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Generalized Nadler G-contraction in cone metric spaces over Banach algebras endowed with a graph

Abstract. In this paper, we introduce the generalized Nadler G-contractions in cone metric spaces endowed with a graph and defined over a Banach algebra. A fixed point result for such mappings is proved. Our result generalizes some known results in metric and cone metric spaces. An example is presented which verifies the significance and usability of the result proved herein.

Keywords. Cone metric space, set-valued mapping, Nadler G-contraction, fixed point.

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1 - Introduction

In 2007, Huang and Zhang [**5**] reviewed the notion of cone metric spaces and gave a generalization of metric spaces. They defined the Cauchy sequences and convergence of sequences in such spaces in terms of interior points of the underlying cone. They proved some basic fixed point theorems with the assumption that the underlying cone is normal. Rezapour and Hamlbarani [**15**] showed that the results obtained by Huang and Zhang remain true if the assumption of normality is dropped. In the papers [**2-4, 9**] authors showed the equivalency of fixed point results in cone metric spaces and the corresponding fixed point results in ordinary metric spaces.

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Recently, Liu and Xu [**12**] introduced the study of cone metric spaces over Banach algebras and proved some fixed point theorems on such spaces. They introduced the generalized Lipschitz mappings on cone metric spaces over Banach algebras. In such mappings the Lipschitz factor (constant) is taken as a vector of underlying cone instead a real constant. They showed that the fixed point results in cone metric spaces over Banach algebras are not equivalent to the corresponding results of metric spaces in view of the existence of the fixed points of the generalized Lipschitz mappings. However, in the fixed point results of Liu and Xu [**12**] a strong condition that the underlying cone is normal was used. Later on, Xu and Radenović [**20**] removed the condition of normality of cone and generalized the results of Liu and Xu [**12**] in cone metric spaces over Banach algebras with the cones not necessarily normal.

Nadler [**14**] generalized the Banach contraction principle for the set-valued mappings, i.e., for the case when the mappings on a metric space can have a value in form of a nonempty subset of space. Wardowski [**19**] (see also, [**11, 18**]) generalized the Nadler's theorem in the setting of cone metric spaces.

In 2008, Jachymski [**6**] considered the Banach contraction principle in the spaces endowed with a graph. He introduced the Banach G-contraction and unified several recent fixed point results on metric space by proving the fixed point result for Banach G-contraction. Motivated by the result of Jachymski [**6**] and the fact that the generalizations of fixed point theorems in cone metric spaces over Banach algebras cannot be obtained by the corresponding results of the ordinary metric spaces, we introduced the Nadler G-contractions in cone metric spaces endowed with a graph and defined over a Banach algebra and prove a fixed point theorem for such contractions. The result of this paper generalizes and unifies the results of Nadler [**14**], Jachymski [**6**], Wardowski [**19**], Liu and Xu [**12**], Xu and Radenović [**20**] and several other results. An example is presented which shows the significance and usability of the result proved herein.

2 - Preliminaries

First, we recall some definitions about cone metric spaces and graphs.

Definition 2.1. Let $\mathfrak{B}$ be a real Banach algebra, i.e., $\mathfrak{B}$ is a real Banach space with a product that satisfies

1. $x(yz) = (xy)z$;
2. $x(y+z) = xy + xz$;

3. $\alpha(xy) = (\alpha x)y = x(\alpha y)$;
4. $\|xy\| \leq \|x\|\|y\|$,

for all $x, y, z \in \mathfrak{B}$, $\alpha \in \mathbb{R}$.

The Banach algebra $\mathfrak{B}$ is said to be unital if there exists an element $e \in \mathfrak{B}$ such that $ex = xe = x$ for all $x \in \mathfrak{B}$. The element e is called the unit. An $x \in \mathfrak{B}$ is said to be invertible if there is a $y \in \mathfrak{B}$ such that $xy = yx = e$. The inverse of x, if it exists, is unique and will be denoted by x^{-1}. For more details, see [**16**].

Proposition 2.2 ([**16**]). *Let $\mathfrak{B}$ be a Banach algebra with a unit e and $x \in \mathfrak{B}$. If the spectral radius $\rho(x)$ of x is less than 1, i.e.,*

$$\rho(x) = \lim_{n\to\infty} \|x^n\|^{1/n} = \inf_{n\in\mathbb{N}} \|x^n\|^{1/n} < 1,$$

then $e - x$ is invertible and $(e-x)^{-1} = \sum_{i=0}^{\infty} x^i$.

