2$, and nonempty closed and convex subsets $C \subset \mathcal{H}_1$ and $Q \subset \mathcal{H}_2$, the split variational inequality problem (SVIP) is the problem of finding a point $x^* \in VI(C, f)$ such that $\mathcal{A}x^* \in VI(Q, g)$, that is,

$$\begin{cases} x^* \in C \quad s.t. \quad \langle f(x^*), x - x^* \rangle \geq 0, \quad \forall x \in C \\ y^* = \mathcal{A}x^* \in Q \quad s.t. \quad \langle g(y^*), y - y^* \rangle \geq 0, \quad \forall y \in Q. \end{cases} \tag{3}$$

SVIP is quite general and should enable split minimization between two spaces so that the image of a solution point of one minimization problem, under a given bounded linear operator, is a solution point of another minimization problem.

Recently, Moudafi [**31**] introduced the following split equality problem. Let $\mathcal{H}_1$, $\mathcal{H}_2$ and $\mathcal{H}_3$ be real Hilbert spaces. Let $\mathcal{A} : \mathcal{H}_1 \to \mathcal{H}_3$, $\mathcal{B} : \mathcal{H}_2 \to \mathcal{H}_3$ be two bounded linear operators, let C and Q be nonempty closed convex subsets of $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively. The split equality problem (SEP) is to find

$$x \in C, \quad y \in Q \qquad \text{such that} \quad \mathcal{A}x = \mathcal{B}y. \tag{4}$$

Obviously, if $\mathcal{B} = I$ and $\mathcal{H}_2 = \mathcal{H}_3$ then (SEP) reduces to (SFP). This kind of split equality problem allows asymmetric and partial relations between the variables x and y. The interest is to cover many situations, such as decomposition methods for PDEs, applications in game theory, and intensity-modulated radiation therapy, (see [**3**, **4**]).

Since, each nonempty closed convex subset of a Hilbert space can be regarded as a set of fixed points of a projection. In [**30**], Moudafi introduced the following split equality fixed point problem:

let $\mathcal{A} : \mathcal{H}_1 \to \mathcal{H}_3$, $\mathcal{B} : \mathcal{H}_2 \to \mathcal{H}_3$ be two bounded linear operators, let $S : \mathcal{H}_1 \to \mathcal{H}_1$ and $T : \mathcal{H}_2 \to \mathcal{H}_2$ be two nonlinear operators such that $Fix(S) \neq \emptyset$ and $Fix(T) \neq \emptyset$. The split equality fixed point problem (SEFP) is to find

$$x \in Fix(S), \quad y \in Fix(T) \qquad \text{such that} \quad \mathcal{A}x = \mathcal{B}y. \tag{5}$$

If $\mathcal{H}_2 = \mathcal{H}_3$ and $\mathcal{B} = I$, then the split equality fixed point problem (5) reduces to the split common fixed point problem (SCFP) originally introduced in Censor and Segal [**11**] which is to find $x \in Fix(S)$ with $\mathcal{A}x \in Fix(T)$. Algorithms for solving the SEP and SCFP receive great attention, (see [**12**, **13**, **15**, **16**, **18**, **19**, **24**–**27**, **35**, **36**, **40**] and references therein).

Moudafi et al. [**30**, **32**] proposed some algorithms for solving the split equality fixed point problem. In these algorithms we need to compute norm of the operators. To solve the split equality fixed point problem for quasi-nonexpansive mappings, Zhao [**42**] proposed the following iteration algorithm which does not require any knowledge of the operator norms:

Theorem 1.1. *Let* $\mathcal{H}_1, \mathcal{H}_2$ *and* $\mathcal{H}_3$*, be real Hilbert spaces,* $\mathcal{A} : \mathcal{H}_1 \to \mathcal{H}_3$ *and* $\mathcal{B} : \mathcal{H}_2 \to \mathcal{H}_3$ *be bounded linear operators. Let* $S : \mathcal{H}_1 \to \mathcal{H}_1$ *and* $T : \mathcal{H}_2 \to \mathcal{H}_2$ *be quasi-nonexpansive mappings such that* $S - I$ *and* $T - I$ *are demiclosed at* 0*. Suppose* $\Omega = \{x \in Fix(S), y \in Fix(T) : \mathcal{A}x = \mathcal{B}y\} \neq \emptyset$*. Let* $\{x_n\}$ *and* $\{y_n\}$ *be sequences generated by* $x_0 \in \mathcal{H}_1, \quad y_0 \in \mathcal{H}_2$ *and by*

$$(6) \qquad \begin{cases} u_n = x_n - \gamma_n \mathcal{A}^*(\mathcal{A}x_n - \mathcal{B}y_n), \\ x_{n+1} = \alpha_n u_n + (1 - \alpha_n) S(u_n), \\ w_n = y_n + \gamma_n \mathcal{B}^*(\mathcal{A}x_n - \mathcal{B}y_n), \\ y_{n+1} = \beta_n w_n + (1 - \beta_n) T(w_n), \qquad \forall n \geq 0. \end{cases}$$

Assume that the step-size γ_n *is chosen in such a way that*

$$\gamma_n \in \left(\epsilon, \frac{2\|\mathcal{A}x_n - \mathcal{B}y_n\|^2}{\|\mathcal{B}^*(\mathcal{A}x_n - \mathcal{B}y_n)\|^2 + \|\mathcal{A}^*(\mathcal{A}x_n - \mathcal{B}y_n)\|^2} - \epsilon\right), \; n \in \Pi$$

otherwise $\gamma_n = \gamma$ (γ *being any nonnegative value), where the index set* $\Pi = \{n : \mathcal{A}x_n - \mathcal{B}y_n \neq 0\}$*. Let* $\{\alpha_n\} \subset (\delta, 1 - \delta)$ *and* $\{\beta_n\} \subset (\eta, 1 - \eta)$ *for small enough* $\delta, \eta > 0$*. Then, the sequences* $\{(x_n, y_n)\}$ *converges weakly to* $(x^\star, y^\star) \in \Omega$*.*

On the other hand, in the last years, many authors studied the problems of finding a common element of the set of fixed points of nonlinear operators and the set of solutions of variational inequality problem. The motivation for studying such a problem is in its possible application to mathematical models whose constraints can be expressed as fixed point problems and/or variational inequality problem: see, for instance, ([**20, 28, 38, 41**]).

Motivated by the above works, the purpose of this paper is to introduce a new algorithm for finding a solution of split equality variational inequality problem for monotone and Lipschitz continuous operators and common fixed points of a finite family of quasi-nonexpansive mappings which does not require any knowledge of the operator norms. Under suitable conditions, we prove that the sequence generated by the proposed new algorithm converges strongly to a solution of the split equality variational inequality and fixed point problem in Hilbert spaces. Our results improve and generalize the results of Moudafi [**30, 32**], Censor et al. [**10**], Zhao [**42**] and many others.

2 - Preliminaries

We use the following notation in the sequel:

- $\rightharpoonup$ for weak convergence and $\to$ for strong convergence.

Given a nonempty closed convex set $C \subset \mathcal{H}$, the mapping that assigns every point $x \in \mathcal{H}$, to its unique nearest point in C is called the metric projection onto C and is denoted by P_C; i.e., $P_C \in C$ and $\|x - P_C x\| = inf_{y \in C}\|x - y\|$. The metric projection P_C is characterized by the fact that $P_C(x) \in C$ and

$$\langle y - P_C(x), x - P_C(x)\rangle \leq 0, \qquad \forall x \in \mathcal{H}, y \in C.$$

The metric projection, P_C, satisfies the nonexpansivity condition with $Fix(P_C) = C$.

