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Solvability of the abstract Regge boundary value problem and asymptotic behavior of eigenvalues of one abstract spectral problem

Abstract. In the paper, we give an abstract formulation of the classical Regge boundary value problem (but with a constant potential) in a Hilbert space and prove an isomorphism result for the problem. This result implies, in particular, maximal L_p-regularity for the problem. We also obtain an estimate of the solution with respect to the spectral parameter. Then, for one homogeneous abstract spectral problem, we find asymptotic behavior of its eigenvalues. A possible application of the abstract results to elliptic partial differential equations is shown at the end of the paper.

Keywords. Differential-operator equations, elliptic equations, isomorphism, spectral parameter, scattering problem, Regge problem, maximal L_p-regularity, eigenvalues.

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1 - Introduction

Boundary value problems for elliptic differential-operator equations with the same linear spectral parameter entering into the equation and boundary conditions

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have been investigated in various aspects in the papers by V. I. Gorbachuk and M. A. Rybak [**12**], M. A. Rybak [**19**], L. A. Oleinik [**18**], V. M. Bruk [8], B. A. Aliev [**2, 3, 4**], B. A. Aliev and Ya. Yakubov [**5**], M. Bairamoglu and N. M. Aslanova [**7**]. In contrast to the above mentioned papers, in this paper, we study solvability of boundary value problems for an elliptic differential-operator equation of the second order when the spectral parameter enters quadratically into the equation and enters linearly into one of the boundary conditions. Such a case has been also previously considered in a few studies but no one of them covers our situation. A prototype of our abstract problem, in the first isomorphism part of the paper, is the classical Regge problem (but with a constant potential), for which there is no a general theory (for the details of the last remarks see below).

So, in a separable Hilbert space H, consider a boundary value problem on the interval [0,1] for a second order elliptic differential-operator equation

$$L(\lambda, D)u := \lambda^2 u(x) - u''(x) + Au(x) = f(x), \quad x \in (0,1), \tag{1.1}$$

$$\begin{aligned} L_1(\lambda)u &:= \alpha u'(1) + \lambda u(1) = f_1, \\ L_2 u &:= u(0) = f_2, \end{aligned} \tag{1.2}$$

where the spectral parameter λ and $\alpha \neq 0$ are some complex numbers from the right-hand side of the complex plane; A is a linear selfadjoint positive-definite operator in H; $D := \dfrac{d}{dx}$. Note that the solvability questions, discreteness of the spectrum and two-fold completeness of a system of root functions of boundary value problems of the form (1.1), (1.2) but for second order ordinary differential equations (replacing A by $Q(x)$, where $Q(x) \in W_p^1(0,1)$, $Q(1) \neq 0$, and $\alpha = 1$) have been investigated in the monograph by S. Yakubov and Ya. Yakubov [**22**] (see also the paper by S. Yakubov [**21**]). In that case, the problem becomes, so called, the Regge problem (just replacing λ by $i\lambda$, that can be done without loss of generality) which arises in the description of scattering by a finite potential. The main difficulty in such case is that the Regge problem is not only Birkhoff-irregular but also Stone-irregular and there is no a general theory for such problems. We decided to keep this name, the Regge problem, for our abstract boundary value problem (1.1), (1.2), at least for the first part of the paper. Let us emphasize again, that we cannot cover a general situation of the classical Regge problem in the framework of ordinary differential equations since our operator A is a constant operator with respect to x variable, i.e., we can just take $A = Q =$const$\neq 0$. This first study of such abstract problems can stimulate the readers (including ourself) to consider in future the general situation of a non-constant operator $A(x)$ in the equation. For the classical Regge ordinary boundary value problem with a non-constant potential, we also refer the readers to [**13**].