Let $\mathfrak{B}$ be a unital Banach algebra. A non-empty closed set $P \subset \mathfrak{B}$ is said to be a cone (see [**12, 13**]) if

1. $e \in P$,
2. $P + P \subset P$,
3. $\alpha P \subset P$ for all $\alpha \geq 0$,
4. $P^2 \subset P$,
5. $P \cap (-P) = \{\theta\}$, where θ is the zero vector of $\mathfrak{B}$.

Given a cone $P \subset \mathfrak{B}$ one can define a partial order $\preceq$ on $\mathfrak{B}$ by $x \preceq y$ if and only if $y - x \in P$. The notation $x \ll y$ will stand for $y - x \in P^\circ$, where P° denotes the interior of P.

The cone P is called normal if there exists a number $K > 0$ such that for all $a, b \in \mathfrak{B}$,

$$a \preceq b \quad \text{implies} \quad \|a\| \leq K\|b\|.$$

The least positive value of K satisfying the above inequality is called the normal constant (see [**5**]). Note that, for any normal cone P we have $K \geq 1$ (see [**15**]).

Lemma 2.3 ([8]). *Let $P \subset \mathfrak{B}$ be a solid cone and $a, b, c \in P$.*

(a) *If $a \preceq b$ and $b \ll c$, then $a \ll c$.*

(b) *If $a \ll b$ and $b \ll c$, then $a \ll c$.*

(c) *If* $\theta \preceq u \ll c$ *for each* $c \in P^\circ$, *then* $u = \theta$.

(d) *If* $c \in P^\circ$ *and* $a_n \to \theta$, *then there exist* $n_0 \in \mathbb{N}$ *such that, for all* $n > n_0$ *we have* $a_n \ll c$.

(e) *If* $\theta \preceq a_n \preceq b_n$ *for each* n *and* $a_n \to a,\ b_n \to b$, *then* $a \preceq b$.

Henceforth, we will assume that the real Banach algebra $\mathfrak{B}$ is unital and that the cone $P \subset \mathfrak{B}$ is a solid cone, i.e., $P^\circ \neq \emptyset$.

Lemma 2.4. *Let* $\mathfrak{B}$ *be a Banach algebra with a unit* e, P *be a cone in* $\mathfrak{B}$ *and* $a, b, c \in P$.

(i) *If* $\rho(a) < 1$, *then* $\rho(a^m) \leq \rho(a) < 1$ *for each* $m \in \mathbb{N}$.

(ii) *If* $\rho(a) < 1$ *and* $b \preceq ac$, *then* $b \preceq c$.

Proof. (i): Since $\rho(a) < 1$ and $m \in \mathbb{N}$ is fixed, it follows that

$$\begin{aligned}\rho(a^m) &= \lim_{n\to\infty} \|(a^m)^n\|^{\frac{1}{n}} = \lim_{n\to\infty} \|(a^n)^m\|^{\frac{1}{n}} \leq \lim_{n\to\infty} \left(\|a^n\|^{\frac{1}{n}}\right)^m \\ &= \left(\lim_{n\to\infty} \|a^n\|^{\frac{1}{n}}\right)^m = (\rho(a))^m < 1.\end{aligned}$$

(ii): Suppose $b \preceq ac$, then we have $ac - b \in P$. Now

$$c - b = c - ac + ac - b = c(e - a) + ac - b.$$

Since $\rho(a) < 1$, we have $e - a \in P$ (see, [**20**]) and so by the above equality $c - b \in P$, i.e., $b \preceq c$. □

Definition 2.5 ([**5, 12, 13**]). Let X be a non-empty set. Suppose that the mapping $d: X \times X \to \mathfrak{B}$ satisfies:

1. $\theta \preceq d(x, y)$ for all $x, y \in X$ and $d(x, y) = \theta$ if and only if $x = y$.
2. $d(x, y) = d(y, x)$ for all $x, y \in X$.
3. $d(x, y) \preceq d(x, z) + d(z, y)$ for all $x, y, z \in X$.

Then d is called a cone metric on X, and (X, d) is called a cone metric space over the Banach algebra $\mathfrak{B}$.

For some examples of cone metric space over a Banach algebra we refer [**12, 13, 20**].

