Riv. Mat. Univ. Parma, Vol. **7** (2016), 217-237

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Formal period integrals and special value formulas

Abstract. Motivated by the conjectures of Gan-Gross-Prasad, we develop a p-adic formalism for placing these conjectures in a p-adic setting which is suited for p-adic interpolation.

Keywords. Special values of L-functions, p-adic L-Functions.

Mathematics Subject Classification (2010): 11F67, 11F33.

1 - Introduction

Suppose that $\eta : \mathbf{H} \subset \mathbf{G}$ is an inclusion of algebraic subgroups (over $\mathbb{Q}$, for simplicity, in this introduction) such that $\mathbf{H}(\mathbb{R})$ and $\mathbf{G}(\mathbb{R})$ are connected and such that $\mathbf{G}(\mathbb{R})/\mathbf{S}_{\mathbf{G}}(\mathbb{R})$ is compact, where $\mathbf{S}_{\mathbf{G}} \subset \mathbf{Z}_{\mathbf{G}}$ is the maximal split torus in the center $\mathbf{Z}_{\mathbf{G}}$ of $\mathbf{G}$. Let ω be a unitary Hecke character and let $\omega^{\eta} : \mathbf{H}(\mathbb{A}) \to \mathbb{C}^{\times}$ be a continuous character trivial on $\mathbf{H}(\mathbb{Q})$ such that $\omega^{\eta}_{|\mathbf{S}_{\mathbf{H}}(\mathbb{A})} = \omega_{|\mathbf{S}_{\mathbf{H}}(\mathbb{A})}$. Motivated by the Gan-Gross-Prasad conjectures recalled below and other special value formulas, the paper [**6**] investigates period integrals of the form

$$I_{\eta}(f) := \int_{[\mathbf{H}(\mathbb{A})]_{\mathbf{S}_{\mathbf{H}}}} f(\eta(x))\omega^{-\eta}(x)d\mu_{[\mathbf{H}(\mathbb{A})]_{\mathbf{S}_{\mathbf{H}}}}(x), \tag{1}$$

where $f \in L^2(\mathbf{G}(\mathbb{A})/\mathbf{G}(F), \omega)$, $[\mathbf{H}(\mathbb{A})]_{\mathbf{S}_{\mathbf{H}}} := \mathbf{S}_{\mathbf{H}}(\mathbb{A})\backslash\mathbf{H}(\mathbb{A})/\mathbf{H}(\mathbb{Q})$ (for $\mathbf{S}_{\mathbf{H}}$ the maximal split torus in the center of $\mathbf{H}$) and the measures are normalized as in § 2. In particular, we refer the reader to (6) for the relation between the measure $\mu_{[\mathbf{H}(\mathbb{A})]_{\mathbf{S}_{\mathbf{H}}}}$, $\mu_{[\mathbf{H}(\mathbb{A}_f)]_{\mathbf{S}_{\mathbf{H}}}}$ on $[\mathbf{H}(\mathbb{A}_f)]_{\mathbf{S}_{\mathbf{H}}} := \mathbf{S}_{\mathbf{H}}(\mathbb{A}_f)\backslash\mathbf{H}(\mathbb{A}_f)/\mathbf{H}(\mathbb{Q})$ and $\mu_{\mathbf{S}_{\mathbf{H}}\backslash\mathbf{H},\infty}$

Received: January 12, 2016; accepted in revised form: March 1, 2016.

on $\mathbf{S_H}(\mathbb{R})\backslash\mathbf{H}(\mathbb{R})$. We set

$$m_{\mathbf{S_H}\backslash\mathbf{H},\infty} := \mu_{\mathbf{S_H}\backslash\mathbf{H},\infty}(\mathbf{S_H}(\mathbb{R})\backslash\mathbf{H}(\mathbb{R})).$$

It is proved there that, if we restrict I_η to a suitable subspace of "algebraic automorphic forms", this rule extends to a morphism of functors from modular forms defined over E-algebras to $\mathbf{A}^1$, where $E/\mathbb{Q}$ is a Galois splitting field of $\mathbf{G}$. Namely, it is possible to express $m^{-1}_{\mathbf{S_H}\backslash\mathbf{H},\infty} I_\eta$ as a functor

$$m^{-1}_{\mathbf{S_H}\backslash\mathbf{H},\infty} I_\eta = J_\eta : M[\mathbf{G},\rho,\omega_0]_{/E(\omega_f)} \to \mathbf{A}^1_{/E(\omega_f)}, \tag{2}$$

where ω_0 is an appropriate twist of ω_f and $M[\mathbf{G},\rho,\omega_0]$ is a suitable space of Gross' style algebraic modular forms, as we are going to explain.

Let us write $A(\mathbf{G}(\mathbb{A}),\omega) \subset L^2(\mathbf{G}(\mathbb{A})/\mathbf{G}(\mathbb{Q}),\omega)$ for the dense $\mathbf{G}(\mathbb{A})$-submodule of finite vectors: it is the space of functions f which are right $\mathbf{G}(\mathbb{R})$-finite and such that there exists an open and compact subgroup $K \subset \mathbf{G}(\mathbb{A}_f)$ such that $f(ux) = f(x)$ for every $u \in K$ (see [**6**, §2]). We may write

$$A(\mathbf{G}(\mathbb{A}),\omega) = \bigoplus_{\pi^u_\infty} A(\mathbf{G}(\mathbb{A}),\omega)[\pi^u_\infty], \tag{3}$$

where π^u_∞ runs over all unitary irreducible representations of $\mathbf{G}(\mathbb{R})$ with central character ω^{-1}_∞. Let us suppose, for simplicity, that $\mathbf{G}(\mathbb{R})$ is compact. Then the Borel-Weil theorem implies the existence of (canonical) rational models ρ of π^u_∞ over E and the $\mathbb{C}$-points of the source of (2) are identified, by means of an adelic Peter-Weyl Theorem (see [**6**]), with

$$M[\mathbf{G},\rho,\omega_0]_{/E(\omega_f)}(\mathbb{C}) \simeq A(\mathbf{G}(\mathbb{A}),\omega)[\pi^u_\infty]. \tag{4}$$

When $\mathbf{G}(\mathbb{R})$ is possibly non-compact (as in our application to the interpolation problem), i.e. $\mathbf{S_G}(\mathbb{R}) \neq \{1\}$, it is important to take into account possible twists π_∞ of π^u_∞.

As already remarked, our interest in (1) is motivated by the Gan-Gross-Prasad conjectures and its refinements (see [**9**]). Roughly, for suitable algebraic groups of the form $\mathbf{G} = \mathbf{G}_V \times \mathbf{G}_W$ with $\mathbf{G}_V$ and $\mathbf{G}_W$ (the connected component of) symplectic, orthogonal or unitary groups, they specify $\eta : \mathbf{H} \subset \mathbf{G}$ for which the equality

$$|I_\eta(f)|^2 = \frac{1}{2^\beta}\frac{\Delta_{\mathbf{G}_V} L(1/2,\pi_V \boxtimes \pi_W)}{L(1,\pi_V,\mathrm{Ad})L(1,\pi_W,\mathrm{Ad})}\prod_v \alpha_v(f_v) \tag{5}$$

should hold. Here $f = \otimes_v f_v \in \pi_V \boxtimes \pi_W$, the α_v's are appropriately regularized integral of matrix coefficients which should be non-zero on $\pi_{V,v} \otimes \pi_{W,v} \otimes \nu_v^{-1}$ up to changing $\mathbf{G}$ by a pure inner form $\mathbf{G}'_\pi$ (see [**9**, Conjecture 2.5 (2)]), $\Delta_{\mathbf{G}_V}$ is a product of abelian L-values (attached to dual Gross motives) and β is an integer. The L-function $L(s,\pi_V\boxtimes\pi_W)$ (resp. $L(s,\pi_V,\mathrm{Ad})$ and $L(s,\pi_W,\mathrm{Ad})$) is the complex L-function attached to $\pi_V\boxtimes\pi_W$ (resp. π_V and π_W) taken with respect to the standard (resp. adjoint) representation of $\mathbf{G}$ (resp. $\mathbf{G}_V$ and $\mathbf{G}_W$). We refer the reader to the introduction of [**6**] for the references where all of these terms are precisely defined. Here we remark that the local terms α_v are "almost" of the form

$$\alpha_v(f_v)=\frac{L(1,\pi_{V,v},\mathrm{Ad})L(1,\pi_{W,v},\mathrm{Ad})}{\Delta_{\mathbf{G}_{V,v}}L(1/2,\pi_{V,v}\boxtimes\pi_{W,v})}\cdot\int_{\mathbf{H}_{V,v}(\mathbb{Q}_v)}\langle\pi_{V,v}(h_v)f_{V,v},f_{V,v}\rangle\overline{\langle\pi_{W,v}(h_v)f_{W,v},f_{W,v}\rangle}\,d\mu_{\mathbf{H}_{V,v}(\mathbb{Q}_v)}$$

when $f_v=f_{V,v}\otimes f_{W,v}$ is a pure tensor. "Almost" means that they are of these form when the integral makes sense, but in general they need to be regularized. Formula (5) is known in many cases (see the references in [**6**]).

The period integral (1) is studied in [**6**] which, together with (5), yields special value formulas which are suitable for p-adic interpolation: p-adic L-functions arise from p-adic variation of $J_\eta=m^{-1}_{\mathbf{S_H}\backslash\mathbf{H},\infty}I_\eta$. More precisely, suppose we have given a p-adic parameter space $\mathcal{X}$, a subset $\Sigma\subset\mathcal{X}$ and a family $\{f_k\}_{k\in\Sigma}$ of automorphic forms on $\mathbf{G}$. Then one may ask if it is possible to extend the association $k\mapsto J_\eta(f_k)$ on Σ to a continuous function $\kappa\mapsto J_\eta(\kappa)$ on $\mathcal{X}$. Guided by this ideas, we develop here a general formalism for placing these formal period integrals J_η in a p-adic setting. As explained at the beginning of § 5, this is obtained as a combination of the special value formulas proved in [**6**] and Proposition 4.1 and yields a p-adic avatar $J_{\eta,p}$ of J_η. Then we can consider $k\mapsto J_\eta(f_k)=J_{\eta,p}(f_k)$ on Σ and the problem is, rather, to extend this association to a continuous function $\kappa\mapsto J_{\eta,p}(\kappa)$ on $\mathcal{X}$. The reason of this change is that, in order to fulfill our proposal, we first need to extend the association $k\mapsto f_k$ to an association $\kappa\mapsto f_\kappa$, i.e. we need p-adic families, and it is $J_{\eta,p}(\kappa):=J_{\eta,p}(f_\kappa)$ which makes sense for a general $\kappa\in\mathcal{X}$, rather than $J_\eta(f_\kappa)$.

