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A note on multiple zeta values in Tate algebras

Abstract. In this note, we discuss a generalization of Thakur's multiple zeta values and allied objects, in the framework of function fields of positive characteristic and more precisely, of periods in Tate algebras.

Keywords. Multiple zeta values, Carlitz module, A-harmonic sums.

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1 - Introduction

Let $A = \mathbb{F}_q[\theta]$ be the ring of polynomials in an indeterminate θ with coefficients in $\mathbb{F}_q$ the finite field with q elements and characteristic p, let K be the fraction field of A and K_∞ the completion of K at the infinity place ∞. For $d \geq 0$ an integer, we denote by $A^+(d)$ the set of monic polynomials of A of degree d. Carlitz studied, in [**10**], the so-called *Carlitz zeta values*:

$$\zeta_A(n) := \sum_{a \in A^+} a^{-n} \in K_\infty, \quad n \geq 1.$$

It is likely that the formal analogy of these objects with the *classical zeta values*

$$\zeta(n) := \sum_{i \geq 1} i^{-n} \in \mathbb{R}$$

with n integer (convergence occurs only if $n \geq 2$) was the main motivation for his study (so that, in some way, "$\zeta = \zeta_{\mathbb{Z}}$"). In a more modern approach, we can say that Carlitz suggested, with his first pioneering papers, to develop an arithmetic theory

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of *periods* over the ring $\mathbb{F}_q[\theta]$ in parallel with the study of the arithmetic theory of periods over $\mathbb{Z}$.

In all the following, if R is a ring, $R^\times$ denotes the group of the multiplicative invertible elements of R. It was proved by Carlitz in [**10**] that, if $n \equiv 0 \pmod{q-1}$,

$$\zeta_A(n) \in K^\times \widetilde{\pi}^n, \tag{1}$$

where $\widetilde{\pi}$ is the value in $\mathbb{C}_\infty = \widehat{K_\infty^{ac}}$ [1] of a convergent infinite product

$$\widetilde{\pi} := -(-\theta)^{\frac{q}{q-1}} \prod_{i=1}^{\infty} (1-\theta^{1-q^i})^{-1} \in (-\theta)^{\frac{1}{q-1}} K_\infty, \tag{2}$$

uniquely defined up to the multiplication by an element of $\mathbb{F}_q^\times = \mathbb{F}_q \setminus \{0\}$ (corresponding to the choice of a root $(-\theta)^{\frac{1}{q-1}}$). It has been proved in a variety of ways (see [**20**] to see the most relevant ones) that $\widetilde{\pi}$ is moreover transcendental over K.

The element $\widetilde{\pi}$ is a fundamental period of the *Carlitz exponential* $\exp_C$ (Goss, [**17**, §3.2]), that is, the unique surjective, entire, $\mathbb{F}_q$-linear function

$$\exp_C : \mathbb{C}_\infty \to \mathbb{C}_\infty$$

of kernel $\widetilde{\pi}\mathbb{F}_q[\theta]$ such that its first derivative satisfies $\exp_C' = 1$ (note that, since we are in a characteristic $p > 0$ environment, a function with constant derivative is not necessarily $\mathbb{C}_\infty$-linear).

In his book [**25**, §5.10], Thakur also consider several variants of classical multiple zeta values in the context of the Carlitzian arithmetic over the ring A. We mention here what we think is the most relevant. For $n_1, \ldots, n_r \in \mathbb{Z}_{\geq 1}$, Thakur defines, as one of the analogues of the classical multiple zeta values in the Carlitzian setting:

$$\zeta_A(n_1, \ldots, n_r) = \sum_{\substack{a_i \in A^+ \\ |a_1| > \cdots > |a_r|}} \frac{1}{a_1^{n_1} \cdots a_r^{n_r}} \in K_\infty. \tag{3}$$

Here, for $x \in \mathbb{C}_\infty^\times$, we write $|x| = q^{-v_\infty(x)}$ where v_∞ is the valuation of $\mathbb{C}_\infty$ (so that $v_\infty(\theta) = -1$) and we define $|0| := 0$. If $r = 0$ we further set the corresponding Thakur multiple zeta value $\zeta_A(\emptyset)$ to be equal to 1.

Classically, one of the reasons we could get interested in multiple zeta values is the need of "enveloping" zeta values in the "simplest" $\mathbb{Q}$-algebra possible. From Euler, it is well known that the zeta values $\zeta(2), \zeta(4), \ldots$ all belong to the $\mathbb{Q}$-algebra $\mathbb{Q}[\zeta(2)]$. In general, the other zeta values are not expected to belong to this algebra. However, they belong to the $\mathbb{Q}$-algebra $Z_{\mathbb{R}} \subset \mathbb{R}$ generated by the multiple zeta

[1] This is the completion of an algebraic closure of K_∞.

values. It is known that the product of two multiple zeta values is a $\mathbb{Q}$-linear combination of multiple zeta values, and this algebra also has a more natural structure.

We expect that $Z_{\mathbb{R}}$ is isomorphic to the algebra $\mathbb{Q}\langle f_3, f_5, \ldots\rangle_{\text{Ш}} \otimes_{\mathbb{Q}} \mathbb{Q}[\zeta(2)]$, where $\mathbb{Q}\langle f_3, f_5, \ldots\rangle_{\text{Ш}}$ is the $\mathbb{Q}$-algebra generated by the non-commutative words in the alphabet with letters $f_3, f_5 \ldots$ with, as a product, the shuffle product Ш (see Brown's [**9**]). A folklore conjecture comes in support of this guess; the number π and the zeta values $\zeta(3), \zeta(5), \ldots$ are expected to be algebraically independent over $\mathbb{Q}$. Multiple zeta values are thus expected to provide a natural basis of this $\mathbb{Q}$-algebra. See also [**18**] for the definition of a $\mathbb{Q}$-algebra of *finite multi-zeta values* which could offer a nice realization of the algebra $\mathbb{Q}\langle f_3, f_5, \ldots\rangle_{\text{Ш}}$.

Similarly, in the Carlitzian setting we note that, after (1), the values $\zeta_A(n)$ with $n > 0$ divisible by $q-1$ are all contained in the K-algebra $K[\zeta_A(q-1)]$, which is isomorphic to $K[X]$ for an indeterminate X. However, the remaining Carlitz zeta values $\zeta_A(1), \ldots$ do not belong to this algebra (if $q > 2$). Indeed, Chang and Yu proved in [**14**] that $\widetilde{\pi}$ and the Carlitz zeta values $\zeta_A(n)$ with $n \geq 1$, $q-1 \nmid n$ and $p \nmid n$ with p the prime number dividing q are algebraically independent (we recall that these authors, in ibid., use the powerful algebraic independence methods introduced by Papanikolas in [**19**]); see also [**11, 12, 13**].

Just as for the algebra $Z_{\mathbb{R}}$, Thakur proved in [**27**] that the product of two multiple zeta values as in (3) is a linear combination (this time with coefficients in $\mathbb{F}_p$) of such multiple zeta values. Thakur also mentioned to the author of the present note that G. Todd's numerical computations have led to a good understanding of (conjectural) relations among Thakur's multiple zeta values; the relations are universal in a sense that is described in ibid. Compared to the classical setting, the difficulty here is to handle the product of the so-called *power sums* (see later). We denote by Z_{K_∞} the K-sub-algebra of K_∞ generated by the multiple zeta values (3); even conjecturally, in spite of the striking results of algebraic independence mentioned above, we know very little about the structure of this algebra. In particular, we presently do not know what could be the analogue structure which could play the role of the algebra $\mathbb{Q}\langle f_3, f_5, \ldots\rangle_{\text{Ш}}$ in this setting.

In this note, we shall discuss of a generalization of the Thakur multiple zeta values which, so far, has no counterpart in the classical setting. For this purpose, we note that A is an algebra over $\mathbb{F}_q$ ($\mathbb{Z}$ is not an algebra over a field). Therefore, a series of advantages occurs in the Carlitzian framework, notably the possibility to use the tensor product over $\mathbb{F}_q$. We consider variables $t_1, \ldots, t_s$ over K and we write $\underline{t}_s$ for the family of variables $(t_1, \ldots, t_s)$. We denote by $\mathbf{F}_s$ the field $\mathbb{F}_q(\underline{t}_s)$, so that $\mathbf{F}_0 = \mathbb{F}_q$.

In all the following, if R is a ring, we denote by R^* the underlying multiplicative monoid (inclusive of the element 0). Note that A^+ is a multiplicative sub-monoid of A^*. We denote by $\mathbb{F}_q^{ac}$ the algebraic closure of $\mathbb{F}_q$ in $\mathbb{C}_\infty$.

Definition 1. A monoid homomorphism $\sigma : A^+ \to \mathbb{F}_q^{ac}(\underline{t})^*$ is called a *semi-character*. The *trivial semi-character* is the map $\mathbf{1} : A^+ \to \{1\}$. Let σ be a semi-character. We say that it is *of Dirichlet type* if there exist $\mathbb{F}_q$-algebra homomorphisms

$$\rho_i : A \to \mathbb{F}_q^{ac}(\underline{t}), \quad i = 1, \ldots, s,$$

such that $\sigma(a) = \rho_1(a) \cdots \rho_s(a)$ for all $a \in A^+$. The integer s is called the *length*. By convention, the semi-character $\mathbf{1}$ is the unique semi-character of Dirichlet type of length 0.

For example, setting $t = t_1$, the map $\chi_t : A^+ \to \mathbb{F}_q[t]^* \subset \mathbb{F}_q^{ac}(t)^*$ defined by $\chi_t(a) = a(t)$ ([2]) is a semi-character of Dirichlet type. Let ζ be an element of $\mathbb{F}_q^{ac}$. The map $a \in A^+ \mapsto \chi_\zeta(a) = a(\zeta) \in \mathbb{F}_q^{ac}$ is also a semi-character of Dirichlet type, and the same can be said if we now pick elements $\zeta_1, \ldots, \zeta_s \in \mathbb{F}_q^{ac}$ and consider the map $\chi_{\underline{\zeta}} : a \mapsto \chi_{\zeta_1}(a) \cdots \chi_{\zeta_s}(a)$ (this is more commonly called a “Dirichlet character”). The map $a \in A^+ \mapsto \mathbb{F}_q[t]^*$ which sends a to $t^{\deg_\theta(a)}$ is a semi-character, but it can be proved that it is not of Dirichlet type.