Let us also note, that our equation (1.1) may be considered as an abstract formulation of the classical scattering equation. On the other side, the scattering problem is usually treated in the whole space or in the half-space (see, e.g., [**10**], [**11**], where our $H = \mathbb{R}^n$ and A is an $n \times n$ matrix), so our boundary value problem (1.1)-(1.2) does not really describes the classical scattering problem but describes the classical Regge boundary value problem (see, e.g., [**13**]) even if in its particular case (the problem with a constant potential).

A few similar isomorphism studies by M. Denche [**9**] and A. Aibeche, A. Favini, and Ch. Mezoued [**1**], treat more general boundary conditions than (1.2) but their abstract equations do not cover our abstract equation (1.1). Moreover, their results cannot be applied to the classical Regge problem, even with $Q = \text{const} \neq 0$. We, in addition to isomorphism questions, obtain also asymptotic behavior of eigenvalues of some connected abstract spectral problems.

In the present paper, some simple sufficient conditions for solvability of the problem (1.1), (1.2) have been found (in fact, an isomorphism theorem has been proved) and some estimates (with respect to u and λ) for the solution of the problem (1.1), (1.2), in the space $L_p((0,1);H)$, $1 < p < \infty$, have been also established. The results imply maximal L_p-regularity for (1.1), (1.2). Then, we study asymptotic behavior of eigenvalues of a homogeneous abstract problem which is obtained from (1.1), (1.2) replacing λ by $i\lambda$ and taking $\alpha = i$. Therefore, the corresponding spectral problem does not cover the classical Regge problem ($\alpha = 1$) but some another spectral problem. First, it is proved that the eigenvalues are real numbers.

Note that asymptotic behavior of eigenvalues and eigenfunctions for second order ordinary (and not abstract) differential equations with the spectral parameter in both the equation and boundary conditions have been studied in many papers. It is not our goal to give here a full reference on ordinary differential equations, but, for example, N. B. Kerimov and Kh. R. Mamedov [**14**] have considered a situation when λ enters quadratically into the equation and enters linearly and quadratically into the boundary condition.

Some simple application of obtained abstract results to boundary value problems for elliptic partial differential equations of the second order, considered in a square, is given at the end of the paper.

Introduce now some necessary definitions and notation used in the paper.

Let E_1 and E_2 be Banach spaces. The set $E_1 \dot{+} E_2$ of all vectors of the form (u, v), where $u \in E_1,\ \ v \in E_2$ with standard coordinatewise linear operations and with the norm

$$\|(u,v)\|_{E_1 \dot{+} E_2} := \|u\|_{E_1} + \|v\|_{E_2}$$

is a Banach space and is said to be a direct sum of Banach spaces E_1 and E_2.

Let E_1 and E be two Banach spaces. Denote by $B(E_1, E)$ a Banach space of all linear bounded operators acting from E_1 into E with the standard operator norm. If $E_1 = E$ then $B(E) := B(E, E)$.

Definition 1. A linear closed operator A, densely defined in a Hilbert space H, is said to be *strongly positive* if, for some $\varphi \in [0, \pi)$ and for all complex numbers μ such that $|\arg \mu| \leq \varphi$ (including $\mu = 0$), the operators $A + \mu I$ are (boundedly) invertible and the estimate

$$\left\|(A+\mu I)^{-1}\right\|_{B(H)} \leq C(1+|\mu|)^{-1}$$

holds, where I is the unit operator in H, $C = \text{const} > 0$. For $\varphi = 0$, the operator A is called *positive*.

A simple example of strongly positive operators are selfadjoint positive-definite operators acting in a Hilbert space. Note that from strong positivity of an operator A it follows strong positivity of the operator $A^{\alpha}, \alpha \in (0,1)$. Let A be a strongly positive operator in H. Since A^{-1} is bounded in H, then

$$H(A^n) := \left\{u : u \in D(A^n),\ \|u\|_{H(A^n)} = \|A^n u\|_H\right\},\quad n \in \mathbb{N},$$

is a Hilbert space whose norm is equivalent to the norm of the graph of the operator A^n. If A is strongly positive in H, it is known that the operator $-A$ is a generating operator of the analytic, for $t > 0$, semigroup e^{-tA} and this semigroup exponentially decreases, i.e., there exist two numbers $C > 0$, $\sigma_0 > 0$ such that $\left\|e^{-tA}\right\| \leq Ce^{-\sigma_0 t}$, $0 \leq t < +\infty$. By virtue of [**15**, theorem 1.5.5], the operator $-A^{1/2}$ generates an analytic semigroup, for $t > 0$, decreasing at infinity.

