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# On canonical heights on endomorphism rings over global function fields

**Abstract.** We present a construction of a canonical height on the endomorphism ring of a finite dimensional vector space over a global function field. We also prove a limit formula analogous to the Tate's formula defining the canonical heights on abelian varieties.

**Keywords.** Heights, adelic vector bundles, endomorphism rings.

**Mathematics Subject Classification (2010):** 11G.

## 1 - Introduction

Canonical heights appeared in the mid 50's independently in the work of A. Néron (cf. [**6**]) and J. Tate (unpublished, see [**7**] where Tate's method first appeared in print) on abelian varieties. Since then canonical heights have been constructed and studied in several settings, such as Drinfeld modules (cf. [**3**]), varieties with morphism (cf. [**2**]), endomorphism rings of vector space (cf. [**9**]) and finite sets of matrices in $\mathrm{GL}_d(\overline{\mathbb{Q}})$ (cf. [**1**]), just to mention a few. While Néron's construction is local, i.e., he defines the canonical height as a product of local factors, Tate's method is global and is more suited to be applied in different contexts. We briefly recall Tate's method. Let $A$ be an abelian variety defined over a number field $\mathbf{F}$. Let $[2] : A \to A$ denote the duplication map on $A$. Let $c$ be a linear equivalence class

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of divisors on $A$ containing a symmetric ample divisor $D$ and choose $\phi : A \to \mathbb{P}_N$ so that $D$ is the pull back of a hyperplane via $\phi$. Let $h_{nw} : \mathbb{P}_N(\overline{\mathbb{Q}}) \to \mathbb{R}$ denote the absolute logarithmic Northcott-Weil height and set $h_\phi(P) := h_{nw}(P)$. Then the limit

$$\widehat{h}_c(P) := \lim_{m\to\infty} \frac{1}{4^m}\, h_\phi([2^m]P) \tag{1}$$

exists for each $P \in A(\overline{\mathbb{Q}})$ and is independent of $\phi$. The function $\widehat{h}_c$ is called the *canonical height* associated to $c$. Not only is $\widehat{h}_c$ independent of $\phi$ but also of the choice of $D$. Moreover, if we perform the limit in (1) using any other multiplication map we obtain the same function (cf. [8, Lemma 3.1.]). The equality between Néron's and Tate's definitions is achieved as follows: first it is shown that there is a unique function $\tilde{h}$ satisfying the following two properties:

a) $h_\phi$ and $\tilde{h}$ are in the same class modulo bounded functions;

b) $\tilde{h}([2]P) = 4\tilde{h}(P)$;

then, it is shown that both Neron's and Tate's height satisfy both a) and b).

Let $E$ be a finite dimensional vector space over a global function field $\mathbf{k}$. In this paper we present a construction of a canonical height (called the *spectral height*) on $\mathrm{End}_{\mathbf{k}}(E)$. Our construction is closer to Neron's one (albeit more elementary) as we define the spectral height as a product of local factors (the local spectral radii, cf. Section 5). We then prove a limit formula (cf. Theorem 5.1), analogous to (1), relating the spectral height and heights on $\mathrm{End}_{\mathbf{k}}(E)$ associated to adelic vector bundles over $\mathbf{k}$.

The paper is organised as follows: in Section 2 we introduce adelic vector bundles and their associated heights. In Section 3 we prove an interesting auxiliary inequality between the height relative to a sub-bundle $\overline{D}$ and the minimum of the height on a coset of $D$ on $E$, cf. Proposition 3.1. Section 4 deals with heights on $\mathrm{End}_{\mathbf{k}}(E)$, while the spectral height is introduced in Section 5 where we prove our main theorem (cf. Theorem 5.1).

Notation. Throughout this paper $\mathbf{k} \supset \mathbb{F}_p(t)$ is a global function field of characteristic $p > 0$. We let $\mathcal{M}_{\mathbf{k}}$ be the set of places of $\mathbf{k}$. Given $v \in \mathcal{M}_{\mathbf{k}}$ we denote by $\mathbf{k}_v$ the completion of $\mathbf{k}$ with respect to $v$ and by $\mathbf{C}_v$ the completion of the algebraic closure of $\mathbf{k}_v$. The maximal compact subring of $\mathbf{k}_v$ is denoted by $\mathcal{O}_v$ and we let $n_v = [\mathbf{k}_v : \mathbb{F}_p(t)_\omega]$, where $\omega$ is the restriction of $v$ to $\mathbb{F}_p(t)$. For each $v \in \mathcal{M}_{\mathbf{k}}$ we fix an absolute value $|\cdot|_v$, in the class of $v$, by requiring that $|a|_v^{n_v}$ coincides with the modulus, with respect to the Haar measure on the locally compact group $\mathbf{k}_v$, of the automorphism $x \mapsto ax$. With these normalizations the product formula reads: $\prod_{v\in\mathcal{M}_{\mathbf{k}}} |a|_v^{n_v} = 1$. Regarding vector spaces we will employ the following notation: given a vector space $E$, $E^\times$ denotes the set of non-zero vectors in $E$ and $\langle \mathbf{e}_1, \dots, \mathbf{e}_r \rangle$ denotes the subspace generated by $\mathbf{e}_1, \dots, \mathbf{e}_r \in E$. Lastly if $E$ is a vector space over $\mathbf{k}$, we set $E_v = E \otimes \mathbf{k}_v$.