Definition 2. Let $\sigma : A^+ \to \mathbb{F}_q^{ac}(\underline{t})^*$ be a semi-character. The associated *twisted power sum* of order k and degree d is the sum:

$$S_d(k;\sigma) = \sum_{a \in A^+(d)} a^{-k}\sigma(a) \in \mathbb{F}_q^{ac}(\theta)(\underline{t}_s).$$

More generally, let $\sigma_1, \ldots, \sigma_r$ be semi-characters, let $n_1, \ldots, n_r$ be integers, and d a non-negative integer. The associated *multiple twisted power sum* of degree d is the sum

$$S_d\begin{pmatrix} \sigma_1 & \sigma_2 & \cdots & \sigma_r \\ n_1 & n_2 & \cdots & n_r \end{pmatrix} = S_d(n_1;\sigma_1) \sum_{d > i_2 > \cdots > i_r \geq 0} S_{i_2}(n_2;\sigma_2) \cdots S_{i_r}(n_r;\sigma_r) \in \mathbb{F}_q^{ac}(\theta)(\underline{t}_s).$$

The integer $\sum_i n_i$ is called the *weight* and the integer r is called its *depth*.

We can write in both ways $S_d(n;\sigma) = S_d\binom{\sigma}{n}$. Observe also that

$$S_d\begin{pmatrix} \mathbf{1} & \mathbf{1} & \cdots & \mathbf{1} \\ n_1 & n_2 & \cdots & n_r \end{pmatrix} = S_d(n_1, n_2, \ldots, n_r) \in K,$$

([2]) The “evaluation at $\theta = t$”, in other words, the map which sends a polynomial $a = a(\theta) = a_0 + a_1\theta + \cdots + a_r\theta^r$ with the coefficients $a_0, \ldots, a_r \in \mathbb{F}_q$ to the polynomial $a(t) = a_0 + a_1 t + \cdots + a_r t^r \in \mathbb{F}_q[\theta]$.

in the notations of Thakur, [**27**, §1.2]. We hope that all these slightly different notations will not bother the reader.

Definition 3. With $n_1,\ldots,n_r\geq 1$ and semi-characters $\sigma_1,\ldots,\sigma_r$ as above, we introduce the associated *multiple zeta value*

$$\zeta_A\begin{pmatrix}\sigma_1 & \sigma_2 & \cdots & \sigma_r\\ n_1 & n_2 & \cdots & n_r\end{pmatrix}:=\sum_{d\geq 0}S_d\begin{pmatrix}\sigma_1 & \sigma_2 & \cdots & \sigma_r\\ n_1 & n_2 & \cdots & n_r\end{pmatrix}\in\mathbb{F}_q^{ac}(\underline{t})((\theta^{-1})).$$

The sum thus converges with respect to the unique valuation extending the ∞-adic valuation of K_∞ and inducing the trivial valuation over $\mathbb{F}_q^{ac}(\underline{t})$. Explicitly, we have:

$$\zeta_A\begin{pmatrix}\sigma_1 & \sigma_2 & \cdots & \sigma_r\\ n_1 & n_2 & \cdots & n_r\end{pmatrix}=\sum_{d\geq 0}\sum_{\substack{a_1,\ldots,a_r\in A^+\\ d=\deg_\theta(a_1)>\cdots>\deg_\theta(a_r)\geq 0}}\frac{\sigma_1(a_1)\cdots\sigma_r(a_r)}{a_1^{n_1}\cdots a_r^{n_r}}.$$

We will say that this is the multiple zeta value associated to the *composition array*

$$\begin{bmatrix}\sigma_1 & \sigma_2 & \cdots & \sigma_r\\ n_1 & n_2 & \cdots & n_r\end{bmatrix}.$$

The integer $\sum_i n_i$ is called the *weight* of the above multiple zeta value and the integer r is called its *depth*. If all the semi-characters $\sigma_1,\ldots,\sigma_r$ are of Dirichlet type, then, for all $1\leq i\leq r$, $\sigma_i=\rho_{i,1}\cdots\rho_{i,n_i}$ for ring homomorphisms $\rho_{i,j}$. Then, we say that the multiple zeta value associated to the above matrix data is *of Dirichlet type*, and the cardinality of the set $\{\rho_{i,j};\, i,j\}$ is called its *length*.

The referee has pointed out that these multiple zeta values can be seen as some kind of analogues, for functional analysis in Tate algebras, of the Dirichlet multiple L-functions as in the paper of Akiyama and Ishikawa [**1**].

Again note that if $\sigma_1=\cdots=\sigma_r=\mathbf{1}$, then, we can write

$$\zeta_A\begin{pmatrix}\sigma_1 & \sigma_2 & \cdots & \sigma_r\\ n_1 & n_2 & \cdots & n_r\end{pmatrix}=\zeta_A(n_1,\ldots,n_r)\in K_\infty$$

with $\zeta_A(n_1,\ldots,n_r)$ as in (3) (of Dirichlet type, depth r and length 0). Further, let us assume that $r=1, n=n_1>0$ and that $\sigma=\chi_{t_1}\cdots\chi_{t_s}$. Then, we have that

$$\zeta_A(n;\sigma)=\zeta_A\binom{\sigma}{n}=\sum_{a\in A^+}\frac{a(t_1)\cdots a(t_s)}{a^n}=\prod_P\left(1-\frac{P(t_1)\cdots P(t_s)}{P^n}\right)^{-1}\in\mathbb{T}_s^\times.$$

These series have been introduced in [**21**] and extensively studied in e.g. [**5, 6, 7**]. The product runs over the irreducible polynomials P of A^+ and the convergence holds in the standard *s-dimensional Tate algebra*, which can be identified with the $\mathbb{C}_\infty$-algebra of the rigid analytic functions $B(0,1)^s\to\mathbb{C}_\infty$, where $B(0,1)=$

$\{z \in \mathbb{C}_\infty; |z| \leq 1\}$. In fact, these functions extend to entire functions $\mathbb{C}_\infty^s \to \mathbb{C}_\infty$ (see [**5**, Corollary 8]). More generally, we assume that for all $i = 1, \ldots, r$, there exists a subset $I_i \subset \{1, \ldots, s\}$ such that

$$\sigma_i = \prod_{j \in I_i} \chi_{t_j}.$$

In particular, if $I_i = \emptyset$, we set $\sigma_i = \mathbf{1}$. In this setting, the multiple zeta value

$$\zeta_A \begin{pmatrix} \sigma_1 & \sigma_2 & \cdots & \sigma_r \\ n_1 & n_2 & \cdots & n_r \end{pmatrix} \tag{4}$$

belongs to $\mathbb{T}_s$.

Proposition 4. *With the above assumption over the semi-characters* $\sigma_1, \ldots, \sigma_r$, *the multiple zeta value* (4)*, hence of Dirichlet type and of length* $\leq s$, *extends to an entire function* $\mathbb{C}_\infty^s \to \mathbb{C}_\infty$.

Proof. Note that there exists a non-negative integer κ, depending of $\sigma_1, \ldots, \sigma_r$, such that if we set

$$\Sigma_d := S_d \begin{pmatrix} \sigma_1 & \sigma_2 & \cdots & \sigma_r \\ n_1 & n_2 & \cdots & n_r \end{pmatrix} \in K[t_1, \ldots, t_s],$$

then the degree in t_i of Σ_d is $\leq \kappa d$ for all $i = 1, \ldots, s$.

If $s > 0$ and $\sigma = \chi_{t_1} \cdots \chi_{t_s}$ then, by [**5**, Lemma 7], we have the following estimate for the Gauss norm $\|S_d(n; \sigma)\|$ of $S_d(n; \sigma)$ associated with our valuation of $\mathbb{F}_q^{ac}(\underline{t})((\theta^{-1}))$ such that $\|\theta\| = q$:

$$\|S_d(n; \sigma)\| \leq q^{-dn} q^{-q^{[d - \frac{s}{q-1}] - 1}}, \quad d > \frac{s-1}{q-1},$$

where $[\,\cdot\,]$ denotes the lower integer part (replace $x = \theta^n$ and $y = -n \in \mathbb{Z}_p$ with $p \mid q$ in the statement of that Lemma). This implies that, if σ_1 is not the trivial semi-character,

$$\|\Sigma_d\| \leq \|S_d(n; \sigma_1)\| \leq q^{-dn_1} q^{-q^{[d - \frac{\mu}{q-1}] - 1}}, \quad d \gg 0,$$

for some constant μ, which immediately implies the proposition. If σ_1 is the trivial semi-character we cannot directly apply the statement of the above lemma but the problem is superficial, as we can more easily conclude by using [**17**, Proposition 8.8.2] as, in this case:

$$\|\Sigma_d\| \leq \|S_d(n_1; \mathbf{1})\| \leq q^{-(q-1)\frac{d(d+1)}{2}}.$$

□

Remark 5. A more general result of this type can now be found in [4].

We set $\mathbf{F}_s := \mathbb{F}_q(\underline{t}_s)$. It is presently a work in progress of the author to show that the product of two multiple zeta values as in Definition 3 is a linear combination, with coefficients in the field $K \otimes_{\mathbb{F}_q} \mathbb{F}_q^{ac} \otimes_{\mathbb{F}_q} \mathbf{F}_s$, of such multiple zeta values (with the various matrices of associated data not including, necessarily, the same semi-characters). We hope this will allow us to exhibit new multiple zeta values algebras $Z_{\widehat{\mathbb{F}_q^{ac}((\theta^{-1}))\otimes_{\mathbb{F}_q}\mathbf{F}_s}}$ containing the algebra Z_{K_∞} and collecting the algebraic relations of Z_{K_∞} in families by specialization.

2 - Content of the present note

Waiting for more general results, in this note we will accomplish a more modest objective, as we will only give a few explicit examples of shuffle products of such multiple zeta values in the following case: $s = 2$, weight ≤ 2, and the semi-characters $\mathbf{1}, \sigma, \psi$ and $\sigma\psi$ of Dirichlet type, where

$$\sigma : a \mapsto a(t_1) \in \mathbb{F}_q[t_1, t_2], \quad \psi : a \mapsto a(t_2) \in \mathbb{F}_q[t_1, t_2], \quad (\sigma\psi)(a) = \sigma(a)\psi(a) = a(t_1)a(t_2).$$

As an advantage of our explicit and restrictive viewpoint, we will see beautiful formulas dropping out from this new theory that we will apply to some new properties of the so-called "Bernoulli-Goss" polynomials.

