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# Rank *r* spanned vector bundles with extremal Chern classes on a smooth surface

**Abstract.** We study spanned vector bundles with extremal Chern classes and large rank on very simple smooth surfaces $X$ (e.g. on $\mathbb{P}^2$, following the rank two case solved by Ph. Ellia). Let $\mathcal{L}$ be a spanned and ample line bundle on $X$. Let $\mathcal{E}$ be a rank $r$ spanned vector bundle with $\det(\mathcal{E}) \cong \mathcal{L}$ and no trivial factor. We prove that $r \le h^0(\mathcal{L}) - 1$ and classify all $\mathcal{E}$ with $h^0(\mathcal{L}) - r - 1 \le \alpha$, where $\alpha$ is the maximal integer $k$ such that the adjoint line bundle $\mathcal{L} \otimes \omega_X$ is $k$-spanned.

**Keywords.** Spanned vector bundles, vector bundles on surfaces, k-spanned line bundle, adjoint line bundle.

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## 1 - Introduction

Let $X$ be a smooth and connected projective surface. Fix $\mathcal{L} \in \mathrm{Pic}(X)$, $c_2 \in \mathbb{Z}$ and an integer $r \ge 2$. We assume that $\mathcal{L}$ is ample and spanned. We are interested in rank $r$ spanned vector bundles $\mathcal{E}$ on $X$ with $\det(\mathcal{E}) \cong \mathcal{L}$, $c_2(\mathcal{E}) = c_2$ and no trivial factor. We have $c_2 \le \mathcal{L}^2$ (self-intersection number), $r \le h^0(\mathcal{L}) - 1$ (Remark 1) and often stronger inequalities may be proved obtaining a classification of all $\mathcal{E}$ with extremal $\mathcal{L}^2 - c_2$ or very small $h^0(\mathcal{L}) - 1 - r$ (see Proposition 3). Now assume $\mathrm{rank}(\mathcal{E}) = 2$ and that $\mathcal{E}$ has no trivial factor, i.e. $\mathcal{E} \ne \mathcal{O}_X \oplus \mathcal{L}$. Fix an integer $r > 2$. We say that a rank $r$ vector bundle $\mathcal{G}$ is an extension of $\mathcal{E}$ if it fits in an exact sequence

$$0 \to \mathcal{O}_X^{\oplus(r-2)} \to \mathcal{G} \to \mathcal{E} \to 0. \tag{1}$$

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We say that $\mathcal{G}$ is a non-trivial extension of $\mathcal{E}$ if it has no trivial factors, i.e. the $r-2$ elements of $H^1(\mathcal{E}^\vee)$ inducing (1) are linearly independent. If $h^1(\mathcal{O}_X) = 0$ the bundle $\mathcal{G}$ is spanned if and only if $\mathcal{E}$ is spanned. Rank two bundles $\mathcal{E}$ and $\mathcal{F}$ with the same Chern classes may have $h^1(\mathcal{F}^\vee) \neq h^1(\mathcal{E}^\vee)$. Hence the classification of all possible $(\mathcal{L}, c_2)$ with $r = 2$ does not give the classification for $r > 2$, too (although for several reasons the rank 2 case is the most important one). Assume $X = \mathbb{P}^2$ and $\mathrm{rank}(\mathcal{E}) = 2$. Ph. Ellia gave the classification of all Chern classes of rank 2 spanned vector bundles on $\mathbb{P}^2$ ([**13**]). In the range $4c_2 > c_1^2$ and $c_2 \le \binom{c_1+2}{2} - 3$ a general rank two stable bundle $\mathcal{E}$ on $\mathbb{P}^2$ with $c_i(\mathcal{E}) = c_i$ is spanned and these are the bundles used by Ph. Ellia to cover this range of pairs $(c_1, c_2)$ and usually these bundles are not extendable (Remark 4). In extremal cases it may be possible to get a full classification. We first prove the following result (the existence/non existence part is a very particular case of [**13**], only the classification part is, we hope, new). In the body of the paper the reader will find the quoted examples and remarks.

Proposition 1. *Fix integers $x, z$ such that $x \ge 4$ and $z \ge x^2 - 2x + 5$. There is a spanned rank 2 vector bundle $\mathcal{E}$ on $\mathbb{P}^2$ with $\det(\mathcal{E}) = \mathcal{O}_{\mathbb{P}^2}(x)$ and $c_2(\mathcal{E}) = z$ if and only if either $z = x^2$ or $z = x^2 - x + 1$ or $z = x^2 - x$. If $z = x^2$ (resp. $z = x^2 - x + 1$, resp. $z = x^2 - x$), then $\mathcal{E} = \mathcal{E}_{\mathcal{L},W}$ with $\mathcal{L} = \mathcal{O}_{\mathbb{P}^2}(x)$, $W \subset H^0(\mathcal{L})$, $\dim(W) = 3$, is as in Remark 3 (resp. Example 1 with $(c_1, t) = (x, 0)$, resp. Example 2 with $(c_1, t, m) = (x, 1, x-1)$). If $z = x^2$ (resp. $z = x^2 - x + 1$, resp. $z = x^2 - x$) there is a rank $r$ spanned vector bundle $\mathcal{F}$ on $\mathbb{P}^2$ with $\det(\mathcal{F}) = \mathcal{O}_{\mathbb{P}^2}(t)$, $c_2(\mathcal{F}) = z$ and no trivial factor if and only if $2 \le r \le \binom{x+2}{2} - 1$ (resp. $2 \le r \le (x^2 + x + 4)/2$, resp. $2 \le r \le (x^2 + x)/2$).*

All the bundles $\mathcal{F}$ coming from Remark 3 and Example 1 are indecomposable, while in Example 2 for each $r \in \{3, \dots, (x^2 + x)/2\}$ some rank $r$ bundle $\mathcal{F}$ is decomposable (always with $\mathcal{F} \cong \mathcal{O}_{\mathbb{P}^2}(1) \oplus A$ with $A$ indecomposable and as in Remark 3), but the general bundle in this family is indecomposable.

To get a related result (just in the range $c_2 \ge x^2 - x + 2$) on an arbitrary surface $X$ with $h^0(\omega_X^\vee) \ge 2$ we use the following definition related to the notion of $k$-spanned line bundle ([**3**], [**4**], [**5**], [**6**], [**24**]).

Let $X$ be a smooth and connected projective surface and let $\mathcal{R}$ be a line bundle on $X$. For each integer $k \ge 0$ we say that $X$ is *weakly $k$-spanned* if $h^0(\mathcal{I}_S \otimes \mathcal{R}) = h^0(\mathcal{R}) - k - 1$ for all finite sets $S \subset X$ with $\sharp(S) = k + 1$. $\mathcal{R}$ is spanned if and only if it is weakly 0-spanned. If $\mathcal{R}$ is very ample, then it is weakly 1-spanned. Let $\alpha(\mathcal{R})$ be the maximal integer $k$ such that $\mathcal{R}$ is weakly k-spanned, with the convention $\alpha(\mathcal{R}) = -\infty$ if $\mathcal{R}$ is not spanned. We have $\alpha(\mathcal{O}_{\mathbb{P}^2}(t)) = t$ for all $t \ge 0$. We prove the following result.

Theorem 1. *Assume that $\mathcal{L}$ is ample and spanned and that $\alpha(\mathcal{L}\otimes\omega_X)\geq 0$. Let $\mathcal{E}$ be a rank $r$ spanned vector bundle on $X$ with* $\det(\mathcal{E})=\mathcal{L}$ *and no trivial factor. If $c_2(\mathcal{E})\geq \mathcal{L}^2-\alpha(\mathcal{L}\otimes\omega_X)-1$, then $h^0(\mathcal{E})=r+1$, $c_2(\mathcal{E})=\mathcal{L}^2$ and $\mathcal{E}$ fits in an exact sequence*

$$0\to\mathcal{L}^\vee\to\mathcal{O}_X^{\oplus(r+1)}\to\mathcal{E}\to 0. \tag{2}$$

On very simple surfaces (e.g. the Hirzebruch surfaces or a K3 surface with $\mathrm{Pic}(X)\cong\mathbb{Z}$ or $\mathbb{P}^2$ blow up at a very small number of points) one knows the integer $\alpha(\mathcal{R})$ for all $\mathcal{R}\in\mathrm{Pic}(X)$. Many authors proved that the adjoint line bundle $\mathcal{R}\otimes\omega_X$ is k-spanned (and hence weakly k-spanned) under certain assumptions on $\mathcal{R}$ and $X$ (**[3]**, **[4]**, **[5]**, **[6]**, **[24]**).

To get better results we consider the following invariants. Take $X=\mathbb{P}^2$. For any rank 2 vector bundle $\mathcal{E}$ on $\mathbb{P}^2$ let $t(\mathcal{E})$ be the maximal integer $t$ such that $h^0(\mathcal{E}(-t))>0$. The integer $t(\mathcal{E})$ is a key step in the determination of $h^1(\mathcal{E}^\vee)$ and hence in the description of the rank $r>2$ bundles associated to non-trivial extensions of $\mathcal{E}$. $\mathcal{E}$ is stable (resp. semistable) if and only if $t(\mathcal{E})<c_1(\mathcal{E})/2$ (resp. $t(\mathcal{E})\leq c_1(\mathcal{E})/2$). Let $t_1(\mathcal{E})$ be the maximal integer $y\leq t(\mathcal{E})$ such that $h^0(\mathcal{E}(-y))>\binom{t(\mathcal{E})-y+2}{2}$. The integer $t(\mathcal{E})$ is defined for every surface $X$ with $\mathrm{Pic}(X)\cong\mathbb{Z}$. The integer $t_1(\mathcal{E})$ is defined for every surface $X$ with $\mathrm{Pic}(X)\cong\mathbb{Z}$ and with a very ample positive generator of $\mathrm{Pic}(X)$ (e.g., on some K3's or several complete intersection surfaces).