Definition 2.1. Let $U : \mathcal{H} \to \mathcal{H}$ be a mapping, then $I - U$ is said to be demiclosed at zero if for any sequence $\{x_n\}$ in $\mathcal{H}$, the conditions $x_n \rightharpoonup x$ and $\lim_{n\to\infty} \|x_n - Ux_n\| = 0$, imply $x = Ux$.

Lemma 2.1 ([**1**]). *Let $F : \mathcal{H} \to \mathcal{H}$ be a monotone and L-Lipschitz operator on C and λ be a positive number. Let $u_n = P_C(x_n - \lambda F(x_n))$ and $v_n = P_C(x_n - \lambda F(u_n))$. Then for all $x^\star \in VI(C, F)$ we have*

$$\|v_n - x^\star\|^2 \leq \|x_n - x^\star\|^2 - (1 - \lambda L)\|u_n - x_n\|^2 - (1 - \lambda L)\|u_n - v_n\|^2.$$

Lemma 2.2 ([**39**]). *Assume that $\{a_n\}$ is a sequence of nonnegative real numbers such that*

$$a_{n+1} \leq (1 - \vartheta_n)a_n + \vartheta_n\delta_n, \qquad n \geq 0,$$

where $\{\vartheta_n\}$ is a sequence in $(0, 1)$ and $\{\delta_n\}$ is a sequence in $\mathbb{R}$ such that

(i) $\sum_{n=1}^{\infty} \vartheta_n = \infty$,

(ii) $\limsup_{n\to\infty} \delta_n \leq 0$ *or* $\sum_{n=1}^{\infty} |\vartheta_n\delta_n| < \infty$.

Then $\lim_{n\to\infty} a_n = 0$.

Lemma 2.3 ([**29**]). *Let $\{t_n\}$ be a sequence of real numbers such that there exists a subsequence $\{n_i\}$ of $\{n\}$ such that $t_{n_i} < t_{n_i+1}$ for all $i \in \mathbb{N}$. Then there exists a nondecreasing sequence $\{\tau(n)\} \subset \mathbb{N}$ such that $\tau(n) \to \infty$ and the following properties are satisfied by all (sufficiently large) numbers $n \in \mathbb{N}$:*

$$t_{\tau(n)} \leq t_{\tau(n)+1}, \qquad t_n \leq t_{\tau(n)+1}.$$

In fact

$$\tau(n) = \max\{k \leq n : t_k < t_{k+1}\}.$$

Lemma 2.4 ([**17**]). *Let $\mathcal{H}$ be a Hilbert space and $x_i \in \mathcal{H}$, $(1 \leq i \leq m)$. Then for any given $\{\lambda_i\}_{i=1}^m \subset (0, 1)$ with $\sum_{i=1}^m \lambda_i = 1$ and for any positive integer k, j with $1 \leq k < j \leq m$, we have*

$$\|\sum_{i=1}^{m} \lambda_i x_i\|^2 \leq \sum_{i=1}^{m} \lambda_i\|x_i\|^2 - \lambda_k\lambda_j\|x_k - x_j\|^2.$$

3 - Algorithm and Convergence Theorem

Now we state and prove our main results of this paper.

Theorem 3.1. *Let $\mathcal{H}_1, \mathcal{H}_2$ and $\mathcal{H}_3$, be real Hilbert spaces, $\mathcal{A} : \mathcal{H}_1 \to \mathcal{H}_3$ and $\mathcal{B} : \mathcal{H}_2 \to \mathcal{H}_3$, be bounded linear operators and let C and Q, be two nonempty closed convex subsets of $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively. Let for $i = 1, 2, ..., m$, $T_i : \mathcal{H}_1 \to \mathcal{H}_1$ and $S_i : \mathcal{H}_2 \to \mathcal{H}_2$, be two finite families of quasi-nonexpansive mappings such that $S_i - I$ and $T_i - I$ are demiclosed at 0. Let, $F : \mathcal{H}_1 \to \mathcal{H}_1$ be a monotone and L- Lipschitz continuous operator on C and $G : \mathcal{H}_2 \to \mathcal{H}_2$ be a monotone and K- Lipschitz continuous operator on Q. Suppose $\Omega = \{x \in \bigcap_{i=1}^m Fix(T_i) \bigcap VI(C,F), \quad y \in \bigcap_{i=1}^m Fix(S_i) \bigcap VI(Q,G) : \mathcal{A}x = \mathcal{B}y\} \neq \emptyset$. Let $\{x_n\}$ and $\{y_n\}$ be sequences generated by $x_0, \vartheta \in \mathcal{H}_1, \quad y_0, \zeta \in \mathcal{H}_2$ and by*

$$(7) \qquad \begin{cases} z_n = x_n - \gamma_n \mathcal{A}^*(\mathcal{A}x_n - \mathcal{B}y_n), \\ u_n = P_C(z_n - \lambda_n F(z_n)), \\ v_n = P_C(z_n - \lambda_n F(u_n)), \\ x_{n+1} = \alpha_n \vartheta + \beta_n v_n + \sum_{i=1}^m \delta_{n,i} T_i v_n, \\ w_n = y_n + \gamma_n \mathcal{B}^*(\mathcal{A}x_n - \mathcal{B}y_n), \\ s_n = P_Q(w_n - \eta_n G(w_n)), \\ t_n = P_Q(w_n - \eta_n G(s_n)), \\ y_{n+1} = \alpha_n \zeta + \beta_n t_n + \sum_{i=1}^m \delta_{n,i} S_i t_n \qquad \forall n \geq 0, \end{cases}$$

where the step-size γ_n is chosen in such a way that

$$\gamma_n \in \left(\epsilon, \frac{2\|\mathcal{A}x_n - \mathcal{B}y_n\|^2}{\|\mathcal{B}^*(\mathcal{A}x_n - \mathcal{B}y_n)\|^2 + \|\mathcal{A}^*(\mathcal{A}x_n - \mathcal{B}y_n)\|^2} - \epsilon\right), \ n \in \Pi$$

otherwise $\gamma_n = \gamma$ (γ being any nonnegative value), where the index set $\Pi = \{n : \mathcal{A}x_n - \mathcal{B}y_n \neq 0\}$. Let the sequences $\{\alpha_n\}$, $\{\beta_n\}$,$\{\delta_{n,i}\}$, $\{\lambda_n\}$ and $\{\eta_n\}$ satisfy the following conditions:

(i) $\alpha_n + \beta_n + \sum_{i=1}^m \delta_{n,i} = 1$, *and* $\liminf_n \beta_n \delta_{n,i} > 0$, *for each* $i \in \{1, 2, ..., m\}$,

(ii) $\lambda_n \subset [a, b] \subset (0, \frac{1}{L})$ *and* $\eta_n \subset [c, d] \subset (0, \frac{1}{K})$,

(iii) $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=0}^\infty \alpha_n = \infty$.

Then, the sequences $\{(x_n, y_n)\}$ converges strongly to $(x^\star, y^\star) \in \Omega$.

