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A note on upper ramification jumps in Abelian extensions of exponent *p*

Abstract. In this paper we present a classification of the possible upper ramification jumps for an elementary Abelian p-extension of a p-adic field. The fundamental step for the proof of the main result is the computation of the ramification filtration for the maximal elementary Abelian p-extension of the base field K. This result generalizes [**3**, Lemma 9, p. 286], where the same result is proved under the assumption that K contains a primitive p-th root of unity. To deal with this general case we use class field theory and the explicit relations between the normic group of an extension and its ramification jumps, and we obtain necessary and sufficient conditions for the upper ramification jumps of an elementary Abelian p-extension of K.

Keywords. Elementary Abelian p-extensions, upper ramification jumps, normic groups, class field theory.

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1 - Introduction

Let K be a finite extension of $\mathbb{Q}_p$. By Hasse-Arf Theorem ([**10**, p. 76]), the upper ramification jumps of a finite Abelian extension L/K are rational integers. The problem of determining the upper ramification jump sequence of an extension, as well as the inverse problem of deciding whether a set of integers could be the ramification jump sequence of an extension with a fixed Galois group, is very difficult in general. However, this problem is completely solved in the case of cyclic extensions: a very neat result, due to Maus [7] in the case where $\zeta_p \notin K$, and to Miki [8] in the case

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where $\zeta_p \in K$, characterizes the sequence of integers which can be the ramification jumps of a totally ramified cyclic p-extension L/K. In this case, the ramification jumps completely determine the sequence of the ramification groups, since the quotients of the filtration are necessarily cyclic of order p.

In this paper we consider another basic case, namely the case of elementary Abelian p-extensions of a p-adic field (some results for biquadratic extensions can be found in [**1**]). In this case, the ramification subgroup sequence depends upon the jumps and the order of the subgroups. In Theorem 13, we characterize the sequences of couples of integers (t, m), where t denotes an upper jump and m its "size" (see Definition 3), which describe the ramification subgroup sequence of an elementary Abelian p-extension of K.

Our main tool is class field theory and the explicit relation, already used in [**11**] for cyclic extensions, between the group of norms of an extension and its ramification jumps. The fundamental step for the proof of Theorem 13 is the computation of the ramification filtration for the maximal elementary Abelian p-extension of the base field K. This result is contained in Theorem 12 and generalises [**3**, Lemma 9, p. 286].

In the case of non-Abelian extensions, Hasse-Arf Theorem can fail and the upper ramification jumps do not have to be integral. However, one can give a classification for the lower ramification jumps. In this setting, very few cases are known. One special case can be found in [**4**] where, to better understand the counterexamples to the conclusion of Hasse-Arf Theorem and as a first step towards an explicit description of wildly ramified Galois module structure, the two authors classify the ramification break numbers of totally ramified quaternion extensions of dyadic number fields.

2 - Notation and preliminary results

Throughout the paper p will be a fixed prime number. If K is a finite extension of $\mathbb{Q}_p$, we shall denote by e_K and f_K the ramification index and the inertial degree of $K/\mathbb{Q}_p$, and by n_K the degree of the extension; hence, we have $n_K = e_K f_K = [K : \mathbb{Q}_p]$.

We shall denote by $\mathcal{O}_K$ the ring of integers of K, by $\pi = \pi_K$ a uniformizer of K (i.e. a generator of the maximal ideal $\mathcal{M}_K$ of $\mathcal{O}_K$) and by v_K the valuation of K normalized so that $v_K(\pi_K) = 1$. We shall indicate the residue field of K by $\overline{K}$; then, $|\overline{K}| = p^{f_K}$.

Let U_K be the group of units of K, and consider its usual filtration $\{U_K^i\}_{i \geq 1}$ given by $U_K^i = 1 + \mathcal{M}_K^i$ for $i \geq 1$.

For a finite extension L/K, we denote by $N_{L/K}$, $\mathcal{D}_{L/K}$ and $\mathrm{Disc}_{L/K}$ the norm, the different and the discriminant of L/K respectively. If L/K is a Galois extension with Galois group G, we consider the filtration of G given by the ramification subgroups:

in our context, instead of the more classical lower numbering G_i for the ramification subgroups, it is useful to use the upper numbering, so, for every $v \geq 0$, we denote the ramification subgroups by G^v (see [**10**, Ch. IV] for the definition and the fundamental properties of ramification subgroups).