The matrix data we are going to handle are:

Four in weight 1

$$\begin{bmatrix} \mathbf{1} \\ 1 \end{bmatrix}, \begin{bmatrix} \sigma \\ 1 \end{bmatrix}, \begin{bmatrix} \psi \\ 1 \end{bmatrix}, \begin{bmatrix} \sigma\psi \\ 1 \end{bmatrix}.$$

Four in weight 2 depth 1

$$\begin{bmatrix} \mathbf{1} \\ 2 \end{bmatrix}, \begin{bmatrix} \sigma \\ 2 \end{bmatrix}, \begin{bmatrix} \psi \\ 2 \end{bmatrix}, \begin{bmatrix} \sigma\psi \\ 2 \end{bmatrix}.$$

Nine in weight 2 depth 2

$$\begin{bmatrix} \mathbf{1} & \mathbf{1} \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} \sigma & \mathbf{1} \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} \mathbf{1} & \sigma \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} \psi & \mathbf{1} \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} \mathbf{1} & \psi \\ 1 & 1 \end{bmatrix},$$

$$\begin{bmatrix} \sigma\psi & \mathbf{1} \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} \sigma & \psi \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} \psi & \sigma \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} \mathbf{1} & \sigma\psi \\ 1 & 1 \end{bmatrix}.$$

We shall show the following Theorem, which provides, taking into account the above tables, a complete picture of all the products of two weight one multiple zeta values in the restrictive context we have prefixed (in two variables t_1, t_2, and with the semi-characters $\mathbf{1}, \sigma, \psi$ and $\sigma\psi$), unveiling partly an extremely complex and mysterious algebra structure. From now on, we suppose that $q > 2$. All the arguments presented below under this restriction can be also developed in the case $q = 2$ with appropriate modifications, but we refrain from giving full details here.

Theorem 6. *The following formulas hold.*

(1) $\zeta_A(1)^2 = \zeta_A(2) + 2\zeta_A(1,1)$,

(2) $\zeta_A(1;\sigma)\zeta_A(1) = \zeta_A(2;\sigma) + \zeta_A\begin{pmatrix}\sigma & \mathbf{1}\\ 1 & 1\end{pmatrix}$,

(3) $\zeta_A(1;\psi)\zeta_A(1) = \zeta_A(2;\psi) + \zeta_A\begin{pmatrix}\psi & \mathbf{1}\\ 1 & 1\end{pmatrix}$,

(4) $\zeta_A(1;\sigma)\zeta_A(1;\psi) = \zeta_A(2;\sigma\psi)$,

(5) $\zeta_A(1;\sigma\psi)\zeta_A(1) = \zeta_A(2;\sigma\psi) - \zeta_A\begin{pmatrix}\sigma & \psi\\ 1 & 1\end{pmatrix} - \zeta_A\begin{pmatrix}\psi & \sigma\\ 1 & 1\end{pmatrix} + \zeta_A\begin{pmatrix}\mathbf{1} & \sigma\psi\\ 1 & 1\end{pmatrix} +$ $\zeta_A\begin{pmatrix}\sigma\psi & \mathbf{1}\\ 1 & 1\end{pmatrix}$.

We observe that the formula (1) can be found in Thakur's [**25**, Theorem 5.10.13]. Further, due to the symmetry of the roles of t_1 and t_2, the formulas (2) and (3) are equivalent. Moreover, the formula (4) is in fact well known (see Perkins' [**23**] for more general formulas of this type). However, we will give a proof of this in the spirit of multiple zeta values. The formulas (2) is, on the other side, new, as far as we can see.

3 - Twisted power sums

We need a few tools in order to obtain our formulas; more precisely, we have to improve our skill in computing twisted powers sums. For this purpose, we are going to use the tools introduced in the recent preprint of Perkins and the author [**22**]; we are going to use for a while the notations of this reference. Let s be an integer ≥ 0. We set, for an integer $d \geq 0$:

$$S_d(n;s) = \sum_{a\in A^+(d)} \frac{a(t_1)\cdots a(t_s)}{a^n} \in K[\underline{t}_s].$$

We are thus considering a special case of Definition 2. We also set, for $d \geq 0$,

$F_0(n;s) = 0$ and

$$F_d(n;s) = \sum_{i=0}^{d-1} S_d(n;s) \in K[\underline{t}_s],$$

so that

$$\lim_{d\to\infty} F_d(n;s) = \zeta_A(n;s) := \prod_P \left(1 - \frac{P(t_1)\cdots P(t_s)}{P^n}\right)^{-1} \in \mathbb{T}_s^\times,$$

where the product runs over the irreducible polynomials of A^+ (in general, all along this note, empty sums are by convention equal to zero). We are using the notation of [**22**]. In particular, if σ is the semi-character $\chi_{t_1}\cdots\chi_{t_s}$ (of Dirichlet type), then, the comparison between the notations of ibid. and those of the present note are:

$$S_d(n;s) = S_d(n;\sigma), \quad \zeta_A(n;s) = \zeta_A(n;\sigma).$$

It is easy to show that, if $0 \leq s' < s$, $S_d(n;s')$ is the coefficient of $(t_{s'+1}\cdots t_s)^d$ in $F_{d+1}(n;s)$. We define inductively $l_0 = 1$ and $l_i = (\theta - \theta^{q^i})l_{i-1}$, and we set $l_{-n} = 0$ for $n > 0$. We denote by $b_i(Y)$ the product $(Y-\theta)\cdots(Y-\theta^{q^{i-1}}) \in A[Y]$ (for an indeterminate Y) if $i > 0$ and we set $b_0(Y) = 1$. We also write $m = \lfloor \frac{s-1}{q-1} \rfloor$ (the brackets denote the integer part so that m is the biggest integer $\leq \frac{s-1}{q-1}$). We set

$$\Pi_{s,d} = \frac{b_{d-m}(t_1)\cdots b_{d-m}(t_s)}{l_{d-1}} \in K[\underline{t}_s], \quad d \geq \max\{1, m\}.$$

Now, we quote [**22**, Theorem 1]:

Theorem 7. *For all integers $s \geq 1$, such that $s \equiv 1 \pmod{q-1}$, there exists a non-zero rational fraction $\mathbb{H}_s \in K(Y, \underline{t}_s)$ such that, for all $d \geq m$, the following identity holds:*

$$F_d(1;s) = \Pi_{s,d} \mathbb{H}_s|_{Y=\theta^{q^{d-m}}}.$$

If $s = 1$, we have the explicit formula

$$\mathbb{H}_1 = \frac{1}{t_1 - \theta}.$$

Further, if $s = 1 + m(q-1)$ for an integer $m > 0$, then the fraction $\mathbb{H}_s$ is a polynomial of $A[Y, \underline{t}_s]$ with the following properties:

(1) *For all i,* $\deg_{t_i}(\mathbb{H}_s) = m - 1$,

(2) $\deg_Y(\mathbb{H}_s) = \dfrac{q^m - 1}{q-1} - m.$

The polynomial $\mathbb{H}_s$ is uniquely determined by these properties.

We apply this Theorem to compute explicitly the twisted power sums associated to the data we have chosen. For this, it suffices to choose $s = q$. In this case $m = 1$ and the polynomial $\mathbb{H}_q$ of Theorem 7 has degree 0 in Y as well as in $t_1, \ldots, t_q$. From [**22**, §2.6] we deduce that $\mathbb{H}_q = 1$ (the same result is given as an example in Florent Demeslay's thesis [**15**]). In particular, to compute most of the twisted power sums associated to our data it suffices to analyze the polynomials

$$F_{d+1}(1;q) = \frac{b_d(t_1)\cdots b_d(t_q)}{l_d}. \tag{5}$$

The coefficients of $(t_3\cdots t_q)^d$, $(t_2\cdots t_q)^d$ and $(t_1\cdots t_q)^d$ in $F_{d+1}(1;q)$ are easily computed, and we get, for all $d \geq 0$ (note that these are well known formulas; see [**23**]):

$$S_d(1;0) = \frac{1}{l_d}, \tag{6}$$

$$S_d(1;1) = \frac{b_d(t_1)}{l_d}, \tag{7}$$

$$S_d(1;2) = \frac{b_d(t_1)b_d(t_2)}{l_d}. \tag{8}$$

To compute $S_d(2;0), S_d(2,1), S_d(2,2)$ ([3]) we observe that, replacing θ with θ^q in (5):

$$F_{d+1}(q;q) = \frac{b_{d+1}(t_1)\cdots b_{d+1}(t_q)}{l_d^q(t_1-\theta)\cdots(t_q-\theta)}.$$

Note that the former is a polynomial in $\underline{t}_s$, written as a reducible fraction. We get that

$$F_{d+1}(2;2) = \left.\frac{b_{d+1}(t_1)\cdots b_{d+1}(t_q)}{l_d^q(t_1-\theta)\cdots(t_q-\theta)}\right|_{t_3=\cdots=t_q=\theta} = \frac{b_{d+1}(t_1)b_{d+1}(t_2)}{l_d^2(t_1-\theta)(t_2-\theta)}.$$

Calculating the coefficients of t_2^d and $(t_1t_2)^d$, and subtracting $F_{d+1}(2;2) - F_d(2;2)$, we easily obtain the formulas, valid for $d \geq 0$ (the first one is well known):

$$S_d(2;0) = \frac{1}{l_d^2},$$

$$S_d(2;1) = \frac{b_d(t_1)}{(t_1-\theta)l_d^2}(t_1 - \theta^{q^d}), \tag{9}$$

$$S_d(2;2) = \frac{b_d(t_1)b_d(t_2)}{(t_1-\theta)(t_2-\theta)l_d^2}(t_1t_2 - \theta^{q^d}(t_1+t_2) + 2\theta^{1+q^d} - \theta^2)$$

$$= \frac{b_d(t_1)b_d(t_2)}{(t_1-\theta)(t_2-\theta)l_d^2}[(t_1-\theta)(t_2-\theta) + (t_1-\theta)(\theta-\theta^{q^d}) + (t_2-\theta)(\theta-\theta^{q^d})]. \tag{10}$$

([3]) In the notations of [**22**]; in our note, we should write $S_d(2;\mathbf{1}), S_d(2;\sigma), S_d(2;\sigma\psi)$.

We will also need the next Lemma which is also well known, where $\tau : A[t] \to A[t]$ is the $\mathbb{F}_q[t]$-linear endomorphism such that $\tau(\theta) = \theta^q$.