Proof. Take $(x^\star, y^\star) \in \Omega$. From Lemma 2.1 we have

$$\|v_n - x^\star\|^2 \leq \|z_n - x^\star\|^2 - (1-\lambda_n L)\|z_n - u_n\|^2 - (1-\lambda_n L)\|u_n - v_n\|^2, \tag{8}$$

and

$$\|t_n - y^\star\|^2 \leq \|w_n - y^\star\|^2 - (1-\eta_n K)\|w_n - s_n\|^2 - (1-\eta_n K)\|t_n - s_n\|^2. \tag{9}$$

Using Lemma 2.4 and inequality (8), for each $i \in \{1, 2, ..., m\}$, we have

(10)
$$\begin{aligned}
\|x_{n+1} - x^\star\|^2 &= \|\alpha_n \vartheta + \beta_n v_n + \textstyle\sum_{i=1}^m \delta_{n,i} T_i v_n - x^\star\|^2 \\
&\leq \alpha_n \|\vartheta - x^\star\|^2 + \beta_n \|v_n - x^\star\|^2 + \textstyle\sum_{i=1}^m \delta_{n,i} \|T_i v_n - x^\star\|^2 \\
&- \beta_n \delta_{n,i} \|T_i v_n - v_n\|^2 \\
&\leq \alpha_n \|\vartheta - x^\star\|^2 + \beta_n \|v_n - x^\star\|^2 + \textstyle\sum_{i=1}^m \delta_{n,i} \|v_n - x^\star\|^2 \\
&- \beta_n \delta_{n,i} \|T_i v_n - v_n\|^2 \\
&\leq \alpha_n \|\vartheta - x^\star\|^2 + (1-\alpha_n)\|z_n - x^\star\|^2 - \beta_n \delta_{n,i} \|T_i v_n - v_n\|^2 \\
&- (1-\alpha_n)(1-\lambda_n L)\|z_n - u_n\|^2 \\
&- (1-\alpha_n)(1-\lambda_n L)\|u_n - v_n\|^2.
\end{aligned}$$

Similarly, from inequality (9) we have

(11)
$$\begin{aligned}
\|y_{n+1} - y^\star\|^2 &= \|\alpha_n \zeta + \beta_n t_n + \textstyle\sum_{i=1}^m \delta_{n,i} S_i t_n - y^\star\|^2 \\
&\leq \alpha_n \|\zeta - y^\star\|^2 + \beta_n \|t_n - y^\star\|^2 \\
&+ \textstyle\sum_{i=1}^m \delta_{n,i} \|S_i t_n - y^\star\|^2 - \beta_n \delta_{n,i} \|S_i t_n - t_n\|^2 \\
&\leq \alpha_n \|\zeta - y^\star\|^2 + (1-\alpha_n)\|w_n - y^\star\|^2 - \beta_n \delta_{n,i} \|S_i t_n - t_n\|^2 \\
&- (1-\alpha_n)(1-\eta_n K)\|w_n - s_n\|^2 \\
&- (1-\alpha_n)(1-\eta_n K)\|t_n - s_n\|^2.
\end{aligned}$$

From algorithm (7) we have that

$$
\begin{aligned}
\|z_n - x^\star\|^2 &= \|x_n - \gamma_n \mathcal{A}^*(\mathcal{A}x_n - \mathcal{B}y_n) - x^\star\|^2 \\
&= \|x_n - x^\star\|^2 + \gamma_n^2 \|\mathcal{A}^*(\mathcal{A}x_n - \mathcal{B}y_n)\|^2 \\
&- 2\gamma_n \langle x_n - x^\star, \mathcal{A}^*(\mathcal{A}x_n - \mathcal{B}y_n)\rangle \\
&= \|x_n - x^\star\|^2 + \gamma_n^2 \|\mathcal{A}^*(\mathcal{A}x_n - \mathcal{B}y_n)\|^2 \\
&- 2\gamma_n \langle \mathcal{A}x_n - \mathcal{A}x^\star, (\mathcal{A}x_n - \mathcal{B}y_n)\rangle \\
&= \|x_n - x^\star\|^2 + \gamma_n^2 \|\mathcal{A}^*(\mathcal{A}x_n - \mathcal{B}y_n)\|^2 - \gamma_n \|\mathcal{A}x_n - \mathcal{A}x^\star\|^2 \\
&- \gamma_n \|\mathcal{A}x_n - \mathcal{B}y_n\|^2 + \gamma_n \|\mathcal{B}y_n - \mathcal{A}x^\star\|^2.
\end{aligned} \tag{12}
$$

By similar way we obtain that

$$
\begin{aligned}
\|w_n - y^\star\|^2 &= \|y_n + \gamma_n \mathcal{B}^*(\mathcal{A}x_n - \mathcal{B}y_n) - y^\star\|^2 \\
&= \|y_n - y^\star\|^2 + \gamma_n^2 \|\mathcal{B}^*(\mathcal{A}x_n - \mathcal{B}y_n)\|^2 - \gamma_n \|\mathcal{B}y_n - \mathcal{B}y^\star\|^2 \\
&- \gamma_n \|\mathcal{A}x_n - \mathcal{B}y_n\|^2 + \gamma_n \|\mathcal{A}x_n - \mathcal{B}y^\star\|^2.
\end{aligned} \tag{13}
$$

By adding the two last inequalities and by taking into account the fact that $\mathcal{A}x^\star = \mathcal{B}y^\star$ we obtain

$$
\begin{aligned}
\|z_n - x^\star\|^2 + \|w_n - y^\star\|^2 &= \|x_n - x^\star\|^2 + \|y_n - y^\star\|^2 \\
&- \gamma_n [2\|\mathcal{A}x_n - \mathcal{B}y_n\|^2 - \gamma_n (\|\mathcal{B}^*(\mathcal{A}x_n - \mathcal{B}y_n)\|^2 \\
&+ \|\mathcal{A}^*(\mathcal{A}x_n - \mathcal{B}y_n)\|^2)] \\
&\leq \|x_n - x^\star\|^2 + \|y_n - y^\star\|^2.
\end{aligned} \tag{14}
$$

This implies that

$$
\begin{aligned}
&\|x_{n+1} - x^\star\|^2 + \|y_{n+1} - y^\star\|^2 \\
&\quad \leq (1-\alpha_n)(\|z_n - x^\star\|^2 + \|w_n - y^\star\|^2) + \alpha_n(\|\vartheta - x^\star\|^2 + \|\zeta - y^\star\|^2) \\
&\quad \leq (1-\alpha_n)(\|x_n - x^\star\|^2 + \|y_n - y^\star\|^2) + \alpha_n(\|\vartheta - x^\star\|^2 + \|\zeta - y^\star\|^2) \\
&\quad \leq \max\{\|x_n - x^\star\|^2 + \|y_n - y^\star\|^2, \|\vartheta - x^\star\|^2 + \|\zeta - y^\star\|^2\} \\
&\qquad \vdots \\
&\quad \leq \max\{\|x_0 - x^\star\|^2 + \|y_0 - y^\star\|^2, \|\vartheta - x^\star\|^2 + \|\zeta - y^\star\|^2\}.
\end{aligned} \tag{15}
$$

Thus $\|x_{n+1} - x^\star\|^2 + \|y_{n+1} - y^\star\|^2$ is bounded. Therefore $\{x_n\}$ and $\{y_n\}$ are bounded. Consequently $\{z_n\}, \{w_n\}, \{v_n\}$ and $\{t_n\}$ are all bounded. From inequalities (10), (11) and (14) we have that