We are interested in studying the filtration of the G^v and, more specifically, the values of v for which these subgroups change.

Definition 1. We say that s is a *lower ramification jump* for the extension L/K if $G_s \neq G_u$ for every $u > s$. Similarly, we say that t is an *upper ramification jump* if $G^t \neq G^u$ for every $u > t$.

The lower ramification jumps of an extension are always integers, whereas in general the upper ones are not. However, the well known theorem of Hasse and Arf [**10**, p. 76] ensures that, in case of Abelian extensions, all the jumps of the filtration G^v are integers.

As already observed in the Introduction, the problem of determining whether a set of integers may be realized as the sequence of upper ramification jumps for an extension L/K can be very difficult in general. The problem is completely solved in the case of cyclic extensions.

A necessary and sufficient condition for m natural numbers $t^1 < \cdots < t^m$ to be upper ramification jumps of a totally ramified cyclic p-extension over K was given by Maus [**7**] in two cases, namely when $\zeta_p \notin K$ and when, if r is the maximal integer such that $\zeta_{p^r} \in K, v_K(\zeta_{p^r} - 1) \not\equiv 0 \bmod p$, and by Miki [8] in the general case. A constructive proof of the existence part of Miki's result was given by Sueyoshi in [**11**].

The general result of Miki is rather technical to state; we at least recall what can happen in the easier case when $\zeta_p \notin K$:

Theorem 2 [Maus, 1973]. *Let $\{t^1 < \cdots < t^m\}$ be a finite set of integers and suppose that $\zeta_p \notin K$; then, there exists a totally ramified cyclic extension L/K of degree p^m with upper ramification jumps $t^1, \ldots, t^m$ if and only if the following conditions hold:*

- $1 \leq t^1 < \dfrac{pe_K}{p-1}$ *and* $(t^1, p) = 1$;
- *if* $t^i < \dfrac{e_K}{p-1}$, *then* $t^{i+1} = pt^i$ *or* $pt^i < t^{i+1} < \dfrac{pe_K}{p-1}$ *and* $(p, t^{i+1}) = 1$;
- *if* $t^i \geq \dfrac{e_K}{p-1}$, *then* $t^{i+1} = t^i + e_K$.

Our aim is to characterize the upper ramification jumps and the ramification subgroups of an elementary Abelian p-extension of K. In this case, ramification subgroups are clearly elementary Abelian p-groups, so the ramification group sequence is completely determined by the jumps and the order of the subgroups.

Definition 3. Let L/K be a Galois extension with Galois group G and let $t \geq 1$ be an upper ramification jump. If $|G^t/G^{t+1}| = p^m$, we call m the *"size"* of the upper jump t.

Given an elementary Abelian p-extension, we can associate to the ramification subgroups a sequence of couples of integers (t, m), where t denotes an upper jump and m its size. We will refer to the couple simply as to the ramification jump.

For the convenience of the reader, we quote the class field correspondence theorem in a form which easily follows from Theorem 6.2 and 6.3 of [**5**, Ch. III, p. 154].

Theorem 4 [Class field correspondence]. *There is a one-to-one correspondence between the finite Abelian extensions of K and the open subgroups of finite index of $K^\times$ given by $L \longleftrightarrow N_{L/K}(L^\times)$. This correspondence is an order reversing bijection between the lattice of finite Abelian extensions of K (with respect to the intersection $L_1 \cap L_2$ and the compositum L_1L_2) and the lattice of open subgroups of finite index in $K^\times$ (with respect to the intersection $N_1 \cap N_2$ and the product N_1N_2).*

Furthermore, if L/K is the extension associated to the subgroup N and G is its Galois group, $K^\times/N \cong G$, hence $|K^\times/N| = [L : K]$.

It is well known [**5**, Rm. 1, p. 156] that, if char$(K) = 0$, one may omit the word "open" in Theorem 4.