Indeed, motivated by the rationality result (2), we expect that these periods could be frequently p-adically interpolated and we hope this formalism could be useful in order to address this issue: we exemplify our philosophy in the classical case [**11**] by constructing a functional on p-adic families of modular forms which interpolates the functional I_η in this case. The result is the p-adic L-function considered in [**2**], with the assumptions on the conductor of the character (arising from use of the explicit Waldspurger's formula by Hui and Hatcher in loc. cit.) removed (because we use [**11**] and [**6**]). Hence we will consider p-adic variation of the weight variable $\mathcal{X}:=Hom_{cts}(\mathbb{Z}_p^\times,\mathbf{G}_m)$ for p-adic families of automorphic forms on $\mathbf{GL}_2$, but other variations, such as those of the Galois variable, are still interesting. Examples of these kind of variations can be obtained (under our assumptions) considering p-adic families of automorphic forms on the algebraic group attached to a quadratic imaginary field, i.e. p-adic families of Hecke characters attached to such a kind of fields. Of course, one can combine the two variations in order to obtain two variable p-adic L-functions. Another example of our methods is provided by [**6**].

2 - Automorphic forms and the period integrals

In this section and the following, we recall the theory from [**6**]. Let $\mathbf{G}$ be a reductive algebraic group over a field F with adele ring $\mathbb{A} = \mathbb{A}_f \times F_\infty$ and let $\mathbf{Z_G}$ be its center. We write $\mathcal{K} = \mathcal{K}(\mathbf{G}(\mathbb{A}_f))$ for the set of open and compact subgroups of $\mathbf{G}(\mathbb{A}_f)$. For a closed, algebraic subgroup $\mathbf{Z}$ of $\mathbf{Z_G}$, set

$$[\mathbf{G}(\mathbb{A})]_{\mathbf{Z}} := \mathbf{Z}(\mathbb{A})\backslash\mathbf{G}(\mathbb{A})/\mathbf{G}(F) \quad \text{and} \quad [\mathbf{G}(\mathbb{A}_f)]_{\mathbf{Z}} := \mathbf{Z}(\mathbb{A}_f)\backslash\mathbf{G}(\mathbb{A}_f)/\mathbf{G}(F).$$

Setting $\mathbf{PG_Z} = \mathbf{G}/\mathbf{Z}$, we make the following assumptions on the pair $(\mathbf{G}, \mathbf{Z})$: (A1) $\mathbf{PG_Z}(F_\infty)$ is compact, (A2) $\Delta_{\mathbf{G},f}$ embeds $\mathbf{G}(F)$ as a discrete subgroup of $\mathbf{G}(\mathbb{A}_f)$ and (A3) $\mathbf{G}(\mathbb{A}_f)/\mathbf{G}(F)$ is compact. Let us remark that, when $\mathbf{Z} = \mathbf{S_G}$, is the maximal split torus in the center of $\mathbf{G}$, and $F = \mathbb{Q}$, it is proved in [8, Proposition 1.4] that (A1) is indeed equivalent to (A2) and to (A3).

For the remainder of this paper, we suppose that we are given two pairs $(\mathbf{H}, \mathbf{Z}^{\mathbf{H}})$ and $(\mathbf{G}, \mathbf{Z}) = (\mathbf{G}, \mathbf{Z}^{\mathbf{G}})$ as above and a morphism of algebraic groups

$$\eta : \mathbf{H} \longrightarrow \mathbf{G}$$

such that $\eta(\mathbf{Z}^{\mathbf{H}}) \subset \mathbf{Z}^{\mathbf{G}}$. We assume (A1), (A2) and (A3) for the pairs $(\mathbf{H}, \mathbf{Z}^{\mathbf{H}})$ and $(\mathbf{G}, \mathbf{Z}^{\mathbf{G}})$. In addition, we impose the following normalizations to the measures obtained from the couple $(\mathbf{H}, \mathbf{Z}^{\mathbf{H}})$ (see [**6**, § 2] for details). We may normalize the non-zero left $\mathbf{H}(\mathbb{A}_f)$-invariant Radon measures $\mu_{\mathbf{H}(\mathbb{A}_f)}$ on $\mathbf{H}(\mathbb{A}_f)$, $\mu_{\mathbf{H}(\mathbb{A}_f)/\mathbf{H}(F)}$ on $\mathbf{H}(\mathbb{A}_f)/\mathbf{H}(F)$ and $\mu_{[\mathbf{H}(\mathbb{A}_f)]_{\mathbf{Z}}}$ on $[\mathbf{H}(\mathbb{A}_f)]_{\mathbf{Z}}$ so that $\mu_{\mathbf{H}(\mathbb{A}_f)}(K) \in \mathbb{Q}$ for some (and hence every) $K \in \mathcal{K}(\mathbf{H}(\mathbb{A}_f))$, $\mu_{\mathbf{H}(\mathbb{A}_f)/\mathbf{H}(F)}$ satisfies

$$\int_{\mathbf{H}(\mathbb{A}_f)} f(g)d\mu_{\mathbf{H}(\mathbb{A}_f)}(g) = \int_{\mathbf{H}(\mathbb{A}_f)/\mathbf{H}(F)} \sum_{\gamma} f(g\gamma)d\mu_{\mathbf{H}(\mathbb{A}_f)}(g)$$

and restricts to $\mu_{[\mathbf{H}(\mathbb{A}_f)]_{\mathbf{Z}}}$ on $\mathbf{Z}(\mathbb{A}_f)$-invariant functions

$$C([\mathbf{H}(\mathbb{A}_f)]_{\mathbf{Z}}) \subset C(\mathbf{H}(\mathbb{A}_f)/\mathbf{H}(F))$$

(here $C(X)$ is the set of continuous $\mathbb{C}$-valued functions on X). Furthermore, it easily follows from (A1) and (A2) that we may normalize the left $\mathbf{H}(\mathbb{A})$-invariant (resp.

$\mathbf{H}(F_\infty)$-invariant) non-zero Radon measure $\mu_{[\mathbf{H}(\mathbb{A})]_{\mathbf{Z}}}$ (resp. $\mu_{\mathbf{Z}\backslash\mathbf{H},\infty}$) on $[\mathbf{H}(\mathbb{A})]_{\mathbf{Z}}$ (resp. $\mathbf{Z}(F_\infty)\backslash\mathbf{H}(F_\infty)$) so that the following formula is satisfied:

$$\begin{aligned}(6)\qquad &\int_{[\mathbf{H}(\mathbb{A})]_{\mathbf{Z}}} f(x)d\mu_{[\mathbf{H}(\mathbb{A})]_{\mathbf{Z}}}(x)\\ &= \int_{[\mathbf{H}(\mathbb{A}_f)]_{\mathbf{Z}}} \left(\int_{\mathbf{Z}(F_\infty)\backslash\mathbf{H}(F_\infty)} f(x_f x_\infty) d\mu_{\mathbf{Z}\backslash\mathbf{H},\infty}(x_\infty)\right) d\mu_{[\mathbf{H}(\mathbb{A}_f)]_{\mathbf{Z}}}(x_f).\end{aligned}$$

Fix once and for all a *continuous* and *unitary* character $\omega : \dfrac{\mathbf{Z}^{\mathbf{G}}(\mathbb{A})}{\mathbf{Z}^{\mathbf{G}}(F)} \to \mathbb{C}^\times$. We write $A(\mathbf{G}(\mathbb{A}), \omega) \subset L^2(\mathbf{G}(\mathbb{A})/\mathbf{G}(F), \omega)$ for the dense submodule of finite vectors, a right $\mathbf{G}(\mathbb{A})$-submodule due to the compactness of $\mathbf{PG}_{\mathbf{Z}}(F_\infty)$ (we follow the conventions of [**6**]). In particular, if $\pi_\infty^u \in \mathrm{Irr}^u\big(\mathbf{G}(F_\infty), \omega_\infty^{-1}\big)$ (the set of isomorphism classes of complex unitary irreducible representations of $\mathbf{G}(F_\infty)$ with central character ω_∞^{-1}), it makes sense to consider the π_∞^u-isotypic component $A(\mathbf{G}(\mathbb{A}), \omega)\big[\pi_\infty^u\big]$ of $A(\mathbf{G}(\mathbb{A}), \omega)$.

We suppose that there is a character $\omega^\eta : \mathbf{H}(\mathbb{A}) \to \mathbb{C}^\times$ such that ω^η is trivial on $\mathbf{H}(F)$ and $\omega^\eta_{|\mathbf{Z}^{\mathbf{H}}(\mathbb{A})} = \omega \circ \eta_{\mathbb{A}|\mathbf{Z}^{\mathbf{H}}(\mathbb{A})}$. We write $\omega^{-\eta} := (\omega^\eta)^{-1}$.

Definition 2.1 (Global period integral). Define the *global period integral*

$$I_\eta : L^2(\mathbf{G}(\mathbb{A})/\mathbf{G}(F), \omega) \longrightarrow \mathbb{C}$$

by the rule

$$(7)\qquad I_\eta(f) := \int_{[\mathbf{H}(\mathbb{A})]_{\mathbf{Z}^{\mathbf{H}}}} f(\eta(x))\omega^{-\eta}(x) d\mu_{[\mathbf{H}(\mathbb{A})]_{\mathbf{Z}^{\mathbf{H}}}}(x).$$

As explained in the introduction, the π_∞^u-isotypic part $A(\mathbf{G}(\mathbb{A}), \omega)\big[\pi_\infty^u\big]$ of $A(\mathbf{G}(\mathbb{A}), \omega)$ has an algebraic interpretation via (4). Let us assume, for simplicity, that $F = \mathbb{Q}$ and suppose that E'/F is a field extension with the property that E' contains the values of ω_f and there is an algebraic representation (ρ, V) defined over E' with the property that the base change $\rho_{\mathbb{C}|\mathbf{G}(\mathbb{R})} = \pi_\infty^u$. Then we say that ρ is a model of π_∞^u over E'. A slightly more general notation which works for arbitrary fields F has been introduced in [**6**]; when F is totally real it is also shown in loc. cit. that one can always take $E' = E\big(\omega_f\big)$ and these models exist. Once again assuming that $F = \mathbb{Q}$ for simplicity, it makes sense to consider, for every $\mathbb{Q}$-algebra R, the R-modules

$$M[\mathbf{G}, \rho, \omega_0]_{/E'}(R) := \rho_R^\vee \otimes_R M\big(\mathbf{G}\big(\mathbb{A}_f\big), \rho_R, \omega_0\big),$$

where $M(\mathbf{G}(\mathbb{A}_f), \rho_R, \omega_0)$ is the space of functions $\varphi : \mathbf{G}(\mathbb{A}_f) \to V$ such that $\varphi(zxg_f) = \omega_0(z)\varphi(x)\rho(g_\infty)$ for every $z \in \mathbf{Z_G}(\mathbb{A}_f)$ and $g \in \mathbf{G}(F)$ and such that there exists $K \in \mathcal{K}$ such that $\varphi(ux) = \varphi(x)$. The character ω_0 is a twist of ω_f (see [**6**]), which equals ω_f in case $\mathbf{G}(\mathbb{R})$ itself is compact. Then (4) is true.

In the next section we will recall the definition of the formal period integral J_η satisfying (2) of the Introduction; the reader should keep in mind that the triple $(\mathbf{G}(F), \mathbf{G}(\mathbb{A}_f), \mathbf{Z_G}(\mathbb{A}_f))$ considered in this section corresponds to the triple (Γ, G_f, Z_f) in the next section.