Theorem 2. *Let $\mathcal{E}$ be a rank 2 spanned vector bundle on $\mathbb{P}^2$ with $t:=t(\mathcal{E})\geq 4$. Set $c_1:=c_1(\mathcal{E})$ and $c_2:=c_2(\mathcal{E})$. We have $t(c_1-t)\leq c_2\leq c_1(c_1-t)$; if $c_1\geq 2t+1$, then $c_2\geq (c_1^2+2t^2-2tc_1+c_1-2t)/2$. Let $r$ be the maximal integer $\geq 2$ such that there is an extension of $\mathcal{E}$ by $\mathcal{O}_{\mathbb{P}^2}^{\oplus(r-2)}$ without trivial factors. We have $r\geq 2+c_2-t(c_1-t)-\binom{c_1-t-1}{2}$ and equality holds if and only if $t_1(\mathcal{E})\leq 2$; equality always holds if $c_2(\mathcal{E})>(c_1-3)(c_1-t)$. We have $r\leq 2+(c_1-t-1)(c_1-t+2)/2$, unless $\mathcal{E}$ is Example 2 with $m=c_1-t$ and $r=2+(c_1-t)^2-\binom{c_1-t-1}{2}$.*

See Proposition 6 for a stronger statement when $c_2\leq(c_1-3)(c_1-t)$ and $r\neq 2+c_2-t(c_1-t)-\binom{c_1-t-1}{2}$.

In this paper we first study the the rank 2 spanned bundles $\mathcal{E}$ and then try to compute $h^1(\mathcal{E}^\vee)$ to see for which rank $r>2$ there is an extension of $\mathcal{E}$ by $\mathcal{O}_X^{\oplus(r-2)}$ without trivial factors. There is another approach to globally generated vector bundles on projective spaces (**[1]**, **[2]**, **[22]**, **[23]**). The quoted authors first classify all rank $r$ spanned bundles $\mathcal{G}$ with no trivial factor (i.e. with $h^0(\mathcal{G}^\vee)=0$), and such that each extension of $\mathcal{G}$ by $\mathcal{O}_{\mathbb{P}^2}$ splits (i.e. with $h^1(\mathcal{G}^\vee)=0$); we say that any such bundle is a *master*. Every spanned bundle $\mathcal{E}$ with no trivial factor, say of rank $s$, has a unique master (up to isomorphisms) and the rank of the master $\mathcal{K}$ of $\mathcal{E}$ is $s+h^1(\mathcal{E}^\vee)$. If

$h^1(\mathcal{E}^\vee) = 0$, then $\mathcal{E}$ is its own master. If $h^1(\mathcal{E}^\vee) > 0$, then $\mathcal{E}$ is isomorphic to a quotient of $\mathcal{K}$ by a trivial subbundle of $\mathcal{K}$ of rank $h^1(\mathcal{E}^\vee)$. So if one has classified all possible masters, then one knows which triple $(r, c_1, c_2)$ is realized by a spanned bundle. In many cases from a resolution of a master $\mathcal{G}$ with direct sum of line bundles (or other homogeneous sheaf, like a twisted tangent bundle) they obtain a similar description of all bundles which are the quotient of the master $\mathcal{G}$ by a trivial subbundle. From this often they know all integers $h^i(\mathcal{E}(t))$, $i \in \mathbb{N}$, $t \in \mathbb{Z}$. However, sometimes for a fixed $s$ a master $\mathcal{G}$ gives both decomposable bundles and indecomposable ones (see Example 2).

We work over an algebraically closed field with characteristic zero.

## 2 - General results

Let $X$ be a smooth and connected projective surface. We often use the following exact sequences with $S$ a locally complete intersection zero-dimensional scheme and $\mathcal{L} \in \mathrm{Pic}(X)$

$$0 \to \mathcal{O}_X \to \mathcal{E} \to \mathcal{I}_S \otimes \mathcal{L} \to 0 \tag{3}$$

$$0 \to \mathcal{O}_X^{\oplus(r-1)} \to \mathcal{E} \to \mathcal{I}_S \otimes \mathcal{L} \to 0. \tag{4}$$

Remark 1. Let $Y$ be an integral projective variety and $\mathcal{E}$ a rank $r$ spanned vector bundle on $Y$. Since the evaluation map $H^0(\mathcal{E}) \otimes \mathcal{O}_Y \to \mathcal{E}$ is surjective, we have $h^0(\mathcal{E}) \ge r$ and equality holds if and only if $\mathcal{E}$ is trivial. Now assume that $Y := X$ is a smooth surface. Taking $r-1$ general sections of $\mathcal{E}$ we get and exact sequence (4) with $\mathcal{L} \cong \det(\mathcal{E})$ and $S \subset X$ a finite set with $\sharp(S) = c_2(\mathcal{E})$. Since $\mathcal{E}$ is spanned, (4) gives that $\mathcal{I}_S \otimes \mathcal{L}$ is spanned. Hence there are $D, D' \in |\mathcal{L}|$ such that the scheme $D \cap D'$ is zero-dimensional and it contains $S$. Hence $\mathcal{L}^2 = D \cdot D' \ge \sharp(S) = c_2(\mathcal{E})$.

Lemma 1. *Let $\mathcal{E}$ be a rank 2 spanned vector bundle on $X$ with no trivial factor and let $S \subset X$ be a zero-dimensional scheme which is the zero-locus of a non-zero section of $\mathcal{E}$. Set $\mathcal{L} := \det(\mathcal{E})$ and $a := h^1(\mathcal{E}^\vee)$. There is a rank $r$ vector bundle $\mathcal{G}$ on $X$ fitting in an exact sequence (1) and with no trivial factor if and only if $2 \le r \le 2 + a$. Assume $h^1(\mathcal{O}_X) = 0$. We have $\dim \mathrm{Ext}^1(\mathcal{I}_S \otimes \mathcal{L}, \mathcal{O}_X) = a + 1$. There is a rank $r$ spanned vector bundle $\mathcal{F}$ on $X$ with $\det(\mathcal{F}) \cong \mathcal{L}$, no trivial factor and with $S$ as the dependency locus of $r-1$ of its sections if and only if $2 \le r \le 2 + a$.*

Proof. The first part follows from the definition of extensions, because $\mathrm{Ext}^1$ commutes with finite direct sums. Now assume $h^1(\mathcal{O}_X) = 0$. In this case any extension of a spanned sheaf by a trivial vector bundle is spanned. Hence we get the " if " part. Since $\mathcal{E}$ is spanned, so is $\mathcal{I}_S \otimes \mathcal{L}$. Since $h^1(\mathcal{O}_X) = 0$, the middle term of any exact sequence (4) or (1) is spanned. We have $\dim \operatorname{Ext}^0(\mathcal{E}, \mathcal{O}_X) = h^0(\mathcal{E}^\vee) = 0$, because $\mathcal{E}$ has no trivial factor. We have $\dim \operatorname{Ext}^0(\mathcal{O}_X, \mathcal{O}_X) = h^0(\mathcal{O}_X) = 1$ and $\dim \operatorname{Ext}^1(\mathcal{O}_X, \mathcal{O}_X) = h^1(\mathcal{O}_X) = 0$. Applying the global Ext functors to (3) we get $\dim \operatorname{Ext}^1(\mathcal{I}_S \otimes \mathcal{L}, \mathcal{O}_X) = \dim \operatorname{Ext}^1(\mathcal{E}, \mathcal{O}_X) + 1$. □

Lemma 2. *Take an exact sequence*

$$0 \to A \to B \to E \to 0 \tag{5}$$

*of vector bundles on X such that* $h^1(B^\vee) = 0$. *Then any extension F of E by* $\mathcal{O}_X^{\oplus k}$ *fits in an exact sequence*

$$0 \to A \to B \oplus \mathcal{O}_X^{\oplus k} \to F \to 0.$$

Proof. Take the exact sequence

$$0 \to \mathcal{O}_X^{\oplus k} \xrightarrow{u} F \to E \to 0. \tag{6}$$

Since $h^1(B^\vee) = 0$, (5) gives a surjection $\rho: H^0(B^\vee \otimes F) \to H^0(B^\vee \otimes E)$. Take a map $v : B \to F$ inducing the surjection $F \to E$ of (5). Take $v' : B \to F$ such that $\rho(v') = v$. The map $f = (v', u): B \oplus \mathcal{O}_X^{\oplus k} \to F$ is surjective and $\ker(f) \cong A$. □

Remark 2. Lemma 2 works (i.e. $h^1(B^\vee) = 0$) in Examples 1 and 2. Hence in these cases it is sufficient to find (5) when $E$ has rank 2.