Definition 2 [**20**, theorem 1.14.5]. *Interpolation spaces* $(H(A^n), H)_{\theta,p}$ of Hilbert spaces $H(A^n)$ and H, where A is a strongly positive operator in H, are defined by the equality

$$(H(A^n), H)_{\theta,p} := \left\{u : u \in H,\ \|u\|_{(H(A^n),H)_{\theta,p}}\right.$$

$$\left. = \int_0^{+\infty} t^{-1+n\theta p} \|A^n e^{-tA} u\|_H^p dt < \infty\right\},\quad \theta \in (0,1),\ p > 1,\ n \in \mathbb{N}.$$

We denote $(H(A^n), H)_{0,p} := H(A^n)$, $(H(A^n), H)_{1,p} := H$.

Denote by $L_p((0,1); H)$ $(1 < p < \infty)$ a Banach space (for $p = 2$ a Hilbert space) of vector-functions $x \to u(x) : [0,1] \to H$ strongly measurable and summable with order p and with the norm

$$\|u\|_{L_p((0,1);H)} := \left(\int_0^1 \|u(x)\|_H^p dx \right)^{1/p} < \infty.$$

Denote by $W_p^{2n}((0,1);H(A^n),H) := \{u : A^n u, u^{(2n)} \in L_p((0,1);H)\}$ a Banach space of vector-functions $x \to u(x) : [0,1] \to H$ strongly measurable and summable with order p and with the norm

$$\|u\|_{W_p^{2n}((0,1);H(A^n),H)} := \|A^n u\|_{L_p((0,1);H)} + \|u^{(2n)}\|_{L_p((0,1);H)} < \infty.$$

It is known [**20**, theorem 1.8.2] (see also [**22**, theorem 1.7.7/1]) that if $u \in W_p^{2n}((0,1); H(A^n),H)$ then, $\forall x_0 \in [0,1]$,

$$u^{(j)}(x_0) \in (H(A^n),H)_{\frac{j+\frac{1}{p}}{2n},p}, \quad j = 0,\dots,2n-1.$$

2 - Homogeneous equations

First, consider the following boundary value problem, in a Hilbert space H,

$$L(\lambda,D)u := \lambda^2 u(x) - u''(x) + Au(x) = 0, \quad x \in (0,1), \tag{2.1}$$

$$\begin{aligned} L_1(\lambda)u &:= \alpha u'(1) + \lambda u(1) = f_1, \\ L_2 u &:= u(0) = f_2. \end{aligned} \tag{2.2}$$

Theorem 1. *Let the following conditions be fulfilled:*

1. *A is a selfadjoint, positive-definite operator $(A = A^* \geq \gamma^2 I)$ in a separable Hilbert space H;*
2. *$\alpha \neq 0$ is some complex number with $|\arg \alpha| \leq \frac{\pi}{2}$.*

Then the problem (2.1), (2.2), *for $f_k \in (H(A),H)_{\theta_k,p}$, where $\theta_1 = \frac{1}{2} + \frac{1}{2p}$, $\theta_2 = \frac{1}{2p}$, $p \in (1,\infty)$, and for $|\arg \lambda| \leq \varphi < \frac{\pi}{2}$, $|\lambda|$ is sufficiently large, has a unique solution $u(x)$ which belongs to $W_p^2((0,1);H(A),H)$ and, for these λ, the following estimate holds for the solution of the problem* (2.1), (2.2)

$$\begin{aligned} &|\lambda|^2 \|u\|_{L_p((0,1);H)} + \|u''\|_{L_p((0,1);H)} + \|Au\|_{L_p((0,1);H)} \\ &\leq C \sum_{k=1}^{2} \left(\|f_k\|_{(H(A),H)_{\theta_k,p}} + |\lambda|^{2(1-\theta_k)} \|f_k\|_H \right). \end{aligned} \tag{2.3}$$