Definition 2.6 ([**5, 12, 13**]). Let (X, d) be a cone metric space over the Banach algebra$\mathfrak{B}$, $x \in X$ and $\{x_n\}$ be a sequence in X. Then:

(i) $\{x_n\}$ is said to be converges to x if for every $c \in \mathfrak{B}$ with $\theta \ll c$ there exists a natural number n_0 such that $d(x_n, x) \ll c$ for all $n > n_0$. We denote this by $\lim_{n\to\infty} x_n = x$ or $x_n \to x$ as $n \to \infty$.

(ii) $\{x_n\}$ is called a Cauchy sequence if for every $c \in \mathfrak{B}$ with $\theta \ll c$ there exists a natural number n_0 such that $d(x_n, x_m) \ll c$ for all $n, m > n_0$.

(iii) (X, d) is called complete if every Cauchy sequence in X converges to some point in X.

Definition 2.7 (see also [**19**]). Let (X, d) be a cone metric space over a Banach algebra $\mathfrak{B}$ and let $\mathcal{A}$ be a collection of nonempty subsets of X. A map $H: \mathcal{A} \times \mathcal{A} \to \mathfrak{B}$ is called a $H-$cone metric with respect to d if for any $A_1, A_2 \in \mathcal{A}$ the following conditions hold:

(H1) $H(A_1, A_2) = 0 \Rightarrow A_1 = A_2$;

(H2) $H(A_1, A_2) = H(A_2, A_1)$;

(H3) $\forall_{c\in E, \theta\ll c}\ \forall_{x\in A_1}\ \exists_{y\in A_2}\ \ d(x, y) \preceq H(A_1, A_2) + c$;

(H4) One of the following is satisfied:

(i) $\forall_{c\in E, \theta\ll c}\ \exists_{x\in A_1}\ \forall_{y\in A_2}\ H(A_1, A_2) \preceq d(x, y) + c$;

(ii) $\forall_{c\in E, \theta\ll c}\ \exists_{x\in A_2}\ \forall_{y\in A_1}\ H(A_1, A_2) \preceq d(x, y) + c$.

It is obvious that each $H-$cone metric depends on the choice of the collection $\mathcal{A}$. The following are some natural examples of $H-$cone metrics on cone metric spaces over a Banach algebra (see also [**19**]).

Example 2.8 ([**19**]). Let (X, d) be a cone metric space over a Banach algebra $\mathfrak{B}$ and let $\mathcal{A} = \{\{x\}: x \in X\}$. Then the mapping $H: \mathcal{A} \times \mathcal{A} \to E$ given by the formula

$$H(\{x\}, \{y\}) = d(x, y) \text{ for all } x, y \in X,$$

is a $H-$cone metric with respect to d.

Example 2.9 ([**19**]). Let (X, d) be a metric space, $\mathfrak{B} = \mathbb{R}$ with usual product and norm, $P = \mathbb{R}^+ = [0, \infty)$ and let $\mathcal{A}$ be the family of all nonempty, closed bounded subsets of X. Then the mapping $H: \mathcal{A} \times \mathcal{A} \to \mathfrak{B}$ given by the formula

$$H(A, B) = \max\left\{\sup_{x\in A} d(x, B),\ \sup_{y\in B} d(y, A), A, B \in \mathcal{A}\right\}$$

which is called a Hausdorff metric, is a $H-$cone metric with respect to d.

The following definition about the graphs can be found in [**6, 7, 17**].

Let X be a nonempty set and by Δ denote the diagonal of the cartesian product $X \times X$. Let $G = (V_G, E_G)$ be a graph such that the set of vertices $V_G = X$ and the set of its edges $E_G \supseteq \Delta$, that is, E_G contains all loops, then we say that X is endowed with the graph $G = (V_G, E_G)$. We assume that G is without parallel edges, and so we can identify G with the pair (V_G, E_G). The conversion of graph G is denoted by G^{-1} and it is defined by:

$$V_{G^{-1}} = V_G \text{ and } E_{G^{-1}} = \{(x, y) \in X \times X : (y, x) \in E_G\}.$$

By $\widetilde{G}$, we denote the undirected graph obtained from G by including all the edges of G^{-1}. More precisely, we define

$$V_{\widetilde{G}} = V_G \text{ and } E_{\widetilde{G}} = E_G \cup E_{G^{-1}}.$$