## 2 - Adelic vector bundles

In this paper we use heights associated to adelic vector bundles. Adelic vector bundles have been recently introduced and studied by É. Gaudron (see [4] and [5]). An adelic vector bundle $\overline{E} = (E, \{\| \cdot \|_{\overline{E},v}\}_{v \in \mathcal{M}_\mathbf{k}})$ (over spec $\mathbf{k}$ or over $\mathbf{k}$ for short) of dimension $n$ consists of the following data (cf. [4, Definition 2.1]): a $\mathbf{k}$-vector space $E$ of dimension $n$ (called the *support* of $\overline{E}$) and a family of ultrametric norms $\| \cdot \|_{\overline{E},v} : E \otimes_\mathbf{k} \mathbf{C}_v \to \mathbb{R}$, satisfying the following conditions:

1) there exists a $\mathbf{k}$-basis $\{\mathbf{e}_1, \dots, \mathbf{e}_n\}$ of $E$ over $\mathbf{k}$, such that for all but finitely many $v \in \mathcal{M}_\mathbf{k}$ we have
$$\left\| \sum_{i=1}^{n} \alpha_i \mathbf{e}_i \right\|_{\overline{E},v} = \max_{1 \le i \le n} \{|\alpha_i|_{\mathbf{C}_v}\} \quad \forall (\alpha_1, \dots \alpha_n) \in \mathbf{C}_v$$
where $| \cdot |_{\mathbf{C}_v}$ is the unique extension of $| \cdot |_v$ to $\mathbf{C}_v$;
2) let $\mathrm{Gal}(\mathbf{C}_v/\mathbf{k}_v)$ denote the set of continuous automorphism of $\mathbf{C}_v$ which leaves the elements of $\mathbf{k}_v$ fixed, then $\| \cdot \|_{\overline{E},v}$ is invariant under the standard action of $\mathrm{Gal}(\mathbf{C}_v/\mathbf{k}_v)$ on $E \otimes_\mathbf{k} \mathbf{C}_v$.

An adelic vector bundle is called *v-pure* if $\|x\|_{\overline{E},v}$ belongs to the value set of $| \cdot |_v$ for all $x \in E$ and it is called *pure*[1] if it is $v$-pure for all $v \in \mathcal{M}_\mathbf{k}$. Let $\overline{E} = (E, \{\| \cdot \|_{\overline{E},v}\}_{v \in \mathcal{M}_\mathbf{k}})$ be a pure adelic vector bundle over $\mathbf{k}$. It is possible to perform several algebraic constructions with adelic vector bundles, such as exterior powers, symmetric powers and so on. We refer the reader to [4, Section 3.3] for details and briefly recall the few that we need. The absence of archimedean places simplifies some definitions. We say that $\overline{D}$ is an adelic sub-bundle of $\overline{E}$ if $D \subset E$, and for every $v$ the norms of $\overline{D}$ are the restriction of those of $\overline{E}$. If $\overline{D} \subset \overline{E}$ is a sub-bundle then $E/D$ inherits an adelic vector bundle structure (denoted by $\overline{E}/\overline{D}$) where the norms are the quotient norms of those of $\overline{E}$. If $\overline{F} = (F, \{\| \cdot \|_{\overline{F},v}\}_{v \in \mathcal{M}_\mathbf{k}})$

[1] It is not difficult to prove that there is a one to one correspondence between pure adelic vector bundles over $\mathbf{k}$ having $E$ as support and coherent systems of $k_v$-lattices belonging to $E$ as defined by A. Weil in [11].

is another adelic vector bundle over $\mathbf{k}$, we set

$$\|T\|_{\overline{E},\overline{F},v} := \sup_{\mathbf{e}\,\in\,E_v^{\times}} \frac{\|T(\mathbf{e})\|_{\overline{F},v}}{\|\mathbf{e}\|_{\overline{E},v}}$$

for all $T \in Hom_{\mathbf{k}}(E, F) \otimes_{\mathbf{k}} \mathbf{C}_v$ and all $v \in \mathcal{M}_{\mathbf{k}}$. It is straightforward to verify that $Hom_{\mathbf{k}}(\overline{E}, \overline{F}) = (Hom_{\mathbf{k}}(E, F), \{\|\cdot\|_{\overline{E},\overline{F},v}\}_{v\in\mathcal{M}_{\mathbf{k}}})$ is an adelic vector bundle having $Hom_{\mathbf{k}}(E, F)$ as support. Note that if $\overline{F}$ is the trivial bundle this gives the structure of adelic vector bundle to $E^*$ the dual of $E$. Next $\overline{E} \otimes_{\mathbf{k}} \overline{F}$ is the adelic vector bundle having support $E \otimes_{\mathbf{k}} F$ and norms induced by the isomorphism $E \otimes_{\mathbf{k}} F \simeq Hom_{\mathbf{k}}(E^*, F)$. Lastly we denote by $\overline{\bigwedge^m E} = (\bigwedge^m E, \|\cdot\|_{\overline{\wedge^m E},v})$ the adelic vector bundle having $\bigwedge^m E$ as support and whose norms are the quotient norms of $\overline{E}^{\otimes m}$.