Proof. Since $A = A^* \geq \gamma^2 I$ in H, by the spectral theorem (see, e.g., [**17**, chapter V, sections 5 and 6, chapter VI section 5]) there exists an operator-valued function $f(A) = \int_{\gamma^2}^{+\infty} f(\mu) dE_\mu$ for any measurable, bounded complex-valued function $f(\mu)$. Furthermore, $f(A)$ is a bounded operator in H and $\| f(A) \|_{B(H)} \leq \operatorname{ess\,sup}_{\gamma^2 \leq \mu < \infty} |f(\mu)|$. Then, from condition 1, it follows that for any ψ, $0 \leq \psi < \pi$ there exists $C_\psi > 0$ such that

$$\|R(\lambda, A)\| \leq C_\psi (1 + |\lambda|)^{-1}, \quad |\arg \lambda| \geq \pi - \psi,$$

where $R(\lambda, A) := (\lambda I - A)^{-1}$ is the resolvent of the operator A. Hence, by virtue of [**22**, lemma 5.4.2/6], for $|\arg \lambda| \leq \varphi < \frac{\pi}{2}$, there exists an analytic, for $x > 0$, and strongly continuous, for $x \geq 0$, semigroup $e^{-x\left(A+\lambda^2 I\right)^{1/2}}$. By virtue of [**22**, lemma 5.3.2/1], for a function $u(x)$ being a solution of the equation (2.1), for $|\arg \lambda| \leq \varphi < \frac{\pi}{2}$, belonging to $W_p^2((0,1); H(A), H)$, $1 < p < \infty$, it is necessary and sufficient that

$$u(x) = e^{-x\left(A+\lambda^2 I\right)^{1/2}} g_1 + e^{-(1-x)\left(A+\lambda^2 I\right)^{1/2}} g_2, \tag{2.4}$$

where $g_k \in (H(A), H)_{\frac{1}{2p}, p}$, $k = 1, 2$.

A function $u(x)$ of the form (2.4) satisfies the boundary condition (2.2) if

$$\begin{cases} \left[-\alpha\left(A+\lambda^2 I\right)^{1/2} + \lambda I\right] e^{-\left(A+\lambda^2 I\right)^{1/2}} g_1 + \left[\alpha\left(A+\lambda^2 I\right)^{1/2} + \lambda I\right] g_2 = f_1, \\ g_1 + e^{-\left(A+\lambda^2 I\right)^{1/2}} g_2 = f_2 \,. \end{cases} \tag{2.5}$$

We rewrite the system (2.5), in the space $\mathbb{H} := (H(A), H)_{\frac{1}{2}+\frac{1}{2p}, p} \dot{+} (H(A), H)_{\frac{1}{2p}, p}$, in the operator form

$$(A(\lambda) + R(\lambda)) \begin{pmatrix} g_1 \\ g_2 \end{pmatrix} = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}, \tag{2.6}$$

where $A(\lambda)$ and $R(\lambda)$ are operator-matrices of dimension 2×2:

$$A(\lambda) := \begin{pmatrix} 0 & \alpha\left(A+\lambda^2 I\right)^{1/2} + \lambda I \\ I & 0 \end{pmatrix},$$

$$D(A(\lambda)) := (H(A), H)_{\frac{1}{2p}, p} \dot{+} (H(A), H)_{\frac{1}{2p}, p}$$