There is a close connection between the ramification groups of an extension L/K and the group $K^\times/N_{L/K}(L^\times)$. In the case of totally ramified extensions of degree p, this is given by the following proposition:

Proposition 5. *Let L/K be a totally ramified extension of degree p and let t be its upper ramification jump. Then*

$$t = \min\{j \in \mathbb{N} \mid U_K^{j+1} \subseteq N_{L/K}(L^\times)\}.$$

Proof. Let t be the upper ramification jump of the extension L/K. From [**10**, Cor. 3, p. 85], we have that $U_K^{t+1} = N_{L/K}(U_L^{\psi(t)+1}) \subseteq N_{L/K}(L^\times)$, where ψ denote the inverse of the Herbrand function (see [**10**, Ch. IV]). Hence,

$$t \geq \min\{j \in \mathbb{N} \mid U_K^{j+1} \subseteq N_{L/K}(L^\times)\}.$$

We have now to show that t is exactly the minimum. If not, we would have $U_K^t \subseteq N_{L/K}(L^\times)$. From [**10**, Thm 1, p. 227] we know that, for $n \geq 0$, the canonical map induced by inclusion and projection $U_K^n / N_{L/K}(U_L^{\psi(n)}) \longrightarrow K^\times / N_{L/K}(L^\times)$ is injective, hence $U_K^n \cap N_{L/K}(L^\times) = N_{L/K}(U_L^{\psi(n)})$. This means that, if $U_K^n \subset N_{L/K}(L^\times)$, we get $U_K^n \subset N_{L/K}(U_L^{\psi(n)})$, that is a contradiction if $t = n$. □

Remark 6. The proposition follows also from [**6**, Theorem 1.4, p. 74] (see also [**6**, Remark 1.6.1, p. 80]).

3 - The compositum of all extensions of degree *p* of *K*

Let $\mathcal{E}_K(p)$ be the set of all the cyclic extensions of K of degree p within a fixed algebraic closure of K and let $\mathcal{C}_K(p)$ be the compositum of all extensions $E \in \mathcal{E}_K(p)$; then, $\mathcal{C}_K(p)$ is the maximal elementary Abelian p-extension of K in this fixed algebraic closure. In this section, following [**3**], we determine the upper ramification jumps of $\mathcal{C}_K(p)/K$.

Proposition 7.

$$[\mathcal{C}_K(p) : K] = \begin{cases} p^{n_K+1} & \textit{if } \zeta_p \notin K \\ p^{n_K+2} & \textit{if } \zeta_p \in K \end{cases}.$$

Proof. This is a classical result that can be easily proved using, for example, [**9**, Ch. V, Prop 5.8 and Thm 5.7][1]. □

In [**3**], the ramification subgroups of $\mathcal{C}_K(p)/K$ are computed, using Kummer theory, in the case where $\zeta_p \in K$. The use of class field theory allows us to generalize this result to a general field K. Also in this general case, the ramification groups can be computed via the study of all the subextensions of degree p.

Let $M \subseteq \mathcal{C}_K(p)$ be an elementary Abelian p-extension of Galois group G and let $L \subseteq M$ a Galois subextension of degree p over K. Denote by $\mathcal{D}_{L/K} = \pi_L^{D_L}$ and by $G_L = \mathrm{Gal}(M/L)$; hence by Galois correspondence, $\mathrm{Gal}(L/K) \cong G/G_L$.

Using the ramification-discriminant formula [**10**, Prop. 4, p. 64], $v_L(\mathcal{D}_{L/K}) = \sum_{v \geq 0}(|(G/G_L)_v| - 1)$. In our case, $|(G/G_L)_v| = p$ if $0 \leq v \leq t$ and $|(G/G_L)_v| = 1$ otherwise, hence $v_L(\mathcal{D}_{L/K}) = (p-1)(t+1)$ and the jump of this extension is

[1] This result holds for every complete field with finite residue field.

$t = \dfrac{D_L}{(p-1)} - 1$. Hence,

$$(G/G_L)^v = (G/G_L)_v = \begin{cases} \mathbb{Z}/p\mathbb{Z} & \text{if } v \leq \dfrac{D_L}{(p-1)} - 1 \\ 0 & \text{if } v > \dfrac{D_L}{(p-1)} - 1 \end{cases}.$$

This information allows us to reconstruct the ramification groups of M/K. In fact, by Herbrand's Theorem [**10**, Lemma 5, p. 75], $(G/G_L)^v = G^vG_L/G_L$, for every $v \geq 0$, hence

$$(1) \qquad (G/G_L)^v = 0 \iff G^v \subseteq G_L \iff v > \frac{D_L}{(p-1)} - 1.$$

Since G_L runs over all subgroups of index p of G as L runs over all normal extension of degree p of K contained in M, it follows that

$$G^v = \bigcap_{\substack{L \subseteq M \\ [L:K]=p \\ D_L/(p-1) < v+1}} G_L.$$

This characterization of the ramification subgroups allow us to easily prove the following proposition:

Proposition 8. *Let M/K be an elementary Abelian p-extension; then t is an upper ramification jump of M/K if and only if there exists a subextension L/K of degree p with upper ramification jump equal to t.*

Proof. Assume that there exists a subextension $L \subseteq M$ such that L/K is cyclic of degree p with upper ramification jump t; we prove that t is an upper ramification jump for M/K. According to the notation introduced before, call G the group $\mathrm{Gal}(M/K)$ and $G_L = \mathrm{Gal}(M/L)$. Since t is the upper ramification jump for L/K, $\mathrm{Gal}(L/K) = (G/G_L)^t$ and $(G/G_L)^{t+1} = \{1\}$, hence $G^{t+1} \subseteq G_L$ and $G^t \not\subseteq G_L$. By (1), this shows that t is an upper ramification jump for M/K.

Assume now that t is an upper ramification jump for M/K. From the previous description, we get

$$G^t = \bigcap_{\substack{L \subseteq M \\ [L:K]=p \\ \frac{D_L}{(p-1)} < t+1}} G_L \quad \text{and} \quad G^{t+1} = \bigcap_{\substack{L \subseteq M \\ [L:K]=p \\ \frac{D_L}{(p-1)} < t+2}} G_L.$$

Since $G^t \neq G^{t+1}$, there exists $L \subset M$ with $[L:M] = p$ and $D_L = (p-1)(t+1)$ and the upper ramification jump of this extension is exactly t. $\square$

We want now to construct a subgroup of $K^\times$ such that the corresponding extension (via class field correspondence) is a subextension of $\mathcal{C}_K(p)$ and its Galois group over K has a given jump. To this aim, we need to describe the structure of the group of units U_K.

Let $I = \left\{i \in \mathbb{Z} \mid 1 \leq i < \dfrac{pe_K}{p-1} \text{ and } (p,i) = 1\right\} = \{t^1, \ldots, t^{e_K}\}$, let $\overline{K}$ be the residue field of K and let us fix a set $C = \{c_1, \ldots, c_{f_K}\}$ of elements of $\mathcal{O}_K$ such that the residues of its elements in $\overline{K}$ form a basis of $\overline{K}$ over $\mathbb{F}_p$. If $\zeta_p \in K$, denote by r the maximal integer such that K contains a p^r-root of unity.

Theorem 9 ([**5**], Ch. I, Prop. 6.4, p. 19). *Every $\alpha \in U_K^1$ can be written as a product*

$$\alpha = \prod_{i \in I} \prod_{j=1}^{f_K} (1 + c_j \pi^i)^{a_{ij}} {\omega_*}^a,$$

where:

- *if $\zeta_p \notin K$, $\omega_* = 1$, $a = 0$ and the above expression for α is unique, hence U_K^1 is a free $\mathbb{Z}_p$-module of rank $n_K = e_K f_K = [K : \mathbb{Q}_p]$;*
- *if $\zeta_p \in K$, then $\omega_* = 1 + c_* \pi^{\frac{pe_K}{p-1}}$ is a principal unit such that $\omega_* \notin K^p$, $c_* \in C$ and $a \in \mathbb{Z}_p$. In this case, the above expression is not unique, and U_K^1 is a product of a free $\mathbb{Z}_p$-module of rank n_K and the p-torsion group μ_{p^r}.*

Let us call $F = \{(x,y) \in \mathbb{Z} \times \mathbb{Z} \mid x \in I,\ 1 \leq y \leq f_K\}$; we put $\eta_{(x,y)} = 1 + c_y \pi^x$ for every $(x,y) \in F$.

It is known that the maximal unramified extension K_{ur} of K contained in $\mathcal{C}_K(p)$ is cyclic of degree p (and it is the one associated to the group $\langle K^{\times p}, U_K^1\rangle$). The following lemma characterizes the maximal subextensions of $\mathcal{C}_K(p)$ with only one ramification jump:

Lemma 10. *Let $t \in I$ and let L_t/K be the extension associated to the group*

$$N_t = \langle K^{\times p},\ \pi,\ \{\eta_{(x,y)}\}_{(x,y) \in F,\ x \neq t},\ \omega_* \rangle.$$

Then, L_t/K is an elementary Abelian extension of degree p^{f_K} with only one ramification jump equal to t.