Lemma 8. *We have the following formula, which holds in* $A[t]$.

$$\tau(b_d(t)) = l_d \sum_{i=0}^{d} \frac{b_i(t)}{l_i}, \quad d \geq 0.$$

Proof. We recall the proof for convenience of the reader. We proceed by induction over d. If $d = 0$, the formula is obvious. If $d > 0$, it suffices to show that

$$b_{d+1}(t) = (t - \theta) l_d \sum_{i=0}^{d} l_i^{-1} b_i(t).$$

Now, we compute easily, by using the induction hypothesis:

$$\begin{aligned} b_{d+2}(t) &= (t - \theta + \theta - \theta^{q^d}) b_{d+1} \\ &= (\theta - \theta^{q^d}) b_{d+1} + (t - \theta) b_{d+1} \\ &= (t - \theta) l_d (\theta - \theta^{q^d}) \left(\sum_{i=0}^{d} l_i^{-1} b_i(t) + l_{d+1}^{-1} b_{d+1}(t) \right) \\ &= (t - \theta) l_{d+1} \sum_{i=0}^{d+1} l_i^{-1} b_i(t). \end{aligned}$$

□

4 - Proof of Theorem 6

Proof. [Proof of Theorem 6, (2)]. We compute, by using (6) and (7):

$$S_d(1; \sigma) S_d(1; \mathbf{1}) = \frac{b_d(t_1)}{l_d^2}.$$

On the other hand, we have seen in (9) that

$$S_d(2; \sigma) = \frac{b_d(t_1)}{l_d^2} \frac{t_1 - \theta^{q^d}}{t_1 - \theta} = \frac{\tau(b_d(t_1))}{l_d^2},$$

where $\tau(b_d(t_1))$ has the obvious meaning. We compute (the third identity follows from Lemma 8):

$$
\begin{aligned}
S_d\begin{pmatrix}\mathbf{1} & \sigma\\ 1 & 1\end{pmatrix} &= S_d(1;\mathbf{1})\sum_{i=0}^{d-1}S_i(1;\sigma)\\
&= \frac{1}{l_d}\sum_{i=0}^{d-1} l_i^{-1}b_i(t_1)\\
&= \frac{\tau(b_{d-1}(t_1))}{l_dl_{d-1}}\\
&= \frac{b_d(t_1)}{l_d^2}\,\frac{\theta-\theta^{q^d}}{t_1-\theta}.
\end{aligned}
$$

Combining the above formulas, we see that

$$
S_d(1;\sigma)S_d(1;\mathbf{1}) = S_d(2;\sigma) - S_d\begin{pmatrix}\mathbf{1} & \sigma\\ 1 & 1\end{pmatrix}. \tag{11}
$$

With the obvious meaning of some new notations introduced below, we deduce:

$$
\begin{aligned}
F_d(1;\sigma)F_d(1;\mathbf{1}) &= \sum_{i=0}^{d-1}S_i(1;\sigma)\sum_{j=0}^{d-1}S_j(1;\mathbf{1})\\
&= F_d\begin{pmatrix}\mathbf{1} & \sigma\\ 1 & 1\end{pmatrix} + F_d\begin{pmatrix}\sigma & \mathbf{1}\\ 1 & 1\end{pmatrix} + \sum_{i=0}^{d-1}S_d(1;\sigma)S_d(1;\mathbf{1})\\
&= F_d\begin{pmatrix}\mathbf{1} & \sigma\\ 1 & 1\end{pmatrix} + F_d\begin{pmatrix}\sigma & \mathbf{1}\\ 1 & 1\end{pmatrix} + \sum_{i=0}^{d-1}\left(S_i(2;\sigma) - S_i\begin{pmatrix}\mathbf{1} & \sigma\\ 1 & 1\end{pmatrix}\right)\\
&= F_d\begin{pmatrix}\mathbf{1} & \sigma\\ 1 & 1\end{pmatrix} + F_d\begin{pmatrix}\sigma & \mathbf{1}\\ 1 & 1\end{pmatrix} - F_d\begin{pmatrix}\mathbf{1} & \sigma\\ 1 & 1\end{pmatrix} + F_d(2;\sigma)\\
&= F_d\begin{pmatrix}\sigma & \mathbf{1}\\ 1 & 1\end{pmatrix} + F_d(2;\sigma),
\end{aligned}
$$

where we have used (11) in the third equality. We rewrite the resulting formula:

$$
F_d(1;\sigma)F_d(1;\mathbf{1}) = F_d\begin{pmatrix}\sigma & \mathbf{1}\\ 1 & 1\end{pmatrix} + F_d(2;\sigma). \tag{12}
$$

Taking the limit $d\to\infty$ in (12) we obtain the required multiple zeta identity. □

The above also implies the formula (3) of Theorem 6 by interchanging t_1 and t_2. To continue, we notice the next Lemma.

Lemma 9. *We have that*

$$S_d(2;\sigma\psi) = \frac{b_d(t_1)b_d(t_s)}{(t_1-\theta)(t_2-\theta)l_d^2} + S_d\begin{pmatrix}\psi & \sigma\\ 1 & 1\end{pmatrix} + S_d\begin{pmatrix}\sigma & \psi\\ 1 & 1\end{pmatrix}.$$

Proof. We compute:

$$\begin{aligned} S_d\begin{pmatrix}\sigma & \psi\\ 1 & 1\end{pmatrix} &= S_d(1;\sigma)\sum_{i=0}^{d-1} S_i(1;\psi)\\ &= \frac{b_d(t_1)}{l_d}\sum_{i=0}^{d-1}\frac{b_i(t_2)}{l_i}\\ &= \frac{b_d(t_1)b_d(t_s)}{(t_1-\theta)(t_2-\theta)l_d^2}[(t_1-\theta)(\theta-\theta^{q^d})], \end{aligned}$$

in virtue of (7) and Lemma 8. Similarly, we have

$$S_d\begin{pmatrix}\psi & \sigma\\ 1 & 1\end{pmatrix} = \frac{b_d(t_1)b_d(t_s)}{(t_1-\theta)(t_2-\theta)l_d^2}[(t_2-\theta)(\theta-\theta^{q^d})].$$

The lemma follows from (10). □

Proof. [Proof of Theorem 6, (4)]. As we have already mentioned, this is a well known formula but we want to provide a new proof. We note, by (7), that

$$S_d(1;\sigma)S_d(1;\psi) = \frac{b_d(t_1)b_d(t_2)}{(t_1-\theta)(t_2-\theta)l_d^2}\,[(t_1-\theta)(t_2-\theta)].$$

Hence, Lemma 9 implies the formula

$$S_d(1;\sigma)S_d(1;\psi) = S_d(2;\sigma\psi) - S_d\begin{pmatrix}\psi & \sigma\\ 1 & 1\end{pmatrix} - S_d\begin{pmatrix}\sigma & \psi\\ 1 & 1\end{pmatrix}. \tag{13}$$

We deduce:

$$\begin{aligned} F_d(1;\sigma)F_d(1;\psi) &= F_d\begin{pmatrix}\psi & \sigma\\ 1 & 1\end{pmatrix} + F_d\begin{pmatrix}\sigma & \psi\\ 1 & 1\end{pmatrix} + \sum_{i=0}^{d-1} S_d(1;\sigma)S_d(1;\psi)\\ &= F_d\begin{pmatrix}\psi & \sigma\\ 1 & 1\end{pmatrix} + F_d\begin{pmatrix}\sigma & \psi\\ 1 & 1\end{pmatrix} + \sum_{i=0}^{d-1}\left(S_d(2;\sigma\psi) - S_d\begin{pmatrix}\psi & \sigma\\ 1 & 1\end{pmatrix} - S_d\begin{pmatrix}\sigma & \psi\\ 1 & 1\end{pmatrix}\right)\\ &= F_d\begin{pmatrix}\psi & \sigma\\ 1 & 1\end{pmatrix} - F_d\begin{pmatrix}\psi & \sigma\\ 1 & 1\end{pmatrix} + F_d\begin{pmatrix}\sigma & \psi\\ 1 & 1\end{pmatrix} - F_d\begin{pmatrix}\sigma & \psi\\ 1 & 1\end{pmatrix} + F_d(2;\sigma\psi)\\ &= F_d(2;\sigma\psi), \end{aligned}$$

where we have applied the formula (13) in the third equality. The formula of the theorem follows by letting $d \to \infty$. □

Proof. [Proof of Theorem 6, (5)]. The identities (6) and (8) imply that

$$S_d(1;\mathbf{1})S_d(1;\sigma\psi)=\frac{b_d(t_1)b_d(t_2)}{(t_1-\theta)(t_2-\theta)l_d^2}[(t_1-\theta)(t_2-\theta)].$$

Lemma 9 then implies that also:

$$S_d(1;\mathbf{1})S_d(1;\sigma\psi)=S_d(2;\sigma\psi)-S_d\begin{pmatrix}\psi & \sigma\\ 1 & 1\end{pmatrix}-S_d\begin{pmatrix}\sigma & \psi\\ 1 & 1\end{pmatrix}. \tag{14}$$

We deduce:

$$\begin{aligned}F_d(1;\mathbf{1})F_d(1;\sigma\psi)&=F_d\begin{pmatrix}\mathbf{1} & \sigma\psi\\ 1 & 1\end{pmatrix}+F_d\begin{pmatrix}\sigma\psi & \mathbf{1}\\ 1 & 1\end{pmatrix}+\sum_{i=0}^{d-1}S_d(1;\sigma)S_d(1;\psi)\\ &=F_d\begin{pmatrix}\mathbf{1} & \sigma\psi\\ 1 & 1\end{pmatrix}+F_d\begin{pmatrix}\sigma\psi & \mathbf{1}\\ 1 & 1\end{pmatrix}+\sum_{i=0}^{d-1}\left(S_d(2;\sigma\psi)-S_d\begin{pmatrix}\psi & \sigma\\ 1 & 1\end{pmatrix}-S_d\begin{pmatrix}\sigma & \psi\\ 1 & 1\end{pmatrix}\right)\\ &=F_d(2;\sigma\psi)+F_d\begin{pmatrix}\mathbf{1} & \sigma\psi\\ 1 & 1\end{pmatrix}+F_d\begin{pmatrix}\sigma\psi & \mathbf{1}\\ 1 & 1\end{pmatrix}-F_d\begin{pmatrix}\psi & \sigma\\ 1 & 1\end{pmatrix}-F_d\begin{pmatrix}\sigma & \psi\\ 1 & 1\end{pmatrix},\end{aligned}$$

so we have reached the formula

$$\begin{aligned}F_d(1;\mathbf{1})F_d(1;\sigma\psi)\\ =F_d(2;\sigma\psi)+F_d\begin{pmatrix}\mathbf{1} & \sigma\psi\\ 1 & 1\end{pmatrix}+F_d\begin{pmatrix}\sigma\psi & \mathbf{1}\\ 1 & 1\end{pmatrix}-F_d\begin{pmatrix}\psi & \sigma\\ 1 & 1\end{pmatrix}-F_d\begin{pmatrix}\sigma & \psi\\ 1 & 1\end{pmatrix}.\end{aligned} \tag{15}$$