If x and y are vertices in a graph G, then a path in G from x to y of length $k \in \mathbb{N}$ is a sequence $\{x^i\}_{i=0}^k$ of $k+1$ vertices such that $x^0 = x, x^k = y$ and $(x^{i-1}, x^i) \in E_G$ for $i = 1, 2, \ldots, k$. A graph G is called connected if, there is a path between any two vertices of G. The graph G is weakly connected if, treating all of its edges as being undirected, there is a path from every vertex to every other vertex. More precisely, G is weakly connected if, $\widetilde{G}$ is connected. Two vertices x and y of a directed graph are connected if there is a path from x to y and a path from y to x. Obviously, in case of undirected graph the existence of path from x to y is sufficient. For $k \in \mathbb{N}$, we denote

$$[x]_G^k = \{y \in X: \text{ there is a directed path from } x \text{ to } y \text{ of length } k.$$

Throughout this paper, we assume that the graphs under consideration are directed and are with nonempty sets of vertices and edges.

3 - Main results

In this section, we define the set-valued G-contractions and prove some fixed point theorems for such contractions in cone metric spaces endowed with a graph and defined over a Banach algebra.

Definition 3.1. Let (X, d) be a cone metric space over a Banach algebra $\mathfrak{B}$ and let $\mathcal{A}$ be a nonempty collection of nonempty subsets of X. A set-valued mapping $T: X \to \mathcal{A}$ is said to be a generalized Nadler G-contraction with contractive vector a, if for all $x, y \in X$ with $(x, y) \in E_G$:

(i) there exists $a \in P$ such that $\rho(a) < 1$ and $H(Tx, Ty) \preceq a d(x, y)$;

(ii) if $u \in Tx$, $v \in Ty$ are such that $d(u, v) \preceq d(x, y)$, then $(u, v) \in E_G$.

Next, we state the main result of this paper.

Theorem 3.2. *Let (X, d) be a complete cone metric space over a Banach algebra $\mathfrak{B}$, P the solid cone in Banach algebra $\mathfrak{B}$ and let $\mathcal{A}$ be a nonempty collection of nonempty closed subsets of X. Suppose, $T: X \to \mathcal{A}$ be a generalized Nadler G-contraction with contractive vector a and there exist $k \in \mathbb{N}$, $x_0 \in X$ such that:*

(I) $[x_0]_G^k \cap Tx_0 \neq \emptyset$;

(II) *for any sequence $\{x_n\} \in X$, if $x_n \to x$ and $x_n \in [x_{n-1}]_G^k \cap Tx_{n-1}$ for all $n \in \mathbb{N}$, then there is a subsequence $\{x_{n_j}\}$ such that $(x_{n_j}, x) \in E_G$ for all $j \in \mathbb{N}$.*

Then there exists a sequence $\{x_n\} \in X$ such that $x_n \in [x_{n-1}]_G^k \cap Tx_{n-1}$ for all $n \in \mathbb{N}$ and it converges to a fixed point of T.

Proof. In view of (I), let $x_1 \in [x_0]_G^k \cap Tx_0$. Then there is a path $(y^i)_{i=0}^k$ in G from x_0 to x_1, i.e., $y^0 = x_0$, $y^k = x_1$ and $(y^{i-1}, y^i) \in E_G$ for $i = 1, 2, \dots, k$. Since $x_1 \in Tx_0 = Ty^0$, from the definition of $H-$cone metric there exists $y_1^1 \in Ty^1$ such that

$$d(x_1, y_1^1) \preceq H(Ty^0, Ty^1) + c_1^1,$$

where $c_1^1 \in P^\circ$ is chosen so that $\rho(c_1^1) < 1$. Since $(y^0, y^1) \in E_G$ and T is a set-valued G-contraction, it follows from the above inequality that

$$d(x_1, y_1^1) \preceq ad(y^0, y^1) + c_1^1.$$

Since P is closed, letting $c_1^1 \to \theta$ we obtain from the above inequality that $d(x_1, y_1^1) \preceq ad(y^0, y^1) \preceq d(y^0, y^1)$, and so, by assumption $(x_1, y_1^1) \in E_G$. Again, since $y_1^1 \in Ty^1$, there exists $y_1^2 \in y^2$ such that