and

$$R(\lambda) := \begin{pmatrix} \left[-\alpha\left(A+\lambda^2 I\right)^{1/2} + \lambda I\right] e^{-\left(A+\lambda^2 I\right)^{1/2}} & 0 \\ 0 & e^{-\left(A+\lambda^2 I\right)^{1/2}} \end{pmatrix},$$

$$D(R(\lambda)) := \mathbb{H}.$$

Show that the operator $A(\lambda)$, in the space $\mathbb{H}$, for λ from the sector $|\arg\lambda| \leq \varphi < \frac{\pi}{2}$, has a bounded inverse $A(\lambda)^{-1}$ acting from $\mathbb{H}$ into $(H(A), H)_{\frac{1}{2p},p} \dot{+} (H(A), H)_{\frac{1}{2p},p}$ and it holds the estimate

$$\left\| A(\lambda)^{-1} \right\|_{B\left(\mathbb{H},(H(A),H)_{\frac{1}{2p},p} \dot{+} (H(A),H)_{\frac{1}{2p},p}\right)} \leq C, \tag{2.7}$$

where $C > 0$ is a constant independent on λ. Since $A(\lambda)^{-1}$ formally has the form

$$A(\lambda)^{-1} = \begin{pmatrix} 0 & I \\ \left[\alpha\left(A + \lambda^2 I\right)^{1/2} + \lambda I\right]^{-1} & 0 \end{pmatrix}$$

then it is sufficient to show that the operator $\left[\alpha\left(A + \lambda^2 I\right)^{1/2} + \lambda I\right]^{-1}$, for $|\arg\lambda| \leq \varphi < \frac{\pi}{2}$, is bounded from $(H(A), H)_{\frac{1}{2}+\frac{1}{2p},p}$ into $(H(A), H)_{\frac{1}{2p},p}$ and it holds the estimate

$$\left\| \left[\alpha\left(A + \lambda^2 I\right)^{1/2} + \lambda I\right]^{-1} \right\|_{B\left((H(A),H)_{\frac{1}{2}+\frac{1}{2p},p},(H(A),H)_{\frac{1}{2p},p}\right)} \leq C, \tag{2.8}$$

where $C > 0$ is a constant independent on λ.

Consider the function $f(\mu) = \left(1 + \alpha^{-1}\lambda(\mu + \lambda^2)^{-1/2}\right)^{-1}$, for a fixed $\alpha \neq 0$ with $|\arg\alpha| \leq \frac{\pi}{2}$. Note that $z^{\frac{1}{2}} = |z|^{\frac{1}{2}} e^{i\frac{\arg z}{2}}$, where $-\pi < \arg z \leq \pi$. Show now that, for $|\arg\lambda| \leq \varphi < \frac{\pi}{2}$,

$$\inf_{\gamma^2 \leq \mu < \infty} |(f(\mu))^{-1}| = \inf_{\gamma^2 \leq \mu < \infty} \left| \left(1 + \alpha^{-1}\lambda\left(\mu + \lambda^2\right)^{-1/2}\right) \right| \geq C, \quad \exists C > 0. \tag{2.9}$$

Since $|\arg\alpha| \leq \frac{\pi}{2}$ then $\left|\arg\alpha^{-1}\right| \leq \frac{\pi}{2}$. If $0 \leq \arg\lambda \leq \varphi < \frac{\pi}{2}$ and $\gamma^2 \leq \mu < \infty$, then $0 \leq \arg\left(\mu + \lambda^2\right) \leq 2\varphi$ and $-2\varphi \leq \arg\left(\mu + \lambda^2\right)^{-1} \leq 0$. Therefore $-\varphi \leq \arg\left(\mu + \lambda^2\right)^{-1/2} \leq 0$. Consequently, $-\varphi \leq \arg\left(\lambda\left(\mu + \lambda^2\right)^{-1/2}\right) \leq \varphi$. If $-\varphi \leq \arg\lambda \leq 0$ and $\gamma^2 \leq \mu < \infty$, we have $-2\varphi \leq \arg\left(\mu + \lambda^2\right) \leq 0$. Hence, $0 \leq \arg\left(\mu + \lambda^2\right)^{-1/2} \leq \varphi$. Consequently, $-\varphi \leq \arg\left(\lambda\left(\mu + \lambda^2\right)^{-1/2}\right) \leq \varphi$. So, for $|\arg\lambda| \leq \varphi < \frac{\pi}{2}$ and $\gamma^2 \leq \mu < \infty$, we have $\left|\arg\left(\lambda\left(\mu + \lambda^2\right)^{-1/2}\right)\right| \leq \varphi < \frac{\pi}{2}$. Therefore,