Proposition 2. *Assume* $h^1(\mathcal{O}_X) = 0$. *Let* $\mathcal{L}$ *be a very ample line bundle on X such that* $h^0(\mathcal{L} \otimes \omega_X) < h^0(\mathcal{L})$. *Fix an integer x such that* $h^0(\mathcal{L} \otimes \omega_S) < x \leq h^0(\mathcal{L}) - 3$. *Let* $S \subset X$ *be a general subset with* $\sharp(S) = x$. *Then* $\mathcal{I}_S \otimes \mathcal{L}$ *is spanned. The set S satisfies the Cayley-Bacharach condition with respect to* $\mathcal{L} \otimes \omega_X$ *and it is the zero-locus of a section of a rank* 2 *spanned bundle* $\mathcal{E}$ *with* $\det(\mathcal{E}) \cong \mathcal{L}$. *For any* $r > 2$ *any rank r bundle* $\mathcal{F}$ *with* $r - 1$ *sections with S as their dependency locus has* $\mathcal{O}_X^{\oplus(r-2)}$ *as a factor.*

Proof. Let $U$ be the set of all subsets of $X$ with cardinality $x$. We claim the existence of a non-empty open subset $\Omega$ of the irreducible variety $U$ such that every $S \in \Omega$ satisfies the theorem. Since $h^0(\mathcal{L}) \geq x$, there is a non-empty open subset $U'$ of $U$ such that $h^0(\mathcal{I}_S \otimes \mathcal{L}) = h^0(\mathcal{L}) - x$ for each $S \in U'$. Since $\mathcal{L}$ is very ample, a general $D \in |\mathcal{L}|$ is smooth. Since $h^0(\mathcal{L}) > x$, there is a non-empty open subset $U''$ of $U'$ such that every $S \in U''$ is contained in a smooth element of $|\mathcal{L}|$. The condition " $\mathcal{I}_S \otimes \mathcal{L}$ is spanned " is an open condition for $S \in U'$, because all sheaves $h^0(\mathcal{I}_S \otimes \mathcal{L})$, $S \in U'$, have the same $h^0$. Hence to prove the existence of a non-empty open subset $U_1$ of $U''$

such that $\mathcal{I}_S \otimes \mathcal{L}$ is spanned for all $S \in U_1$ it is sufficient to find one $S \in U''$ with $\mathcal{I}_S \otimes \mathcal{L}$ spanned. Fix a smooth $C \in |\mathcal{L}|$ and take a general $S \subset C$ with $\sharp(S) = x$. Since $h^1(\mathcal{O}_X) = 0$, the exact sequence

$$0 \to \mathcal{O}_X \to \mathcal{L} \to \mathcal{L}|C \to 0$$

gives $h^0(C, \mathcal{L}|C) = h^0(\mathcal{L}) - 1$. Since $h^0(C, \mathcal{L}|C) = h^0(\mathcal{L}) - 1$, $x \le h^0(\mathcal{L}) - 3$, $\mathcal{L}$ is very ample, $S$ is general and we are in characteristic zero, the line bundle $(\mathcal{L}|C)(-S)$ is spanned ([**16**, Lemma 2.1]). Since the section of $\mathcal{L}$ with $C$ has its zero-locus has $C$ as its scheme theoretic zero and vanishes at each point of $S$, the scheme-theoretic zero-locus of $\mathcal{I}_S \otimes \mathcal{L}$ is contained in $C$. Since $(\mathcal{L}|C)(-S)$ is spanned and the restriction map $H^0(\mathcal{L}) \to H^0(C, \mathcal{L}|C)$ is surjective, the sheaf $\mathcal{I}_S \otimes \mathcal{L}$ is spanned. Hence there is a non-empty open subset $U_1$ of $U''$ such that $\mathcal{I}_S \otimes \mathcal{L}$ is spanned for all $S \in U_1$. Since $h^0(\mathcal{L} \otimes \omega_X) < x$, there is a non-empty open subset $\Omega$ of $U_1$ such that for every $S \in \Omega$ we have $h^0(\mathcal{I}_{S'} \otimes \mathcal{L} \otimes \omega_X) = 0$ for each $S' \subset S$ with $\sharp(S') = x - 1$. Each $S \in \Omega$ satisfies the Cayley-Bacharach condition with respect to $\mathcal{L} \otimes \omega_X$ and hence it gives a rank 2 vector bundle $\mathcal{E}$ fitting in the exact sequence (3). Since $h^1(\mathcal{O}_X) = 0$ and $\mathcal{I}_S \otimes \mathcal{L}$ is spanned, $\mathcal{E}$ is spanned. Since $h^1(\mathcal{O}_X) = 0$, Lemma 1 gives the statement for rank $r > 2$ bundles. □

Let $X$ be a smooth and connected projective surface and let $\mathcal{L}$ be a spanned and ample line bundle on $X$. Since $\mathcal{L}$ is ample and spanned, we have $h^0(\mathcal{L}) \ge 3$.

Remark 3. Since $\mathcal{L}$ is spanned, the evaluation map $\rho_{\mathcal{L},H^0(\mathcal{L})}: H^0(\mathcal{L}) \otimes \mathcal{O}_X \to \mathcal{E}$ is surjective. Fix an integer $r$ such that $2 \le r \le h^0(\mathcal{L}) - 1$. Since $\dim(X) = 2$, a general $(r+1)$-dimensional linear subspace $W$ of $H^0(\mathcal{L})$ spans $\mathcal{L}$. For each $(r+1)$-dimensional linear subspace $W$ of $H^0(\mathcal{L})$ spanning $\mathcal{L}$ let $\rho_{\mathcal{L},W}: W \otimes \mathcal{O}_X \to \mathcal{L}$ denote the evaluation map. Since $\rho_{\mathcal{L},W}$ is surjective, the sheaf $\ker(\rho_{\mathcal{L},W})$ is locally free. By construction $E_{\mathcal{L},W} := \ker(\rho_{\mathcal{L},W})^\vee$ is a rank $r$ spanned vector bundle with $\det(E_{\mathcal{L},W}) \cong \mathcal{L}$, $c_2(E_{\mathcal{L},W}) = \mathcal{L}^2$ and no trivial factor. It fits in an exact sequence

$$0 \to \mathcal{L}^\vee \xrightarrow{u} \mathcal{O}_X^{\oplus(r+1)} \to E_{\mathcal{L},W} \to 0 \tag{7}$$

Kodaira's vanishing gives $h^0(E_{\mathcal{L},W}) = r + 1$. The bundle $E_{\mathcal{L},W}$ has no trivial factor if and only if the map $u^\vee{}_*: H^0(\mathcal{O}_X^{r+1}) \to H^0(\mathcal{L})$ obtained dualizing (16) is injective. Conversely, any rank $r$ spanned vector bundle $\mathcal{E}$ with rank $r$ and $h^0(\mathcal{E}) = r + 1$ fits in (16) with $W = H^0(\mathcal{E})$. For any vector bundle $\mathcal{F}$ on $X$ let $c(\mathcal{F}) = 1 + c_1(\mathcal{F})h + c_2(\mathcal{F})h^2$ be its total Chern class. If $\mathcal{L}$ is a line bundle, we have $c(\mathcal{L}^\vee) = 1 - c_1(\mathcal{L})h$. Hence for any vector bundle $\mathcal{E}$ fitting in (2) we have $c(\mathcal{L}^\vee)c(\mathcal{E}) = 1$ in the Chow ring of $X$, i.e. $c_1(\mathcal{E}) = c_1(\mathcal{L})$ and $c_2(\mathcal{E}) = \mathcal{L}^2$.

Note that the bundles $E_{\mathcal{L},W}$ are exactly the spanned bundles $E$ with $\det(E) \cong \mathcal{L}$, $h^0(E) = \text{rank}(E) + 1$ and no trivial factor. The set of isomorphism classes of rank $r$ bundles arising in this way is parametrized (not necessarily finite to one) by an irreducible variety (a non-empty open subset of a Grassmannian).

If the morphism associated to $|\mathcal{L}|$ does not map $X$ onto a curve, then this construction does not work when $r = 1$, i.e. in this case we would have $E_{\mathcal{L},W}$ a rank 1 torsion free sheaf which is not locally free (i.e. $E_{\mathcal{L},W} \neq \mathcal{L}$); this is seen also because the morphism associated to $|\mathcal{L}|$ has not a curve as its image if and only if $\mathcal{L}^2 > 0$.

Claim. Each bundle $E_{\mathcal{L},W}$ is indecomposable.

Proof of Claim. Assume $E_{\mathcal{L},W} \cong A_1 \oplus A_2$ with $A_1, A_2$ spanned vector bundles. Since $A_i$ has no trivial factor, we have $h^0(A_i) \geq \text{rank}(A_i) + 1$, $i = 1, 2$. Hence $h^0(E_{\mathcal{L},W}) \geq r + 2$, a contradiction.

Proposition 3. *There is a rank $r$ spanned vector bundle $\mathcal{E}$ on $X$ with* $\det(\mathcal{E}) \cong \mathcal{L}$ *and no trivial factor if and only if* $2 \leq r \leq h^0(\mathcal{L}) - 1$.