$$d(y_1^1, y_1^2) \preceq H(Ty^1, Ty^2) + c_1^2,$$

where $c_1^2 \in P^\circ$ is chosen so that $\rho(c_1^2) < 1$. Since $(y^1, y^2) \in E_G$ and T is a set-valued G-contraction, it follows from the above inequality that

$$d(y_1^1, y_1^2) \preceq ad(y^1, y^2) + c_1^2.$$

Letting $c_1^2 \to \theta$ we obtain from the above inequality that $d(y_1^1, y_1^2) \preceq ad(y^1, y^2) \preceq d(y^1, y^2)$, and so, by assumption $(y_1^1, y_1^2) \in E_G$. In a similar way, we obtain: for $i = 2, 3, \dots, k$ there exists $y_1^i \in Ty^i$ such that $(y_1^{i-1}, y_1^i) \in E_G$ and

$$d(y_1^{i-1}, y_1^i) \preceq ad(y^{i-1}, y^i) + c_1^i, \tag{1}$$

where $c_1^i \in P^\circ$ is chosen so that $\rho(c_1^i) < 1$. Thus, $\{y_1^i\}_{i=0}^k$ is a path from $x_1 = y_1^0$ to $y_1^k = x_2$ (say), where $y_1^k = x_2 \in Ty^k = Tx_1$. Thus, $x_2 \in [x_1]_G^k \cap Tx_1$.

Now, since $x_2 \in [x_1]_G^k \cap Tx_1$, there exists $y_2^1 \in Ty_1^1$ such that

$$d(x_2, y_2^1) \preceq H(Tx_1, Ty_1^1) + c_2^1 = H(Ty_1^0, Ty_1^1) + c_2^1,$$

where $c_2^1 \in P^\circ$ is chosen so that $\rho(c_2^1) < 1$. Since $(y_1^0, y_1^1) \in E_G$ and T is a set-valued G-contraction, it follows from the above inequality that

$$d(x_2, y_2^1) \preceq ad(y_1^0, y_1^1) + c_2^1.$$

Letting $c_2^1 \to \theta$ we obtain from the above inequality that $d(x_2, y_2^1) \preceq ad(y_1^0, y_1^1) \preceq d(y_1^0, y_1^1)$ and so $d(x_2, y_2^1) \in E_G$.

In a similar way, we obtain: for $i = 2, 3, \ldots, k$ there exists $y_2^i \in Ty_1^i$ such that $(y_2^{i-1}, y_2^i) \in E_G$ and

$$d(y_2^{i-1}, y_2^i) \preceq ad(y_1^{i-1}, y_1^i) + c_2^i, \tag{2}$$

where $c_2^i \in P^\circ$ is chosen so that $\rho(c_2^i) < 1$. Thus, $\{y_2^i\}_{i=0}^k$ is a path from $x_2 = y_2^0$ to $y_2^k = x_3$ (say), where $y_2^k = x_3 \in Ty_1^k = Tx_2$. Thus, $x_3 \in [x_2]_G^k \cap Tx_2$.

The above process leads us to a sequence $\{x_n\} \in X$ such that $x_{n+1} \in [x_n]_G^k \cap Tx_n$ with a path $\{y_n^i\}$ from $x_n = y_n^0$ to $x_{n+1} = y_n^k$ and the following inequality is satisfied: for $i = 1, 2, \ldots, k$

$$d(y_n^{i-1}, y_n^i) \preceq ad(y_{n-1}^{i-1}, y_{n-1}^i) + c_n^i, \tag{3}$$

where $c_n^i \in P^\circ$ is chosen so that $\rho(c_n^i) < 1$.

For simplicity, set $d_0^i = d(y^{i-1}, y^i)$ and $d_n^i = d(y_n^{i-1}, y_n^i)$. Then, successive applications of (3) yield

$$\begin{aligned}
d_n^i = d(y_n^{i-1}, y_n^i) &\preceq ad(y_{n-1}^{i-1}, y_{n-1}^i) + c_n^i \\
&= ad_{n-1}^i + c_n^i \\
&\preceq a^2 d_{n-2}^i + ac_{n-1}^i + c_n^i \\
&\preceq a^3 d_{n-3}^i + a^2 c_{n-2}^i + ac_{n-1}^i + c_n^i \\
&\vdots \\
&\preceq a^n d_0^i + a^{n-1} c_1^i + a^{n-2} c_2^i + \cdots + ac_{n-1}^i + c_n^i.
\end{aligned}$$