3 - Profinite groups I: the ∞-adic formalism

3.1 - *Vector valued modular forms and the formal period integral*

In this section, we consider a data of the form

$$(\Gamma, G_f, Z_f) = (\Gamma, G_f, Z_f^G)$$

subject to the following assumptions. We suppose that G_f is a locally profinite unimodular group, let $\Gamma \subset G_f$ be a discrete subgroup such that G_f/Γ is compact and let $Z_f \subset Z_{G_f}$ be a closed subgroup. We may normalize the Haar measures following the conventions of [**6**, § 3.1]. We write $\mathcal{K} = \mathcal{K}(G_f)$ to denote the set of open and compact subgroups of G_f. If V is a G_f-module, then we define $V^{\mathcal{K}} := \bigcup_{K \in \mathcal{K}} V^K$.

Let G_∞ be a group and let $\Gamma \to G_\infty$ be a group homomorphism, so that $\Gamma \subset G_f \times G_\infty =: G$. If $g \in G$, we write $g_f \in G_f$ and $g_\infty \in G_\infty$ for its components. Let (V, ρ) be a right representation of G_∞ with coefficients in some commutative ring R. When ρ is understood, we simply write vg_∞ for $v\rho(g_\infty)$. If $\omega_0 : Z_f \longrightarrow R^\times$ is a character, we let $S(G_f, \rho, \omega_0)$ be the space of maps $\varphi : G_f \to V$ such that $\varphi(zx) = \omega_0(z)\varphi(x)$ for every $z \in Z_f$ and $x \in G_f$, endowed with the (G, G_f)-action given by

$$(g\varphi u)(x) := \varphi(uxg_f)\rho(g_\infty^{-1}), \quad \text{where } g \in G \text{ and } u \in G_f.$$

We set

$$S\Big(G_f/\Gamma, \rho_{/\Gamma}, \omega_0\Big) := S(G_f, \rho, \omega_0)^{(\Gamma,1)}$$

and, the group Γ usually being understood,

$$M(G_f, \rho, \omega_0) = M_\Gamma(G_f, \rho, \omega_0) := S(G_f/\Gamma, \rho_{/\Gamma}, \omega_0)^{\mathcal{K}}.$$

We omit ω_0 from the notation when $Z_f = 1$ and write $M(Z_f \backslash G_f, \rho) := M(G_f, \rho, \omega_0)$ when ω_0 is the trivial character of Z_f.

The following remark is easily verified.

Remark 3.1. Suppose that $\chi_0 : G_f \to R^\times$ is a character with the property that $\chi_0(K) = 1$ for some $K \in \mathcal{K}$ and that $\chi_\infty : G_\infty \to R^\times$ is a character with the property that $\chi_{0|\Gamma} = \chi_{\infty|\Gamma}$.

(1) If $\varphi \in M(G_f, \rho, \omega_0)$, then the rule $(\chi_0\varphi)(x) := \chi_0(x)\varphi(x)$ defines an element $\chi_0\varphi \in M(G_f, \rho(\chi_\infty), \chi_{0|Z}\omega_0)$.

(2) We have $\chi_0 \in M(G_f, R(\chi_\infty), \chi_{0|Z})$.

The formation of these spaces satisfies obvious functoriality properties. If $\psi : \rho \to \rho'$ is a morphism of representations of Γ, then we get

$$\psi_* : M(G_f, \rho, \omega_0) \to M(G_f, \rho', \omega_0) \tag{8}$$

by the rule $\psi_*(\varphi) := \psi \circ \varphi$. In the opposite direction, suppose that we are given another triple (Δ, H_f, H) satisfying the same assumptions made for (Γ, G_f, G_∞).

Definition 3.1. A period morphism

$$\eta : \left(\Delta, H_f, H_\infty, Z_f^H\right) \to \left(\Gamma, G_f, G_\infty, Z_f^G\right)$$

is a couple $\eta = (\eta_f, \eta_\infty)$ of group morphisms $\eta_f : H_f \to G_f$ and $\eta_\infty : H_\infty \to G_\infty$ both mapping Δ to Γ and such that η_f is continuous and maps Z_f^H to Z_f^G.

Writing $\eta_\infty^*(\rho)$ for the H_∞-representation obtained by restriction from η_∞ and setting $\eta_f^*(\omega_0) = \omega_0 \circ \eta_{f|Z_f^H}$, we get

$$\eta^* = (\eta_f, \eta_\infty)^* : M_\Gamma(G_f, \rho, \omega_0) \to M_\Delta\Big(H_f, \eta_\infty^*(\rho), \eta_f^*(\omega_0)\Big). \tag{9}$$

3.1.1 - Trace maps

For $x \in G_f$ and $K \in \mathcal{K}$, define $\Gamma_K(x) = \Gamma \cap x^{-1}Kx$. Being discrete (as Γ is) and compact (as K is), the set $\Gamma_K(x)$ is finite. For each $K \in \mathcal{K}$ and each set $R_K \subset G_f$ of representatives of $K\backslash G_f/\Gamma$, define

$$T_K = T_{R_K} : M(G_f, R)^K \longrightarrow R \quad \text{by} \quad T_{R_K}(f) := \mu_{G_f}(K) \sum_{x \in R_K} \frac{f(x)}{|\Gamma_K(x)|}. \tag{10}$$

It is easy to see that this is a well defined quantity, i.e. it does not depend from the choice of the double coset representatives R_K, and that it is independent from the choice of K, i.e. we have $T_{R_{K_1}}(f) = T_{R_{K_2}}(f)$ if $K_1 \subset K_2$ and

$$f \in M(G_f, R)^{K_2} \subset M(G_f, R)^{K_1}$$

(see [**6**, § 3.3.1] for details). We can therefore define R-linear functionals

$$T_{G_f/\Gamma} : M(G_f, R) \to R \text{ and } T_{Z\backslash G_f/\Gamma} : M(Z\backslash G_f, R) \to R \tag{11}$$

where $T_{G_f/\Gamma}(f) = T_K(f)$ for $f \in M(G_f, R)^K$ and $T_{Z\backslash G_f/\Gamma} := T_{G_f/\Gamma|M(Z\backslash G_f,R)}$.

Observe that

$$M_\Gamma(G_f, \mathbb{C}) = C(G_f/\Gamma)^{\mathcal{K}} \quad \text{and} \quad M_\Gamma(G_f, \mathbb{C}, 1) = C(Z\backslash G_f/\Gamma)^{\mathcal{K}}.$$

Since in [**6**, § 3.1] one of the assumptions on the measure was that $\mu_{Z\backslash G_f/\Gamma}$ is normalized so that it agrees with $\mu_{G_f/\Gamma}$ on $C(Z\backslash G_f/\Gamma) \subset C(G_f/\Gamma)$, we see that

$$T(f) = \int_{Z\backslash G_f/\Gamma} f(g_f) d\mu_{Z\backslash G_f/\Gamma}(g_f) \tag{12}$$

for all $f \in M(G_f, \mathbb{C}, 1)$ (see [**6**, § 3.3.1]).

3.1.2 - Pairings, n-linear forms and the formal period integral

We also have a natural map

$$\otimes : M(G_f, \rho, \omega_0) \otimes_R M(G_f, \rho', \omega_0') \to M(G_f, \rho \otimes_R \rho', \omega_0\omega_0') \tag{13}$$

defined by the rule $(\varphi \otimes \varphi')(x) := \varphi(x) \otimes \varphi'(x)$. In particular, writing $\rho^\vee$ for the R-dual representation $(v^\vee\gamma)(v) = v^\vee(v\gamma^{-1})$, we may define

$$\begin{aligned} \langle \cdot, \cdot \rangle \quad : \quad & M(G_f, \rho, \omega_0) \otimes_R M(G_f, \rho^\vee, \omega_0^{-1}) \overset{\otimes}{\to} M(G_f, \rho \otimes_R \rho^\vee) \\ \to \quad & M(Z\backslash G_f, R) \xrightarrow{T_{Z\backslash G_f/\Gamma}} R. \end{aligned} \tag{14}$$

Definition 3.2. We let $X(G_f, G_\infty, \omega_0) = X_\Gamma(G_f, G_\infty, Z_f, \omega_0)$ be the set of couples (χ_0, χ_∞) with the property that $\chi_0 : G_f \to R^\times$ is a character such that $\chi_0(K) = 1$ for some $K \in \mathcal{K}$, $\chi_{0|Z_f} = \omega_0$ and $\chi_\infty : G_\infty \to R^\times$ is a character such that $\chi_{0|\Gamma} = \chi_{\infty|\Gamma}$.

Suppose that we are given a period morphism $\eta: (\Delta, H_f, H_\infty) \to (\Gamma, G_f, G_\infty)$, say $\eta = (\eta_f, \eta_\infty)$, that (V, ρ) is a representation of G_∞ with coefficients in some ring R and that $\omega_0 : Z \to R^\times$ is a character. If $(\chi_0, \chi_\infty) \in X_\Delta(H_f, H_\infty, \eta_f^*(\omega_0))$ and we are given

$$\Lambda \in Hom_{R[H_\infty]}(\eta_\infty^*(\rho), R(\chi_\infty)) = \rho^\vee(\chi_\infty)^{H_\infty}$$

then we get

$$M_\eta^{\chi_0,\chi_\infty}(\Lambda) \in Hom_R(M(G_f, \rho, \omega_0), R)$$

by the rule

$$M_\eta^{\chi_0,\chi_\infty}(\Lambda)(\varphi) := \mu_{H_f}(K) \sum_{x\in K\backslash H_f/\Delta} \frac{\Lambda(\varphi(\eta_f(x)))\chi_0^{-1}(x)}{|\Delta_K(x)|}$$

if $\varphi \in M(G_f,\rho,\omega_0)^K$. Alternatively, we have

$$\begin{aligned} M_\eta^{\chi_0,\chi_\infty}(\Lambda) \quad : \quad & M_\Gamma(G_f,\rho,\omega_0)\eta^* \to M_\Delta(H_f,\eta_\infty^*(\rho),\eta_f^*(\omega_0)) \\ & \xrightarrow{\Lambda_*} M(H_f,R(\chi_\infty),\eta_f^*(\omega_0)=\chi_{0|Z}) \xrightarrow{\langle\cdot,\chi_0^{-1}\rangle} R, \end{aligned} \tag{15}$$

where $\langle \cdot,\chi_0^{-1}\rangle$ is the pairing (14), which makes sense thanks to Remark 3.1 (2): it follows from this description that $M_\eta^{\chi_0,\chi_\infty}(\Lambda)$ is well defined. We write $M^{\chi_0,\chi_\infty} := M_\eta^{\chi_0,\chi_\infty}$.