Proof. The " if " part is covered by Remark 3. The " only if " part is true by Remark 1. □

Proof of Theorem 1. Let $\Omega$ be the non-empty open subset of $X$ such that $|\mathcal{L}|$ has injective differential. Let $S$ be the dependency locus of $r - 1$ general sections of $\mathcal{E}$. $S$ is a finite set contained in $\Omega$ and $\mathcal{E}$ fits in (4). We have $\sharp(S) = c_2(\mathcal{E})$. Since $\mathcal{E}$ is spanned, $\mathcal{I}_S \otimes \mathcal{L}$ is spanned. Fix general $T, T' \in |\mathcal{I}_S \otimes \mathcal{L}|$. By Bertini's theorem $T \cap T'$ is reduced outside $S$. Since $S \subset \Omega$ and $\mathcal{I}_S \otimes \mathcal{L}$ is spanned, we also see that $T \cap T'$ is reduced at each point of $S$. Set $A := T \cap T' \setminus S$. We just checked that $A$ is a set with cardinality $\mathcal{L}^2 - c_2(\mathcal{E})$.

(a) First assume $A = \emptyset$, i.e. assume that $S$ is the complete intersection of $T$ and $T'$. Let $U \subseteq H^0(\mathcal{I}_S \otimes \mathcal{L})$ be the image of $H^0(\mathcal{E})$ by the map induced by (4). Since $\mathcal{E}$ is spanned, $U$ spans $\mathcal{I}_S \otimes \mathcal{L}$. Since $\mathcal{L} \neq \mathcal{O}_X$, we have $\dim(U) \geq 2$. Fix general $s, s' \in U$. Since $s, s'$ are general and $U$ spans $\mathcal{I}_S \otimes \mathcal{L}$, the divisors $\{s = 0\}$ and $\{s' = 0\}$ have no common component. Since $S$ is the complete intersection of 2 elements of $|\mathcal{L}|$, we get that the linear span of $s$ and $s'$ spans $\mathcal{I}_S \otimes \mathcal{L}$. Take $\sigma, \sigma' \in H^0(\mathcal{E})$ whose image is $s, s'$, respectively. From (4) we get an $(r+1)$-dimensional linear subspace $W$ of $\mathcal{E}$ containing $\sigma, \sigma'$ and spanning $\mathcal{E}$. The kernel of the evaluation map $W \otimes \mathcal{O}_X \to \mathcal{E}$ is isomorphic to $\mathcal{L}$. Hence we are in the set-up of Remark 3 with $\mathcal{E}_{\mathcal{L},W} = E$ and in particular $W = H^0(\mathcal{E})$.

(b) Assume $A \neq \emptyset$, i.e. $c_2(\mathcal{E}) \neq \mathcal{L}^2$. We have $\sharp(A) = \mathcal{L}^2 - c_2(\mathcal{E})$. Note that $A$ and $S$ are linked by the complete intersection $T \cap T'$. We proceed as in the proof of [**17**, Lemma in § 3]. Let $\pi\colon \widetilde{X} \to X$ denote the blowing up of $X$ at the points of $A$. Let $E := \pi^{-1}(A)$ denote the union of the exceptional divisors.

Let $D, D'$ be the strict transforms of $T, T'$ and $S' := \pi^{-1}(S)$. We have $\mathcal{O}_{\widetilde{X}}(D) \cong \mathcal{O}_{\widetilde{X}}(D') \cong \pi^*(\mathcal{L})(-E)$. Since $A \cap S = \emptyset$, we have $E \cap S' = \emptyset$. We have $\sharp(S') = \sharp(S)$ and $S'$ is the complete intersection of $D$ and $D'$. We have an exact sequence on $\widetilde{X}$:

$$0 \to \mathcal{O}_{\widetilde{X}}(-D-D') \to \mathcal{O}_{\widetilde{X}}(-D) \oplus \mathcal{O}_{\widetilde{X}}(-D') \to \mathcal{I}_{S'} \to 0. \tag{8}$$

Since $\mathcal{O}_{\widetilde{X}}(D) \cong \mathcal{O}_{\widetilde{X}}(D') \cong \pi^*(\mathcal{L})(-E)$, tensoring (8) with $\pi^*(\mathcal{L})$ we get an exact sequence

$$0 \to \mathcal{O}_{\widetilde{X}}(2E) \otimes \pi^*(\mathcal{L})^\vee \to \mathcal{O}_{\widetilde{X}}(E) \oplus \mathcal{O}_{\widetilde{X}}(E) \to \mathcal{I}_{S'} \otimes \pi^*(\mathcal{L}) \to 0. \tag{9}$$

We have $h^0(\mathcal{I}_{S'} \otimes \pi^*(\mathcal{L})) = h^0(\mathcal{I}_S \otimes \mathcal{L})$ by the projection formula. Since $A \neq \emptyset$ and $\mathcal{I}_S \otimes \mathcal{L}$ is spanned, we have $h^0(\mathcal{I}_S \otimes \mathcal{L}) > 2$. Since $E$ is a disjoint union of exceptional divisors, we have $h^0(\mathcal{O}_{\widetilde{X}}(E)) = 1$. Hence (9) gives $h^1(\mathcal{O}_{\widetilde{X}}(2E) \otimes \pi^*(\mathcal{L})^\vee) > 0$. Since $\omega_{\widetilde{X}} \cong \pi^*(\omega_X) \otimes E$, Serre duality gives $h^1(\mathcal{O}_{\widetilde{X}}(2E) \otimes \pi^*(\mathcal{L})^\vee) = h^1(\mathcal{O}_{\widetilde{X}}(-E) \otimes \pi^*(\mathcal{L} \otimes \omega_X))$. Since $R^1\pi_*(\mathcal{O}_{\widetilde{X}}(-E)) = 0$, $\pi_*(\mathcal{O}_{\widetilde{X}}(-E)) \cong \mathcal{I}_A$, $R^1\pi_*(\mathcal{O}_{\widetilde{X}}) = 0$ and $\pi_*(\mathcal{O}_{\widetilde{X}}) \cong \mathcal{O}_X$, then $h^1(\mathcal{O}_{\widetilde{X}}(-E) \otimes \pi^*(\mathcal{L} \otimes \omega_X)) = h^1(\mathcal{I}_A \otimes \mathcal{L} \otimes \omega_X)$ (use the Leray spectral sequence of $R^i\pi_*$ and the projection formula). Hence $h^1(\mathcal{I}_A \otimes \mathcal{L} \otimes \omega_X) > 0$. Therefore $\sharp(A) \geq \alpha(\mathcal{L} \otimes \omega_X) + 2$, i.e. $c_2(\mathcal{E}) \leq \mathcal{L}^2 - \alpha(\mathcal{L} \otimes \omega_X) - 2$, a contradiction. □

Assume $\alpha(\mathcal{L} \otimes \omega_X) \geq 0$ and that we have a good description of all finite sets $A \subset X$ with $\sharp(A) = \alpha(\mathcal{L} \otimes \omega_X) + 2$ and $h^1(\mathcal{I}_A \otimes \mathcal{L} \otimes \omega_X) > 0$. We may hope to get a description of all spanned vector bundles $\mathcal{E}$ on $X$ with $\det(\mathcal{E}) \cong \mathcal{L}$, no trivial factor and $c_2(\mathcal{E}) = \mathcal{L}^2 - \alpha(\mathcal{L} \otimes \omega_X) - 2$. We are able to do this when $X = \mathbb{P}^2$.

## 3 - The projective plane

If $\mathcal{E}$ is a rank 2 spanned bundle on $\mathbb{P}^2$, then $t(\mathcal{E}) \geq 0$. Taking $t := t(\mathcal{E})$ and $c_1 := c_1(\mathcal{E})$ we get an exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^2}(t) \to \mathcal{E} \to \mathcal{I}_Z(c_1 - t) \to 0 \tag{10}$$

with $Z$ a locally complete intersection zero-dimensional scheme of degree $c_2(\mathcal{E}) - t(c_1 - t)$. A bundle in (10) is spanned if and only if $\mathcal{I}_Z(c_1 - t)$ is spanned. Duality gives

$h^1(\mathcal{E}^\vee) = h^1(\mathcal{E}(-3))$. Hence if $t > 0$, then $h^1(\mathcal{E}^\vee) = h^1(\mathcal{I}_Z(c_1 - t - 3))$. If $t = 0$ we have $h^1(\mathcal{E}^\vee) = h^1(\mathcal{I}_Z(c_1 - 3)) - 1$ if $\mathcal{E}$ is spanned, because for a spanned bundle $\mathcal{G}$ on a surface $X$ with $h^2(\mathcal{O}_X) = 0$ we have $h^2(\mathcal{G}) = 0$.