Let $\overline{E} = (E, \{\|\cdot\|_{\overline{E},v}\}_{v\in\mathcal{M}_{\mathbf{k}}})$ be an adelic vector bundle over spec $\mathbf{k}$. The height function $H_{\overline{E}} : E \to \mathbb{R}$, relative to $\overline{E}$ is defined by setting:

$$H_{\overline{E}}(\mathbf{e}) = \prod_{v\,\in\,\mathcal{M}_{\mathbf{k}}} \|\mathbf{e}\|_{\overline{E},v}^{n_v} \tag{2}$$

for all $\mathbf{0} \neq \mathbf{e} \in E$. As usual we set $H_{\overline{E}}(\mathbf{0}) = 1$. It follows from the product formula that $H_{\overline{E}}$ is constant on one dimensional subspaces of $E$. The height of a subspace $D \subset E$, is defined as follows: choose a basis $\mathbf{d}_1, \ldots, \mathbf{d}_m$ of $D$ over $\mathbf{k}$ and set $H_{\overline{E}}(D) = H_{\overline{\wedge^m E}}(\mathbf{d}_1 \wedge \cdots \wedge \mathbf{d}_m)$, which does not depend on the choice of the basis by the product formula (see [**5**, Introduction]). Lastly if $B$ is a subset of $E$ we set

$$\lambda_1^{\overline{E}}(B) = \inf_{\mathbf{x}\,\in\,B} H_{\overline{E}}(\mathbf{x}).$$

## 3 - Comparison between $H_{\overline{E}/\overline{D}}([\mathbf{e}]_D)$ and $\lambda_1^{\overline{E}}([\mathbf{e}]_D)$

Let $D$ be a subspace of $E$. If $\mathbf{e} \in E - D$ we denote by $\langle D, \mathbf{e}\rangle$ the subspace generated by $D$ and $\mathbf{e}$, and by $[\mathbf{e}]_D$ the coset of $\mathbf{e}$ modulo $D$. In this section we obtain a comparison result (Proposition 3.1 below) for $H_{\overline{E}/\overline{D}}\big([\mathbf{e}]_D\big)$ and $\lambda_1^{\overline{E}}([\mathbf{e}]_D)$. One of the main constituents of the proof of Proposition 3.1 is the uniform Sigel's lemma recently proved by É. Gaudron in [**5**]. The lower bound obtained in Proposition 3.1 is a key ingredient for the results of the next section.

The quotient height $H_{\overline{E}/\overline{D}}$ is defined as

$$H_{\overline{E}/\overline{D}} : E/D \longrightarrow \mathbb{R}$$
$$[\mathbf{e}]_D \longmapsto H_{\overline{E}/\overline{D}}([\mathbf{e}]_D) = \prod_{v\,\in\,\mathcal{M}_{\mathbf{k}}} \inf_{\mathbf{e}'\,\in\,[\mathbf{e}]_{D_v}} \|\mathbf{e}'\|_{\overline{E},v}^{n_v}$$

if $[\mathbf{e}]_D \neq [0]_D$, as usual we set $H_{\overline{E}/\overline{D}}([0]_D) = 1$.

Proposition 3.1. *Let $\overline{E}$ be a pure adelic vector bundle over* $\mathbf{k}$. *Let $D$ be a subspace of dimension $d$ and suppose $\mathbf{e} \in E - D$. Then*

$$\frac{\lambda_1^{\overline{E}}(\langle D, \mathbf{e}\rangle)^d}{q^{2(d+1)g(\mathbf{k})} H_{\overline{E}}(D)}\, \lambda_1^{\overline{E}}([\mathbf{e}]_D) \leq H_{\overline{E}/\overline{D}}([\mathbf{e}]_D) \leq \lambda_1^{\overline{E}}([\mathbf{e}]_D), \tag{3}$$

*where $q$ is the cardinality of the constant field of* $\mathbf{k}$ *and $g(\mathbf{k})$ is the genus of* $\mathbf{k}$.

Proof. The inequality $H_{\overline{E}/\overline{D}}([\mathbf{e}]_D) \leq \lambda_1^{\overline{E}}([\mathbf{e}]_D)$ follows immediately from the definitions. To prove the other inequality we need the following lemma which gives a decomposition for the heights of a subspace[2].