(16)

$$\begin{aligned}
&\|x_{n+1} - x^\star\|^2 + \|y_{n+1} - y^\star\|^2\\
&\leq (1-\alpha_n)(\|z_n - x^\star\|^2 + \|w_n - y^\star\|^2) + \alpha_n(\|\vartheta - x^\star\|^2 + \|\zeta - y^\star\|^2)\\
&- \beta_n\delta_{n,i}\|T_i v_n - v_n\|^2 - \beta_n\delta_{n,i}\|S_i t_n - t_n\|^2\\
&- (1-\alpha_n)(1-\lambda_n L)\|z_n - u_n\|^2 - (1-\alpha_n)(1-\lambda_n L)\|u_n - v_n\|^2\\
&- (1-\alpha_n)(1-\eta_n K)\|w_n - s_n\|^2 - (1-\alpha_n)(1-\eta_n K)\|t_n - s_n\|^2\\
&\leq (1-\alpha_n)(\|x_n - x^\star\|^2 + \|y_n - y^\star\|^2) + \alpha_n(\|\vartheta - x^\star\|^2 + \|\zeta - y^\star\|^2)\\
&- (1-\alpha_n)\gamma_n[2\|\mathcal{A}x_n - \mathcal{B}y_n\|^2 - \gamma_n(\|\mathcal{B}^*(\mathcal{A}x_n - \mathcal{B}y_n)\|^2 + \|\mathcal{A}^*(\mathcal{A}x_n - \mathcal{B}y_n)\|^2)]\\
&- \beta_n\delta_{n,i}\|T_i v_n - v_n\|^2 - \beta_n\delta_{n,i}\|S_i t_n - t_n\|^2\\
&- (1-\alpha_n)(1-\lambda_n L)\|z_n - u_n\|^2 - (1-\alpha_n)(1-\lambda_n L)\|u_n - v_n\|^2\\
&- (1-\alpha_n)(1-\eta_n K)\|w_n - s_n\|^2 - (1-\alpha_n)(1-\eta_n K)\|t_n - s_n\|^2.
\end{aligned}$$

From above inequality we have that

$$(17)\qquad \begin{aligned}
(1-\alpha_n)(1-\lambda_n L)\|z_n - u_n\|^2 &\leq (1-\alpha_n)(\|x_n - x^\star\|^2 + \|y_n - y^\star\|^2)\\
&- \|x_{n+1} - x^\star\|^2 - \|y_{n+1} - y^\star\|^2\\
&+ \alpha_n(\|\vartheta - x^\star\|^2 + \|\zeta - y^\star\|^2).
\end{aligned}$$

By our assumption that

$$\gamma_n \in \left(\epsilon, \frac{2\|\mathcal{A}x_n - \mathcal{B}y_n\|^2}{\|\mathcal{B}^*(\mathcal{A}x_n - \mathcal{B}y_n)\|^2 + \|\mathcal{A}^*(\mathcal{A}x_n - \mathcal{B}y_n)\|^2} - \epsilon\right),$$

we have that

$$(\gamma_n + \epsilon)(\|\mathcal{B}^*(\mathcal{A}x_n - \mathcal{B}y_n)\|^2 + \|\mathcal{A}^*(\mathcal{A}x_n - \mathcal{B}y_n)\|^2) \leq 2\|\mathcal{A}x_n - \mathcal{B}y_n\|^2.$$

From above inequality and inequality (16) we have that

(18)

$$\begin{aligned}
&(1-\alpha_n)\gamma_n^2(\|\mathcal{B}^*(\mathcal{A}x_n - \mathcal{B}y_n)\|^2 + \|\mathcal{A}^*(\mathcal{A}x_n - \mathcal{B}y_n)\|^2)\\
&\leq (1-\alpha_n)\gamma_n[2\|\mathcal{A}x_n - \mathcal{B}y_n\|^2 - \gamma_n(\|\mathcal{B}^*(\mathcal{A}x_n - \mathcal{B}y_n)\|^2 + \|\mathcal{A}^*(\mathcal{A}x_n - \mathcal{B}y_n)\|^2)]\\
&\leq (1-\alpha_n)(\|x_n - x^\star\|^2 + \|y_n - y^\star\|^2) - \|x_{n+1} - x^\star\|^2 - \|y_{n+1} - y^\star\|^2\\
&+\alpha_n(\|\vartheta - x^\star\|^2 + \|\zeta - y^\star\|^2).
\end{aligned}$$

Put $\Gamma_n = \|x_n - \vartheta^\star\|^2 + \|y_n - \zeta^\star\|^2$ for all $n \in \mathbb{N}$, where $\vartheta^\star = P_\Omega \vartheta$ and $\zeta^\star = P_\Omega \zeta$. We finally analyze the inequalities (17) and (18) by considering the following two cases.

Case A. Suppose that $\Gamma_{n+1} \leq \Gamma_n$ for all $n \geq n_0$ (for n_0 large enough). In this case, since Γ_n is bounded, the limit $\lim_{n\to\infty} \Gamma_n$ exists. Since $\lim_{n\to\infty} \alpha_n = 0$, from (18) and by our assumption that on $\{\gamma_n\}$ we have

$$\lim_{n\to\infty} (\|\mathcal{B}^*(\mathcal{A}x_n - \mathcal{B}y_n)\|^2 + \|\mathcal{A}^*(\mathcal{A}x_n - \mathcal{B}y_n)\|^2) = 0.$$

So we obtain that $\lim_{n\to\infty} \|\mathcal{B}^*(\mathcal{A}x_n - \mathcal{B}y_n)\| = 0$ and $\lim_{n\to\infty} \|\mathcal{A}^*(\mathcal{A}x_n - \mathcal{B}y_n)\| = 0$. This implies that $\lim_{n\to\infty} \|\mathcal{A}x_n - \mathcal{B}y_n\| = 0$. Also from (17) we deduce

$$\lim_{n\to\infty} (1-\alpha_n)(1-\lambda_n L)\|z_n - u_n\|^2 = 0.$$

By our assumption that $\lambda_n \subset [a,b] \subset (0, \frac{1}{L})$, we obtain that

$$\lim_{n\to\infty} \|z_n - u_n\| = 0. \tag{19}$$

By similar argument we get that

$$\lim_{n\to\infty} \|u_n - v_n\| = \lim_{n\to\infty} \|w_n - s_n\| = \lim_{n\to\infty} \|t_n - s_n\| = 0; \tag{20}$$

and

$$\lim_{n\to\infty} \|S_i t_n - t_n\| = \lim_{n\to\infty} \|T_i v_n - v_n\| = 0, \quad i \in \{1,2,...,m\}. \tag{21}$$

Since $\|z_n - x_n\| = \gamma_n \|\mathcal{A}^*(\mathcal{A}x_n - \mathcal{B}y_n)\|$ and $\{\gamma_n\}$ is bounded, we have

$$\lim_{n\to\infty} \|z_n - x_n\| = 0. \tag{22}$$

From (19), (20) and (22) we have

$$\|x_n - v_n\| \leq \|x_n - z_n\| + \|z_n - u_n\| + \|u_n - v_n\| \to 0, \quad \text{as} \quad n \to \infty.$$

Therefore

$$\|x_{n+1} - x_n\| \leq \alpha_n \|\vartheta - x_n\| + \beta_n \|v_n - x_n\| + \sum_{i=1}^m \delta_{n,i} \|T_i v_n - x_n\| \to 0, \quad \text{as} \ n \to \infty. \tag{23}$$

Similarly we get that $\lim_{n\to\infty} \|y_{n+1} - y_n\| = 0$.