Fix integers $t$, $c_1$, $z$. Let $U(t, c_1, z)$ be the set of all rank 2 vector bundles $\mathcal{E}$ on $\mathbb{P}^2$ with $c_1(\mathcal{E}) = c_1$, $t(\mathcal{E}) = t$ and $c_2(\mathcal{E}) = z + t(c_1 - t)$. Each $\mathcal{E} \in U(t, c_1, z)$ fits in the exact sequence (10) with $Z$ a locally complete intersection zero-dimensional scheme of degree $z$. Hence $U(t, c_1, z) = \emptyset$ if $z < 0$, while $U(t, c_1, 0) = \{\mathcal{O}_{\mathbb{P}^2}(t) \oplus \mathcal{O}_{\mathbb{P}^2}(c_1 - t)\}$. We have $t(\mathcal{E}) = t$ if and only if $h^0(\mathcal{I}_Z(c_1 - 2t - 1)) = 0$. Hence if $t(\mathcal{E}) = t$, then either $c_1 \le 2t$ or $z \ge \binom{c_1 - 2t + 1}{2}$. $\mathcal{E}$ fits in a unique extension (10) (for some $Z$) if and only if $h^0(\mathcal{I}_Z(c_1 - 2t)) = 0$. If $t \ge 0$, any $\mathcal{E}$ fitting in (10) is spanned if and only if $\mathcal{I}_Z(c_1 - t)$ is spanned. Set $U'(t, c_1, z) := \{\mathcal{E} \in U(t, c_1, z) : h^1(\mathcal{E}) = 0\}$. Notice that $\mathcal{E} \in U'(t, c_1, z)$ with $t(\mathcal{E}) = t$ if and only if $\mathcal{E}$ fits in (10) with $Z$ a locally complete intersection, $h^1(\mathcal{I}_Z(c_1 - t)) = 0$ and $h^0(\mathcal{I}_Z(c_1 - 2t - 1)) = 0$.

Lemma 3. *Fix integers $x > 0$, $z \ge 0$. Let $S \subset \mathbb{P}^2$ be a general subset of $\mathbb{P}^2$ with $\sharp(S) = z$. The sheaf $\mathcal{I}_S(x)$ is spanned if and only if either $x \ge 3$ and $z \le \binom{x+2}{2} - 3$ or $x \le 2$ and $z \le x^2$.*

Proof. Since $S$ is general, $h^0(\mathcal{I}_S(x)) = \max\{0, \binom{x+2}{2} - z\}$. Since the cases $x = 1, 2$ are obvious, we assume $x \ge 3$. Since $S$ is general, $S$ is not the complete intersection of two curves of degree $z \ge 3$. Hence if $\mathcal{I}_S(x)$ is spanned, then $z \le \binom{x+2}{2} - 3$. The property “ globally generated ” is an open property for sheaves with constant cohomology. Fix a smooth plane curve $C$ with $\deg(C) = x$ and let $S \subset C$ be a general subset of $C$ with cardinality $z$. As in the proof of Proposition 2 we see that $\mathcal{I}_S(x)$ is spanned. □

Proposition 4. *Fix positive integers $z, t, c_1$ such that $c_1 > t > 0$ and either $c_1 - t = 1, 2$ and $z \le (c_1 - t)^2$ or $c_1 - t \ge 3$ and $z \le \binom{c_1 - t + 2}{2} - 3$; if $c_1 \ge 2t + 3$ assume $z > \binom{c_1 - 2t - 1}{2}$. Then there are $\mathcal{E} \in U'(t, c_1, z)$ fitting in (10) with $Z$ general in $\mathbb{P}^2$ and $\mathcal{E}$ spanned. For any such bundle $\mathcal{E}$ we have $h^1(\mathcal{E}^\vee) = 0$ if $z \le \binom{c_1 - t - 1}{2}$ and $h^1(\mathcal{E}^\vee) = z - \binom{c_1 - t - 1}{2}$ if $z \ge \binom{c_1 - t - 1}{2}$.*

Proof. Since $Z \subset \mathbb{P}^2$ is general, $\mathcal{I}_Z(c_1 - t)$ is spanned (Lemma 3). Since $Z$ is general and either $c_1 \le 2t + 2$ or $z > \binom{c_1 - 2t - 1}{2}$, we have $h^0(\mathcal{I}_{Z \setminus \{o\}}(c_1 - 2t - 3)) = 0$ for each $o \in Z$, i.e. the finite set $Z$ satisfies the Cayley-Bacharach condition in degree $c_1 - 2t - 3$. Thus there is a bundle $\mathcal{E}$ fitting in (10). Since $Z$ is general and $z \le \binom{c_1 - t + 2}{2}$, we have $h^1(\mathcal{I}_Z(c_1 - t)) = 0$. Since $\mathcal{I}_Z(c_1 - t)$ is spanned, $\mathcal{E}$ is spanned. Since $t > 0$, we have $h^2(\mathcal{O}_{\mathbb{P}^2}(t - 3)) = 0$ and hence $h^1(\mathcal{E}^\vee) = h^1(\mathcal{E}(-3)) = h^1(\mathcal{I}_Z(c_1 - t - 3))$. Use that $Z$ is general. □

Example 1. Fix integers $t \geq 0$ and $c_1 \geq t+2$. Let $U$ be the set of all locally complete intersection zero-dimensional schemes $Z \subset \mathbb{P}^2$, which are linked by two curves of degree $c_1 - t$ to a degree $c_1 - t - 1$ zero-dimensional subscheme $A$ contained in a line $D$. Fix plane curves $T, T'$ without common components and set $u := \deg(T)$, $u' := \deg(T')$ Let $S_1$ and $A_1$ be any two zero-dimensional subschemes of $\mathbb{P}^2$ linked by $T \cap T'$. By [**20**, Theorem 3] or ([**12**, Theorem CB7]), or if $S_1$ and $A_1$ are reduced and disjoint, [**17**, Lemma in § 3] we have

$$h^0(\mathcal{I}_{S_1}(y)) = h^1(\mathcal{I}_{A_1}(u+u'-y-3)) + h^0(\mathcal{I}_{T\cap T'}(y)) \quad \forall\, y \in \mathbb{Z}. \tag{11}$$

Look at (16) with $S_1 := Z$, $A_1 := A$, $u = u' = c_1 - t$. Taking $y = c_1 - t$ we get $h^0(\mathcal{I}_Z(c_1 - t)) = 3$. Taking $y = c_1 - t - 1$, we get $h^0(\mathcal{I}_Z(c_1 - t - 1)) = 0$. Since $h^0(\mathcal{I}_Z(c_1 - 2t - 2)) = 0$, $Z$ satisfies the Cayley-Bacharach condition in degree $c_1 - 2t - 3$ and hence we get bundles $\mathcal{E}$ fitting in (10) and with $c_1(\mathcal{E}) = c_1$, $c_2(\mathcal{E}) = t(c_1 - t) + (c_1 - t)^2 - (c_1 - t - 1) = c_1^2 - c_1 t - c_1 + t - 1$. Since $h^0(\mathcal{I}_Z(c_1 - 2t - 3)) = 0$, the family of bundles obtained for a fixed $Z$ is irreducible and its dimension does not depend on $Z$. Hence varying $Z$ we get an irreducible family of bundles. Since $h^0(\mathcal{I}_Z(c_1 - t - 1)) = 0$, we have $t(\mathcal{E}) = t$ for all $\mathcal{E}$. Now we check that each $\mathcal{E}$ is spanned, i.e. that $\mathcal{I}_Z(c_1 - t)$ is spanned, at least for a general $Z$. Since $Z$ is linked to $A$ by a complete intersection of two curves of degree $c_1 - t$, $\mathcal{I}_Z(c_1 - t)$ is spanned outside the support of $A$. Since $\deg(A) = c_1 - t - 1$, $\mathcal{I}_A(c_1 - t - 1)$ is spanned and hence for a general linkage (say by curves $T, T'$) we get $Z$ with $Z \cap A = \emptyset$. We claim that if $Z \cap A = \emptyset$, then $\mathcal{I}_Z(c_1 - t)$ is spanned, i.e. it is spanned at each point of $A_{\mathrm{red}}$. Fix $O \in A_{\mathrm{red}}$ and call $A'$ the zero-dimensional scheme linked to $Z \cup \{O\}$ by $T \cap T'$. $A'$ is a colength 1 subscheme of $A$ and so $\deg(A') = c_1 - t - 2$. Therefore $h^1(\mathcal{I}_{A'}(c_1 - t - 3)) = 0$. Using (16) with $y = c_1 - t, S_1 := Z \cup \{O\}$ and $A_1 := A'$ we get $h^0(\mathcal{I}_{Z\cup\{O\}}(c_1 - t)) = 2$. Hence $O$ is not a base point of $\mathcal{I}_Z(c_1 - t)$. Since $h^0(\mathcal{I}_Z(c_1 - t - 3)) = 0$, we have $h^1(\mathcal{E}^\vee) = h^1(\mathcal{E}(-3)) = c_1^2 - c_1 t - c_1 + t - 1 - \binom{c_1 - t - 1}{2}$. By Lemma 1 there is a spanned rank $r$ vector bundle $\mathcal{F}$ on $\mathbb{P}^2$ with $c_1(\mathcal{F}) = c_1$, $c_2(\mathcal{F}) = c_1^2 - c_1 t - c_1 + t + 1$ and no trivial factor if and only if $2 \leq r \leq c_1^2 - c_1 t - c_1 + t + 1 - \binom{c_1 - t - 1}{2} + 2$.

We claim that for fixed $r, c_1, t$ the set of all isomorphism classes of rank $r$ bundles $\mathcal{F}$ obtained in this way is parametrized (perhaps not finite to one) by an irreducible variety. Indeed, the set of all zero-dimensional schemes $Z$ is irreducible and the integer $h^1(\mathcal{I}_Z(c_1 - t - 3))$ is the same for all $Z$'s. For a fixed $Z$ we use as a parameter space a non-empty open subset of the Grassmannian of all $(r-1)$-dimensional linear subspaces of $H^1(\mathcal{I}_Z(c_1 - t - 3))$.