Lemma 3.1. *Let $\overline{E} = (E, \{\|\cdot\|_{\overline{E},v}\}_{v \in \mathcal{M}_{\mathbf{k}}})$ be a pure adelic vector bundle. Let $\overline{D} \subset \overline{E}$ be a sub-bundle of dimension $d$. Then*

$$H_{\overline{E}}(\langle D, \mathbf{e}\rangle) = H_{\overline{E}}(D) H_{\overline{E}/\overline{D}}([\mathbf{e}]_D).$$

Proof. Let $\mathbf{d}_1, \ldots, \mathbf{d}_d$ be a basis for $D$. Clearly it suffices to show that

$$\|\mathbf{d}_1 \wedge \cdots \wedge \mathbf{d}_d \wedge \mathbf{e}\|_{\overline{\wedge^{d+1} E},v} = \|\mathbf{d}_1 \wedge \cdots \wedge \mathbf{d}_d\|_{\overline{\wedge^d E},v} \inf_{\mathbf{e}' \in [\mathbf{e}]} \|\mathbf{e}'\|_{\overline{E},v}$$

for all $v \in \mathcal{M}_{\mathbf{k}}$. Fix $v \in \mathcal{M}_{\mathbf{k}}$. Since $\overline{E}$ is pure we can find, by [**11**, Ch. II-2 Thm. 1], a basis $\mathbf{f}_1, \ldots, \mathbf{f}_n$ of $E_v$ such that

(i) $\|\gamma_1 \mathbf{f}_1 + \cdots + \gamma_n \mathbf{f}_n\|_{\overline{E},v} = \sup_{1 \leq i \leq n} |\gamma_i|_v$, for all $\gamma_1, \ldots, \gamma_n \in \mathbf{k}_v$

(ii) $\mathbf{d}_{k+1} \in \langle \mathbf{f}_n, \ldots, \mathbf{f}_{n-k}\rangle$ for all $k = 0, \ldots, d-1$ and $\mathbf{e} \in \langle \mathbf{f}_n, \ldots, \mathbf{f}_{n-d}\rangle$.

Write $\mathbf{d}_k = \sum_{i=1}^{k} \alpha_{ki} \mathbf{f}_{n-i+1}$ for $k = 1, \ldots, d$ and $\mathbf{e} = \sum_{i=1}^{d+1} \beta_i \mathbf{f}_{n-i+1}$, then an easy calculation shows that (ii) implies that

$$\|\mathbf{d}_1 \wedge \cdots \wedge \mathbf{d}_d\|_{\overline{\wedge^d E},v} = |\alpha_{11}\alpha_{22} \cdots \alpha_{dd}|_v$$

and

$$\|\mathbf{d}_1 \wedge \cdots \wedge \mathbf{d}_d \wedge \mathbf{e}\|_{\overline{\wedge^{d+1} E},v} = |\alpha_{11}\alpha_{22} \cdots \alpha_{dd}\beta_{d+1}|_v.$$

It remains to show that $|\beta_{d+1}|_v = \inf_{\mathbf{e}' \in [\mathbf{e}]} \|\mathbf{e}'\|_{\overline{E},v}$. To this end, note that by construction

[2] The same result, although stated in terms of orthogonal projection was first proven over number fields for the standard $L^2$-height by J. Vaaler, see [**10**, Lemma 4].

$[\mathbf{e}]_{D_v} = [\beta_{d+1}\mathbf{f}_{n-d}]_{D_v}$ and hence $\inf\limits_{\mathbf{e}' \in [\mathbf{e}]} \|\mathbf{e}'\|_{\overline{E},v} \le |\beta_{d+1}|_v$. On the other hand any $\mathbf{d} \in D_v$ can be written as $\sum\limits_{i=1}^{d} \gamma_i \, \mathbf{f}_{n-i+1}$ and so

$$\|\mathbf{e} - \mathbf{d}\|_{\overline{E},v} = \Big\| \sum_{i=1}^{d+1} \beta_i \, \mathbf{f}_{n-i+1} - \sum_{i=1}^{d} \gamma_i \, \mathbf{f}_{n-i+1} \Big\|_{\overline{E},v} \ge |\beta_{d+1}|_v \, .$$

□

Now we can quickly finish the proof of Proposition 3.1. By the uniform Sigel's lemma for global function fields, see [**5**, Corollary 3.3], there exists $\mathbf{f}$ belonging to $\langle D, \mathbf{e} \rangle$ but not belonging to $D$ such that

$$H_{\overline{E}}(\mathbf{f}) \le \frac{q^{2(d+1)g(\mathbf{k})} H_{\overline{E}}(\langle D, \mathbf{e} \rangle)}{\lambda_1^{\overline{E}}(\langle D, \mathbf{e} \rangle)^d}.$$

By definition $\lambda_1^{\overline{E}}([\mathbf{e}]_D) \le H_{\overline{E}}(\mathbf{f})$ and hence Lemma 3.1, yields

$$\lambda_1^{\overline{E}}([\mathbf{e}]_D) \le \frac{q^{2(d+1)g(\mathbf{k})} H_{\overline{E}}(D) H_{\overline{E}/\overline{D}}([\mathbf{e}]_D)}{\lambda_1^{\overline{E}}(\langle D, \mathbf{e} \rangle)^d} \, .$$