Thus, for $i = 1, 2, \ldots, k$ we have

$$d_n^i \preceq a^n d_0^i + \sum_{j=0}^{n-1} a^j c_{n-j}^i\,. \tag{4}$$

We shall show that $\{x_n\}$ is a Cauchy sequence. For $n \in \mathbb{N}$ we have,

$$d(x_n, x_{n+1}) \preceq d(y_n^0, y_n^k) \preceq \sum_{i=1}^k d(y_n^{i-1}, y_n^i) = \sum_{i=1}^k d_n^i\,.$$

Using (4) in the above inequality we obtain

$$d(x_n, x_{n+1}) \preceq \sum_{i=1}^{k} a^n d_0^i + \sum_{i=1}^{k} \sum_{j=0}^{n-1} a^j c_{n-j}^i \,. \tag{5}$$

Note that, since $\rho(a) < 1$, by Lemma 2.4, we have $\rho(a^n) < 1$ for all $n \in \mathbb{N}$. Therefore, for $i = 1, 2, \ldots, k$ we can choose $c_n^i = a^{2n}$ for all $n \in \mathbb{N}$, $i = 1, 2, \ldots, k$ and so by (5) we obtain

$$\begin{aligned} d(x_n, x_{n+1}) &\preceq \sum_{i=1}^{k} a^n d_0^i + \sum_{i=1}^{k} \sum_{j=0}^{n-1} a^{2n-j} = a^n \sum_{i=1}^{k} d_0^i + k a^n \sum_{j=0}^{n-1} a^{n-j} \\ &\preceq a^n \sum_{i=1}^{k} d_0^i + k a^n \sum_{j=0}^{\infty} a^j. \end{aligned}$$

Since $\rho(a) < 1$, it follows from Proposition 2.2 and the above inequality that

$$d(x_n, x_{n+1}) \preceq a^n D_0 + (e - a)^{-1} k a^n, \tag{6}$$

where $D_0 = \sum\limits_{i=1}^{k} d_0^i$.

Let $n, m \in \mathbb{N}$ and $m > n$. Then, using (6) we obtain

$$\begin{aligned} d(x_n, x_m) &\preceq \sum_{i=n}^{m-1} d(x_i, x_{i+1}) \\ &\preceq \sum_{i=n}^{m-1} [a^i D_0 + (e - a)^{-1} k a^i] \\ &= D_0 \sum_{i=n}^{m-1} a^i + (e - a)^{-1} k \sum_{i=n}^{m-1} a^i \\ &\preceq D_0 a^n \sum_{i=0}^{\infty} a^i + (e - a)^{-1} k a^n \sum_{i=0}^{\infty} a^i. \end{aligned}$$

Again, since $\rho(a) < 1$, it follows from Proposition 2.2 and the above inequality that

$$d(x_n, x_m) \preceq D_0 a^n (e - a)^{-1} + (e - a)^{-2} k a^n. \tag{7}$$

Since, $\rho(a) < 1$ we have $\|a^n\| \to 0$, i.e., $a^n \to \theta$ as $n \to \infty$. Hence, $D_0 a^n (e - a)^{-1} + (e - a)^{-2} k a^n \to \theta$ as $n \to \infty$. Now by part (a) and (d) of Lemma 2.3 it follows that, for given $c \in P$, $\theta \ll c$ there exists $n_0 \in \mathbb{N}$ such that $d(x_n, x_m) \ll c$ for all $n, m > n_0$. Thus, $\{x_n\}$ is a Cauchy sequence in X. By completeness of X, there exists $x^* \in X$ such that $x_n \to x^*$ as $n \to \infty$.