(a) Take $t = 0$ and set $x := c_1$. We have $c_2(\mathcal{E}) = x^2 - x$. In this case $h^1(\mathcal{E}^\vee) = (x^2 + x)/2$. A referee suggested that each such bundle occurs in an exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^2}(1-x) \oplus \mathcal{O}_{\mathbb{P}^2}(-1) \to \mathcal{O}_{\mathbb{P}^2}^{\oplus 4} \to \mathcal{E} \to 0 \tag{12}$$

Indeed, we proved that $h^0(\mathcal{E}) = 4$. Let $\mathcal{A}$ denote the kernel of the evaluation map $H^0(\mathcal{E}) \otimes \mathcal{O}_{\mathbb{P}^2} \to \mathcal{E}$. Since $h^1(\mathcal{I}_A(x-4)) = 2$, we have $h^0(\mathcal{I}_S(x+1)) = 8$ (see (16) with $y = x+1$, $S_1 = S$, $A_1 = A$). Thus $h^0(\mathcal{E}(1)) = 11$. Hence $h^0(\mathcal{A}(1)) > 0$. Let $\mathcal{B}$ be the cokernel of an injective map $\mathcal{O}_{\mathbb{P}^2} \to \mathcal{A}(1)$. Since $h^0(\mathcal{A}) = 0$, then $\mathcal{B} \cong \mathcal{I}_W(-x)$ for some zero-dimensional scheme $W$. Since $c_2(\mathcal{E}) = x(x-1)$, (12) gives $\deg(W) = 0$, i.e. $W = \emptyset$. Proposition 1 gives that all bundles in (12) arises as in (3) for $\mathcal{L} \cong \mathcal{O}_{\mathbb{P}^2}(x)$ and some complete intersection $S$.

Claim. Let $F$ be any extension of $\mathcal{E}$ by a trivial factor. Assume that $F$ has no trivial factor. Then $F$ is indecomposable.

Proof of Claim. Assume $F \cong A_1 \oplus A_2$ with $A_1, A_2$ non-trivial vector bundles. Set $x_i := c_1(A_i)$ and $z_i := c_2(A_i)$. We have $x = x_1 + x_2$ and $x^2 - x + 1 = z_1 + z_2 + x_1x_2$. Since $h^0(\mathcal{E}) = 4$, we have $h^0(F) = \mathrm{rank}(F) + 2$. Thus $h^0(A_i) = \mathrm{rank}(A_i) + 1$. Hence $A_i$ is as in Remark 3 and in particular $z_i = x_i^2$, $\mathrm{rank}(A_i) \geq 2$ and $x_i \geq 2$. Since $x_2 = x - x_1$, we get $x - 1 = x_1(x - x_1)$, i.e. either $x_1 = 1$ or $x_2 = 1$, a contradiction.

Example 2. Fix integers $c_1, t, m$ such that $t \geq 0$, $c_1 - t \geq m \geq \max\{1, c_1 - 2t\}$ and $c_1 \geq t+2$. Take any zero-dimensional scheme $Z \subset \mathbb{P}^2$ which is the complete intersection of a curve of degree $c_1 - t$ and a curve of degree $m$. Since $m \geq c_1 - 2t - 1$, we have $h^0(\mathcal{I}_Z(c_1 - 2t - 2)) = 0$ and hence $Z$ satisfies the Cayley-Bacharach condition in degree $c_1 - 2t - 3$. Hence there are bundles $\mathcal{E}$ fitting in (10). Fix any $\mathcal{E}$ in (10). We have $\deg(Z) = (c_1 - t)m$ and $\mathcal{I}_Z(c_1 - t)$ is spanned. Since $m \geq c_1 - 2t$, we have $h^0(\mathcal{I}_Z(c_1 - 2t - 1)) = 0$ and hence $t(\mathcal{E}) = t$. We have $t_1(\mathcal{E}) = c_1 - t - m$. Since $t \geq 0$ and $\mathcal{I}_Z(c_1 - t)$ is spanned, we get in this way an irreducible family of spanned vector bundle with $c_1(\mathcal{E}) = c_1$, $c_2(\mathcal{E}) = t(c_1 - t) + (c_1 - t)m$. Since $h^1(\mathcal{O}_{\mathbb{P}^2}(t)) = 0$ and $h^1(\mathcal{O}_{\mathbb{P}^2}(t + m - c_1)) = 0$, lifting the sections of $\mathcal{I}_Z(m)$ and $\mathcal{I}_Z(c_1 - t)$ we get that $\mathcal{E}$ fits in the exact sequence (suggested by a referee)

$$0 \to \mathcal{O}_{\mathbb{P}^2}(-m) \xrightarrow{u} \mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(c_1 - t - m) \oplus \mathcal{O}_{\mathbb{P}^2}(t) \to \mathcal{E} \to 0. \tag{13}$$

Conversely, by Bertini's theorem a general map $u$ in (13) is injective with locally free cokernel ([**9**]) and $\mathcal{E}$ is spanned and with the same Chern classes as the bundle in (10). All bundles in (13) fits in (10) with $Z$ determined by the first two entries of $u$.

We have $h^0(\mathcal{I}_Z(c_1 - t - 3)) = 0$ if $m \geq c_1 - t - 2$ and $h^0(\mathcal{I}_Z(c_1 - t - 3)) = \binom{c_1 - t - m - 1}{2}$ if $m \leq c_1 - t - 3$. Therefore (10) gives $h^1(\mathcal{E}^\vee) = h^1(\mathcal{E}(-3)) =$

$(c_1-t)m-\binom{c_1-t+1}{2}$ if $m \ge c_1-t-2$ and $h^1(\mathcal{E}^\vee)=\binom{c_1-t-m-1}{2}+(c_1-t)m-\binom{c_1-t+1}{2}$ if $m \le c_1-t-2$.

We claim that for fixed $r, c_1, t, m$ the set of all isomorphism classes of rank $r$ bundles $\mathcal{F}$ obtained in this way is parametrized (perhaps not finite to one) by an irreducible variety. Indeed, the set of all zero-dimensional schemes $Z$ is irreducible and the integer $h^1(\mathcal{I}_Z(c_1-t-3))$ is the same for all $Z$'s. For a fixed $Z$ we use as a parameter space a non-empty open subset of the Grassmannian of all $(r-1)$-dimensional linear subspaces of $H^1(\mathcal{I}_Z(c_1-t-3))$.

(a) Taking $t=1$, $m=c_1-t$ and $x=c_1$ we get the case quoted in Proposition 1 with $z=x^2-x$. In this case we have $h^1(\mathcal{E}^\vee)=h^1(\mathcal{I}_Z(x-4))=(x-1)^2-\binom{x-2}{2}=(x^2+x)/2-2$. Now we prove which non-trivial extensions are decomposable.

Claim. Fix an integer $a$ such that $3 \le a \le (x^2+x)/2$ and any extension $F$ of $\mathcal{E}$ by $\mathcal{O}_{\mathbb{P}^2}^{\oplus(a-2)}$ with no trivial factor. $F$ is decomposable if and only if $F \cong \mathcal{O}_{\mathbb{P}^2}(1)\oplus A$ with $A$ as in Remark 3. This case occurs if and only if $3 \le a \le (x^2+x)/2$, but for each $a$ the general extension of $\mathcal{E}$ by $\mathcal{O}_{\mathbb{P}^2}^{\oplus(a-2)}$ is indecomposable.

Proof of Claim. By Lemma 2 the bundle $F$ fits in an exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^2}(1-x) \xrightarrow{u} \mathcal{O}_{\mathbb{P}^2}^{\oplus a} \oplus \mathcal{O}_{\mathbb{P}^2}(1) \xrightarrow{v} F \to 0. \tag{14}$$

Since $\mathcal{E}$ is indecomposable (e.g. by (14) and the assumption $x \ge 2$), we may assume $a \ge 3$. Write $u=(u_1,u_2)$ with $u_1\colon \mathcal{O}_{\mathbb{P}^2}(1-x)\to \mathcal{O}_{\mathbb{P}^2}^{\oplus a}$, $u_2\colon \mathcal{O}_{\mathbb{P}^2}(1-x)\to\mathcal{O}_{\mathbb{P}^2}(1)$ and $v=(v_1,v_2)$ with $v_1\colon \mathcal{O}_{\mathbb{P}^2}(1)\to F$ and $v_2\colon \mathcal{O}_{\mathbb{P}^2}^{\oplus a}\to A$. If $a\ge 3$ we may find $u$ with $u_2=0$ and hence $F\cong \mathcal{O}_{\mathbb{P}^2}(1)\oplus A$ with $A$ a spanned vector bundle with $h^0(A)=\operatorname{rank}(A)+1$. Hence $A$ as in Remark 3. Therefore if $a\ne 2$ we may find decomposable examples with $\mathcal{O}_{\mathbb{P}^2}(1)$ as one factor and an indecomposable bundle $A$ as in Remark 3 with $c_1(A)=x-1$. We claim that these decomposable bundles are not the general ones. Fix any $F\cong\mathcal{O}_{\mathbb{P}^2}(1)\oplus A$ in (14). Write $v_1=(v_{1,1},v_{1,2})$ with $v_{1,1}\colon\mathcal{O}_{\mathbb{P}^2}(1)\to\mathcal{O}_{\mathbb{P}^2}(1)$ and $v_{1,2}\colon\mathcal{O}_{\mathbb{P}^2}(1)\to A$. Since $h^0(A(-1))=0$, then $v_{1,2}=0$ and hence $v_{1,1}$ is an isomorphism. Since $v_{1,1}\circ u_1=0$, we get $u_1=0$. Hence $u$ is not the general map in (14). Now assume $F\cong A_1\oplus A_2$ with $\operatorname{rank}(A_i)\ge 2$ for all $i$. With no loss of generality we may assume $h^0(A_1(-1))=1$ and $h^0(A_2(-1))=0$. Since $h^0(A_1(-2))=0$, a non-zero section of $A_1(-1)$ induces an exact sequence

$$0\to\mathcal{O}_{\mathbb{P}^2}(1)\to A_1\to G\to 0$$

with $G$ a torsion free spanned sheaf. Since $A_1$ has no trivial factors, $G$ is not trivial and hence $h^0(G)\ge\operatorname{rank}(G)+1=\operatorname{rank}(A_1)$. Hence $h^0(A_1)\ge\operatorname{rank}(A_1)+3$. Since $h^0(F)=$

$a+3$, we get $h^0(A_2) = \mathrm{rank}(A_2)$ and hence $A_2$ is trivial, so $F$ has a trivial factor, a contradiction.