In this case, we may define

$$J^{\rho,\chi_0,\chi_\infty} : \rho^\vee(\chi_\infty)^{G_\infty} \otimes_R M(G_f,\rho,\omega_0) \to R \tag{16}$$

by the rule

$$J^{\rho,\chi_0,\chi_\infty}(\Lambda\otimes_R \varphi) := M^{\chi_0,\chi_\infty}(\Lambda)(\varphi).$$

Finally, suppose that we are given a family $\{\rho_i\}_{i\in I}$ for representations, characters $\{\omega_{0,i}\}_{i\in I}$ and $\Lambda \in Hom_{R[G_\infty]}(\rho,R(\chi_\infty))$, where $\rho := \otimes_{R,i\in I}\rho_i$. Then, assuming that $\prod_i \omega_{0,i} = \omega_0$ we generalize (14) as follows:

$$\Lambda_M^{\chi_0,\chi_\infty} : \otimes_{R,i\in I} M(G_f,\rho_i,\omega_{0,i}) \xrightarrow{\otimes} M(G_f,\rho,\omega_0) \xrightarrow{M^{\chi_0,\chi_\infty}(\Lambda)} R. \tag{17}$$

If $p^{\rho,\chi_\infty} : V^\vee \to (\eta_\infty^*(\rho))^\vee(\chi_\infty)^{H_\infty}$ is a projection, setting $p_\eta^{\rho,\chi_\infty} := p^{\rho,\chi_\infty}\otimes_R \eta^*$ and $M_\Gamma[G_f,\rho,\omega_0] := V^\vee \otimes_R M_\Gamma(G_f,\rho,\omega_0)$, we can define the formal period integral:

$$\begin{aligned} J_\eta^{\rho,\chi_0,\chi_\infty} \quad : \quad & M_\Gamma[G_f,\rho,\omega_0] \xrightarrow{p^{\rho,\chi_\infty}_{\langle -,-\rangle_{V^\vee},\eta}} \\ & (\eta_\infty^*(\rho))^\vee(\chi_\infty)^{H_\infty} \otimes_R M_\Delta\Big(H_f,\eta_\infty^*(\rho),\eta_f^*(\omega_0)\Big) \xrightarrow{J^{\eta_\infty^*(\rho),\chi_0,\chi_\infty}} R. \end{aligned} \tag{18}$$

3.2 - *The special value formulas and an example*

Let us assume now for simplicity that $F = \mathbb{Q}$ and that ρ is a model over $E \subset \mathbb{C}$ of the unitary and irreducible representation π_∞^u which appears in the decomposition (3). Then, up to changing $\mathbf{G}$ by a pure inner form $\mathbf{G}'_\pi$, in the setting of the Gan-Gross-Prasad conjectures, (5) is in force and, hence, (7) becomes interesting.

Let $X\big(\mathbf{H}(\mathbb{A}_f),\mathbf{H},\eta^*_{\mathbb{A}_f}(\omega_0)\big)$ be the set of pairs

$$\big(\chi_0:\mathbf{H}(\mathbb{A}_f)\to E^\times,\ \chi:\mathbf{H}_{/E}\to\mathbf{G}_{m/E}\big)$$

such that $(\chi_0,\chi_E)\in X_{\mathbf{H}(F)}\big(\mathbf{H}(\mathbb{A}_f),\mathbf{H}(E),\eta^*_{\mathbb{A}_f}(\omega_0)\big)$. Setting $\chi_{0,R}:=\varphi\circ\chi_0$ and $\omega_{0,R}:=\varphi\circ\omega_0$ gives, for every E-algebra $\varphi:E\to R$, an element $\big(\chi_{0,R},\chi_R\big)\in X_{\mathbf{H}(F)}\big(\mathbf{H}(\mathbb{A}_f),\mathbf{H}(R),\omega_{0,R}\big)$. Since $\mathbf{H}_{/E}$ is a reductive group over a characteristic zero field, there is a canonical projection $p^{\rho_R,\chi_R}:V_R^\vee\to\eta_R^*(\rho_R)^\vee(\chi_R)^{\mathbf{H}(R)}$ for every E-algebra R. We write $J_\eta^{\rho_R,\chi_{0,R},\chi_R}$ for the period morphism (18) obtained from p^{ρ_R,χ_R}. It is proved in [**6**, Cor. 7.3] that, for a suitable choice of $(\chi_0,\chi_\infty)\in X\big(\mathbf{H}(\mathbb{A}_f),\mathbf{H},\eta^*_{\mathbb{A}_f}(\omega_0)\big)$, we have the equality

$$m^{-1}_{\mathbf{S_H}\backslash\mathbf{H},\infty}I_\eta=J_\eta^{\pi^u_\infty,\chi_{0,\mathbb{C}},\chi_{\infty,\mathbb{C}}}=J_\eta^{\rho_{\mathbb{C}},\chi_{0,\mathbb{C}},\chi_{\infty,\mathbb{C}}}. \tag{19}$$

Without further details, let us remark that (χ_0,χ_∞) is of the form $\chi_0=\omega_f^\eta\mathbf{N}_f^\eta$ and $\chi_\infty=\omega_\infty^{-\eta}\mathbf{N}_\infty^\eta$ (we set $\kappa^\eta:=\kappa\circ\eta$), with $\mathbf{N}=1$ when $\mathbf{G}(F_\infty)$ itself is compact. It turns out that, in the setting of Gan-Gross-Prasad conjectures, $\eta^*(\rho)^\vee(\chi)^{\mathbf{H}}$ should always be one dimensional.

Let $j:K\hookrightarrow B$ be an embedding of a quadratic imaginary field K in a definite quaternion $\mathbb{Q}$-algebra B (so that $\mathbf{B}^\times(\mathbb{R})/\mathbf{S}_{\mathbf{B}^\times}(\mathbb{R})$ is compact). This embedding induces $\mathbf{j}^\times:\mathrm{Res}_{K/\mathbb{Q}}(\mathbf{K}^\times)\subset\mathbf{B}^\times$, where $\mathbf{B}^\times$ (resp. $\mathbf{K}^\times$) is the algebraic group attached to B (resp. K). We consider

$$\eta:=\mathbf{j}^\times\times 1:\mathbf{H}:=\mathrm{Res}_{K/\mathbb{Q}}(\mathbf{K}^\times)\subset\mathbf{B}^\times\times\mathrm{Res}_{K/\mathbb{Q}}(\mathbf{K}^\times)=:\mathbf{G}$$

(so that $\mathbf{S_H}=\mathbf{G}_m$). Let π_g be the automorphic representation in the space $A(\mathbf{B}^\times(\mathbb{A}),\varepsilon)\big[\pi^u_{g,\infty}\big]$ obtained as the Jacquet-Langlands lift of the representation π'_g of $\mathbf{GL}_2$ attached to a modular form g of weight $k+2$ and let $\chi:\mathrm{Res}_{K/\mathbb{Q}}(\mathbf{K}^\times)(\mathbb{A})\to\mathbb{C}^\times$ be a Hecke character of K. Let the assumptions be as in [**11**, III, §3]: π_g is unitary, $\chi_{|\mathbf{G}_m(\mathbb{A})}=\varepsilon=1$ (i.e. g has trivial nebentype) and χ is a finite order character. Then $\pi_g\times\chi^{-1}\in A(\mathbf{G}(\mathbb{A}),1)\big[\pi^u_\infty\big]$ where $\pi_\infty=\pi_{g,\infty}\times\chi_\infty^{-1}=\pi_{g,\infty}$. We fix $\mathbf{B}_{/K}\simeq\mathbf{M}_{2/K}$ inducing $\mathbf{B}^\times_{/K}\simeq\mathbf{GL}_{2/K}$ and can take $E/\mathbb{Q}$ any Galois extension such that $K\subset E$.

If $k\in 2\mathbb{N}$ we let $\mathbf{P}_{k/E}$ be the left $\mathbf{GL}_{2/E}$-representation on two variables polynomials of degree k, the action being defined by the rule $(gP)(X,Y)=P((X,Y)g)$. We write $\mathbf{V}_{k/E}$ for the dual right representation, that we may also view as a representation of $\mathbf{G}_{/E}$ letting $\mathbf{H}_{/E}$ acts trivially.

Consider the (normalized) absolute value functions $|-|_v:\mathbb{Q}_v^\times\to\mathbb{R}_+^\times$, $|-|_{\mathbb{A}_f}:\mathbb{A}_f^\times\to\mathbb{Q}_+^\times$ and $|-|_{\mathbb{A}}:\mathbb{A}^\times\to\mathbb{R}_+^\times$ and define $\mathrm{N}:=|-|^{-1}_{\mathbb{A}_f}|-|_\infty$. The maximal split toric quotient of $\mathbf{G}$ (resp. $\mathbf{H}$) is

$$\mathrm{nr}_{\mathbf{G}}:=\big(\mathrm{nrd},\mathrm{nr}_{K/\mathbb{Q}}\big):\mathbf{G}=\mathbf{B}^\times\times\mathrm{Res}_{K/\mathbb{Q}}(\mathbf{K}^\times)\to\mathbf{G}_m\times\mathbf{G}_m\ (\text{resp. } \mathrm{nr}_{K/\mathbb{Q}}).$$

Hence the algebraic characters of $\mathbf{G}$ (resp. $\mathbf{H}$) can be describes as follows: if $(k,l) \in \mathbb{Z}^2 = Hom(\mathbf{G}_m^2, \mathbf{G}_m)$, we set $\mathrm{nr}_{\mathbf{G}}^{k,l}(g) := \mathrm{nr}_{\mathbf{G}}(g)^{(k,l)}$ (resp. $\mathrm{nr}_{K/\mathbb{Q}}^{l}(h) := \mathrm{nr}_{K/\mathbb{Q}}(h)^{l}$). We define $\mathrm{Nr}_{\mathbf{G}}^{k,l} := \mathrm{N} \circ \mathrm{nr}_{\mathbf{G}}^{k,l}$ (resp. $\mathrm{Nr}_{K/\mathbb{Q}}^{l} := \mathrm{N} \circ \mathrm{nr}_{K/\mathbb{Q}}^{l}$), so that $\mathrm{Nr}_{\mathbf{G}}^{k,l} \circ \eta = \mathrm{Nr}_{K/\mathbb{Q}}^{k+l}$. Then [**6**, Thm 7.6] applied to $\pi_\infty = \mathbf{V}_k(\mathbb{C})$ implies that (19) is in force with $\mathbf{N} = \mathrm{Nr}_{\mathbf{G}}^{k/2,0}$, so that $(\chi_0, \chi_\infty) = \left(\mathrm{Nr}_{K/\mathbb{Q},f}^{k/2}, \mathrm{nr}_{K/\mathbb{Q}}^{k/2}\right)$.