□

## 4 - Heights of linear transformations

Let us start by recalling the definition of the operator height for linear transformations and compare it with $H_{\overline{E},\overline{F}} := H_{Hom_{\mathbf{k}}(\overline{E},\overline{F})}$. So let $\overline{E}$ and $\overline{F}$ be two adelic vector bundles over $\mathbf{k}$-vector spaces. Given $T \in Hom_{\mathbf{k}}(E, F)$, set:

$$H^{op}_{\overline{E},\overline{F}}(T) := \sup_{\mathbf{e} \in E} \frac{H_{\overline{F}}\big(T(\mathbf{e})\big)}{H_{\overline{E}}(\mathbf{e})} = \sup_{[\mathbf{e}]_D \in E/D} \frac{H_{\overline{F}}\big(T(\mathbf{e})\big)}{\lambda_1^{\overline{E}}(D + \mathbf{e})}$$

where $D = \ker(T)$. The function $H^{op}_{\overline{E},\overline{F}}$ is the *operator height* on $Hom_{\mathbf{k}}(E, F)$ associated to $\overline{E}$ and $\overline{F}$. If $\overline{E} = \overline{F}$ we will use $H^{op}_{\overline{E}}$ (respectively $H_{\overline{E}}$) instead of $H^{op}_{\overline{E},\overline{E}}$ (respectively $H_{\overline{E},\overline{E}}$). The main goal of this section is to prove a comparison result between $H^{op}_{\overline{E},\overline{F}}$ and $H_{\overline{E},\overline{F}}$, which will be used in the proof of Theorem 5.1. Clearly $H^{op}_{\overline{E},\overline{F}}(T) \le H_{\overline{E},\overline{F}}(T)$, so our next objective is to prove a reverse inequality where, for non invertible linear transformations, some arithmetic constants, such as the height of the kernel, will appear, see Proposition 4.2. We start with a preparatory result that not only establishes a useful alternative description for $H_{\overline{E},\overline{F}}$ but also proves that $H_{\overline{E},\overline{F}}(T) = H^{op}_{\overline{E},\overline{F}}(T)$ if $T$ is an injective linear transformation.

Proposition 4.1. *Let $\overline{E}$ and $\overline{F}$ be pure adelic vector bundles over* $\mathbf{k}$. *Given* $T$ *in* $Hom_{\mathbf{k}}(E, F)$, *set* $D = \ker T \subset E$. *Then:*

$$H_{\overline{E},\overline{F}}(T) = \sup_{[\mathbf{e}] \in E/D} \frac{H_{\overline{F}}(T(\mathbf{e}))}{H_{\overline{E/D}}([\mathbf{e}]_D)}.$$

*In particular if* $T$ *is injective we have* $H_{\overline{E},\overline{F}}(T) = H^{op}_{\overline{E},\overline{F}}(T)$.

Proof. Clearly

$$H_{\overline{E},\overline{F}}(T) = \prod_{v \in \mathcal{M}_{\mathbf{k}}} \sup_{\mathbf{e} \in E_v^\times} \frac{\|T(\mathbf{e})\|^{n_v}_{\overline{F},v}}{\|\mathbf{e}\|^{n_v}_{E,v}} \geq \sup_{[\mathbf{e}]_D \in \overline{E}/\overline{D}} \frac{H_{\overline{F}}(T(\mathbf{e}))}{H_{\overline{E/D}}([\mathbf{e}]_D)}.$$

To prove the reverse inequality we need the following:

Lemma 4.1. *Under the hypotheses of Proposition* 4.1 *there exists a finite set of places* $\mathcal{S} \subset \mathcal{M}_{\mathbf{k}}$ *and a subspace* $G \subset E$ *of dimension equal to the rank of* $T$ *such that for all* $v \notin \mathcal{S}$ *we have*

(a) $\inf_{\mathbf{g}' \in [\mathbf{g}]_{D_v}} \|\mathbf{g}'\|_{\overline{E},v} = \|\mathbf{g}\|_{\overline{E},v}$ for all $\mathbf{g} \in G_v$;

(b) $\|T(\mathbf{g})\|_{\overline{F},v} = \|\mathbf{g}\|_{E,v}$ for all $\mathbf{g} \in G_v$.

Proof. From the definition of adelic vector bundles and the fact that we are proving a statement for all but finitely many places, it follows that we can assume that $\overline{E} = (\mathbf{k}^n, \{\|\cdot\|_v\}_{v \in \mathcal{M}_{\mathbf{k}}})$, $\overline{F} = (\mathbf{k}^m, \{\|\cdot\|_v\}_{v \in \mathcal{M}_{\mathbf{k}}})$, where $\|\cdot\|_v$ is the sup norm on $\mathbf{k}_v^n$ and $\mathbf{k}_v^m$. If $m = n$ and $T$ is invertible (a) is trivial and (b) is equivalent to say that an invertible $n \times n$ matrix with coefficients in $\mathbf{k}$ actually belongs to $\mathrm{GL}_n(\mathcal{O}_v)$ for all but finitely many $v \in \mathcal{M}_{\mathbf{k}}$. In general let $r = \mathrm{rank}\,(T)$, and choose $\phi$ to be an automorphism of $\mathbf{k}^n$ such that $\ker T = \phi(U)$ where $U$ is the subspace generated the last $n - r$ vectors of the standard basis of $\mathbf{k}^n$. Moreover choose $\psi$ to be an automorphism of $\mathbf{k}^m$ mapping $W = \mathrm{Im}\,(T)$ onto the subspace generated by the first $r$ vectors of the standard basis of $\mathbf{k}^m$. Finally we let $G$ be the image via $\phi$ of the subspace generated by the first $r$ vectors of the standard basis of $\mathbf{k}^n$. Since $\phi$ is invertible there exists a finite set $\mathcal{S}_\phi \subset \mathcal{M}_{\mathbf{k}}$ such that $\phi$ preserves $\|\cdot\|_v$ for all $v \notin \mathcal{S}_\phi$. For $\mathbf{g} \in G_v$ and $v \notin \mathcal{S}_\phi$, we have:

$$\inf_{\mathbf{g}' \in [\mathbf{g}]_{D_v}} \|\mathbf{g}'\|_v = \inf_{\mathbf{d} \in D_v} \|\phi^{-1}(\mathbf{g}) - \phi^{-1}(\mathbf{d})\|_v = \inf_{\mathbf{u} \in U_v} \|\phi^{-1}(\mathbf{g}) - \mathbf{u}\|_v = \|(\mathbf{g})\|_v$$

proving (a). To prove (b) let $\mathcal{S}_\psi \subset \mathcal{M}_{\mathbf{k}}$ be the finite subset such that $\psi$ preserves $\|\cdot\|_v$

for all $v \notin \mathcal{S}_\psi$, and set $\mathcal{S} = \mathcal{S}_\phi \cup \mathcal{S}_\psi$. Given $\mathbf{g} \in G$ and $v \in \mathcal{M}_{\mathbf{k}} - \mathcal{S}$, we have:

$$\|\mathbf{g}\|_= \|T(\mathbf{g})\|_{M_v} \Longleftrightarrow \|\phi^{-1}(\mathbf{g})\|_v = \|\left(\psi_{|_{T(G)}} \circ T \circ \phi_{|_{\phi^{-1}(G)}}\right)(\phi^{-1}(\mathbf{g}))\|_v.$$

But $\psi_{|_{T(G)}} \circ T \circ \phi_{|_{\phi^{-1}(G)}} : \phi^{-1}(G) \to \psi(W)$ is an invertible linear transformation between vector spaces of the same dimension, and so (b) follows. □

Let $G \subset E$ and $\mathcal{S} \subset \mathcal{M}_{\mathbf{k}}$ be as in the conclusion of Lemma 4.1. Given $\mathbf{e} \in E$ write $\mathbf{e} = \mathbf{d} + \mathbf{g}$ with $\mathbf{g} \in G$ and $\mathbf{d} \in D$. By Lemma 4.1 we have that

$$\frac{\|T(\mathbf{e})\|_{\overline{F},v}}{\|\mathbf{e}\|_{E,v}} = \frac{\|T(\mathbf{g})\|_{\overline{F},v}}{\|\mathbf{e}\|_{E,v}} \leq \frac{\|T(\mathbf{g})\|_{\overline{F},v}}{\inf_{\mathbf{g}' \in [\mathbf{g}]_{D_v}} \|\mathbf{g}'\|_{\overline{E},v}} = \frac{\|T(\mathbf{g})\|_{\overline{F},v}}{\|\mathbf{g}\|_{\overline{E},v}} = 1$$

for all $v \notin \mathcal{S}$. Hence $\|T\|_{\overline{E},\overline{F},v} = 1$ for all $v \notin \mathcal{S}$, and so

$$H_{\overline{E},\overline{F}}(T) = \prod_{v \in \mathcal{S}} \|T\|^{n_v}_{\overline{E},\overline{F},v}. \tag{4}$$

A second consequence of Lemma 4.1 is that for all $\mathbf{e} \notin D$ we have

$$\frac{H_{\overline{F}}(T(\mathbf{e}))}{H_{\overline{E}/\overline{D}}([\mathbf{e}]_D)} = \prod_{v \in \mathcal{S}} \frac{\|T(\mathbf{e})\|^{n_v}_{\overline{F},v}}{\inf_{\mathbf{d} \in D_v} \|\mathbf{e} + \mathbf{d}\|^{n_v}_{\overline{E},v}}. \tag{5}$$

Now given $\varepsilon > 0$ choose $\delta > 0$ so that $\prod_{v \in \mathcal{S}} \|T\|^{n_v}_{\overline{E},\overline{F},v} < \varepsilon + \prod_{v \in \mathcal{S}} \left(\|T\|^{n_v}_{\overline{E},\overline{F},v} - \delta\right)$. By the strong approximation theorem we can find $\mathbf{e} \in E$ such that

$$\|T\|^{n_v}_{\overline{E},\overline{F},v} - \delta \leq \frac{\|T(\mathbf{e})\|^{n_v}_{\overline{F},v}}{\|\mathbf{e}\|^{n_v}_{\overline{E},v}} \leq \frac{\|T(\mathbf{e})\|^{n_v}_{\overline{F},v}}{\inf_{\mathbf{d} \in D_v} \|\mathbf{e} + \mathbf{d}\|^{n_v}_{\overline{E},v}}.$$