Letting d tend to ∞ in (15) concludes the proof. □

Remark 10. In fact, it is trivial that

$$\begin{aligned}\zeta_A(1)\zeta_A(1;\sigma\psi)-\zeta_A(1;\sigma)\zeta_A(1,\psi)\\ =\zeta_d\begin{pmatrix}\mathbf{1} & \sigma\psi\\ 1 & 1\end{pmatrix}+\zeta_d\begin{pmatrix}\sigma\psi & \mathbf{1}\\ 1 & 1\end{pmatrix}-\zeta_d\begin{pmatrix}\psi & \sigma\\ 1 & 1\end{pmatrix}-\zeta_d\begin{pmatrix}\sigma & \psi\\ 1 & 1\end{pmatrix},\end{aligned}$$

and that the corresponding identity for the sums F_d holds as well. Indeed, setting $\alpha_i=S_i(1;\mathbf{1})=l_i^{-1}$, $\beta_i=S_i(1;\sigma)=b_i(t_1)l_i^{-1}$, $\gamma_i=S_i(1;\psi)=b_i(t_2)l_i^{-1}$ and $\delta_i=S_i(1;\sigma\psi)=b_i(t_1)b_i(t_2)l_i^{-1}$, we see that

$$\begin{aligned}&\sum_{i\geq 0}\alpha_i\sum_{j\geq 0}\delta_j-\sum_{i\geq 0}\beta_i\sum_{j\geq 0}\gamma_j\\ &\quad=\sum_{i>j\geq 0}\alpha_i\delta_j+\sum_{j>i\geq 0}\alpha_i\delta_j-\sum_{i>j\geq 0}\beta_i\gamma_j-\sum_{j>i\geq 0}\beta_i\gamma_j+\\ &\quad+\sum_{i\geq 0}\alpha_i\delta_i-\sum_{i\geq 0}\beta_i\gamma_i.\end{aligned}$$

But, of course, $\alpha_i\delta_i = \beta_i\gamma_i$ for all i, from which the identity follows. Up to this simple trick, the identity (5) of Theorem 6 can be considered as equivalent to the identity (4) of the same result.

5 - Some consequences

In virtue of Proposition 4 or by direct verification, the identities of Theorem 6 involve entire functions in two variables t_1, t_2. Hence, specializing the variables, we are able to recover identities in $\mathbb{C}_\infty$ or in some intermediate Tate algebra. We are going to show several results arising from the formula (2). We recall the formula (2) of Theorem 6 for convenience:

$$\zeta_A(1;\sigma)\zeta_A(1) = \zeta_A\begin{pmatrix} \sigma & \mathbf{1} \\ 1 & 1 \end{pmatrix} + \zeta_A(2;\sigma). \tag{16}$$

First of all, we can replace t_1 by θ, but this does not give anything interesting; we mention it only to show how the substitution works. We recall the following formula that the author proved in [**21**]:

$$\zeta_A(1;\sigma) = \frac{\widetilde{\pi}}{(\theta - t_1)\omega(t_1)}, \tag{17}$$

where

$$\omega(t_1) = (-\theta)^{\frac{1}{q-1}} \prod_{i\geq 0}\left(1 - \frac{t_1}{\theta^{q^i}}\right)^{-1} \in \mathbb{T}_1^\times,$$

is the *Anderson-Thakur function* (note that $\Omega(t_1) = \dfrac{1}{(t_1 - \theta)\omega(t_1)}$ is an entire function; see [**6**], containing a recent overview on the known properties of this function). The function $\omega(t_1)$ having a simple pole of residue $-\widetilde{\pi}$ at $t_1 = \theta$, we see that $\zeta_A(1;\sigma)|_{t_1=\theta} = 1$. Now, it is easy to see that

$$\zeta_A\begin{pmatrix} \mathbf{1} & \sigma \\ 1 & 1 \end{pmatrix} = \sum_{d\geq 0} S_d(1;\sigma)\sum_{i=0}^{d-1} l_i^{-1} = \sum_{d\geq 0} l_d^{-1} b_d(t_1)\sum_{i=0}^{d-1} l_i^{-1} \tag{18}$$

vanishes at $t_1 = \theta$. Further,

$$\zeta_A(2;\sigma) = \sum_{d\geq 0} S_d(2;\sigma)$$

takes the value $\zeta_A(1)$ at $t_1 = \theta$. Hence, with this evaluation, we only get the tautological identity $\zeta_A(1) = \zeta_A(1)$.

5.1 - *A family of multiple zeta identitities*

We can also evaluate this identity at $t_1 = \theta^{q^{-k}}$ with $k > 0$ and raise the obtained identity to the power q^k. Working out the intermediate details, the reader will easily recover the following sum shuffle formula:

$$\zeta_A(q^k)\zeta_A(q^k-1) = \zeta_A(2q^k-1) + \zeta_A(q^k-1, q^k), \quad k \geq 1.$$

5.2 - *Evaluation at trivial zeros*

Now, we evaluate the second identity of Theorem 6 at t_1 equal to a trivial zero of the function $\zeta_A(1;\sigma)$ which, as it appears from the computation of the poles of the gamma factor of (17), means that we replace t_1 with θ^{q^d} with $d > 0$. This implies the following result.

Theorem 11. *The following formula holds*

$$\mathrm{BG}_{q^d-2} = -\sum_{d \geq i > j \geq 0} \frac{b_i(\theta^{q^d})}{l_i l_j}, \quad d \geq 1. \tag{19}$$

Proof. In fact, to make things more transparent, we make a step back to the identity (12) that we rewrite as

$$F_k(2;\sigma) = F_k(1;\sigma)F_k(1;\mathbf{1}) - F_k\begin{pmatrix} \sigma & \mathbf{1} \\ 1 & 1 \end{pmatrix}, \quad k \geq 0.$$

By Proposition 4, we see that all the sequences $F_k(\cdots)$ involved tend, for $k \to \infty$, to entire functions of the variable t_1. However, we already know from [**5**, Proposition 6] that $F_k(2;\sigma)$ and $F_k(1;\sigma)F_k(1;\mathbf{1})$ tend to entire functions, so we immediately obtain that $F_k\begin{pmatrix} \sigma & \mathbf{1} \\ 1 & 1 \end{pmatrix}$ tends to an entire function as $k \to \infty$ without using Proposition 4 (in fact, this can be also seen directly). Replacing $t_1 = \theta^{q^d}$ with $d > 0$ yields the value zero for the limit $\lim_{k\to\infty} F_k(1;\sigma)F_k(1;\mathbf{1}) = \zeta_A(1;\sigma)\zeta_A(1)$ evaluated at $t_1 = \theta^{q^d}$. Indeed, after (17), θ^{q^d} is a trivial zero of $\zeta_A(1;\sigma)$. Further, we see that

$$\zeta_A(2;\sigma)\big|_{t_1=\theta^{q^d}} = \lim_{k\to\infty} F_k(2;\sigma)\big|_{t_1=\theta^{q^d}} = \sum_{k\geq 0}\sum_{a\in A^+(k)} a^{q^d-2} = \mathrm{BG}_{q^d-2} \in A.$$

Moreover, by (18) and evaluating at $t_1 = \theta^{q^d}$,

$$\zeta_A\begin{pmatrix} \sigma & \mathbf{1} \\ 1 & 1 \end{pmatrix}_{t_1=\theta^{q^d}} = \lim_{k\to\infty} F_k\begin{pmatrix} \sigma & \mathbf{1} \\ 1 & 1 \end{pmatrix}_{t_1=\theta^{q^d}} = \sum_{i>j\geq 0} \frac{b_i(\theta^{q^d})}{l_i l_j} = \sum_{d\geq i>j\geq 0} \frac{b_i(\theta^{q^d})}{l_i l_j},$$

because $b_i(\theta^{q^d})$ vanishes for all $i > d$. □

One nice aspect of the formula (19) is that it is easy to reduce it modulo an irreducible polynomial of A of degree d. The following family of congruences is an immediate consequence of our result, and was first observed by Thakur in [**24**], and Anglès and Ould Douh in [**3**]:

Corollary 12. *For all P an irreducible polynomial of $A^+(d)$, we have (recall that $|P| = q^d$):*

$$\mathrm{BG}_{|P|-2} \equiv \sum_{i=0}^{d-1} \frac{1}{l_i} \equiv F_d(1;\mathbf{1}) \pmod{P}.$$

Proof. For all i,j with $d \geq i > j \geq 0$, the fraction $b_i(\theta^{q^d})(l_i l_j)^{-1}$ is P-integral for any P irreducible of degree d. If $i < d$, we have that $b_i(\theta^{q^d})l_i^{-1} \equiv 0 \pmod{P}$, because $b_i(\theta^{q^d})$ is divisible, in A, by $\theta^{q^d} - \theta$. Further, $b_d(\theta^{q^d})/l_d \equiv -1$, from which the congruence follows. $\square$

Remark 13. The reader can do similar computations with other formulas; more results will appear elsewhere. Observe, however, that manipulating in the same way the formula (5) of Theorem 6 returns relatively less interesting identities. The reason seems to be that this formula is trivially equivalent to the formula (4), as it follows from Remark 10. At least, we deduce, specializing $t_1 = \theta^{q^{-k}}$ and $t_2 = \theta^{q^{-h}}$, the following strange sum shuffle identity, valid for $h, k \geq 0$ with $h + k > 0$:

$$\begin{aligned}&\zeta_A(1)^{q^k}\zeta_A(q^{k+h} - q^h - 1)\\ &\quad = \zeta_A(2q^{h+k} - q^h - 1) + \zeta_A(q^{k+h}, q^{k+h} - q^h - 1) + \zeta_A(q^{k+h} - q^h - 1, q^{k+h})\\ &\qquad - \zeta_A(q^{k+h} - q^h, q^{k+h} - 1) - \zeta_A(q^{k+h} - 1, q^{k+h} - q^h).\end{aligned}$$

The formula (2) of Theorem 6 and especially the formula of Theorem 19 can be seen as some kind of analogue of Euler's formula $\zeta(3) = \zeta(2,1)$ (for classical Euler multiple zeta values).