Let $Q_{j/E} \in \mathbf{P}_{2/E}$ be defined as in [**7**, § 2.3.2] (which applies with no changes when K is imaginary), then the evaluation at $Q_j^{k/2} \in \mathbf{P}_{k/E}$ gives (see [**7**, (3.5)])

$$\varLambda_{j,k/E} \in Hom_{\mathbf{H}_{/E}}(\mathbf{V}_k, \mathbf{1}(k/2)).$$

It follows from [**7**, § 2.3.2] that there are models $\mathbf{V}_k$ and $\varLambda_{j,k}$ over $\mathbb{Q}$ for the representation $\mathbf{V}_{k/E}$ and $\varLambda_{j,k/E}$. In this case, the identification (4) becomes

$$f : \mathbf{V}_{k,\mathbb{C}}^{\vee} \otimes_{\mathbb{C}} M\left(B_f^\times, \mathbf{V}_{k,\mathbb{C}}, \mathrm{N}_f^k\right)\left[\mathrm{Nrd}_f^{-k/2}\pi_{g,f}\right] \simeq A(\mathbf{B}^\times(\mathbb{A}), 1)\left[\pi_g\right],$$

where $(-)[\theta]$ means taking the θ-component. Hence, if $K_{\varphi,\chi} \in \mathcal{K}(\mathbf{H}(\mathbb{A}_f))$ is such that $\eta(K_{\varphi,\chi}) \subset K_\varphi \times K_\chi$ and

$$\begin{aligned} \varphi \times \chi^{-1} &\in M\left(\mathbf{B}^\times, \mathbf{V}_{k,\mathbb{C}}, \mathrm{N}_f^k\right)^{K_\varphi} \otimes M(\mathbf{B}^\times, \mathbf{1}, 1)^{K_\chi} \\ &\subset M\left(\mathbf{G}(\mathbb{A}_f), \mathbf{V}_{k,\mathbb{C}}, \mathrm{Nr}_{\mathbf{G},f|\mathbf{G}_m(\mathbb{A}_f)}^{k/2,0}\right)^{K_{\varphi,\chi}}, \end{aligned}$$

we have

$$\begin{aligned} J_{\chi^{-1}}(\varphi) &:= J_\eta^{\mathbf{V}_{k,\mathbb{C}}, \mathrm{Nr}_{K/\mathbb{Q},f}^{k/2}, \mathrm{nr}_{K/\mathbb{Q}}^{k/2}}\left(\varLambda_{j,k} \otimes (\varphi \times \chi^{-1})\right) \\ &= \mu_{\mathbf{H}(\mathbb{A}_f)}(K_{\varphi,\chi}) \sum_{x \in K_{\varphi,\chi}\backslash \mathbf{H}(\mathbb{A}_f)/\mathbf{H}(\mathbb{Q})} \frac{\chi^{-1}(x)\varLambda_{j,k}(\varphi(j(x)))}{|\varGamma_{K_{\varphi,\chi}}(x)|\mathrm{Nr}_{K/\mathbb{Q},f}^{k/2}(x)}. \end{aligned}$$

Let $\pi_{\chi^{-1}}$ be the representation attached to the theta lift $\theta_{\chi^{-1}}$ of χ^{-1}. Then (19) together with (5) (which is [**11**, Prop. 7], in this case) gives (note that $J_\chi(\varphi) = \overline{J_{\chi^{-1}}(\varphi)}$):

$$J_{\chi^{-1}}(\varphi) J_\chi(\varphi) = \frac{1}{2^\beta m_{\mathbf{S_H}\backslash\mathbf{H},\infty}} \frac{\varDelta_{\mathbf{G}_V} L(1/2, \pi_g' \times \pi_{\chi^{-1}})}{L(1, \pi_g', \mathrm{Ad}) L(1, \pi_{\chi^{-1}}, \mathrm{Ad})} \prod_v \alpha_v(\varphi_v). \tag{20}$$

In the next section we introduce the general p-adic formalism which is needed to place (18) in a p-adic setting. We will exemplify (20) in § 5.

4 - Profinite groups II: the p-adic formalism

In this section, we consider a data of the form

$$(\varGamma, G_f, G_p, Z_f, K_p^\diamond) = (\varGamma, G_f, G_p, Z_f^G, K_p^{\diamond,G})$$

where (Γ, G_f, Z_f) is a triple as in § 3.1 and the following further assumptions are satisfied. We suppose that we may write $G_f = G_f^p \times G_p$ and $Z_f = Z_f^p \times Z_p$ topologically, where $Z_f^p \subset Z_{G_f^p}$ and $Z_p \subset Z_{G_p}$ are such that $G_f \to G_f^p$ (resp. $G_f \to G_p$) maps Z_f to Z_f^p (resp. to Z_p). If $g \in G_f$, we let $g_p \in G_p$ be its component. With an abuse of notation, we will implicitly consider $\Gamma \subset G_f \to G_p$ as an inclusion. We also suppose that we are given an open subgroup $K_p^\diamond \subset G_p$. Let $\mathcal{K}^\diamond := \mathcal{K}(G_f, K_p^\diamond)$ be the set of open and compact subgroups $K = K^p \times K_p$ with $K^p \subset G_f^p$ and $K_p \subset K_p^\diamond$ open and compact. If V is a $K_p^\diamond(G_f) := G_f^p \times K_p^\diamond$-module, then we define

$$V^{\mathcal{K}^\diamond} := \bigcup_{K \in \mathcal{K}^\diamond} V^K.$$

Suppose now that (V, ρ) is a right Σ_p-representation with coefficients in R for some subsemigroup $K_p^\diamond \subset \Sigma_p \subset G_p$ and set $\Sigma_p(G_f) := G_f^p \times \Sigma_p$. If $\omega_{0,p} : Z_f \to R^\times$ is a character, we let $S_p(G_f, \rho, \omega_{0,p})$ be the space of maps $\varphi : G_f \to V$ such that

$$\varphi(zx) = \omega_{0,p}(z)\varphi(x), \text{ for every } z \in Z_f \text{ and } x \in G_f,$$

endowed with the $(G, \Sigma_p(G_f))$-action defined by the rule

$$(g\varphi u)(x) := \varphi(uxg_f)\rho(u_p), \text{ where } g \in G \text{ and } u \in \Sigma_p(G_f).$$

When ρ is understood, we simply write $vu_p := v\rho(u_p)$. We set

$$S_p(G_f/\Gamma, \rho_{/\Gamma}, \omega_{0,p}) := S(G_f, \rho, \omega_{0,p})^{(\Gamma,1)}$$

and, the group Γ usually being understood,

$$M_p(G_f, \rho, \omega_{0,p}) = M_{p,\Gamma}(G_f, \rho, \omega_{0,p}) := S_p(G_f/\Gamma, \rho_{/\Gamma}, \omega_{0,p})^{\mathcal{K}^\diamond}.$$

We omit $\omega_{0,p}$ from the notation when $Z_f = 1$ and write $M_p(Z_f \backslash G_f, \rho) := M_p(G_f, \rho, \omega_{0,p})$ when $\omega_{0,p}$ is the trivial character of Z_f. Sometimes we will abusively replace ρ with the underlying subspace V in the notation. These spaces are naturally right $\Sigma_p(G_f)$-submodules of $S_p(G_f, \rho, \omega_{0,p})$. By a p-adic period morphism

$$\eta_f : (\Delta, H_f, H_p, Z_f^H, K_p^{\diamond,H}) \to (\Gamma, G_f, G_p, Z_f^G, K_p^{\diamond,G})$$

we mean a continuous group morphism $\eta_f : H_f \to G_f$ of the form $\eta_f = \eta_f^p \times \eta_p$ with $\eta_f^p : H_f^p \to G_f^p$ and $\eta_p : H_p \to G_p$ such that η_f maps Δ to Γ, η_f^p maps $Z_f^{p,H}$ to $Z_f^{p,G}$ and η_p maps $Z_{p,H}$ to $Z_{p,G}$ and $K_p^{\diamond,H}$ to $K_p^{\diamond,G}$. Then there are analogues of (8) and (9), the first

$$\psi_* : M_p(G_f, \rho, \omega_{0,p}) \to M_p(G_f, \rho', \omega_{0,p}) \tag{21}$$

being induced by any morphism of $K_p^\diamond$-modules. We note that ψ_* respects the K-level structures $M_p(G_f, -, \omega_{0,p})^K$ obtained from any $K \in \mathcal{K}^\diamond$.

We remark that, since $G_f = G_f^p \times G_p$ topologically, $\mathcal{K}^\diamond \subset \mathcal{K}$ is a cofinal family, from which we deduce that, when $(V,\rho) = R$, we have $M(G_f, R) = M_p(G_f, R)$ (resp. $M(Z_f \backslash G_f, R) = M_p(Z_f \backslash G_f, R)$) on which we have defined a trace map $T_{G_f/\Gamma}$ (resp. $T_{Z_f \backslash G_f/\Gamma}$) by means of (10). The formula $(\varphi \otimes \varphi')(x) := \varphi(x) \otimes \varphi'(x)$ defines an analogue of (13), so that we may define the analogue of (14):

$$\langle \cdot, \cdot \rangle \quad : \quad M_p(G_f, \rho, \omega_{0,p}) \otimes_R M_p(G_f, \rho^\vee, \omega_{0,p}^{-1}) \xrightarrow{\otimes} M_p(G_f, \rho \otimes_R \rho^\vee)$$

$$\to M_p(Z_f \backslash G_f, R) \xrightarrow{T_{Z_f \backslash G_f/\Gamma}} R. \tag{22}$$

In order to define an analogue of (15) in this setting, it is convenient to give the analogue of Remark 3.1.

Remark 4.1. Suppose that $\chi_{0,p} : G_f \to R^\times$ and $\chi_p : K_p^\diamond \to R^\times$ are characters such that there is some $K \in \mathcal{K}^\diamond$ satisfying the condition $\chi_{0,p}(u) = \chi_p(u_p)^{-1}$ for every $u \in K$ and $\chi_{0,p|\Gamma} = 1$.

(1) If $\varphi \in M_p(G_f, \rho, \omega_{0,p})$, then the rule $(\chi_{0,p}\varphi)(x) := \chi_{0,p}(x)\varphi(x)$ defines an element $\chi_0\varphi \in M_p(G_f, \chi_p\rho, \chi_{0,p|Z_f}\omega_{0,p})$.

(2) We have $\chi_{0,p} \in M_p(G_f, R(\chi_p), \chi_{0,p|Z_f})$.

Definition 4.1. We let $X_p(G_f, K_p^\diamond, \omega_{0,p}) = X_{p,\Gamma}(G_f, K_p^\diamond, Z_f, \omega_{0,p})$ be the set of couples $(\chi_{0,p}, \chi_p)$ with the property that $\chi_{0,p} : G_f \to R^\times$ and $\chi_p : K_p^\diamond \to R^\times$ are characters for which there is some $K \in \mathcal{K}^\diamond$ satisfying $\chi_{0,p}(u) = \chi_p(u_p)^{-1}$ for every $u \in K$, $\chi_{0,p|\Gamma} = 1$ and $\chi_{0,p|Z_f} = \omega_{0,p}$.