Taking the product over $v \in \mathcal{S}$ and using (4) and (5) yields

$$H_{\overline{E},\overline{F}}(T) = \prod_{v \in \mathcal{S}} \|T\|^{n_v}_{\overline{E},\overline{F},v} < \varepsilon + \prod_{v \in \mathcal{S}} \left(\|T\|^{n_v}_{\overline{E},\overline{F},v} - \delta\right) \leq \varepsilon + \frac{H_{\overline{F}}(T(\mathbf{e}))}{H_{\overline{E}/\overline{D}}([\mathbf{e}]_D)}$$

completing the proof of the proposition. □

Corollary 4.1. *Let $\overline{E}$ and $\overline{F}$ be pure adelic vector bundles over* $\mathbf{k}$*. Suppose $T$ in $Hom_{\mathbf{k}}(E, F)$ is injective. Then $H_{\overline{E},\overline{F}}(T) = H^{op}_{\overline{E},\overline{F}}(T)$.*

We are now in the position to prove the main result of this section.

Proposition 4.2. *Let $\overline{E}$ and $\overline{F}$ be pure adelic vector bundles over* $\mathbf{k}$*. Given $T \in Hom_{\mathbf{k}}(E, F)$ let $D = \ker T$ and $d = \dim_{\mathbf{k}} D$.*

(a) *If* $1 \leq d < n-1$, *then*

$$H^{op}_{\overline{E},\overline{F}}(T) \leq H_{\overline{E},\overline{F}}(T) \leq \frac{q^{2(d+1)g(\mathbf{k})} H_{\overline{E}}(D)}{\lambda_1^{\overline{E}}(E)^d} H^{op}_{\overline{E},\overline{F}}(T).$$

(b) *If* $d = n-1$, *then* $H_{\overline{E},\overline{F}}(T) = \frac{\lambda_1^{\overline{E}}(E-D) H_{\overline{E}}(D)}{H_{\overline{E}}(E)} H^{op}_{\overline{E},\overline{F}}(T)$.

Proof. (a) We have:

$$\begin{aligned} H_{\overline{E},\overline{F}}(T) &= \sup_{[\mathbf{e}]_D \in E/D} \frac{H_{\overline{F}}\big(T(\mathbf{e})\big)}{H_{\overline{E}/\overline{D}}([\mathbf{e}]_D)} && \text{by Proposition 4.1} \\ &\leq \sup_{[\mathbf{e}]_D \in E/D} \frac{q^{2(d+1)g(\mathbf{k})} H_{\overline{E}}(D)}{\lambda_1^{\overline{E}}(\langle D, \mathbf{e}\rangle)^d} \frac{H_{\overline{F}}\big(T(\mathbf{e})\big)}{\lambda_1^{\overline{E}}([\mathbf{e}]_D)} && \text{by Proposition 3.1} \\ &\leq \frac{q^{2(d+1)g(\mathbf{k})} H_{\overline{E}}(D)}{\lambda_1^{\overline{E}}(\langle E\rangle)^d} \sup_{[\mathbf{e}]_D \in E/D} \frac{H_{\overline{F}}\big(T(\mathbf{e})\big)}{\lambda_1^{\overline{E}}([\mathbf{e}]_D)} && \text{for } \lambda_1^{\overline{E}}(E) \leq \lambda_1^{\overline{E}}(\langle D, \mathbf{e}\rangle) \\ &= \frac{q^{2(d+1)g(\mathbf{k})} H_{\overline{E}}(D)}{\lambda_1^{\overline{E}}(E)^d} H^{op}_{\overline{E},\overline{F}}(T) \end{aligned}$$

proving (a). To prove (b) note that since $\dim_{\mathbf{k}}(D) = n-1$ we have $\langle D, \mathbf{e}\rangle = E$ for any $\mathbf{e} \notin D$. It follows from Proposition 3.1 and Proposition 4.1 that for any $\mathbf{e} \in E - D$ we have $H_{\overline{E}}(E) H_{\overline{E},\overline{F}}(T) = H_{\overline{F}}\big(T(\mathbf{e})\big) H_{\overline{E}}(D)$. On the other hand $\lambda_1^{\overline{E}}(E-D) H^{op}_{\overline{E},\overline{F}}(T) = H_{\overline{F}}\big(T(\mathbf{e})\big)$, proving (b). □

## 5 - The spectral height

The goal of this section is to define the spectral height on the endomorphism ring of a $\mathbf{k}$-vector space $E$ and prove the analogue of the spectral radius formula for operator heights associated to adelic vector bundles over $\operatorname{spec} \mathbf{k}$ having $E$ as support. Let $(X, \|\cdot\|)$ be a finite dimensional normed space over $\mathbf{C}_v$. Given $T \in \operatorname{End}_{\mathbf{C}_v}(X)$, the *spectral radius of* $T$ is:

$$\rho_v(T) = \sup_{\lambda \in \operatorname{sp}(T)} |\lambda|_{\mathbf{C}_v}$$

where $\operatorname{sp}(T)$ denotes the set of roots of the minimal polynomial of $T$.