We shall show that x^* is a fixed point of X. By part (II) of assumption there exists a subsequence $\{x_{n_j}\}$ such that $(x_{n_j}, x^*) \in E_G$ for all $j \in \mathbb{N}$. Since $x_n \in T x_{n-1}$ for all

$n \in \mathbb{N}$, therefore for all $j \in \mathbb{N}$ there exists $y_j \in Tx^*$ such that

$$d(x_{n_j+1}, y_j) \preceq H(Tx_{n_j}, Tx^*) + c_j,$$

where $c_j \in P^\circ$ is chosen such that $c_j \to \theta$ as $j \to \infty$. As $(x_{n_j}, x^*) \in E_G$ for all $j \in \mathbb{N}$ we obtain from the above inequality that

$$d(x_{n_j+1}, y_j) \preceq ad(x_{n_j}, x^*) + c_j. \tag{8}$$

Now,

$$\begin{aligned} d(y_j, x^*) &\preceq d(y_j, x_{n_j+1}) + d(x_{n_j+1}, x^*) \\ &\preceq d(x_{n_j+1}, x^*) + ad(x_{n_j}, x^*) + c_j. \end{aligned}$$

Since $x_j \to x^*$ and $c_j \to \theta$ as $j \to \infty$, for any $c \in P, \theta \ll c$ and each $m \in \mathbb{N}$ there exists $m_0 \in \mathbb{N}$ such that $d(x_{n_j+1}, x^*) + ad(x_{n_j}, x^*) + c_j \ll c$ for all $j > m_0$. By part (a) of Lemma 2.3 we have,

$$d(y_j, x^*) \ll c \quad \text{for all } j > m_0\,.$$

It shows that $y_j \to x^*$ as $j \to \infty$. Since $y_j \in Tx^*$ for all $j \in \mathbb{N}$ and Tx^* is closed we must have $x^* \in Tx^*$. Thus, x^* is a fixed point of T. $\square$

Next, we give an example which shows that how the above result is different from the existing ones.

Example 3.3. Let $\mathfrak{B} = \mathbb{R}^2$ with the norm defined by $\|(x_1, x_2)\| = |\,x_1\,| + |\,x_2\,|$ and the multiplication defined by

$$(x_1, x_2)(y_1, y_2) = (x_1y_1, x_1y_2 + x_2y_1)$$

for all $(x_1, x_2), (y_1, y_2) \in \mathfrak{B}$. Then $\mathfrak{B}$ is a Banach algebra with unit $e = (1, 0)$. Let $P = \{(x_1, x_2) \in \mathfrak{B} \colon x_1, x_2 \geq 0\}$. Then $P \subset \mathfrak{B}$ is a cone and $P^\circ \neq \emptyset$.

By $\lfloor \theta, (x_1, x_2) \rfloor$ we denote an ordered interval in $\mathbb{R}^2$ and it contains all points $(a_1, a_2) \in \mathbb{R}^2$ such that $0 \leq a_1 \leq x_1, 0 \leq a_2 \leq x_2$, i.e.,

$$\lfloor \theta, (x_1, x_2) \rfloor = \{(a_1, a_2) \colon 0 \leq a_1 \leq x_1, 0 \leq a_2 \leq x_2\}.$$

Let $X = \mathbb{R}^+ \times \mathbb{R}^+$ and the the function $d \colon X \times X \to \mathfrak{B}$ be defined by

$$d((x_1, x_2), (y_1, y_2)) = (\,|\,x_1 - y_1\,|, |\,x_2 - y_2\,|\,).$$

Then (X, d) is a cone metric space. Let $\mathcal{A}$ be the family of subsets of X of the following form:

$$\mathcal{A} = \big\{\lfloor \theta, (x_1, x_2) \rfloor \colon (x_1, x_2) \in X\big\}.$$

For some fixed $\alpha, \beta > 1$, $\gamma > 0$ and $\alpha \leq 1 + \beta^2$, define the mappings $T \colon X \to \mathcal{A}$ and

$H: \mathcal{A} \times \mathcal{A} \to \mathfrak{B}$ by:

$$T(x_1, x_2) = \lfloor \theta, \big(\ln(\alpha + x_1), \tan^{-1}(\beta + x_2) + \gamma x_1\big) \rfloor \ \ \forall\, (x_1, x_2) \in X \quad \text{and}$$

$$H(A, B) = \big(\mid x_1 - y_1 \mid, \mid x_2 - y_2 \mid \big), A = \lfloor \theta, (x_1, x_2) \rfloor, B = \lfloor \theta, (y_1, y_2) \rfloor \in \mathcal{A}.$$

Define a graph $G = (V_G, E_G)$ by $V_G = X$ and $E_G = X \times X$.