Now we claim that $(\omega_w(x_n), \omega_w(y_n)) \subset \Omega$, where

$$\omega_w(x_n) = \{x \in \mathcal{H}_1 : x_{n_i} \rightharpoonup x \, \text{for some subsequence} \, \{x_{n_i}\} \, of \, \{x_n\}\}.$$

Since the sequences $\{x_n\}$ and $\{y_n\}$ are bounded we have $\omega_w(x_n)$ and $\omega_w(y_n)$ are nonempty. Now, take $\widehat{x} \in \omega_w(x_n)$ and $\widehat{y} \in \omega_w(y_n)$. Thus, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ which converges weakly to $\widehat{x}$. Without loss of generality, we can assume that $x_n \rightharpoonup \widehat{x}$. Since $\lim_{n\to\infty} \|z_n - x_n\| = 0$, we have $z_n \rightharpoonup \widehat{x}$. From $u_n = P_C(z_n - \lambda_n F(z_n))$, for each $x \in C$ we have that

$$\langle x - u_n, z_n - \lambda_n F(z_n) - u_n\rangle \leq 0. \tag{24}$$

Since, F is monotone, for each $x \in C$ we have

$$\langle \lambda_n F(x), z_n - x\rangle \leq \langle \lambda_n F(z_n), z_n - x\rangle. \tag{25}$$

Utilizing the inequalities (24) and (25) we have

$$\begin{aligned}\langle \lambda_n F(x), z_n - x\rangle &\leq \langle \lambda_n F(z_n), z_n - x\rangle\\ &= \langle \lambda_n F(z_n), z_n - u_n\rangle + \langle \lambda_n F(z_n), u_n - x\rangle\\ &= \langle \lambda_n F(z_n), z_n - u_n\rangle + \langle \lambda_n F(z_n) - z_n + u_n, u_n - x\rangle\\ &+ \langle z_n - u_n, u_n - x\rangle\\ &\leq \lambda_n\langle F(z_n), z_n - u_n\rangle + \langle z_n - u_n, u_n - x\rangle\\ &\leq \lambda_n\|F(z_n)\|\|z_n - u_n\| + \|z_n - u_n\|\|u_n - x\|.\end{aligned} \tag{26}$$

Hence

$$\langle Fx, z_n - x\rangle \leq \|F(z_n)\|\|z_n - u_n\| + \frac{1}{\lambda_n}\|z_n - u_n\|\|u_n - x\|.$$

Since $\{F(z_n)\}$ is bounded, $z_n - u_n \to 0$ and $z_n \rightharpoonup \widehat{x}$, we have

$$\langle F(x), \widehat{x} - x\rangle = \lim_{n\to\infty} \langle F(x), z_n - x\rangle \leq 0, \qquad \forall x \in C.$$

This implies that $\widehat{x} \in VI(C, F)$. By similar argument we can obtain that $\widehat{y} \in VI(Q, G)$. Next we show that $\widehat{x} \in \bigcap_{i=1}^m Fix(T_i)$ and $\widehat{y} \in \bigcap_{i=1}^m Fix(S_i)$. Since $\lim_{n\to\infty} \|v_n - x_n\| = 0$, we have $v_n \rightharpoonup \widehat{x}$. From inequality (21), and the demiclosedness of $T_i - I$ in 0, for each $i \in \{1, 2, ..., m\}$, we get that $\widehat{x} \in Fix(T_i)$. By similar argument we obtain that $\widehat{y} \in \bigcap_{i=1}^m Fix(S_i)$. On the other hand, $\mathcal{A}\widehat{x} - \mathcal{B}\widehat{y} \in \omega_w(\mathcal{A}x_n - \mathcal{B}y_n)$ and weakly lower semi continuity of the norm imply that

$$\|\mathcal{A}\widehat{x} - \mathcal{B}\widehat{y}\| \leq \liminf_{n\to\infty} \|\mathcal{A}x_n - \mathcal{B}y_n\| = 0.$$

Thus $(\widehat{x}, \widehat{y}) \in \Omega$. We also have the uniqueness of the weak cluster point of $\{x_n\}$ are $\{y_n\}$, (see [**42**] for details) which implies that the whole sequences $\{(x_n, y_n)\}$

weakly convergence to a point $(\widehat{x}, \widehat{y}) \in \Omega$. Next we prove that the sequences $\{(x_n, y_n)\}$ converges strongly to $(\vartheta^\star, \zeta^\star)$ where $\vartheta^\star = P_\Omega \vartheta$ and $\zeta^\star = P_\Omega \zeta$. First we show that

$$\lim sup_{n\to\infty}\langle \vartheta - \vartheta^\star, x_n - \vartheta^\star\rangle \le 0. \tag{27}$$

To show this inequality, we choose a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that

$$\lim_{k\to\infty}\langle \vartheta - \vartheta^\star, x_{n_k} - \vartheta^\star\rangle = \limsup_{n\to\infty}\langle \vartheta - \vartheta^\star, x_n - \vartheta^\star\rangle.$$

Since $\{x_{n_k}\}$ converges weakly to $\widehat{x}$, it follows that

$$\limsup_{n\to\infty}\langle \vartheta - \vartheta^\star, x_n - \vartheta^\star\rangle = \lim_{k\to\infty}\langle \vartheta - \vartheta^\star, x_{n_k} - \vartheta^\star\rangle = \langle \vartheta - \vartheta^\star, \widehat{x} - \vartheta^\star\rangle \le 0. \tag{28}$$

By similar argument we obtain that

$$\lim sup_{n\to\infty}\langle \zeta - \zeta^\star, y_n - \zeta^\star\rangle \le 0. \tag{29}$$

From the inequality, $\|x+y\|^2 \le \|x\|^2 + 2\langle y, x+y\rangle, \qquad (\forall x, y \in \mathcal{H}_1)$, we find that

$$\begin{aligned}
\|x_{n+1} - \vartheta^\star\|^2 &\le \|\beta_n v_n + \textstyle\sum_{i=1}^\infty \delta_{n,i} T_i v_n - (1-\alpha_n)\vartheta^\star\|^2 \\
&+ 2\alpha_n\langle \vartheta - \vartheta^\star, x_{n+1} - \vartheta^\star\rangle \\
&= (1-\alpha_n)^2\|\frac{\beta_n}{(1-\alpha_n)} v_n + \frac{\sum_{i=1}^\infty \delta_{n,i}}{(1-\alpha_n)} T_i v_n - \vartheta^\star\|^2 \\
&+ 2\alpha_n\langle \vartheta - \vartheta^\star, x_{n+1} - \vartheta^\star\rangle \\
&\le \beta_n(1-\alpha_n)\|v_n - \vartheta^\star\|^2 + \textstyle\sum_{i=1}^\infty \delta_{n,i}(1-\alpha_n)\|T_i v_n - \vartheta^\star\|^2 \\
&+ 2\alpha_n\langle \vartheta - \vartheta^\star, x_{n+1} - \vartheta^\star\rangle \\
&\le (1-\alpha_n)^2\|v_n - \vartheta^\star\|^2 \\
&+ 2\alpha_n\langle \vartheta - \vartheta^\star, x_{n+1} - \vartheta^\star\rangle.
\end{aligned}$$