Suppose that we are given a p-adic period morphism η_f as above and that $\omega_0 : Z_f^H \to R^\times$ is a character. If $(\chi_{0,p}, \chi_p) \in X_p(H_f, K_p^{\diamond,H}, \eta_f^*(\omega_{0,p}))$ and we are given

$$\Lambda \in Hom_{R[K_p^{\diamond,H}]}(\eta_p^*(\rho), R(\chi_p))$$

then we get

$$M_{p,\eta_f}^{\chi_{0,p},\chi_p}(\Lambda) \in Hom_R(M(G_f, \rho, \omega_{0,p}), R)$$

by the rule

$$M_{p,\eta_f}^{\chi_{0,p},\chi_p}(\Lambda)(\varphi) := \mu_{H_f}(K) \sum_{x \in K \backslash G_f/\Gamma} \frac{\Lambda(\varphi(\eta_f(x)))\chi_{0,p}^{-1}(x)}{|\Delta_K(x)|} \tag{23}$$

if $\varphi \in M_p\big(G_f, \rho, \omega_0\big)^K$. Alternatively, we have

$$M_{p,\eta_f}^{\chi_{0,p},\chi_p}(\Lambda) : M_p\big(G_f, \rho, \omega_{0,p}\big) \xrightarrow{\eta_f^*} M_p\big(H_f, \eta_p^*(\rho), \eta_0^*\big(\omega_{0,p}\big) = \chi_{0,p|Z_f^H}\big)$$

$$\text{(24)} \qquad \xrightarrow{\Lambda_*} M_p\big(H_f, R\big(\chi_p\big), \omega_{0,p}\big) \xrightarrow{\langle\,\cdot\,,\chi_{0,p}^{-1}\,\rangle} R,$$

where $\langle\,\cdot\,,\chi_{0,p}^{-1}\rangle$ is the pairing (22), which makes sense thanks to Remark 4.1 (2) asserting that $\chi_{0,p}^{-1} \in M_p\big(H_f, R\big(\chi_p^{-1}\big), \chi_{0,p|Z_f^H}^{-1}\big)$: it follows from this description that $M_{p,\eta_f}^{\chi_{0,p},\chi_p}(\Lambda)$ is well defined. We set $M_{p,\eta_f}^{\chi_{0,p},\chi_p} := M_{p,1}^{\chi_{0,p},\chi_p}$.

In this case, we may define

$$\text{(25)} \qquad J_p^{\rho,\chi_{0,p},\chi_p} : \rho^\vee\big(\chi_p\big)^{K_p^\diamond,H} \otimes_R M_p\big(G_f, \rho, \omega_{0,p}\big) \to R$$

by the rule

$$J_p^{\rho,\chi_{0,p},\chi_p}(\Lambda \otimes_R \varphi) := M_p^{\chi_{f,p},\chi_p}(\Lambda)(\varphi).$$

Finally, suppose that we are given a family $\{\rho_i\}_{i\in I}$ of representations, characters $\{\omega_{0,p,i}\}_{i\in I}$ and $\Lambda \in Hom_{R[K_p^\diamond]}\big(\rho, R\big(\chi_p\big)\big)$, where $\rho := \otimes_{R,i\in I}\,\rho_i$. Then, assuming that $\prod_i \omega_{0,p,i} = \omega_{0,p}$ we generalize (22) as follows:

$$\text{(26)} \qquad \Lambda_{M_p}^{\chi_{0,p},\chi_p} : \otimes_{R,i\in I} M_p\big(G_f, \rho_i, \omega_{0,i}\big) \xrightarrow{\otimes} M_p\big(G_f, \rho, \omega_0\big) \xrightarrow{M_p^{\chi_{0,p},\chi_p}(\Lambda)} R.$$

Starting with a projection $p^{\rho,\chi_p} : \rho \to \big(\eta_p^*(\rho)\big)^\vee\big(\chi_p\big)^{K_p^\diamond,H}$ and $\big(\chi_{0,p},\chi_p\big) \in X_\Delta\big(H_f, K_{H,p}^\diamond, \eta_f^*(\omega_0)\big)$, we can define the analogue of (18) (where the source is defined in a similar way):

$$J_{p,\eta_f}^{\rho,\chi_{0,p},\chi_p} : M_{p,\Gamma}\big[G_f, \rho, \omega_{0,p}\big] \xrightarrow{p_{\eta_f}^{\rho,\chi_p}}$$

$$\text{(27)} \qquad \big(\eta_p^*(\rho)\big)^\vee\big(\chi_p\big)^{K_p^\diamond,H} \otimes_R M_{p,\Delta}\big(H_f, \eta_p^*(\rho), \eta_f^*\big(\omega_{0,p}\big)\big) \xrightarrow{J^{\eta_p^*(\rho),\chi_{0,p},\chi_p}} R.$$

We note that Hecke operators act on $V^{\mathcal{K}^\diamond}$ for any $\Sigma_p\big(G_f\big)$-module by double cosets. If $K_1, K_2 \in \mathcal{K}^\diamond$ and $\pi \in \Sigma_p\big(G_f\big)$, the space $K_1\backslash K_1\pi K_2$ is finite[1] and we may write $K_1\pi K_2 = \bigsqcup_{x\in K_1\backslash K_1\pi K_2} K_1 x$. As usual we may define

$$\text{(28)} \qquad [K_1\pi K_2] : V^{K_1} \to V^{K_2}$$

[1] Indeed note that $K_1\pi K_2$ is compact, being the image of $K_1 \times K_2$ by means of the continuous map given by $(x,y) \mapsto x\pi y$. Since K_1 is open, $K_1\pi K_2 = \bigsqcup_i K_1\pi_i$ is an open covering which, by compactness, admits a finite refinement.

by the rule $v \mid [K_1\pi K_2] = \sum_{x\in K_1\backslash K_1\pi K_2} vx$. We can define in this way an action of the Hecke algebra $\mathcal{H}(\Sigma_p(G_f))$ of double cosets on $\Sigma_p(G_f)$. When $K_p^\diamond = G_p$ we have $V^{\mathcal{K}^\diamond} = V^{\mathcal{K}}$ and we have an action of $\mathcal{H}(\Sigma_p(G_f)) = \mathcal{H}(G_f)$. Let $\mathcal{K}^{\diamond\diamond} \subset \mathcal{K}^\diamond$ be the subset of those groups such that $K_p = K_p^\diamond$ and write $\mathcal{H}(\Sigma_p)$ for the Hecke algebra of double cosets $K\pi_p K$ with $\pi \in \Sigma_p$ and $K \in \mathcal{K}^{\diamond\diamond}$. Then (28) defines an operator on $V^{\mathcal{K}^{\diamond\diamond}} = \left(V^{\mathcal{K}^\diamond}\right)^{K_p^\diamond}$ by means of the formula $vT_\pi := v \mid [K\pi K]$ if $v \in V^K$ where $K \in \mathcal{K}^{\diamond\diamond}$, i.e. it does not depend on $K \in \mathcal{K}^{\diamond\diamond}$. It follows that $V^{\mathcal{K}^{\diamond\diamond}}$ is endowed with an action of $G_f^p \times \mathcal{H}(\Sigma_p)$.

We now investigate the relationship between the ∞-adic and the p-adic formalism, assuming that we are given a period morphism η_f as above. Suppose now that, as in § 3.1, we have also given a group morphism $\Gamma \to G_\infty$ (resp. $\Delta \to H_\infty$), so that $\Gamma \subset G_f^p \times G_p \times G_\infty =: G$ and that we are given $\omega_0 : Z_f \to R^\times$ and coefficient rings $i_\infty : R \subset R_\infty$ and $i_p : R \subset R_p$. For a character χ of some group with values in $R^\times$, we let $i_{p*}(\chi) := i_p \circ \chi$ and $i_{\infty*}(\chi) := i_\infty \circ \chi$. Suppose we are given characters $\chi_0 : H_f \to R^\times$, $\chi_p : H_p \to R_p^\times$ and $\chi_\infty : H_\infty \to R_\infty^\times$ such that $\chi := \chi_{p|\Delta} = \chi_{\infty|\Delta} : \Delta \to R^\times$. We also assume that we are given a representation ρ_p (resp. ρ_∞) of G_p (resp. G_∞) with coefficients in R_p (resp. R_∞) with the property that

$$\rho := \rho_{p|\Gamma} = \rho_{\infty|\Gamma} \subset \rho_p, \rho_\infty$$

with coefficients in R. Finally, suppose that $(\Lambda_p, \Lambda_\infty)$ is a couple of elements $\Lambda_p \in Hom_{R_p[H_p]}\big(\eta_p^*(\rho_p), R_p(\chi_p)\big)$ and $\Lambda_\infty \in Hom_{R_\infty[H_\infty]}\big(\eta_\infty^*(\rho_\infty), R_\infty(\chi_\infty)\big)$ with the property that $\Lambda := \Lambda_{p|\rho} = \Lambda_{\infty|\rho}$ with coefficients in R.

Proposition 4.1. *With the above notations the following facts hold.*

(1) *The rules*

$$\begin{aligned} M(G_f, \rho_p) &\to M_p(G_f, \rho_p) & \qquad M_p(G_f, \rho_p) &\to M(G_f, \rho_p) \\ \varphi &\mapsto \psi_\varphi : \psi_\varphi(x) := \varphi(x)x_p^{-1} & \qquad \psi &\mapsto \varphi_\psi : \varphi_\psi(x) := \psi(x)x_p \end{aligned}$$

set up a right $\Sigma_p(G_f)$-equivariant bijection and $M(G_f, \rho) \subset M(G_f, \rho_p)$ is identified with the submodule of those $\psi \in M_p(G_f, \rho_p)$ such that $\psi(x) \in \rho \subset \rho_p$ for every $x \in G_f$. Furthermore, if ρ_p has central character ω_{ρ_p} and $(-)_p$: $Z_f \to Z_p$ is the projection induced by $G_f \to G_p$, then the bijection induces

$$M(G_f, \rho, \omega_0) \subset M(G_f, \rho_p, i_{p*}(\omega_0)) \simeq M_p(G_f, \rho_p, \omega_{0,p})$$

with $\omega_{0,p} := i_{p*}(\omega_0)\omega_{\rho_p}^{-1}((-)_p)$. These identifications and inclusions are $\mathcal{H}(\Sigma_p(G_f))$-equivariant.

(2) *With* $\eta_f = 1$ *we have* $(i_{\infty *}(\chi_0), \chi_\infty) \in X(G_f, G_\infty, i_{\infty *}(\omega_0))$ *if and only if* $(\chi_{0,p}, \chi_p) \in X_p(G_f, G_p, \omega_{0,p})$, *where* $\omega_{0,p}(z) := i_{p*}(\omega_0)(z)\chi_p^{-1}(z_p)$ *and* $\chi_{0,p}(x) := i_{p*}(\chi_0)(x)\chi_p^{-1}(x_p)$. *In this case,*

$$\chi_0 \in M(G_f, R(\chi), \omega_0) \subset M(G_f, R_\infty(\chi_\infty), i_{\infty *}(\omega_0))$$

has image $i_{\infty *}(\chi_0)$ *and*

$$\begin{aligned}\chi_0 &\in M(G_f, R(\chi), \omega_0) \subset M(G_f, R_p(\chi_p), i_{p*}(\omega_0))\\ &\simeq M_p(G_f, R_p(\chi_p), \omega_{0,p})\end{aligned}$$

has image $i_{p*}(\chi_p)$ *via the first inclusion and this maps to* $\chi_{0,p}$ *via the bijection in* (1).

(3) *Suppose that the conditions in* (2) *are satisfied for*

$$(\chi_{0,p}, \chi_p) \in X_p(H_f, K_p^{\diamond, H}, \eta_f^*(\omega_{0,p})).$$

Then (15) *is compatible with* (24), *i.e. they both agree on*

$$\begin{aligned}&M(G_f, \rho, \omega_0) \subset M(G_f, \rho_\infty, i_{\infty *}(\omega_0)),\\ &M(G_f, \rho, \omega_0) \subset M_p(G_f, \rho_p, \omega_{0,p})\end{aligned}$$

and (17) *is compatible with* (26), *meaning a similar statement. If we are given projections* $p^{\rho_\infty, \chi_\infty}$ *(see the lines before* (18)) *and* p^{ρ_p, χ_p} *which agree on* $\rho^\vee$, *then* (18) *agree with* (27) *on* $M(G_f, \rho, \omega_0)$.