Proof. By Theorem 4, we have $[L_t : K] = |K^\times/N_t|$. Now, $K^\times/N_t \cong U_K^t/U_K^{t+1}$ and, since $t \in I$, by Theorem 9 we have that $U_K^t/U_K^{t+1} \cong \overline{K}$. It follows that $[L_t : K] = |\overline{K}| = p^{f_K}$. Moreover, since $\langle K^{\times p}, \pi\rangle \subset N_t$, then L_t/K is totally ramified.

Using the previous proposition, it easy to see that t is a ramification jump for L_t. In fact, we can consider the extension L/K associated to the soubgroup

$$N = \langle K^{\times p},\ \pi,\ \{\eta_{(x,y)}\}_{(x,y)\in F-\{(t,1)\}},\ \omega_*\rangle;$$

L is a subextension of L_t/K since $N_t \subset N$, has degree p (because $|K^\times/N| = p$) and ramification jump equal to t (this follows easily from Proposition 5). By Proposition 8, we have that t is also an upper ramification jump for L_t/K.

We want to show that t is the only possible jump. In fact, let L be any subextension of L_t/K of degree p over K; then, the group $N_L = N_{L/K}(L^\times)$ is a subgroup of $K^\times$ of index p and contains N_t, so $U_K^{t+1} \subset N_t \subset N_L$.

On the other hand, $K^\times = \langle U_K^t, N_t\rangle$ and $N_L \subsetneq K^\times$, so $U_K^t \not\subset N_L$ and, applying Proposition 5, its ramification jump is t. Using Proposition 8, we get that t is the only ramification jump of L_t/K. □

If $\zeta_p \in K$, the field $\mathcal{C}_K(p)$ has also a totally ramified subextension with jump not in the set I:

Lemma 11. *If $\zeta_p \in K$, let $t' = \dfrac{pe_K}{p-1}$ and let $L_{t'}$ be the extension associated to the subgroup*

$$N_{t'} = \langle K^{\times p}, \pi,\ \{\eta_{(x,y)}\}_{(x,y)\in F}\rangle.$$

Then, $L_{t'}/K$ is a cyclic extension of degree p with ramification jump equal to t'.[2]

Proof. The argument of the proof is the same of the previous lemma. By class field theory (Theorem 4), we have that $[L_{t'} : K] = |K^\times/N_{t'}|$ and $|K^\times/N_{t'}|$ is a cyclic group generated by $\omega_* N_{t'}$ that has order p (recall that ω_* is, by Theorem 9, a principal unit of $U_K^{\frac{pe_K}{p-1}}$ such that $\omega_* \notin K^p$). The fact that the ramification jump is exactly t' follows easily from Proposition 5, since $U_K^{t'} \not\subseteq N_{t'}$ (by construction $\omega_* \notin N_{t'}$) and $U_K^{t'+1} \subseteq K^{\times p} \subseteq N_{t'}$.

Theorem 12. *If $\zeta_p \notin K$, the upper ramification groups of $\mathcal{C}_K(p)/K$ are the following:*

1. $G = G^{-1} = (\mathbb{Z}/p\mathbb{Z})^{n_K+1}$;
2. $G^0 = \ldots = G^{t^1} = (\mathbb{Z}/p\mathbb{Z})^{n_K}$;

[2] We recall that, if $\zeta_p \in K$, $(p-1) \mid e_K$ so $\dfrac{pe_K}{p-1}$ is an integer.

3. $G^{t^i+1} = \ldots = G^{t^{i+1}} = (\mathbb{Z}/p\mathbb{Z})^{n_K - if_K}$ *for every* $i = 1, \ldots, e_K - 1$;

4. $G^{t^{e_K}+1} = \{e\}$;

so, the upper ramification jumps are exactly -1 *of size* 1 *and* $t^1 \ldots\ t^{e_K}$ *of size* f_K.

If $\zeta_p \in K$, *the upper ramification groups of* $\mathcal{C}_K(p)/K$ *are the following:*

1. $G = G^{-1} = (\mathbb{Z}/p\mathbb{Z})^{n_K+2}$;

2. $G^0 = \ldots = G^{t^1} = (\mathbb{Z}/p\mathbb{Z})^{n_K+1}$;

3. $G^{t^i+1} = \ldots = G^{t^{i+1}} = (\mathbb{Z}/p\mathbb{Z})^{n_K+1-if_K}$ *for every* $i = 1, \ldots, e_K - 1$;

4. $G^{t^{e_K}+1} = G^{\frac{pe_K}{p-1}} = \{\mathbb{Z}/p\mathbb{Z}\}$;

5. $G^{\frac{pe_K}{p-1}+1} = \{e\}$;

so, the upper ramification jumps are exactly -1 *and* $\dfrac{pe_K}{p-1}$ *of size* 1 *and* $t^1 \ldots\ t^{e_K}$ *of size* f_K.