Similarly we obtain that

$$\|y_{n+1} - \zeta^\star\|^2 \le (1-\alpha_n)^2\|w_n - \zeta^\star\|^2 + 2\alpha_n\langle \zeta - \zeta^\star, y_{n+1} - \zeta^\star\rangle. \tag{30}$$

By adding the two last inequalities we have that

$$\begin{aligned}
\|x_{n+1} - \vartheta^\star\|^2 &+ \|y_{n+1} - \zeta^\star\|^2 \\
&\le (1-\alpha_n)^2(\|x_n - \vartheta^\star\|^2 + \|y_n - \zeta^\star\|^2) \\
&+ 2\alpha_n(\langle \vartheta - \vartheta^\star, x_{n+1} - \vartheta^\star\rangle + \langle \zeta - \zeta^\star, y_{n+1} - \zeta^\star\rangle).
\end{aligned} \tag{31}$$

It immediately follows that

$$
\begin{aligned}
\Gamma_{n+1} &\leq (1-\alpha_n)^2\Gamma_n + 2\alpha_n\eta_n \\
&= (1-2\alpha_n)\Gamma_n + \alpha_n^2\Gamma_n + 2\alpha_n\eta_n \\
&\leq (1-2\alpha_n)\Gamma_n + 2\alpha_n\left(\frac{\alpha_n N}{2} + \eta_n\right) \\
&\leq (1-\rho_n)\Gamma_n + \rho_n\delta_n,
\end{aligned} \tag{32}
$$

where $\varsigma_n = \langle \vartheta - \vartheta^\star, x_{n+1} - \vartheta^\star\rangle + \langle \zeta - \zeta^\star, y_{n+1} - \zeta^\star\rangle$, $N = sup\{\|x_n - x^\star\|^2 + \|y_n - y^\star\|^2 : n \geq 0\}$, $\rho_n = 2\alpha_n$ and $\delta_n = \dfrac{\alpha_n N}{2} + \varsigma_n$. It is easy to see that $\rho_n \to 0, \sum_{n=1}^{\infty}\rho_n = \infty$ and $\limsup_{n\to\infty}\delta_n \leq 0$. Hence, all conditions of Lemma 2.2 are satisfied. Therefore, we immediately deduce that $\lim_{n\to\infty}\Gamma_n = 0$. Consequently $\lim_{n\to\infty}\|x_n - \vartheta^\star\| = \lim_{n\to\infty}\|y_n - \zeta^\star\| = 0$, that is $(x_n, y_n) \to (\vartheta^\star, \zeta^\star)$.

Case B. Assume that $\{\Gamma_n\}$ is not a monotone sequence. Then, we can define an integer sequence $\{\tau(n)\}$ for all $n \geq n_0$ (for some n_0 large enough) by

$$\tau(n) = max\{k \leq n : \Gamma_k < \Gamma_{k+1}\}.$$

Clearly, τ is a nondecreasing sequence such that $\tau(n) \to \infty$ as $n \to \infty$ and for all $n \geq n_0$, $\Gamma_{\tau(n)} < \Gamma_{\tau(n)+1}$. Now, it follows from (16) that

$$\Gamma_{\tau(n)+1} - \Gamma_{\tau(n)} \leq \alpha_n(\|\vartheta - \vartheta^\star\|^2 + \|\zeta - \zeta^\star\|^2) - \alpha_n\Gamma_{\tau(n)}.$$

Since $\lim_{n\to\infty}\alpha_n = 0$ and $\{x_n\}$ and $\{y_n\}$ are bounded, we derive that

$$\lim_{n\to\infty}(\Gamma_{\tau(n)+1} - \Gamma_{\tau(n)}) = 0. \tag{33}$$

Following an argument similar to that in *Case A* we have

$$\Gamma_{\tau(n)+1} \leq (1-\rho_{\tau(n)})\Gamma_{\tau(n)} + \rho_{\tau(n)}\delta_{\tau(n)},$$

where $\limsup_{n\to\infty}\delta_{\tau(n)} \leq 0$. Since $\Gamma_{\tau(n)} < \Gamma_{\tau(n)+1}$, we have

$$\rho_{\tau(n)}\Gamma_{\tau(n)} \leq \rho_{\tau(n)}\delta_{\tau(n)}.$$

Since $\rho_{\tau(n)} > 0$ we deduce that

$$\Gamma_{\tau(n)} \leq \delta_{\tau(n)}.$$

From $\limsup_{n\to\infty}\delta_{\tau(n)} \leq 0$ we get that $\lim_{n\to\infty}\Gamma_{\tau(n)} = 0$. This together with (33), implies that $\lim_{n\to\infty}\Gamma_{\tau(n)+1} = 0$. Thus by Lemma 2.3, we have

$$0 \leq \Gamma_n \leq \max\{\Gamma_{\tau(n)}, \Gamma_n\} \leq \Gamma_{\tau(n)+1}.$$

Therefore $(x_n, y_n) \to (\vartheta^\star, \zeta^\star)$. This completes the proof. □

Remark 3.1. In [**10**], the authors present an algorithm for solving split variational inequality problem for inverse strongly monotone operators, but in this paper we consider a new algorithm for solving split equality variational inequality problem for monotone and Lipschitz continuous operators which does not require any knowledge of the operator norms. We also present a strong convergence theorem which is more desirable than weak convergence.

Remark 3.2. In [**11**, **18**, **19**, **25**], the authors present some algorithms for solving split fixed point problem but in this paper we introduce a new algorithm for solving split equality variational inequality and fixed point problem which does not require any knowledge of the operator norms.

Remark 3.3. Our main theorem, generalized the main result of Moudafi [**30**, **32**] from firmly quasi-nonexpansive mapping to a finite family of quasi-nonexpansive mappings. Our algorithm does not require any knowledge of the operator norms. We also present a strong convergence theorem which is more desirable than weak convergence.

Remark 3.4. In [**42**], Zhao present a weak convergence theorem for solving split equality fixed point problem of quasi-nonexpansive mapping, (see Theorem 1.1 of this paper). In this paper we extend the result for solving split equality common fixed problem of a finite family of quasi-nonexpansive mappings and variational inequality problem of monotone and Lipschitz continuous operator. We also present a strong convergence theorem which is more desirable than weak convergence.

4 - Results

In [**2**], Aoyoma, Iemoto, Kohsaka, and Takahashi introduced an important class of mapping which called $\lambda-$ hybrid mapping as following: Let λ be a real number. A mapping $T : C \to C$ is called $\lambda-$ hybrid if

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + \lambda\langle x - Tx, y - Ty\rangle, \qquad \forall x, y \in C.$$

In particular, if $\lambda = 0$, then T is a nonexpansive mapping. If $\lambda = 1$, then T is called a hybrid mapping [**37**], and if $\lambda = 2$, then T is called a non-spreading mapping [**22**, **23**]. It is obvious that every $\lambda-$ hybrid mapping is quasi-nonexpansive.

Lemma 4.1 ([**2**]). *Let C be nonempty closed convex subset of a real Hilbert space $\mathcal{H}$, and let $T : C \to C$ be $\lambda-$ hybrid mapping. Then $I - T$ is demiclosed at* 0. *Also $Fix(T)$ is closed and convex.*

From Theorem 3.1 and above lemma we obtain the following result.