Proof. Assertion (1) is checked by a direct computation and (2) is a special case of it, in view of the definitions of $X(G_f, G_\infty, i_{\infty *}(\omega_0))$, $X_p(G_f, G_p, \omega_{0,p})$ and Remarks 3.1 and 4.1. As far as the statement (3) is concerned, let us remark that the definitions of (15) (resp. (24)), (17) (resp. (26)), (18) (resp. (27)) relies on (14) (resp. (22)). In view of the compatibility that we have assumed before the statement and that between $p^{\rho_\infty, \chi_\infty}$ and p^{ρ_p, χ_p}, we are reduced to comparing (14) and (22) up to the identification provided by (1). But it follows from their definition that we only need to compare the $\otimes$-product (13) and its p-adic analogue. It is easy to see that the identification in (1) preserves this operations. □

4.1 - *Pairings and adjointness*

We now state a generalization of the adjointness formula of [**5**, § 2.3], which plays a crucial role in loc. cit. It will play no role in the sequel and could be skipped. Suppose that we have $(\chi_{0,p}, \chi_p) \in X_p(G_f, K_p^{\diamond}, \omega_{0,p})$ and that D (resp. E)

is a Σ_D (resp. Σ_E) module, where Σ_D (resp. Σ_E) satisfies the assumption that was done on Σ_p, and we let $\omega_{0,p,D}, \omega_{0,p,E} : Z_f \to R^\times$ be characters such that $\omega_{0,p,D}\omega_{0,p,E} = \omega_{0,p}$.

Suppose that we are given

$$\langle -,- \rangle \in Hom_{R[K_p^\diamond]}(D \otimes E, R(\chi_p)).$$

Then (26) gives

$$\langle -,- \rangle_{M_p}^{\chi_{0,p},\chi_p} : M_p(G_f, D, \omega_{0,p,D}) \otimes_R M_p(G_f, E, \omega_{0,p,E}) \to R.$$

Suppose that we are given an anti-automorphism ι of G_f which respects the decomposition $G_f = G_f^p \times G_p$, i.e. a homeomorphism of G onto itself such that $(g_1 g_2)^\iota = g_2^\iota g_1^\iota$ and $1^\iota = 1$. We further assume that $g^\iota g = g g^\iota \in Z_f$ for every $g \in G$ (so that the same is true for the p-component) and then we note that the rule $\mathrm{n}(g) := g^\iota g$ (resp. $\mathrm{n}_p(g_p) := g_p^\iota g_p$) defines a continuous character $\mathrm{n} : G_f \to Z_f$ (resp. $\mathrm{n}_p : G_p \to Z_p$). Then we suppose $\Sigma_D = \Sigma_p$, $\Sigma_E = \Sigma_p^\iota$ and $(K_p^\diamond)^\iota = K_p^\diamond \subset \Sigma_p \cap \Sigma_p^\iota$. Another piece of data that we need is an open and compact subgroup $Z_f^\diamond = Z^{\diamond p} \times Z_p^\diamond \subset Z_f \cap K_p^\diamond$ such that $\mathrm{n}_{p|K_p^\diamond} : K_p^\diamond \to Z_p^\diamond$. Assuming that E has central character κ_E, we can consider the second of the following compositions:

$$\mathrm{n}^{\omega_{0,p,E}} : G_f \xrightarrow{\mathrm{n}} Z_f \xrightarrow{\omega_{0,p,E}} R^\times \text{ and } \mathrm{n}_p^{\kappa_E} : K_p^\diamond \xrightarrow{\mathrm{n}_{p|K_p^\diamond}} Z_p^\diamond \xrightarrow{\kappa_{E|Z_p^\diamond}} R^\times$$

Suppose that $\chi_p : K_p^\diamond \to R^\times$ (resp. κ_E) extends to a character $\widetilde{\chi_p} : G_p \to R^\times$ (resp. $\widetilde{\kappa_E} : Z_f \to R^\times$). Then $\mathrm{n}_p^{\widetilde{\kappa_E}} := \widetilde{\kappa_E} \circ \mathrm{n}_p$ is an extension of $\mathrm{n}_p^{\kappa_E}$ and we let $Hom_{R[\Sigma_p,\Sigma_p^\iota]}(D \otimes E, R(\widetilde{\chi_p}))$ be the set of those pairings such that

$$\langle v\sigma, w \rangle = \widetilde{\chi_p}(\sigma)\mathrm{n}_p^{-\widetilde{\kappa_E}}(\sigma)\langle v, w\sigma^\iota \rangle \text{ for every } \sigma \in \Sigma_p.$$

We remark that, for every element $u \in K_p^\diamond$,

$$\langle vu, wu \rangle = \chi_p(u)\mathrm{n}_p^{-\kappa_E}(u)\langle v, wuu^\iota \rangle = \chi_p(u)\langle v, w \rangle,$$

so that $Hom_{R[\Sigma_p,\Sigma_p^\iota]}(D \otimes E, R(\widetilde{\chi_p})) \subset Hom_{R[K_p^\diamond]}(D \otimes E, R(\chi_p))$.

Remark 4.2. Suppose now that $D \subset \widetilde{D}$ and $E \subset \widetilde{E}$, where $\widetilde{D}$ and $\widetilde{E}$ are G_p-modules, the above inclusions are Σ_p and, respectively, Σ_p^ι-equivariant and that $\widetilde{E}$ has central character $\kappa_{\widetilde{E}} = \widetilde{\kappa_E}$ extending κ_E. If

$$\langle \cdot,\cdot \rangle \in Hom_{\mathcal{O}[K_p^\diamond]}(D \otimes E, R(\chi_p))$$

extends to $\langle \cdot,\cdot \rangle^\sim \in Hom_{\mathcal{O}[G_p]}(\widetilde{D} \otimes \widetilde{E}, R(\widetilde{\chi_p}))$, then

$$\langle \cdot,\cdot \rangle \in Hom_{\mathcal{O}[\Sigma_p,\Sigma_p^\iota]}(D \otimes E, R(\widetilde{\chi_p})).$$

Indeed we have[2]:

$$\langle v\sigma, w\rangle = \langle v\sigma, w\sigma^{-1}\sigma\rangle^{\sim} = \widetilde{\chi}_p(\sigma)\langle v, w\sigma^{-1}\rangle^{\sim}$$
$$= \widetilde{\chi}_p(\sigma)\mathrm{n}_p^{-\widetilde{\kappa_E}}(\sigma)\langle v, w\sigma'\rangle^{\sim} = \widetilde{\chi}_p(\sigma)\mathrm{n}_p^{-\widetilde{\kappa_E}}(\sigma)\langle v, w\sigma'\rangle.$$

In the following proposition we suppose that we are placed in the setting outlined above: it then follows from Definition 4.1 that we have $(\chi_{0,p}, \widetilde{\chi}_p) \in X_p(G_f, G_p, \omega_0)$. Furthermore, we need to assume that $\pi \in \Sigma_p$, that $\Gamma \to G_f^p$ is injective (it will be $\mathbf{G}(F) \to \mathbf{G}(\mathbb{A}_f^p)$ in our applications) and that $\chi_{0,p}u = \chi_{0,p}$ for every $u \in K_p^\diamond$ (indeed we will have $\chi_{0,p}u = \chi_{0,p}$ for every $K \in \mathcal{K}(\mathbf{G}(\mathbb{A}_f))$). Also, we suppose that $f \in M_p(G_f, D, \omega_{0,p,D})^{K_p^\diamond}$ and $g \in M_p(G_f, E, \omega_{0,p,E})^{K_p^\diamond}$ (and make a similar assumption in the ∞-adic case). Finally, we suppose that

$$\langle -, -\rangle \in Hom_{R[\Sigma_p, \Sigma_p']}(D \otimes E, R(\widetilde{\chi}_p))$$

(resp. $\langle -, -\rangle \in Hom_{R[B^\times]}(D \otimes E, R(\chi_p))$ in the ∞-adic case and E does not need to have central character κ_E). Having introduced the appropriate setting, the proof of the following result can be copied from [**5**, Prop. 2.6].

Proposition 4.2. *We have the following formulas, in the p-adic case:*

$$\langle f \mid T_\pi, g\rangle = \widetilde{\chi}_p(\pi_p)\chi_{0,p}(\pi)\mathrm{n}_p^{-\widetilde{\kappa_E}}(\pi_p)\mathrm{n}^{-\omega_{0,p,E}}(\pi)\langle f, g \mid T_{\pi'}\rangle.$$

In the ∞-adic case $\langle f \mid T_\pi, g\rangle = \chi_0(\pi)\langle f, g \mid T_{\pi^{-1}}\rangle$ and, whenever E has central character κ_E, $\langle f \mid T_\pi, g\rangle = \chi_0(\pi)\mathrm{n}^{-\kappa_E}(\pi)\langle f, g \mid T_{\pi'}\rangle$.

5 - p-adic automorphic forms and p-adic interpolation

The theory we have developed allows us to place the general period integrals (7) in a p-adic setting by means of (19) and Proposition 4.1. More precisely, fix embeddings $\sigma_\infty : E \to \mathbb{C}$ and $\sigma_p : E \to E_p$, with $E_p \subset \mathbb{C}_p$ the completion where E is as in § 3.2. We have inclusions

$$M[\mathbf{G}, \rho, \omega_0](E) \subset M[\mathbf{G}, \rho, \omega_0](\mathbb{C}) \simeq M[\mathbf{G}(\mathbb{A}_f), \pi_\infty, \omega_0],$$
$$M[\mathbf{G}, \rho, \omega_0](E) \subset M[\mathbf{G}, \rho, \omega_0](E_p) \simeq M_p[\mathbf{G}(\mathbb{A}_f), \rho_{E_p}, \omega_{0,p}].$$

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[2] Note that $\sigma^{-1} = \mathrm{n}_p(\sigma)^{-1}\sigma'$ because

$$\mathrm{n}_p(\sigma)^{-1}\sigma'\sigma = \mathrm{n}_p(\sigma)^{-1}\mathrm{n}_p(\sigma) = 1.$$

This is used in the third of the following equalities.

Then (19) and Proposition 4.1 (3) imply that

$$m_{\mathbf{S_H}\backslash\mathbf{H},\infty}^{-1} I_\eta = J_{p,\eta_f}^{\rho_{E_p},(\omega_f^\eta \mathbf{N}_f^\eta)_{0,p},(\omega^{-\eta}\mathbf{N}^\eta)_p}$$

on $M[\mathbf{G},\rho,\omega_0](F)$. Here

$$\left((\omega_f^\eta \mathbf{N}_f^\eta)_{0,p},(\omega^{-\eta}\mathbf{N}^\eta)_p\right) \text{ and } \left(\omega_f^\eta \mathbf{N}_f^\eta,(\omega^{-\eta}\mathbf{N})_\infty^\eta\right)$$

are associated as in Proposition 4.1. Hence, the p-adic L-functions can be obtained by deforming the quantities appearing in the definition of

$$J_{p,\eta_f}^{\rho_{E_p},(\omega_f^\eta \mathbf{N}_f^\eta)_{0,p},(\omega^{-\eta}\mathbf{N}^\eta)_p},$$

as illustrated in the example below.