Proposition 5.1 (Local spectral formula). *Let $(X, \|\cdot\|)$ be a finite dimensional normed space over $\mathbf{C}_v$. For all $T \in \mathrm{End}_{\mathbf{C}_v}(X)$ we have:*

$$\lim_{m \to \infty} \|T^m\|^{1/m} = \rho_v(T). \tag{6}$$

Proof. This result should be well known, but since we could not find a reference for it, we provide a sketch of its proof. First, note that if $T$ is nilpotent both sides are 0, and so there is nothing to prove. Hence we may assume that $T$ is not nilpotent. Since all norms on $X$ are equivalent we only have to prove the limit formula for one norm. We are going to use the operator norm relative to the sup norm attached to a basis of $X$. The key point being that for such a norm the corresponding operator norm of $T$ is simply the maximum of the absolute value of the entries of the matrix representing T with respect to the chosen basis. If $\rho_v(T) > 1$ we choose a basis $\mathcal{B}$ having the property that the matrix of $T$ with respect to $\mathcal{B}$ is the Jordan normal form. Clearly (6) follows. If $\rho_v(T) < 1$, we choose a basis in such a way that non-zero entries not on the diagonal have absolute value strictly smaller than $\rho_v(T)$. Again (6) follows at once. □

Now we go back to global function fields and define the spectral height:

Definition 5.1. Let $T \in \mathrm{End}_{\mathbf{k}}(E)$, where $E$ is a finite dimensional $\mathbf{k}$-vector space. Let $T_v$ denote the linear transformation induced by $T$ on $E \otimes_{\mathbf{k}} \mathbf{C}_v$. If $T$ is not nilpotent then the *spectral height* of $T$ is

$$H_s(T) = \prod_{v \in \mathcal{M}_{\mathbf{k}}} \rho_v(T_v)^{n_v} \tag{7}$$

while if $T$ is nilpotent we set $H_s(T) = 0$.

It is straightforward to verify that the spectral height enjoys the following properties, as their proof follows directly from the analogous properties of the spectrum of linear transformations:

(S1) $H_s(\lambda T) = H_s(T)$, for all $\lambda \in \mathbf{k}^\times$;

(S2) $H_s(T) \geq 1$;

(S3) $H_s(T^m) = H_s(T)^m$, for all $m \geq 1$;

(S4) if $T, T' \in \mathrm{End}_{\mathbf{k}}(X)$ commute, then $H_s(TT') \leq H_s(T)H_s(T')$;

(S5) $H_s$ is invariant under conjugation.

As it is apparent from (S5) the Northcott finiteness theorem does not hold for $H_s$. The main result of this section is the following:

Theorem 5.1. *Let* $\mathbf{k}$ *be a global function field, and* $\overline{E} = (E, \{\|\cdot\|_{\overline{E},v}\}_{v\in\mathcal{M}_{\mathbf{k}}})$ *be an adelic vector bundle over* $\mathbf{k}$*. Let* $T$ *belong to* $\mathrm{End}_{\mathbf{k}}(E)$*, then*

(a) $\lim_{m\to\infty} H_{\overline{E}}(T^m)^{1/m} = H_s(T)$;

(b) $\lim_{m\to\infty} H^{op}_{\overline{E}}(T^m)^{1/m} = H_s(T)$.

Proof. First of all note that (a) follows directly from the local spectral formula. (b) If $T$ is nilpotent there is nothing to prove since both sides are zero. Let $D_m = \ker T^m$ and $d_m = \dim_{\mathbf{k}} D_m$. If $d_1 = 0$ then $H^{op}_{\overline{E}}(T^m) = H_{\overline{E}}(T^m)$ for all $m$, and so (b) follows. If $d_1 = n-1$ then by Proposition 4.2.(b) we have

$$\frac{H_{\overline{E}}(T^m)H_{\overline{E}}(E)}{\lambda_1^{\overline{E}}(E-D_m)H_{\overline{E}}(D_m)} = H^{op}_{\overline{E}}(T^m)$$

for all $m \geq 1$. Since $D_k = D_h$ for $h, k \geq n$ we have that (b) follows from (a). Lastly suppose that $0 < d < n-1$. By Proposition 4.2.(a) we have

$$H_{\overline{E}}(T^m)\frac{\lambda_1^{\overline{E}}(E)^d}{q^{2(d_m+1)g}H_{\overline{E}}(D_m)} \leq H^{op}_{\overline{E}}(T^m) \leq H_{\overline{E}}(T^m)$$

for all $s \geq 1$. As we noted before $D_h = D_k$ for all $h, k \geq n$ and hence

$$\lim_{m\to\infty}\left(\frac{\lambda_1^{\overline{E}}(E)^d}{q^{2(d_m+1)g}H_{\overline{E}}(D_m)}\right)^{1/m} = 1.$$

Again (b) follows from (a). □

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