By Lagrange mean value theorem we have

$$\ln\left(\frac{\alpha + x}{\alpha + y}\right) \leq \frac{x - y}{\alpha} \quad \text{and} \quad \tan^{-1}(\beta + x) - \tan^{-1}(\beta + y) \leq \frac{x - y}{1 + \beta^2}$$

for all $0 \leq y \leq x$. Therefore, for all $(x_1, x_2), (y_1, y_2) \in X$ we obtain

$$\begin{aligned}
&H(T(x_1, x_2), T(y_1, y_2)) \\
&= \left(\left| \ln\left(\frac{\alpha + x_1}{\alpha + y_1}\right) \right|, \mid \tan^{-1}(\beta + x_2) - \tan^{-1}(\beta + y_2) + \gamma(x_1 - y_1) \mid \right) \\
&\preceq \left(\left| \frac{x_1 - y_1}{\alpha} \right|, \left| \frac{x_2 - y_2}{1 + \beta^2} \right| + \gamma \mid x_1 - y_1 \mid \right) \\
&\preceq \left(\frac{1}{\alpha}, \gamma \right) d((x_1, x_2), (y_1, y_2)).
\end{aligned}$$

Also, $\left\| \left(\frac{1}{\alpha}, \gamma \right)^n \right\|^{1/n} = \left\| \left(\frac{1}{\alpha^n}, \frac{n\gamma}{\alpha^{n-1}} \right) \right\|^{1/n} \to \frac{1}{\alpha}(<1)$ as $n \to \infty$, and so, $\rho\left(\frac{1}{\alpha}, \gamma\right) < 1$. Therefore, the condition (i) of Definition 3.1 is satisfied with $a = \left(\frac{1}{\alpha}, \gamma\right)$. All the other conditions of Theorem 3.2 are satisfied and T has a fixed point $(0, 0)$ in X.

Remark 3.4. In the above example, for sufficient large $\gamma > 0$ the mapping T is not a set-valued contraction (in the sense of Nadler [**14**]) and so the existence of fixed point of T can not be concluded by the corresponding results from usual metric spaces. Also, for $\gamma > 1$ we have $\left(\frac{1}{\alpha}, \gamma\right) \not\preceq (1, 0) = e$ and $\left\| \left(\frac{1}{\alpha}, \gamma\right) \right\| = \frac{1}{\alpha} + \gamma > 1$, which shows that our result is a proper extension of Wardowski [**19**].

Next, we prove an ordered version of result of Wardowski [**19**] on ordered complete cone metric space over a Banach algebra.

Definition 3.5. Let $(X, \sqsubseteq, d)$ be an ordered complete cone metric space over a Banach algebra $\mathfrak{B}$ and let $\mathcal{A}$ be a nonempty collection of nonempty subsets of X. A set-valued mapping $T: X \to \mathcal{A}$ is said to be an ordered generalized Nadler contraction with contractive vector a, if for all $x, y \in X$ with $x \sqsubseteq y$:

(i) there exists $a \in P$ such that $\rho(a) < 1$ and $H(Tx, Ty) \preceq ad(x, y)$;

(ii) if $u \in Tx$, $v \in Ty$ are such that, $d(u, v) \preceq d(x, y)$, then $u \sqsubseteq v$.

The following corollary is a fixed point result for ordered generalized Nadler contractions and can be obtained by using Theorem 3.2 with the set of edges $E_G = \{(x, y) \in X \times X : x \sqsubseteq y\}$.

Corollary 3.6. *Let $(X, \sqsubseteq, d)$ be an ordered complete cone metric space over a Banach algebra $\mathfrak{B}$, P the solid cone in Banach algebra $\mathfrak{B}$ and let $\mathcal{A}$ be a nonempty collection of nonempty closed subsets of X. Suppose, $T: X \to \mathcal{A}$ be an ordered generalized Nadler contraction with contractive vector a and there exists $x_0 \in X$ such that:*

(I) $x_0 \sqsubseteq Tx_0$;

(II) *for any sequence $\{x_n\} \in X$, if $x_n \to x$, $x_{n-1} \sqsubseteq x_n$ and $x_n \in Tx_{n-1}$ for all $n \in \mathbb{N}$, then there is a subsequence $\{x_{n_j}\}$ such that $x_{n_j} \sqsubseteq x$ for all $j \in \mathbb{N}$.*

Then there exists a sequence $\{x_n\} \in X$ such that $x_{n-1} \sqsubseteq x_n$, $x_n \in Tx_{n-1}$ for all $n \in \mathbb{N}$ and it converges to a fixed point of T.

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