A referee suggested the exact sequences (10) and (13), which immediately give spanned bundles with easily computed cohomology groups. More generally, take "general" maps $g: A \to B$, where $A, B$ are easy vector bundles (e.g., direct sums of line bundles) and $A^\vee$ is spanned, so that $\ker(g)^\vee$ is a spanned vector bundle. This is the approach of [**14**], [**11**], [**21**] in $\mathbb{P}^3$ for a very difficult problem (see also [**11**], [**15**], [**21**]). On $\mathbb{P}^2$ it gives many examples, but we did not found a set of data (e.g., $r, c_1, c_2$) for which all spanned bundles with these invariants arise from the same exact sequence varying the entries of the matrix $g$, except Examples 1 and 2.

Proof of Proposition 1. By Remark 1 we have $z \le x^2$ and $z = x^2$ if and only if $\mathcal{E}$ is as in Remark 3, i.e. if and only if it fits in an exact sequence

(15) $$0 \to \mathcal{O}_{\mathbb{P}^2}(-x) \to \mathcal{O}_{\mathbb{P}^2}^{\oplus 3} \to \mathcal{E} \to 0.$$

We also saw in Remark 3 that in this case we may find $\mathcal{F}$ if and only if $2 \le r \le \binom{x+2}{2} - 1$.

Now assume $z < x^2$ and the existence of $\mathcal{E}$. Let $S \subset \mathbb{P}^2$ be the dependency locus of a general section of $\mathcal{E}$. We have $\sharp(S) = z$ and $\mathcal{I}_S(x)$ is spanned. Fix general $T, T' \in |\mathcal{I}_S(t)|$. The scheme $T \cap T'$ is zero-dimensional, $\deg(T \cap T') = x^2$ and $T \cap T'$ is smooth outside $S$. Fix $P \in S$. Since $\mathcal{I}_S(x)$ is spanned and $\mathcal{O}_{\mathbb{P}^2}(x)$ is very ample, $T \cap T'$ is smooth at $P$. Since $S$ is finite, we get $T \cap T' = S \cup A$ with $A \subset \mathbb{P}^2$ a finite set, $\sharp(A) = x^2 - z$ and $A \cap S = \emptyset$. By the way we defined Examples 1 and 2 it is sufficient to prove that either $z = x^2 - x + 1$ and $S, A$ are as in Example 1 with $t = 0$ and $c_1 = x$ or $z = x^2 - x$ and $S, A$ are as in Example 2 with $t = 1$, $c_1 = x$ and $m = c_1 - t = x - 1$. Let $S_1$ and $A_1$ be any two finite sets such that $S_1 \cap A_1 = \emptyset$ and $S_1 \cup A_1 = T \cap T'$ (as schemes). By [**20**, Theorem 3] or [**17**, Lemma in § 3] or [**12**, Theorem CB7] we have

(16) $$h^0(\mathcal{I}_{S_1}(y)) = h^1(\mathcal{I}_{A_1}(2x - y - 3)) + h^0(\mathcal{I}_{T\cap T'}(y)) \quad \forall\, y \in \mathbb{Z}.$$

We have $h^0(\mathcal{I}_{T\cap T'}(x)) = 2$. Take $y = x$, $S_1 := S$ and $A_1 := A$ in (16). Since $\mathcal{I}_S(t)$ is spanned and $S \subsetneq T \cap T'$, we have $h^0(\mathcal{I}_S(x)) \ge 3$. Hence (16) gives $h^1(\mathcal{I}_A(x-3)) > 0$.

Claim 1. There is a line $D \subset X$ such that $\sharp(A \cap D) \in \{x-1, x\}$, $\sharp(S \cap D) + \sharp(A \cap D) \le x$ and $h^1(\mathcal{I}_A(x-3)) = \sharp(A \cap D) - x + 2$.

Proof of Claim 1. Since $\sharp(A) \le 2(x-3) + 1$ and $h^1(\mathcal{I}_A(x-3)) > 0$, there is a line $D \subset \mathbb{P}^2$ such that $\sharp(A \cap D) \ge x - 1$ ([**7**, Lemma 34]). We have a residual exact sequence

$$(17) \qquad 0 \to \mathcal{I}_{A\setminus A\cap D}(x-4) \to \mathcal{I}_A(x-3) \to \mathcal{I}_{D\cap A,D}(x-3) \to 0.$$

Since $A \setminus A \cap D$ is zero-dimensional, we have $h^2(\mathcal{I}_{A\setminus A\cap D}(x-4)) = h^2(\mathcal{O}_{\mathbb{P}^2}(x-4)) = 0$. Since $\sharp(A \setminus A \cap D) \leq 2x-5-(x-1)$, we have $h^1(\mathcal{I}_{A\setminus A\cap D}(x-4)) = 0$. Hence (17) gives $h^1(\mathcal{I}_A(x)) = h^1(D, \mathcal{I}_{A\cap D}(x-3))$. Since $\sharp(A\cap D) \geq x-1$, we have $h^1(D, \mathcal{I}_{A\cap D}(x-3)) = \sharp(A\cap D) - x + 2$. Since $A \cup S = T \cap T'$, we have $\sharp((A\cup S)\cap D) \leq x$. Hence $\sharp(A\cap D) \in \{x-1, x\}$.

By Claim 1 we have $z \leq x^2 - x + 1$.

(a) Assume $z = x^2 - x + 1$, i.e. $\sharp(A) = x - 1$. By Claim 1 there is a line $D \subset \mathbb{P}^2$ such that $A \subset D$. Hence $\mathcal{E}$ is as in Example 2 with $t = 1$ and $c_1 = x$.

(b) Assume $z = x^2 - x$. Take $D$ as in Claim 1. First assume $A \subset D$. From (16) with $S_1 = S$ and $A_1 = A$ we get $h^0(\mathcal{I}_S(x-1)) > 0$, and $h^0(\mathcal{I}_S(x-2)) = 0$ and $h^0(\mathcal{I}_S(x)) = 5$. Hence $S$ is a complete intersection of a curve of degree $x-1$ and a curve of degree $x$. Hence $S$ and $A$ are as in the case $t = 1$, $c_1 = x$ of Example 2. Now assume $\sharp(A\cap D) = x-1$ and hence $A \setminus A\cap D$ is a unique point. Call $q$ this point. By Claim 1 we have $h^1(\mathcal{I}_{A\cap D}(x-3)) = h^1(\mathcal{I}_A(x-3))$. Taking $y := x$, $S_1 := S$ and $A_1 := A$ in (16) we get $h^1(\mathcal{I}_S(x)) = 2 + h^1(\mathcal{I}_{A\cap D}(x-3))$. The sets $S \cup \{q\}$ and $A \cap D$ are linked by $T \cap T'$. Taking $y := x$, $S_1 := S \cup \{q\}$ and $A_1 := A \cap D$ in (16) we get $h^1(\mathcal{I}_{S\cup\{q\}}(x)) = 2 + h^1(\mathcal{I}_{A\cap D}(x-3))$. Hence $q$ is in the base locus of $\mathcal{I}_S(x)$, a contradiction.