5.2.1 - A degree computation

In contrast with the universal formulas obtained in [**22**] for the sums $F_d(n;s)$ in the case $s \equiv n \pmod{q-1}$, there seems to be no such a formula for BG_{q^d-2}, for $d \geq 1$. At least, we have a "universal formula" for its degree, and this can be deduced from (19) as we are going to see in the next result, where it is supposed, again (as we did until now), that $q > 2$.

Theorem 14. *We have*

$$\deg_\theta(\mathrm{BG}_{q^d-2})=(d-1)q^d-\frac{2q(q^{d-1}-1)}{q-1}.$$

This result should be compared with more classical degree computations by Wan, Diaz-Vargas, Poonen, Sheats, as well as Böckle's [8, Theorem 1.2] where the interested reader can find all the necessary references to the work of these authors on this topic.

Theorem 14 seems to be new. Thomas [**29**, Theorem 2] already proved an explicit formula to compute, not only the degree of BG_{q^d-2} for $d>0$ but also the degree of BG_n for any $n>0$ with $q-1\nmid n$ in case q is a prime number ([4]). It follows from Thomas' Theorem 1 and Corollary 1 ibid. that if $q-1\nmid n$, and q is a prime, then $\mathrm{BG}_n=1$ if and only if $\ell_q(n)<q$, where $\ell_q(n)$ is the sum of the digits of the base-q expansion of n. However, Thomas formula contains an iterative process and for this reason, the identity of Theorem 14 is not immediately recognizable in it, and even if it was, it would have been valid only for q a prime number. Bruno Anglès has communicated to the author a simple proof of Theorem 14 in the case of $q=p>2$ a prime number by using Sheats' method. Also, Dinesh Thakur has pointed out to the author that this result, for general q, can be more simply deduced from an application of his *duality formula* [**28**, Theorem 2, (5)].

Remark 15. Note that for all $d\geq 1$, q^d-2 is a *dual magic number* in the sense of [**16**, §5.7] (see also [**17**]). In this paper, Goss points out a result of Thomas which exhibits a computation of the degree of BG_n when $q-1\nmid n$ and when n is a *magic number* ([**17**, §8.22]), in terms of the the degree of the Carlitz factorial. Our computation involves certain dual magic numbers which are not magic numbers, and this could also be a new instance of the conjectural functional equation for the Goss zeta function associated to the algebra A.

More results of the type of Theorem 14 can be obtained from more general consequences of the sum shuffle relations for our multi-zeta values in the Tate algebras, but they will be described in another work (the present paper can be considered as a first of more general results that will appear elsewhere). Before proving the Theorem, we need some notation and Lemmas. For commodity, we set

$$\alpha_i=\frac{b_i(\theta^{q^d})}{l_i},\quad \beta_j=l_j^{-1},$$

([4]) He obtained a more general result in this direction, also involving "first derivatives" of the Goss zeta function of A at its "trivial zeroes", the negative integers divisible by $q-1$.

so that the formula (19) rewrites as

$$\mathrm{BG}_{q^d-2} = -\sum_{d\geq i>j\geq 0} \alpha_i \beta_j.$$

Then, we have

$$\delta_{i,j} := \deg_\theta(\alpha_i\beta_j) = iq^d - \sum_{n=1}^{i} q^n - \sum_{m=1}^{j} q^m. \tag{20}$$

We recall the convention that an empty sum is zero. Moreover, the degree of 0 in θ is set to be $-\infty$. We have the following Lemma.

Lemma 16. *Assuming that $d \geq i > j \geq 0$, $d \geq i' > j' \geq 0$, we have that*

$$\delta_{i,j} = \deg_\theta(\alpha_i\beta_j) = \deg_\theta(\alpha_{i'}\beta_{j'}) = \delta_{i',j'}$$

if and only if the following cases occur.

1. $i = i', j = j'$,
2. $i = d$, $i' = d-1$ *and* $j = j'$,
3. $i' = d, i = d-1$ *and* $j = j'$.

Proof. For symmetry of the roles of i, i', we can assume that $i \geq i'$. First of all, if $i = i'$ we have that

$$\delta_{i,j} - \delta_{i',j'} = \sum_{m'=1}^{j'} q^{m'} - \sum_{m=1}^{j} q^m,$$

which equals zero if and only if $j = j'$; if $i = i'$, $\delta_{i,j} = \delta_{i',j'}$ if and only if $j = j'$. Now, let us suppose that $i > i'$. From (20) we deduce that

$$\delta_{i,j} - \delta_{i',j'} = (i-i')q^d - \psi_{i,i',j,j'},$$

where

$$\psi_{i,i',j,j'} = \sum_{n=1}^{i} q^n - \sum_{n'=1}^{i'} q^{n'} + \sum_{m=1}^{j} q^m - \sum_{m'=1}^{j'} q^{m'} \in \mathbb{Z}.$$

Since $i > i' > j'$ and $i > j$, we have that $\psi_{i,i',j,j'} \geq 0$ and we can find integers $c_r \in \{0,1,2\}$, unique, such that $\psi_{i,i',j,j'} = \sum_{r=0}^{i} c_r q^r$, so we see that

$$0 \leq \psi_{i,i',j,j'} < q^{i+1}$$

(recall that $q > 2$). If $i < d$, we see that

$$\delta_{i,j} - \delta_{i',j'} = (i - i')q^d - \psi_{i,i',j,j'} > q^d - q^{i+1} \geq 0$$

so that $\delta_{i,j} \neq \delta_{i',j'}$ in this case. It remains to study the case in which $i = d$. In this case, $j' < i' \leq d - 1$ and we can write:

$$\delta_{i,j} - \delta_{i',j'} = (d - i' - 1)q^d - \rho_{i',j,j'},$$

where

$$\rho_{i',j,j'} = \sum_{n=i'+1}^{d-1} q^n + \sum_{m=1}^{j} q^m - \sum_{m'=1}^{j'} q^{m'} \in \mathbb{Z}.$$

This number is obviously ≥ 0 and the following estimate holds

$$0 \leq \rho_{i',j,j'} < q^d.$$

If $i' \leq d - 2$, we thus have that $\delta_{d,j} > \delta_{i',j'}$ for any choice of $j < d$ and $j' < i'$. If $i' = d - 1$, we have

$$\delta_{i,j} - \delta_{i',j'} = \sum_{m'=1}^{j'} q^{m'} - \sum_{m=1}^{j} q^m$$

which equals zero if and only if $j = j'$. □

In view of the above Lemma, to compute the degree of BG_{q^d-2}, we rearrange the sum (19) in the following way:

$$\mathrm{BG}_{q^d-2} = -\overbrace{\alpha_d\beta_{d-1}}^{U} - \underbrace{(\alpha_d + \alpha_{d-1})\sum_{j=0}^{d-2}\beta_j}_{V} - \underbrace{\sum_{i=0}^{d-2}\sum_{j=0}^{i-1}\alpha_i\beta_j}_{W}$$
$$=: -(U + V + W).$$

Lemma 17. *We have:*

1. $\deg_\theta(U) = (d-1)q^d - 2(q + \cdots + q^{d-1})$,
2. $\deg_\theta(V) = (d-2)q^d - (q + \cdots + q^{d-2})$,
3. $\deg_\theta(W) = (d-2)q^d - (q + \cdots + q^{d-2})$.

Proof. We compute the degree of U:

$$\deg_\theta(U) = dq^d - \sum_{n=1}^{d} q^n - \sum_{m=1}^{d-1} q^m = (d-1)q^d - 2\sum_{n=1}^{d-1} q^n.$$

To compute the degree of V, we observe:

$$\begin{aligned}
\alpha_d + \alpha_{d-1} &= \frac{b_d(\theta^{q^d})}{l_d} - \frac{b_{d-1}(\theta^{q^d})}{l_{d-1}} \\
&= \frac{b_d(\theta^{q^d}) - (\theta^{q^d} - \theta) b_{d-1}(\theta^{q^d})}{l_d} \\
&= \frac{b_{d-1}(\theta^{q^d})[\theta^{q^d} - \theta^{q^{d-1}} - \theta^{q^d} + \theta]}{l_d} \\
&= \frac{(\theta - \theta^{q^{d-1}}) b_{d-1}(\theta^{q^d})}{l_d}.
\end{aligned}$$

Hence,

$$\begin{aligned}
\deg_\theta(V) &= \deg_\theta(\alpha_d + \alpha_{d-1}) + \deg_\theta\left(\sum_{j=0}^{d-2} \beta_j\right) \\
&= \deg_\theta(\alpha_d + \alpha_{d-1}) \\
&= q^{d-1} + (d-1)q^d - \sum_{n=1}^{d} q^n \\
&= (d-2)q^d - \sum_{n=1}^{d-2} q^n.
\end{aligned}$$

To compute the degree of W, we first notice, by Lemma 20, that all the terms involved in the double sum have different degrees. The term with the largest degree is the one corresponding to $i = d-2$ and $j = 0$, which is equal to α_{d-2}, and which has the expected degree. □

Proof. [Proof of Theorem 14]. By Lemma 17 and by the assumption $q > 2$, we have

$$\deg_\theta(U) > \deg_\theta(V), \deg_\theta(W),$$

and

$$\deg_\theta(\mathrm{BG}_{q^d-2}) = \deg_\theta\left(\frac{b_d(\theta^{q^d})}{l_d l_{d-1}}\right) = (d-1)q^d - \frac{2q(q^{d-1}-1)}{q-1}.$$

□

Remark 18. Anglès and Ould Douh have proved, in [3], that there exist infinitely many irreducible elements of A^+ such that $\mathrm{BG}_{|P|-2} \not\equiv 0 \pmod{P}$ (recall that $|P| = q^d$ in (19) and that $q > 2$ all along the present note). As a consequence we see, by the fact that

$$F_{\deg_\theta(P)}(1;\sigma) \not\equiv 0 \pmod{P}$$

for all irreducible P of A^+ (easily checked), that the right-hand sides of (12) and (15) determine non-zero elements of the ring $\mathcal{A}_s$ defined in [**22**]. This result is an easy consequence of their formula that we have re-obtained in our Corollary 12.