Proof. Let us consider the case $\zeta_p \in K$ (the case $\zeta_p \notin K$ is the same without taking into account the "special subextension" $L_{t'}$). As done before, call $t' = \dfrac{pe_K}{p-1}$ and $I' = I \cup \{t'\}$. Firstly, we show that $\mathcal{C}_K(p) = K_{ur} L_{t'} \prod_{t \in I} L_t$. In fact, each extension on the right-hand side is an elementary Abelian p-extension, so $\mathcal{C}_K(p) \supseteq K_{ur} \prod_{t \in I'} L_t$. On the other hand, K_{ur} is linearly disjoint from $\prod_{t \in I'} L_t$: in fact, by class field theory and the previous constructions, the extension K_{ur}/K is associated to the subgroup $\langle K^{\times p}, U_K^1 \rangle$, while $\prod_{t \in I'} L_t/K$ is associated to the subgroup $\bigcap_{t \in I'} N_t = \langle K^{\times p}, \pi \rangle$. Hence, the intersection of these extensions is the field associated to the soubgroup $\langle K^{\times p}, \pi, U_K^1 \rangle = K^\times$, namely K.

With the same argument, one can show that, for every $\bar{t} \in I'$, the extension $L_{\bar{t}}$ is linearly disjoint from $\prod_{t \in I \setminus \{\bar{t}\}} L_t$. It follows that

$$[K_{ur} \prod_{t \in I'} L_t : K] = [K_{ur} : K] \prod_{t \in I'} [L_t : K] = p^{n_K+2} = [\mathcal{C}_K(p) : K],$$

so $\mathcal{C}_K(p) = K_{ur} \prod_{t \in I'} L_t$.

It is clear that all the integers $\{-1,\ t^1, \ldots,\ t^{e_K},\ t'\}$ are upper ramification jumps for the extension $\mathcal{C}_K(p)/K$, as all of them are upper ramification jumps for a cyclic subextension of $\mathcal{C}_K(p)/K$ of degree p (see proofs of Lemma 10 and 11).

To see that these are the only upper ramification jumps, it is enough to prove that each of the jumps in $\{t^1, \ldots, t^{e_K}\}$ has at least size p^{f_K} and t' has size at least one. In this case, in fact, we get:

$$|G/G^0||G^{t'}/G^{t'+1}|\prod_{t\in I}|G^t/G^{t+1}|\geq p^{2+f_Ke_K}=|G|,$$

and this yields $|G^t/G^{t+1}|=p^{f_K}$ for every $t\in I$, $|G^{t'}/G^{t'+1}|=p$ and no more jumps are possible.

We already know that $|G/G^0|=p$. Let $t\in I$ and call $H=\mathrm{Gal}(\mathcal{C}_K(p)/L_t)$; by Galois correspondence, $G/H\cong\mathrm{Gal}(L_t/K)$. From the previous lemma, we know that L_t/K has only one upper ramification jump equal to t, so $(G/H)^t=(\mathbb{Z}/p\mathbb{Z})^{f_K}$ and $(G/H)^{t+1}=\{e\}$. On the other hand, Herbrand's Theorem ensures that, for each $s>0$, we have $(G/H)^s\cong G^sH/H$; moreover, $G^sH/H\cong G^s/G^s\cap H$, so we get $G^{t+1}\subseteq H$, and

$$|G^t/G^{t+1}|=|G^t/G^t\cap H|\cdot|G^t\cap H/G^{t+1}|=|(G/H)^t|\cdot|G^t\cap H/G^{t+1}|\geq p^{f_K},$$

as wanted. The same argument holds if we take $t'=\dfrac{pe_K}{p-1}$ and the extension $L_{t'}$ constructed in Lemma 11. □

4 - The General Result

The following theorem classifies the sequence of couples of integers which can be the upper ramification jumps of an elementary Abelian p-extension of K.