Let the notations be as in the example of § 3.2 and fix a prime p such that $B\otimes_{\mathbb{Q}}\mathbb{Q}_p$ is split and which is prime to the discriminant of $K/\mathbb{Q}$. Setting $\left(\mathrm{Nr}_{K/\mathbb{Q},f}\right)_{0,p} := \mathrm{Nr}_{K/\mathbb{Q},f,p}$, by Proposition 4.1

$$J_{\eta,\chi^{-1}}(\varphi) := J_\eta^{\mathbf{V}_{k,\mathbb{C}},\mathrm{Nr}_{K/\mathbb{Q},f}^{k/2},\mathrm{nr}_{K/\mathbb{Q}}^{k/2}}\left(\Lambda_{j,k}\otimes(\varphi\times\chi)\right)$$

corresponds to

$$\begin{aligned} J_{\chi^{-1},p}(\varphi) &:= J_{\eta,p}^{\mathbf{V}_{k,\mathbb{C}},\mathrm{Nr}_{K/\mathbb{Q},f,p}^{k/2},\mathrm{nr}_{K/\mathbb{Q}}^{k/2}}\left(\Lambda_{j,k}\otimes\left(\varphi\times\chi^{-1}\right)\right) \\ &= \mu_{\mathbf{H}(\mathbb{A}_f)}\left(K_{\varphi,\chi}\right)\sum_{x\in K_{\varphi,\chi}\backslash\mathbf{H}(\mathbb{A}_f)/\mathbf{H}(\mathbb{Q})}\frac{\chi^{-1}(x)\Lambda_{j,k}(\varphi(j(x)))}{|\Gamma_{K_{\varphi,\chi}}(x)|\mathrm{Nr}_{K/\mathbb{Q},f,p}^{k/2}(x)}. \end{aligned} \tag{29}$$

This is the p-adic avatar of (20) produced by Proposition 4.1 (3). Writing $\varphi=\varphi_k$ in order to emphasize the dependence from the weight, we have to interpolate the functions $k\mapsto\varphi_k$, $k\mapsto\mathrm{Nr}_{K/\mathbb{Q},f,p}^{k/2}(x)$ and $k\mapsto\Lambda_{j,k}$. The first problem is solved using the theory of p-adic modular forms as developed in [**3**] or, for more general algebraic groups, replacing the coefficients valued in representations (the above $\mathbf{V}_{k_i,\mathbb{Q}_p}$'s) with the Ash-Stevens modules $\mathcal{D}$ constructed in [**1**] (see also [**4**]). The second problem is easy because $\mathrm{Nr}_{K/\mathbb{Q},f,p}(x)\in\mathbb{Z}_p^\times$ and the solution of the third problem yields the p-adic L-functions.

Set $\mathcal{X} := Hom_{cts}\left(\mathbb{Z}_p^\times,\mathbf{G}_m\right)$, the rigid analytic weight space which contains $\mathbb{Z}$ via $k\mapsto\left[t\mapsto t^k\right]$, and fix a character $\mathbf{k}:\mathbb{Z}_p^\times\to\mathcal{O}^\times$ which corresponds to some open affinoid $U\subset\mathcal{X}$, so that $\mathcal{O}=\mathcal{O}_{\mathcal{X}}(U)$ (we use the exponential notation $t^{\mathbf{k}}:=\mathbf{k}(t)$). We refer the reader to [**6**, § 5] for the notations employed here. In particular, we take $\Sigma_p := \Sigma_0\left(p\mathbb{Z}_p\right)$ (resp. $K_p^\diamond=\Gamma_0\left(p\mathbb{Z}_p\right)$) of loc. cit. and we form the space of locally analytic distributions $\mathcal{D}_{\mathbf{k}}(W)$ on $W:=\mathbb{Z}_p^\times\times\mathbb{Z}_p$ which are homogeneous of degree $\mathbf{k}$. Noticing that $\mathrm{N}_p(x)\in\mathbb{Z}_p^\times$, we see that we can define the spaces of p-adic families of modular forms on $\mathbf{B}^\times$ via (see [**6**, Examples 5.1 and 5.2]):

$$M_{p,\mathbf{k}}=M_p\left(\mathcal{D}_{\mathbf{k}}(W),\mathrm{N}_p^{\mathbf{k}}\right) := M_p\left(B_f^\times,\mathcal{D}_{\mathbf{k}}(W),\mathrm{N}_p^{\mathbf{k}}\right).$$

The elements of $\mathcal{D}_{\mathbf{k}}(W)$ naturally integrates locally analytic functions $F : W \to \mathcal{O}$ such that $F(tw) = t^{\mathbf{k}}F(w)$ (where $t^{\mathbf{k}} := \mathbf{k}(t)$), that we denote $\mathcal{A}_{\mathbf{k}}(W)$. If $k \in U$ is an integer, there is a specialization map $\mathcal{D}_{\mathbf{k}}(W) \to \mathcal{D}_k(W)$; the inclusion $\mathbf{P}_{k,\mathbb{Q}_p} \subset \mathcal{A}_k(W)$ yields $\mathcal{D}_k(W) \to \mathbf{V}_{k,\mathbb{Q}_p}$ by duality. Setting $M_p(\mathbf{V}_k, \mathrm{N}_p^{\mathbf{k}}) := M_p(B_f^\times, \mathbf{V}_{k,\mathbb{Q}_p}, \mathrm{N}_p^{\mathbf{k}})$ (and similarly for M), the resulting arrow $\mathcal{D}_{\mathbf{k}}(W) \to \mathbf{V}_{k,\mathbb{Q}_p}$ yields

$$\rho_k : M_p\Big(\mathcal{D}_{\mathbf{k}}(W), \mathrm{N}_p^{\mathbf{k}}\Big) \to M_p\Big(\mathbf{V}_{k,\mathbb{Q}_p}, \mathrm{N}_p^{k}\Big) \simeq M\Big(\mathbf{V}_{k,\mathbb{Q}_p}, \mathrm{N}^{k}\Big)$$

for every $k \in U$ even and classical. This solves the first interpolation problem and $\mathrm{Nr}_{K/\mathbb{Q},f,p}^{\mathbf{k}/2}(x) := \mathrm{Nr}_{K/\mathbb{Q},f,p}(x)^{\mathbf{k}/2}$, which makes sense because $\mathrm{Nr}_{K/\mathbb{Q},f,p}(x) \in \mathbb{Z}_p^\times$, solves the second interpolation problem (see [**6**, § 4.3] for the definition of $\mathbf{k}/2$).

From now on we let f be a Coleman family of finite slope and let $\varphi \in M_{p,\mathbf{k}}$ be its Jacquet-Langlands lift: we set $f_k := \rho_k(f)$ (resp. $\varphi_k := \rho_k(\varphi)$), which is the p-stabilization of a unique $f_k^{\#}$ (resp. $\varphi_k^{\#}$) for almost every classical (i.e. even integer) weight k. We let π_k be the automorphic representation of $\mathbf{GL}_2$ attached to $f_k^{\#}$. We suppose the tame level of f and the conductor of χ to be prime to p.

We now explain how to interpolate $k \mapsto \varLambda_{j,k}$. We view $Q_j \in \mathcal{A}_0(W)$ and set $W_j := Q_j^{-1}(\mathbb{Z}_p^\times) \subset W$; then $Q_{j|W_j} \in \mathcal{A}_0(W_j)$ take value in $\mathbb{Z}_p^\times$ and we can consider the element $Q_j^{\mathbf{k}} \in \mathcal{A}_{\mathbf{k}}(W_j)$. We may write $W = W_j \sqcup W_j^c$ as the disjoint union of open and compact subsets, inducing $\mathcal{D}_{\mathbf{k}}(W) = \mathcal{D}_{\mathbf{k}}(W_j) \oplus \mathcal{D}_{\mathbf{k}}(W_j^c)$. If $\mu \in \mathcal{D}_{\mathbf{k}}(W)$, we let $\mu_{|W_j} \in \mathcal{D}_{\mathbf{k}}(W_j)$ be its component. We may normalize j in such a way that $K_p^\diamond$ stabilizes j so that, setting $K_p^{\diamond,H} := j^{-1}(K_p^\diamond)$, the association $\mu \mapsto \mu_{|W_j}(P_j^{\mathbf{k}}) \in \mathcal{O}$ defines

$$\varLambda_{j,\mathbf{k}}^\circ \in Hom_{\mathcal{O}[K_p^{\diamond,H}]}(\mathcal{D}_{\mathbf{k}}(W), \mathcal{O}(\mathbf{k}/2)).$$

We can define

$$\mathcal{L}_{p,\chi^{-1}} = J_{\eta,p}^{\mathcal{D}_{\mathbf{k}}(W),\mathrm{Nr}_{K/\mathbb{Q},f,p}^{\mathbf{k}/2},\mathrm{nr}_{K/\mathbb{Q}}^{\mathbf{k}/2}}\big(\varLambda_{j,\mathbf{k}}^\circ \otimes (-\times\chi^{-1})\big) : M_p(\mathcal{D}_{\mathbf{k}}(W), \mathrm{N}_p^{\mathbf{k}}) \to \mathcal{O}.$$

When p is inert in K we have $W = W_j$ and $\mathcal{L}_{p,\chi^{-1}}(\varphi)(k) = J_{\chi^{-1},p}(\varphi_k)$; when p is split we have $W_j \approx \mathbb{Z}_p^\times \times \mathbb{Z}_p^\times$ (see [**7**, Lemma 5.3]). For an element $\alpha \in \mathcal{O}^\times$, let $M_{p,\mathbf{k}}^{\alpha} \subset M_{p,\mathbf{k}}$ be the submodule of those φ such that $U_p\varphi = \alpha\varphi$ (there are plenty, by the theory of eigencurve). When p is split the same computations carried on in [**2**, Lemma 3.9] allow one to relate $\mathcal{L}_{p,\chi^{-1}}(\varphi)(k)$ and $J_{\chi^{-1},p}(\varphi_k)$ assuming that $\varphi \in M_{p,\mathbf{k}}^{\alpha}$. Further expressing $J_{\chi^{-1},p}(\varphi_k)$ in terms of $J_{\chi^{-1},p}(\varphi_k^{\#})$ as in [**2**] yields, together with (20), the interpolation formula

$$\mathcal{L}_{p,\chi^{-1}}(\varphi)(k)\mathcal{L}_{p,\chi}(\varphi)(k) = \frac{\mathcal{E}_{p,\chi^{-1}}(k)}{2^\beta m_{\mathbf{S_H}\backslash\mathbf{H},\infty}} \frac{\varDelta_{\mathbf{G}_V} L(1/2, \pi_k \times \pi_{\chi^{-1}})}{L(1, \pi_g', \mathrm{Ad})L(1, \pi_{\chi^{-1}}, \mathrm{Ad})} \prod_v \alpha_v(\varphi_{k,v})$$

for every $\varphi \in M^{\alpha}_{p,\mathbf{k}}$, $k \in U$ even and classical and the Euler factor $\mathcal{E}_{p,\chi^{-1}}(k)$ as in [**10,** Thm. 4.17 and 4.18].

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