Let us recall the elegant proof of this property in [**3**]. Since $\mathrm{BG}_{q^d-2} \equiv \sum_{i=0}^{d-1} l_i^{-1} (\mathrm{mod}\, P)$ (for P irreducible of degree d), we have $\mathrm{BG}_{q^d-2} \equiv 0 (\mathrm{mod}\, P)$ if and only if P divides the polynomial

$$V(d) = l_{d-1} \sum_{i=0}^{d-1} l_i^{-1} \in A$$

which has degree $\sum_{n=1}^{d-1} q^n = \dfrac{q^d - q}{q-1}$, so that we have at most

$$\frac{q^d - q}{d(q-1)}$$

monic irreducible polynomials P of degree d dividing $V(d)$. Now, the number of monic irreducible polynomials P of degree d is equal to the the necklace polynomial (where μ designates Moebius' function)

$$M_d(q) = \frac{1}{d} \sum_{l|d} \mu(l) q^{\frac{d}{l}},$$

which is known to have an asymptotic behavior, as $d \to \infty$, which is of a strictly bigger magnitude than that of the above fraction if $q > 2$. For example, if $d = p'$ is a prime number, the necklace polynomial $M_{p'}(q)$ equals $\dfrac{q^{p'} - q}{p'}$ and we have

$$\frac{q^{p'} - q}{p'} > \frac{q^{p'} - q}{p'(q-1)},$$

because $q > 2$.

The formula (19) does not seem to immediately imply the result of Anglès and Ould Douh (without using the intermediate congruence with the polynomial $V(d)$), but we have not tried to rearrange the terms of the sum completely.

We also point out that the proof of the identity (6) can be easily generalized to give the identity

$$\zeta_A(1;\sigma)\zeta_A(n) = \zeta_A(n+1;\sigma) + \zeta_A\begin{pmatrix} \sigma & \mathbf{1} \\ 1 & n \end{pmatrix}, \quad 1 \leq n \leq q-1.$$

This almost immediately implies a generalization of Theorem 19, Corollary 12, Theorem 14 and the result of Anglès and Ould Douh. For all $m = 2, \ldots, q-1$,

$$\mathrm{BG}_{q^d-m} = -\sum_{k \geq d > i \geq 0} \frac{b_d(\theta^{q^k})}{l_d\, l_i^{m-1}}.$$

In particular,

$$\mathrm{BG}_{q^d-m} \equiv \sum_{i=0}^{d-1} l_i^{1-m} \pmod{P}, \quad \deg_\theta(P) = d, \quad m = 2, \ldots, q-1$$

and the same proof as [3] can be used to show that, for all $2 \leq m \leq q-1$ we have

$$\mathrm{BG}_{q^d-m} \not\equiv 0 \pmod{P},$$

for infinitely many P.

6 - Looking for more relations

We gave above some hints of a variant of the shuffle product for the multiple zeta values:

$$\zeta_A\begin{pmatrix} \sigma_1 & \sigma_2 & \cdots & \sigma_r \\ n_1 & n_2 & \cdots & n_r \end{pmatrix}$$

in the simplest non-trivial cases (weight 2). We shall complete our note by suggesting some other tools to develop, in order to compute other kinds of relations.

We denote by $K\{\tau\}$ the skew polynomial ring of finite sums $\sum_i c_i\tau^i$, with $c_i \in K$, with the non-commutative product uniquely determined by the rule $\tau c = c^q\tau$ for $c \in K$. Additionally, let t be a variable (we can set $t = t_1$ to get compatibility with the first part of the note). We have an isomorphism of K-vector spaces:

$$K[t] \xrightarrow{\eta} K\{\tau\}$$

defined by $\eta(t^i) = C_{\theta^i} = (\theta + \tau)^i$ for $i > 0$ and $\eta(1) = 1$. Here $C_\theta = \theta + \tau$ is the multiplication by θ of the Carlitz module. The inverse of this isomorphism sends 1 to 1 and, for $j > 0$, τ^j to $b_j(t)$, where we recall that

$$b_j(t) = (t - \theta) \cdots (t - \theta^{q^{j-1}}).$$

To check that η is an isomorphism, one uses the *evaluation* at the Anderson-Thakur function. The evaluation $f(\omega)$ of an element $f = f_0 + f_1\tau + \cdots + f_r\tau^r \in K\{\tau\}$ at ω is by definition the expression $(f_0 + f_1b_1 + \cdots + f_rb_r)\omega$. It is easy to see that $C_a(\omega) = a(t)\omega$, so that, for all $f(t) \in K[t]$, we have

$$\eta(f)(\omega) = f(t).$$

This isomorphism η is useful to construct certain identities for finite sums. We recall, as a first example, the formula (7):

$$S_d(1;1) = S_d(1;\sigma) = \sum_{a \in A^+(d)} a^{-1}a(t) = \frac{b_d(t)}{l_d}, \quad d \geq 0.$$

It is easy to show that $\eta(a(t)) = C_a \in K\{\tau\}$. Therefore, the isomorphism η yields the identity:

$$\eta(S_d(1;1)) = \sum_{a\in A^+(d)} a^{-1}C_a = l_d^{-1}\tau^d, \quad d \geq 0.$$

This picture can be generalized. We can use variables $t_1,\ldots,t_s$, indeterminates $\tau_1,\ldots,\tau_s$, the rings $K[t_1,\ldots,t_s]$ (commutative) and $K\{\tau_1,\ldots,\tau_s\}$ (non commutative, with multiplication rules: $\tau_i\tau_j = \tau_j\tau_i$ and $\tau_i c = c^q\tau_i$ for $c \in K$), and the isomorphism

$$K[t_1,\ldots,t_s] \xrightarrow{\eta} K\{\tau_1,\ldots,\tau_s\} \tag{21}$$

uniquely defined by $\eta(t_i^j) = (\theta+\tau_i)^j$ (we write η instead of the more precise expression η_s we should have used, to simplify our notations). Then, any time we can show a formula for power sums in $K[t_1,\ldots,t_s]$, we obtain a similar formula in the ring $K\{\tau_1,\ldots,\tau_s\}$.

Florent Demeslay proved, in his Thesis [**15**], the following result.

Theorem 19. *Assume that $s > 0$. There exists a rational fraction $Q_{k,s} \in K(t_1,\ldots,t_s)(Y)$ such that*

$$S_d(k;s) = \frac{b_d(t_1)\cdots b_d(t_s)}{l_d} Q_{k,s}(\theta^{q^{d-m}}), \quad d \geq 0$$

where $m = \max\{0, \lfloor\frac{s-1}{q-1}\rfloor\}$.

The case $s = 0$ (no variables) was already known to Anderson and Thakur [**2**]. We would like to apply this Theorem for $s > 0$ to produce identities in the non-commutative indeterminates $\tau_1,\ldots,\tau_s$ by means of the isomorphism η. For example, if $k = 1$ and $s = 1$, we are reduced to the formula (7) with $Q_{1,1} = 1$. However, there is no reason to expect that $Q_{k,s}$ is a polynomial and in fact, in general, this is false. For example, it is easy to check that $Q_{q,1} = \frac{t-Y}{t-\theta}$, which is not a polynomial. It is of course possible to compute the rational fractions $Q_{k,s}$ by using the polynomials $\mathbb{H}_s$ of Theorem 7, but even with that in mind, we cannot escape this problem.

A partial solution is given by Lemma 8. We now denote by $\boldsymbol{\tau}$ (we do not want to mix it up with τ which is now an indeterminate!) the $\mathbb{F}_q[t]$-algebra endomorphism of $K[t]$ defined by $\boldsymbol{\tau}(c) = c^q$. Then, Lemma 8 and induction imply the following result.

Proposition 20. *For all $n > 0$ and $d \geq 0$, the following formula holds:*

$$\boldsymbol{\tau}^n(b_d(t)) = l_d^{q^{n-1}} \sum_{d\geq i_1\geq i_2\geq\cdots\geq i_{n-1}\geq i_n\geq 0} l_{i_1}^{q^{n-2}-q^{n-1}} l_{i_2}^{q^{n-3}-q^{n-2}} \cdots l_{i_{n-1}}^{1-q} l_{i_n}^{-1} b_{i_n}(t).$$

In particular, for all $n > 0$ and $d \geq 0$:

$$S_d(q^n;1) = l_d^{q^{n-1}-q^n} \sum_{d \geq i_1 \geq i_2 \geq \cdots \geq i_{n-1} \geq i_n \geq 0} l_{i_1}^{q^{n-2}-q^{n-1}} l_{i_2}^{q^{n-3}-q^{n-2}} \cdots l_{i_{n-1}}^{1-q} l_{i_n}^{-1} b_{i_n}(t).$$

Although the rational fraction $Q_{q,1}$ and more generally the fractions $Q_{q^n,1}$ are certainly not polynomials, the above formulas in $K[t]$ can be transferred to identities in the ring $K\{\tau\}$. We obtain, by applying the map η:

Corollary 21. *For all $d \geq 0$,*

$$\mathfrak{S}_d(q^n;1) := \sum_{a \in A^+(d)} a^{-q^n} C_a = l_d^{q^{n-1}-q^n} \sum_{d \geq i_1 \geq i_2 \geq \cdots \geq i_{n-1} \geq i_n \geq 0} l_{i_1}^{q^{n-2}-q^{n-1}} l_{i_2}^{q^{n-3}-q^{n-2}} \cdots l_{i_{n-1}}^{1-q} l_{i_n}^{-1} \tau^n.$$

Let $\sigma_1, \ldots, \sigma_r$ be semi-characters, let $n_1, \ldots, n_r$ be integers, and d a non-negative integer. We set, for convenience:

$$S_d^\star\begin{pmatrix} \sigma_1 & \sigma_2 & \cdots & \sigma_r \\ n_1 & n_2 & \cdots & n_r \end{pmatrix} = S_d(n_1;\sigma_1) \sum_{d \geq i_2 \geq \cdots \geq i_r \geq 0} S_{i_2}(n_2;\sigma_2) \cdots S_{i_r}(n_r;\sigma_r) \in \mathbb{F}_q^{ac} \otimes_{\mathbb{F}_q} K(\underline{t}_s)$$

(we have introduced non-strict inequalities in the sum). Further, we set:

$$\zeta_A^\star\begin{pmatrix} \sigma_1 & \sigma_2 & \cdots & \sigma_r \\ n_1 & n_2 & \cdots & n_r \end{pmatrix} := \sum_{d \geq 0} S_d^*\begin{pmatrix} \sigma_1 & \sigma_2 & \cdots & \sigma_r \\ n_1 & n_2 & \cdots & n_r \end{pmatrix} \in K_\infty \otimes_{\mathbb{F}_q} \widehat{\mathbb{F}_q^{ac}} \otimes_{\mathbb{F}_q} \mathbf{F}_s\,.$$