Theorem 4.1. *Let $\mathcal{H}_1, \mathcal{H}_2$ and $\mathcal{H}_3$, be real Hilbert spaces, $\mathcal{A} : \mathcal{H}_1 \to \mathcal{H}_3$ and $\mathcal{B} : \mathcal{H}_2 \to \mathcal{H}_3$ be bounded linear operators and let C and Q, be two nonempty closed convex subsets of $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively. Let for $i = 1, 2, ..., m$, $T_i : \mathcal{H}_1 \to \mathcal{H}_1$ be a finite family of $\lambda-$ hybrid mappings and $S_i : \mathcal{H}_2 \to \mathcal{H}_2$ be a finite family of $\varsigma-$ hybrid mappings. Let, $F : \mathcal{H}_1 \to \mathcal{H}_1$ be a monotone and L- Lipschitz continuous operator on C and $G : \mathcal{H}_2 \to \mathcal{H}_2$ be a monotone and K- Lipschitz continuous operator on Q. Suppose $\Omega = \{x \in \bigcap_{i=1}^m Fix(T_i) \bigcap VI(C, F), \quad y \in \bigcap_{i=1}^m Fix(S_i) \bigcap VI(Q, G) : \mathcal{A}x = \mathcal{B}y\} \neq \emptyset$. Let $\{x_n\}$ and $\{y_n\}$ be sequences generated by $x_0, \vartheta \in \mathcal{H}_1, \quad y_0, \zeta \in \mathcal{H}_2$ and by*

$$(34) \qquad \begin{cases} z_n = x_n - \gamma_n \mathcal{A}^*(\mathcal{A}x_n - \mathcal{B}y_n), \\ u_n = P_C(z_n - \lambda_n F(z_n)), \\ v_n = P_C(z_n - \lambda_n F(u_n)), \\ x_{n+1} = \alpha_n \vartheta + \beta_n v_n + \sum_{i=1}^m \delta_{n,i} T_i v_n, \\ w_n = y_n + \gamma_n \mathcal{B}^*(\mathcal{A}x_n - \mathcal{B}y_n), \\ s_n = P_Q(w_n - \eta_n G(w_n)), \\ t_n = P_Q(w_n - \eta_n G(s_n)), \\ y_{n+1} = \alpha_n \zeta + \beta_n t_n + \sum_{i=1}^m \delta_{n,i} S_i t_n \qquad \forall n \geq 0, \end{cases}$$

where the step-size γ_n is chosen in such a way that

$$\gamma_n \in \left(\epsilon, \frac{2\|\mathcal{A}x_n - \mathcal{B}y_n\|^2}{\|\mathcal{B}^*(\mathcal{A}x_n - \mathcal{B}y_n)\|^2 + \|\mathcal{A}^*(\mathcal{A}x_n - \mathcal{B}y_n)\|^2} - \epsilon \right), \quad n \in \Pi$$

otherwise $\gamma_n = \gamma$ (γ being any nonnegative value), where the index set $\Pi = \{n : \mathcal{A}x_n - \mathcal{B}y_n \neq 0\}$. Let the sequences $\{\alpha_n\}$, $\{\beta_n\}$,$\{\delta_{n,i}\}$, $\{\lambda_n\}$ and $\{\eta_n\}$ satisfy the following conditions:

(i) $\alpha_n + \beta_n + \sum_{i=1}^m \delta_{n,i} = 1$, *and* $\liminf_n \beta_n \delta_{n,i} > 0$, *for each* $i \in \{1, 2, ..., m\}$,

(ii) $\lambda_n \subset [a, b] \subset (0, \frac{1}{L})$ *and* $\eta_n \subset [c, d] \subset (0, \frac{1}{K})$,

(iii) $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=0}^\infty \alpha_n = \infty$.

Then, the sequences $\{(x_n, y_n)\}$ converges strongly to $(x^\star, y^\star) \in \Omega$.

As a corollary of our main result we obtain the following strong convergence theorem for solving the split equality common fixed point problem for $\lambda-$ hybrid mappings.

Theorem 4.2. *Let $\mathcal{H}_1, \mathcal{H}_2$ and $\mathcal{H}_3$, be real Hilbert spaces, $\mathcal{A} : \mathcal{H}_1 \to \mathcal{H}_3$ and $\mathcal{B} : \mathcal{H}_2 \to \mathcal{H}_3$ be bounded linear operators. Let for for $i = 1, 2, ..., m$, $T_i : \mathcal{H}_1 \to \mathcal{H}_1$ be a finite family of $\lambda-$ hybrid mappings and $S_i : \mathcal{H}_2 \to \mathcal{H}_2$ be a finite family of $\varsigma-$ hybrid mappings. Suppose $\Omega = \{x \in \bigcap_{i=1}^m Fix(T_i), \quad y \in \bigcap_{i=1}^m Fix(S_i) : \mathcal{A}x = \mathcal{B}y\} \neq \emptyset$. Let $\{x_n\}$ and $\{y_n\}$ be sequences generated by $x_0, \vartheta \in \mathcal{H}_1, \quad y_0, \zeta \in \mathcal{H}_2$ and by*

$$
\begin{cases} z_n = x_n - \gamma_n \mathcal{A}^*(\mathcal{A}x_n - \mathcal{B}y_n), \\ x_{n+1} = \alpha_n \vartheta + \beta_n z_n + \sum_{i=1}^m \delta_{n,i} T_i z_n, \\ w_n = y_n + \gamma_n \mathcal{B}^*(\mathcal{A}x_n - \mathcal{B}y_n), \\ y_{n+1} = \alpha_n \zeta + \beta_n w_n + \sum_{i=1}^m \delta_{n,i} S_i w_n \qquad \forall n \geq 0, \end{cases} \tag{35}
$$

where the step-size γ_n is chosen in such a way that

$$
\gamma_n \in \left(\epsilon, \frac{2\|\mathcal{A}x_n - \mathcal{B}y_n\|^2}{\|\mathcal{B}^*(\mathcal{A}x_n - \mathcal{B}y_n)\|^2 + \|\mathcal{A}^*(\mathcal{A}x_n - \mathcal{B}y_n)\|^2} - \epsilon \right), \quad n \in \Pi
$$

otherwise $\gamma_n = \gamma$ (γ being any nonnegative value), where the index set $\Pi = \{n : \mathcal{A}x_n - \mathcal{B}y_n \neq 0\}$. Let the sequences $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\delta_{n,i}\}$, satisfy the following conditions:

(i) $\alpha_n + \beta_n + \sum_{i=1}^m \delta_{n,i} = 1$, *and* $\liminf_n \beta_n \delta_{n,i} > 0$, *for each* $i \in \{1, 2, ..., m\}$,

(iii) $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=0}^\infty \alpha_n = \infty$.

Then, the sequences $\{(x_n, y_n)\}$ converges strongly to $(x^\star, y^\star) \in \Omega$.

As another corollary we obtain the following result for split equality variational inequality problem for monotone and Lipschitz continuous operators.