Theorem 13. *Let K be a finite extension of $\mathbb{Q}_p$. Let $(t^{i_1},m_1),\dots,(t^{i_h},m_h)$ be couples of integers with $t^{i_1}<\dots<t^{i_h}$; there exists an elementary Abelian p-extension M/K with upper ramification jumps $(t^{i_1},m_1),\dots,(t^{i_h},m_h)$ if and only if the following conditions hold:*

1. *for every $j=1,\dots,h$, we have $1\leq t^{i_j}<pe_K/(p-1)$ and $(t^{i_j},p)=1$ with only two possible exceptions, namely $t^{i_1}=-1$ and, in the case when $\zeta_p\in K$, $t^{i_h}=\dfrac{pe_K}{p-1}$;*
2. *$1\leq m_j\leq f_K$ for every $j=1,\dots h$, $m_1=1$ if $t^{i_1}=-1$ and $m_h=1$ if $t^{i_h}=\dfrac{pe_K}{p-1}$.*

In this case, $[M:K]=\sum_{j=1}^{h}m_j$.

Proof. Let M be a subextension of $\mathcal{C}_K(p)/K$ and let $(t^{i_1},m_1),\dots,(t^{i_h},m_h)$ be its ramification jumps. From Proposition 8, we know that the jumps of M/K are among those of $\mathcal{C}_K(p)/K$, hence $t^{i_1},\dots,t^{i_h}$ verify condition 1. Moreover, denoting by H the subgroup of $G=\mathrm{Gal}(\mathcal{C}_K(p)/K)$ fixing M, we have that the ramification filtration of

M/K is $(G/H)^{t^{i_j}}$ for some $j = 1, \ldots, h$ and $|(G/H)^{t^{i_j}}/(G/H)^{t^{i_j}+1}| = p^{m_j}$. Arguing as in the proof of Theorem 12, we have

$$\left|\frac{(G/H)^{t^{i_j}}}{(G/H)^{t^{i_j}+1}}\right| = \left|\frac{G^{t^{i_j}}/G^{t^{i_j}+1}}{G^{t^{i_j}} \cap H/G^{t^{i_j}+1} \cap H}\right| = \frac{p^{f_K}}{|G^{t^{i_j}} \cap H/G^{t^{i_j}+1} \cap H|},$$

hence $1 \leq m_j \leq f_K$ if $t^{i_j} \in I$ and $m_1 = 1$ if $t^{i_1} = -1$. Moreover, if $\zeta_p \in K$ and $t^{i_h} = \dfrac{pe_K}{p-1}$, then $|G^{t^{i_h}}/G^{t^{i_h}+1}| = p$ as seen before, so $m_h = 1$, namely, the m_j verify condition 2.

On the other hand, let $(t^{i_1}, m_1), \ldots, (t^{i_h}, m_h)$ be a sequence verifying conditions 1 and 2; we construct an extension M/K with these ramification jumps. For each $j = 1, \ldots, h$ let M_j/K be any subextension of degree p^{m_j} of $L_{t^{i_j}}/K$, where $L_{t^{i_j}}$ is the extension defined in Lemma 10 when $t^{i_j} \in I$, $L_{t^{i_h}} = L_{t'}$ (the extension defined in Lemma 11) if $t^{i_h} = \dfrac{pe_K}{p-1}$ and $L_{t^{i_1}} = K_{ur}$ if $t^{i_1} = -1$. Put $M = \prod_{j=1}^{h} M_j$; then, using the same techniques applied in Theorem 12, it is easy to see that the ramification jumps of M/K are exactly $(t^{i_1}, m_1), \ldots, (t^{i_h}, m_h)$. □

Remark 14. If $\zeta_p \in K$, we can prove the same results using another approach, namely using Kummer theory (see [**2**, p. 89]). This method is more explicit as, in this way, we can construct an extension with fixed ramification jumps giving explicit generators.

In fact, by Kummer theory, we know that every p-extension is Galois and of the form $L = K(\sqrt[p]{a})$, where a can be chosen as a p-power free representative of a class in $K^\times/K^{\times p}$. With this choice of the generator, the K-valuation of a gives the ramification jump of the extension L/K. More precisely, the following proposition holds:

Proposition 15. *Let $L = K(\sqrt[p]{a})$ with $a \in K^\times$, a p-power free; call t the ramification jump of L/K. Then,*

- *if $0 < v_K(a) < p$, then $t = \dfrac{pe_K}{p-1}$;*
- *if $v_K(a) = 0$ and $v_K(a-1) = l$ with $1 \leq l < \dfrac{pe_K}{p-1}$ and $(l, p) = 1$, then $t = \dfrac{pe_K}{p-1} - l$;*
- *if $v_K(a) = 0$ and $v_K(a-1) = \dfrac{pe_K}{p-1}$, then $t = -1$ (and the extension is unramified).*

Proof. If $0 < v_K(a) < p$ and z is a p-th root of a, then, for every h such that $(h, p) = 1$, $K(z) = K(z^h)$. We can restrict to the case $v_K(a) = 1$.