We observe that $S_d(j;\mathbf{1}) = l_d^{-j}$ if $j = kq^l$ with $l \geq 0$ and $k = 1, \ldots, q-1$. Hence, the second identity of Proposition 20 can be rewritten, with $\sigma = \chi_t$, in the following way:

$$S_d(q^n;\sigma) = S_d^\star\begin{pmatrix} \mathbf{1} & \mathbf{1} & \cdots & \mathbf{1} & \sigma \\ q^{n-1}(q-1) & q^{n-2}(q-1) & \cdots & q-1 & 1 \end{pmatrix}.$$

Summing over $d = 0, 1, \ldots$, we obtain the formula:

$$\zeta_A(q^n;\sigma) = \zeta_A^\star\begin{pmatrix} \mathbf{1} & \mathbf{1} & \cdots & \mathbf{1} & \sigma \\ q^{n-1}(q-1) & q^{n-2}(q-1) & \cdots & q-1 & 1 \end{pmatrix}. \tag{22}$$

We observe that the evaluation at $t = \theta^{q^k}$ gives:

$$\underbrace{\ldots, \theta^{q^{n-1}}}_{\text{special values}\neq 0}\ ,\ \underbrace{\theta^{q^n}}_{\text{value one}}\ ,\ \underbrace{\theta^{q^{n+1}}, \theta^{q^{n+2}}, \ldots}_{\text{trivial zeroes}}\ .$$

Evaluating e.g. at $t = \theta$ returns us the following identity, with the obvious meaning of the second sum:

$$\zeta_A(q^n-1) = \zeta_A^\star(\underbrace{q^{n-1}(q-1), q^{n-2}(q-1), \ldots, q-1}_{n \text{ terms}}).$$

We can rewrite the identity of our Corollary 21 as follows:

$$\mathfrak{S}_d(q^n;1)=\sum_{a\in A^+(d)}a^{-q^n}C_a$$

$$=S_d(q^{n-1}(q-1))\sum_{d\geq i_1\geq i_2\geq\cdots\geq i_{n-1}\geq i_n\geq 0}S_{i_1}(q^{n-2}(q-1))S_{i_2}(q^{n-3}(q-1))\cdots S_{i_{n-1}}(q-1)S_{i_n}(1)\tau^n,$$

with $S_d(n):=S_d(n;\mathbf{1})$. If $f=f_0+f_1\tau+\cdots+f_r\tau^r\in K\{\tau\}$, the *evaluation at one* $f(1)$ of f is the element $f_0+f_1+\cdots+f_r\in K$. It is easy to see that the series $\sum_{d\geq 0}\sum_{a\in A^+(d)}a^{-q^n}C_a(1)$ converges in K_∞. We obtain the formula:

$$\sum_{d=0}^{\infty}\mathfrak{S}_d(q^n;1)(1)=\zeta_C^\star(\underbrace{q^{n-1}(q-1),q^{n-2}(q-1),\ldots,q-1}_{n\text{ terms}},1). \tag{23}$$

These formulas can be easily related to Thakur's multiple zeta values ζ_A (without the $\star$ mark), by means of simple manipulations. We illustrate this in the case $n=1$. We observe that

$$\sum_{d=0}^{\infty}\mathfrak{S}_d(q;1)(1)=\zeta_C^\star(q-1,1)=\zeta_A(q-1,1)+\sum_{i\geq 0}S_i(q-1;\mathbf{1})S_i(1,\mathbf{1}).$$

Now, since $S_i(q-1;\mathbf{1})=l_i^{1-q}$ and $S_i(1;\mathbf{1})=l_i$, we get

$$\sum_{i\geq 0}S_i(q-1;\mathbf{1})S_i(1;\mathbf{1})=\sum_{i\geq 0}l_i^{-q}=\log_C(1)^q,$$

where $\log_C(z)=\sum_{i\geq 0}l_i^{-1}z^{q^i}$ is the *Carlitz logarithm* of $z\in\mathbb{C}_\infty$, well defined for $|z|<q^{q/(q-1)}$ (we recall that $|\cdot|$ is the unique norm of $\mathbb{C}_\infty$ such that $|\theta|=q$) and in particular, well defined at $z=1$. It is plain that $\log_C(1)=\zeta_A(1)$, an identity which was, essentially, first noticed by Carlitz. Thus we have that

$$\zeta_C^\star(q-1,1)=\zeta_A(q-1,1)+\zeta_A(1)^q.$$

The shuffle product of $\zeta_A(s_1)$ and $\zeta_A(s_2)$ yields, for $s_1,s_2\in\mathbb{N}^+$ such that $s_1+s_2\leq q$ (see Thakur, [**26**, Theorem 1]), the simple formula:

$$\zeta_A(1)\zeta_A(q-1)=\zeta_A(q)+\zeta_A(q-1,1)+\zeta_A(1,q-1)=\zeta_A(1)^q+\zeta_A(q-1,1)+\zeta_A(1,q-1).$$

This means that

$$\sum_{d=0}^{\infty}\mathfrak{S}_d(q;1)(1)=\zeta_C^\star(q-1,1)=\zeta_A(1)\zeta_A(q-1)-\zeta_A(1,q-1).$$

We do not know how to evaluate the sum $\sum_{d=0}^{\infty}\mathfrak{S}_d(q;1)(1)$ (and more generally, similar sums we do not want to introduce in this paper) directly, and it would be nice to develop a technique to do so independently of the shuffle product, in order to re-obtain the shuffle product formula. Note also that Thakur demonstrated the formula (see [**26**, Theorem 5]):

$$\zeta_A(m, m(q-1)) = \frac{\zeta_A(mq)}{(\theta-\theta^q)^m}, \quad m = 1, \dots, q-1.$$

Hence, we compute easily, with $m = 1$:

$$\begin{aligned}\zeta_A^\star(q-1,1) &= \zeta_A(q-1,1) + \zeta_A(1)^q \\ &= \zeta_A(q-1)\zeta_A(1) - \zeta_A(1,q-1) \\ &= \zeta_A(1)\left(\zeta_A(q-1) - \frac{\zeta_A(1)^{q-1}}{\theta-\theta^q}\right).\end{aligned}$$

Remark 22. In the examples we have studied above, the semi-characters are all of Dirichlet type but for no reason this should be considered as a necessary condition for the existence of shuffle-like formulas. For example, if $\nu : A^+ \to \mathbb{F}_q[t]$ is the semi-character which which associates $a \in A^+$ to $t^{\deg_\theta(a)}$ (this is not of Dirichlet type), then the following formula holds in the Tate algebra $\mathbb{T}$, as the reader can easily check:

$$\zeta_A(1;\nu)\zeta_A(1;\mathbf{1}) = \zeta_A(2;\nu) + \zeta_A\begin{pmatrix}\nu & \mathbf{1} \\ 1 & 1\end{pmatrix} + \zeta_A\begin{pmatrix}\mathbf{1} & \nu \\ 1 & 1\end{pmatrix}.$$

Evaluating at $t = 1$ we deduce the formula (1) of Theorem 6.

Remark 23. The referee pointed out a possible link between certain series modeled after the finite sums of Corollary 21 (that we also see in the left-hand side of (23), before evaluation at one) and Chieh-Yu Chang's *multiple polylogarithms* (see [**11**]) which can also be viewed as formal series:

$$\mathrm{Li}_{(n_1,\dots,n_s)} = \sum_{i_1>i_2>\cdots>i_s\geq 0} l_{i_1}^{-n_1}\cdots l_{i_s}^{-n_s}\tau_1^{i_1}\cdots\tau_r^{i_s} \in K\{\{\tau_1,\dots,\tau_s\}\},$$

in the completion of the non-commutative polynomial ring $K\{\tau_1,\dots,\tau_s\}$ with respect to the left ideal generated by $\tau_1,\dots,\tau_s$. It is easy to see that the K-isomorphism η of (21) induces an isomorphism between the K-vector space spanned by the multiple zeta values (4) with $\sigma_i = \chi_{t_i}$ for all i and $1 \leq n_i \leq q$ for all i, and the K-vector space spanned by the multiple polylogarithms $\mathrm{Li}_{(n_1,\dots,n_s)}$ with this same condition $1 \leq n_i \leq q$. More precisely, we get:

$$\eta\left(\zeta_A\begin{pmatrix}\chi_{t_1} & \chi_{t_2} & \cdots & \chi_{t_s} \\ n_1 & n_2 & \cdots & n_s\end{pmatrix}\right) = \mathrm{Li}_{(n_1,\dots,n_s)}, \quad 1 \leq n_1,\dots,n_s \leq q$$

because

$$\eta\left(\sum_{a\in A^+(d)} a^{-n}a(t_i)\right) = l_d^{-n}\tau_i^d, \quad 1\leq n\leq q, \quad 1\leq i\leq s, \quad d\geq 0.$$

Note also that the identity (22) is rewritten, by means of η, as

$$\mathrm{Li}_{q^s}(\tau_s) = \mathrm{Li}^{\star}_{q^{s-1}(q-1),\ldots,q-1}(1,\ldots,1,\tau_s) \in K\{\{\tau_s\}\},$$

with the obvious meaning of the symbol $\star$ and where the list of 1's means that we are replacing $\tau_i = 1$ formally for $i = 1,\ldots,s-1$, which gives well defined series for the τ_s-adic topology.

This suggests to explore, in parallel of the multiple zeta values (4), also the "non-commutative" counterparts provided by Drinfeld modules themselves, which can also be viewed, formally, as some kind of semi-characters $A^+ \to A\{\tau_1,\ldots,\tau_s\}^*$. For example, with this viewpoint, we have, with the above conditions on the parameters,

$$\mathrm{Li}_{(n_1,\ldots,n_s)} = \sum_{i_1>\cdots>i_s\geq 0}\ \sum_{\substack{a_1,\ldots,a_s\in A^+\\ \deg_\theta(a_j)=i_j}} a_1^{-n_1}\cdots a_s^{-n_s}C_{a_1}(\tau_1)\cdots C_{a_s}(\tau_s).$$

It seems that the image in $K_\infty\{\tau_1,\ldots,\tau_s\}$ of the K_∞-vector space spanned by the multiple zeta values (4) requires more general functions than Chang's polylogarithms to be generated, and it is an interesting question to see if an appropriate notion of "non-commutative" multiple zeta values associated to the Drinfeld modules of rank one of [**7**] would suffice for this purpose.

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