(c) Assume $x^2 - 2x + 1 \leq z < x^2 - x$. We have $\sharp(A) \geq x+1$ and hence there is $O \in A \setminus A \cap D$. Since $O \notin D$, the proof of Claim 1 gives $h^1(\mathcal{I}_A(x-3)) = h^1(\mathcal{I}_{A\setminus\{O\}}(x-3))$. As in step (b) we see that $O$ is a base point of $\mathcal{I}_S(x)$, a contradiction. □

Proposition 5. *Fix integers $t \geq 0$, $c_1 \geq t+2$ and $c_2 \geq (c_1 - t)c_1 - c_1 + t + 1$. Let $\mathcal{E}$ be a rank 2 spanned vector bundle on $\mathbb{P}^2$ with $c_1(\mathcal{E}) = c_1$, $t(\mathcal{E}) = t$ and $c_2(\mathcal{E}) = c_2$. Then either $\mathcal{E}$ is as in Example 2 with $m = c_1 - t$ or it is as in Example 1.*

Proof. By assumption $\mathcal{E}$ fits in an exact sequence (10) for some locally complete intersection scheme $Z$ with $\mathcal{I}_Z(c_1 - t)$ spanned and $\deg(Z) = c_2 - t(c_1 - t) \geq (c_1 - t)^2 - c_1 + t + 1$. By assumption $\mathcal{I}_Z(c_1 - t)$ is spanned. Fix two general $T, T' \in |\mathcal{I}_Z(c_1 - t)|$. We have $Z = T \cap T'$ if and only if $\mathcal{E}$ is as in Example 2. Now assume $Z \nsubseteq T \cap T'$ and let $A \subset \mathbb{P}^2$ be the scheme linked to $Z$ by $T \cap T'$. We have $\deg(A) = (c_1 - t)^2 - \deg(Z) \leq c_1 - t - 1$. Since $Z \nsubseteq T \cap T'$ and $\mathcal{I}_Z(c_1 - t)$ is spanned, then $h^0(\mathcal{I}_Z(c_1 - t)) \geq 3$. Using (16) with $y = c_1 - t$, $S_1 = Z$ and $A_1 = A$ we get $h^1(\mathcal{I}_A(c_1 - t - 3)) > 0$. Since $\deg(A) \leq c_1 - t - 1$, we get $\deg(A) = c_1 - t - 1$ and that $A$ is contained in a line ([7, Lemma 34]). Hence $\mathcal{E}$ is as in Example 1. □

Proposition 6. *Take $\mathcal{E}$ as in Theorem 2 and assume $h^1(\mathcal{E}^\vee) \neq c_2 - t(c_1 - t) - \binom{c_1-t-1}{2}$.*

(a) *If $c_2 = (c_1 - 3)(c_1 - t)$, then $\mathcal{E}$ is as in Example 2 with $m = c_1 - t - 3$.*

(b) *If $c_2 < (c_1 - 3)(c_1 - t)$ and $c_1 \geq t + 6$, then $(c_1 - 3)(c_1 - t) - c_2 \geq c_1 - t - 4$.*

Proofs of Theorem 2 and Proposition 6. Fix a non-zero a section of $\mathcal{E}(-t)$ so that $\mathcal{E}$ fits in (10) with $\deg(Z) = c_2 - t(c_1 - t) \geq 0$. Since $t(\mathcal{E}) = t$, we have $h^0(\mathcal{I}_Z(c_1 - 2t - 1)) = 0$. If $c_1 \geq 2t + 1$ we get $\deg(Z) \geq \binom{c_1-2t+1}{2}$ and hence $c_2 \geq t(c_1 - t) + \binom{c_1-2t+1}{2} = (c_1^2 + 2t^2 - 2tc_1 + c_1 - 2t)/2$. Lemma 1 gives $r - 2 = h^1(\mathcal{E}^\vee) = h^1(\mathcal{E}(-3)) = h^1(\mathcal{I}_Z(c_1 - t - 3))$. We have $h^1(\mathcal{I}_Z(c_1 - t - 3)) \geq c_2 - t(c_1 - t) - \binom{c_1-t-1}{2}$. Since $\mathcal{I}_Z(c_1 - t)$ is spanned, we have $\deg(Z) \leq (c_1 - t)^2$ and hence $c_2(\mathcal{E}) \leq c_1(c_1 - t)$.

(a) Assume $c_2 \geq (c_1 - t)c_1 - c_1 + t + 1$. Proposition 5 gives that $\mathcal{E}$ is as either as in Example 2 with $m = c_1 - t$ or as in Example 1. In the latter case we have $h^1(\mathcal{E}^\vee) = (c_1 - t)^2 - (c_1 - t - 1) - \binom{c_1-t-1}{2} = (c_1 - t - 1)(c_1 - t + 2)/2$.

(b) Assume $c_2 \leq (c_1 - t)c_1 - c_1 + t$. We assumed the inequality $c_1 - t - 3 \geq -1$. If $h^0(\mathcal{I}_Z(c_1 - t - 3)) = 0$, then $h^1(\mathcal{I}_Z(c_1 - t - 3)) = c_2 - t(c_1 - t) - \binom{c_1-t-1}{2} \leq (c_1 - t)^2 - c_1 + t - \binom{c_1-t-1}{2} = (c_1 - t - 1)(c_1 - t + 2)/2$. Now assume $h^0(\mathcal{I}_Z(c_1 - t - 3)) > 0$. Take any $D \in |\mathcal{I}_Z(c_1 - t - 3)|$. Since $\mathcal{I}_Z(c_1 - t)$ is spanned, there is $T \in |\mathcal{I}_Z(c_1 - t)|$ containing no irreducible component of $D$. Hence the scheme $W := T \cap D$ is a complete intersection of a curve of degree $c_1 - t$ and a curve of degree $c_1 - t - 3$. Since $Z \subseteq W$ we have $h^1(\mathcal{I}_Z(c_1 - t - 3)) \leq h^1(\mathcal{I}_W(c_1 - t - 3))$. Since $h^0(\mathcal{I}_W(c_1 - t - 3)) = 1$, we have $h^1(\mathcal{I}_W(c_1 - t - 3)) = (c_1 - t)(c_1 - t - 3) - \binom{c_1-t-1}{2} - 1 = \binom{c_1-t-1}{2} - 1$. Since $Z \subseteq W$, then $c_2 \leq t(c_1 - t) + (c_1 - t)(c_1 - t - 3) = (c_1 - 3)(c_1 - t)$.

(c) Assume $c_2 \geq (c_1 - t)(c_1 - 3) - c_1 + t + 5$ and $h^0(\mathcal{I}_Z(c_1 - t - 3)) > 0$. If $c_2 \neq (c_1 - t)(c_1 - 3)$, then assume $c_1 \geq t + 6$. Take $D$, $T$ and $W \supseteq Z$ as in step (b). If $c_2 = (c_1 - 3)(c_1 - t)$, then $Z = W$ and hence $\mathcal{E}$ is as in Example 2 with $m = c_1 - t - 3$. Now assume $c_2 \neq (c_1 - 3)(c_1 - t)$ and $c_1 \geq t + 6$. Let $A \subset W$ be the residual of $Z$ inside $W$, i.e. the closed subscheme of $W$ with $\mathcal{I}_A = \mathrm{Ann}(\mathcal{I}_Z/\mathcal{I}_W)$. Since $c_1 - t \leq (c_1 - t) + (c_1 - t - 3) - 3$, we have $h^0(\mathcal{I}_Z(c_1 - t)) = h^1(\mathcal{I}_A(c_1 - t - 6)) + h^0(\mathcal{I}_W(c_1 - t))$ ([**12**, Theorem CB7]). Since $\deg(A) \leq c_1 - t - 5$, we have $h^1(\mathcal{I}_A(c_1 - t - 6)) = 0$. Since $W \supsetneq Z$, we get that $\mathcal{I}_Z(c_1 - t)$ is not spanned, a contradiction. □

Remark 4. Fix positive integers $c_1, c_2$ such that $4c_2 > c_1^2 \geq c_2$, $c_1(c_1 + 3)/2 - c_2 \geq 2$ and let $\mathcal{E}$ be a general rank 2 stable vector bundle on $\mathbb{P}^2$. $\mathcal{E}$ is spanned. If $(c_1 - 1)(c_1 - 2)/2 \geq c_2$, then $h^1(\mathcal{E}(-3)) = 0$ ([8, 5.1], [**19**, 3.4]) and hence $\mathcal{E}$ is not extendable to a higher rank spanned bundle with no trivial factor. Now we show that for many of these pairs $(c_1, c_2)$ there are extendable rank 2 spanned vector bundles $\mathcal{F}$. Fix an integer $t$ such that $2 \leq t < c_1/2$ and assume $c_2 > t(c_1 - t) + \binom{c_1-t-1}{2}$ (e.g. for $t = 2$ take $c_2 \geq (c_1^2 + c_1 + 6)/2$). Set $z := c_2 - t(c - t)$ and let $S \subset \mathbb{P}^2$ be a general subset with cardinality $z$. Let $G$ be a general extension of $\mathcal{I}_S(c_1 - t)$ by $\mathcal{O}_{\mathbb{P}^2}(t)$. $G$ is a vector bundle and $h^1(G^\vee) = h^1(G(-3)) = c_2 - t(c_1 - t) - \binom{c_1-t-1}{2}$.

Remark 5. The existence part in the determination of the Lüroth semigroup of a smooth plane curve $C$ of degree $d$ are constructive ([**16**, Lemma 2.1 and Corollary 2.6], [**10**]). In the corresponding cases it is often easy to compute $h^1(\mathcal{I}_S(d-3))$ (e.g. if $S$ is general in $C$ as in [**16**]). However, we are unable to get from this a uniqueness statement like Proposition 1. The difficulties in the set-up of Theorem 2 are very different. Since $t > 0$ and $h^0(\mathcal{I}_Z(c_1 - 2t - 1)) = 0$, $Z$ satisfies for free the Caylay-Bacharach condition in degree $c_1 - 2t - 3$. We are unable to construct $Z$ covering large numbers of pairs $(\deg(Z), h^0(\mathcal{I}_Z(c_1 - t))$ with the restriction $h^0(\mathcal{I}_Z(c_1 - 2t - 1)) = 0$.

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