In this case, we can choose $\pi_L = z$ as uniformizing parameter for $\mathcal{O}_{\mathcal{L}}$; if, moreover, g is the generator of $\mathrm{Gal}(L/K)$ defined by $\pi_L \to \zeta_p \pi_L$, we have $v_L(g(\pi_L) - \pi_L) = v_L(\zeta_p \pi_L - \pi_L) = v_L(\zeta_p - 1) + v_L(\pi_L) = \dfrac{pe_K}{p-1} + 1$, hence $t = \dfrac{pe_K}{p-1}$ as wanted.

If $a \in U_K^l \setminus U_K^{l+1}$, from the proof of [**3**, Lemma 6, p. 15], $v_L(\mathcal{D}_{L/K}) = \left(\dfrac{pe_K}{p-1} - l + 1\right)(p-1)$. On the other hand, from the ramification-discriminant formula used in Section 3, $v_L(\mathcal{D}_{L/K}) = (p-1)(t+1)$. Comparing this with the previous expression, we have $t = \dfrac{pe_K}{p-1} - l$. Finally, if $a \in U_K^{\frac{pe_K}{p-1}}$, using Hensel's Lemma [**2**, p. 84], it is easy to see that the extension L/K is unramified, hence the ramification jump is $t = -1$. □

With this relation between the upper ramification jump and the valuation of the generator, one can easily give generators for the subextensions $L_t \subset \mathcal{C}_K(p)$ constructed in Lemma 10. Hence, the following proposition holds:

Proposition 16. *Take $t \in I$ and call $l = \dfrac{pe_K}{p-1} - t$; the extension $L_t = K\left(\sqrt[p]{\eta_{(l,y)}},\ y = 1, \dots, f_K\right)$ is an elementary Abelian extension of degree p^{f_K} over K with just one ramification jump equal to t.*

If $t = \dfrac{pe_K}{p-1}$, the extension $L_t = K(\sqrt[p]{\pi})$ is a totally ramified extension of degree p with ramification jump equal to $\dfrac{pe_K}{p-1}$.

If $t = -1$, the extension $L_{-1} = K(\sqrt[p]{\omega_*})$ is the only unramified extension of K of degree p and has ramification jump equal to -1.

References

[**1**] N. P. Byott and G. G. Elder, *Biquadratic extensions with one break*, Canad. Math. Bull. **45** (2002), 168-179.

[**2**] J. W. S. Cassels and A. Fröhlich, (eds.), *Algebraic number theory*, Academic Press, London and New York 1967.

[**3**] I. Del Corso and R. Dvornicich, *The compositum of wild extensions of local fields of prime degree*, Monatsh. Math. **150** (2007), 271-288.

[**4**] G. G. Elder and J. J. Hooper, *On wild ramification in quaternion extensions*, J. Théor. Nombres Bordeaux **19** (2007), 101-124.

[**5**] I. B. Fesenko and S. V. Vostokov, *Local fields and their extensions*, Transl. Math. Monogr., **121**, 2nd ed., American Mathematical Society, Providence, RI 2002.

[**6**] G. GRAS, *Class field theory. From theory to practice*, Springer Monographs in Mathematics, Springer-Verlag, Berlin 2003.

[**7**] E. MAUS, *Relationen in Verzweigungsgruppen* (German), J. Reine Angew. Math. **258** (1973), 23-50.

[**8**] H. MIKI, *On the ramification numbers of cyclic p-extensions over local fields*, J. Reine Angew. Math. **328** (1981), 99-115.

[**9**] W. NARKIEWICZ, *Elementary and analytic theory of algebraic numbers*, 2nd ed., Springer-Verlag, Berlin; PWN-Polish Scientific Publishers, Warsaw, 1990.

[**10**] J.-P. SERRE, *Local fields*, Graduate Texts in Mathematics, **67**, Springer-Verlag, New York-Berlin 1979.

[**11**] Y. SUEYOSHI, *On ramification of p-extensions of p-adic number fields*, Mem. Fac. Sci. Kyushu Univ. Sez. A **32** (1978), 199-204.

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