Theorem 4.3. *Let $\mathcal{H}_1, \mathcal{H}_2$ and $\mathcal{H}_3$, be real Hilbert spaces, $\mathcal{A} : \mathcal{H}_1 \to \mathcal{H}_3$ and $\mathcal{B} : \mathcal{H}_2 \to \mathcal{H}_3$ be bounded linear operators and let C and Q, be two nonempty closed convex subsets of $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively. Let, $F : \mathcal{H}_1 \to \mathcal{H}_1$ be a monotone and L- Lipschitz continuous operator on C and $G : \mathcal{H}_2 \to \mathcal{H}_2$ be a monotone and K- Lipschitz continuous operator on Q. Suppose $\Omega = \{x \in VI(C, F), \quad y \in VI(Q, G) : \mathcal{A}x = \mathcal{B}y\} \neq \emptyset$. Let $\{x_n\}$ and $\{y_n\}$ be sequences* *generated by* $x_0, \vartheta \in \mathcal{H}_1, \quad y_0, \zeta \in \mathcal{H}_2$ *and by*

$$\begin{cases} z_n = x_n - \gamma_n \mathcal{A}^*(\mathcal{A}x_n - \mathcal{B}y_n), \\ u_n = P_C(z_n - \lambda_n F(z_n)), \\ v_n = P_C(z_n - \lambda_n F(u_n)), \\ x_{n+1} = \alpha_n \vartheta + (1-\alpha_n)v_n \\ w_n = y_n + \gamma_n \mathcal{B}^*(\mathcal{A}x_n - \mathcal{B}y_n), \\ s_n = P_Q(w_n - \eta_n G(w_n)), \\ t_n = P_Q(w_n - \eta_n G(s_n)), \\ y_{n+1} = \alpha_n \zeta + (1-\alpha_n)t_n \qquad \forall n \geq 0, \end{cases} \tag{36}$$

where the step-size γ_n *is chosen in such a way that*

$$\gamma_n \in \left(\epsilon, \frac{2\|\mathcal{A}x_n - \mathcal{B}y_n\|^2}{\|\mathcal{B}^*(\mathcal{A}x_n - \mathcal{B}y_n)\|^2 + \|\mathcal{A}^*(\mathcal{A}x_n - \mathcal{B}y_n)\|^2} - \epsilon \right), \quad n \in \Pi$$

otherwise $\gamma_n = \gamma$ *(*γ *being any nonnegative value), where the index set* $\Pi = \{n : \mathcal{A}x_n - \mathcal{B}y_n \neq 0\}$*. Let the sequences* $\{\alpha_n\}$*,* $\{\lambda_n\}$ *and* $\{\eta_n\}$ *satisfy the following conditions:*

(i) $\alpha_n \in (0,1)$, $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=0}^{\infty} \alpha_n = \infty$.

(ii) $\lambda_n \subset [a,b] \subset (0, \frac{1}{L})$ *and* $\eta_n \subset [c,d] \subset (0, \frac{1}{K})$,

Then, the sequences $\{(x_n, y_n)\}$ *converges strongly to* $(x^\star, y^\star) \in \Omega$.

From our main result, we can easily obtain the following result.

Theorem 4.4. *Let* C *be a nonempty closed convex subset of a Hilbert space* $\mathcal{H}$*. Let for* $i = 1, 2, ..., m$, $T_i : \mathcal{H} \to \mathcal{H}$ *be a finite family of* $\lambda-$ *hybrid mappings and* $F : \mathcal{H} \to \mathcal{H}$ *be a monotone and* L*- Lipschitz continuous operator on* C*. Suppose* $\Omega = \bigcap_{i=1}^m Fix(T_i) \bigcap VI(C,F) \neq \emptyset$*. Let* $\{x_n\}$ *be sequence generated by* $x_0, \vartheta \in \mathcal{H}$ *and by*

$$\begin{cases} u_n = P_C(x_n - \lambda_n F(x_n)), \\ v_n = P_C(x_n - \lambda_n F(u_n)), \\ x_{n+1} = \alpha_n \vartheta + \beta_n v_n + \sum_{i=1}^m \delta_{n,i} T_i v_n, \qquad \forall n \geq 0. \end{cases} \tag{37}$$

Let the sequences $\{\alpha_n\}$*,* $\{\beta_n\}$*,*$\{\delta_{n,i}\}$*and* $\{\lambda_n\}$ *satisfy the following conditions:*

(i) $\alpha_n + \beta_n + \sum_{i=1}^m \delta_{n,i} = 1$*, and* $\liminf_n \beta_n \delta_{n,i} > 0$*, for each* $i \in \{1, 2, ..., m\}$,

(ii) $\lambda_n \subset [a,b] \subset (0, \frac{1}{L})$,

(iii) $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=0}^{\infty} \alpha_n = \infty$.

Then, the sequences $\{x_n\}$ *converges strongly to* $x^\star \in \Omega$ *.*

4.1 - *Numerical Example*

Let $\mathcal{H}_1 = \mathcal{H}_2 = \mathcal{H}_3 = \mathbb{R}$, $C = [0,1]$ and $Q = [0,3]$. We define the operators $\mathcal{A}x = 2x$, $\mathcal{B}x = 3x$, $Fx = \frac{3}{2}x$ and $Gx = \frac{5}{4}x$. It is easy to see that $\mathcal{A}$ and $\mathcal{B}$ are bounded linear operators. We observe that F and G are monotone and Lipschitz continuous operators. We have $\mathcal{A}^\star x = 2x$ and $\mathcal{B}^\star x = 3x$. We also define the nonexpansive mappings $Tx = \frac{x}{2}$ and $Sx = x$. Put $\lambda_n = \frac{1}{3}$, $\eta_n = \frac{2}{5}$, $\gamma_n = \frac{1}{8}$, $\vartheta = \zeta = \frac{1}{3}$, $\alpha_n = \frac{1}{n+1}$, $\beta_n = \frac{n}{2n+2}$ and $\delta_n = \frac{n}{2n+2}$. Then these sequences satisfy the conditions of Theorem 3.1. Now we have the following algorithm

$$\text{(38)} \qquad \begin{cases} z_n = x_n - \gamma_n \mathcal{A}^*(\mathcal{A}x_n - \mathcal{B}y_n) = \frac{1}{2}x_n + \frac{3}{4}y_n, \\ u_n = P_C(z_n - \lambda_n F(z_n)) = P_C\left(\frac{z_n}{2}\right), \\ v_n = P_C(z_n - \lambda_n F(u_n)) = P_C\left(z_n - \frac{u_n}{2}\right), \\ x_{n+1} = \alpha_n \vartheta + \beta_n v_n + \delta_n T v_n = \frac{1}{3n+3} + \frac{3n}{4n+4}v_n, \\ w_n = y_n + \gamma_n \mathcal{B}^*(\mathcal{A}x_n - \mathcal{B}y_n) = \frac{3}{4}x_n - \frac{1}{8}y_n, \\ s_n = P_Q(w_n - \eta_n G(w_n)) = P_Q\left(\frac{w-n}{2}\right), \\ t_n = P_Q(w_n - \eta_n G(s_n)) = P_Q\left(w_n - \frac{s_n}{2}\right), \\ y_{n+1} = \alpha_n \zeta + \beta_n t_n + \delta_n S t_n = \frac{1}{3n+3} + \frac{n}{n+1}t_n, \qquad \forall n \geq 0, \end{cases}$$

Taking $(x_0, y_0) = \left(\frac{1}{2}, \frac{1}{2}\right)$, we obtain the following algorithm:

$$\text{(39)} \qquad \begin{cases} x_{n+1} = \frac{1}{3n+3} + \frac{9n}{32n+32}x_n + \frac{27n}{64n+64}y_n, \\ y_{n+1} = \frac{1}{3n+3} + \frac{9n}{16n+16}x_n - \frac{3n}{32n+32}y_n \qquad \forall n \geq 0, \end{cases}$$

We observe that, $\{(x_n, y_n)\}$ is convergent to $(0, 0)$. We note that $\Omega = \{(0, 0)\}$.

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