$$\left|\arg\left(\alpha^{-1}\lambda\left(\mu + \lambda^2\right)^{-1/2}\right)\right| \leq \frac{\pi}{2} + \varphi < \pi. \tag{2.10}$$

If (2.9) is not true then there necessarily exist sequences μ_n and λ_n such that $\gamma^2 \leq \mu_n < \infty$, $|\arg\lambda_n| \leq \varphi$ and $\alpha^{-1}\lambda_n\left(\mu_n + \lambda_n^2\right)^{-1/2} + 1 \to 0$ or $\alpha^{-1}\lambda_n\left(\mu_n + \lambda_n^2\right)^{-1/2} \to -1$ and this contradicts with (2.10). Consequently, (2.9) holds. It means that $f(\mu)$ is a bounded function on $[\gamma^2, \infty)$, uniformly on λ, $|\arg\lambda| \leq \varphi < \frac{\pi}{2}$. Then, according to the remark at the beginning of the proof, $\exists C > 0$ such that

$$\left\| \left[I + \alpha^{-1}\lambda\left(A + \lambda^2 I\right)^{-1/2} \right]^{-1} \right\|_{B(H)} \leq \sup_{\gamma^2 \leq \mu < \infty} \left| 1 + \alpha^{-1}\lambda\left(\mu + \lambda^2\right)^{-1/2} \right|^{-1} \leq C \tag{2.11}$$

uniformly on λ, $|\arg\lambda| \leq \varphi < \frac{\pi}{2}$. Note that "*ess* sup"="sup" since $f(\mu)$ is a continuous function.

On the other hand, by the same remark at the beginning of the proof, for λ from the sector $|\arg\lambda| \leq \varphi < \frac{\pi}{2}$, we have

$$\left\| \left(A + \lambda^2 I\right)^{-1} \right\|_{B(H)} \leq \frac{C\varphi}{1 + |\lambda|^2}. \tag{2.12}$$

Similarly,

$$\left\| \left(A + \lambda^2 I\right)^{-1/2} \right\|_{B(H)} \leq \frac{C\varphi}{1 + |\lambda|}. \tag{2.13}$$

Then, from the representation

$$\left[\alpha\left(A + \lambda^2 I\right)^{1/2} + \lambda I\right]^{-1} = \frac{1}{\alpha}\left[I + \alpha^{-1}\lambda\left(A + \lambda^2 I\right)^{-1/2} \right]^{-1}\left(A + \lambda^2 I\right)^{-1/2}, \tag{2.14}$$

by virtue of (2.11) and (2.13), for λ from the sector $|\arg\lambda| \leq \varphi < \frac{\pi}{2}$, we have

$$\left\| \left[\alpha\left(A + \lambda^2 I\right)^{1/2} + \lambda I\right]^{-1} \right\|_{B(H)} \leq \frac{C}{1 + |\lambda|}, \quad \exists C > 0. \tag{2.15}$$

Now prove the estimate (2.8). According to (2.14), it is sufficient to show that

a) the operator $\left(A + \lambda^2 I\right)^{-1/2}$ for λ from the sector $|\arg\lambda| \leq \varphi < \frac{\pi}{2}$ is bounded from $(H(A), H)_{\frac{1}{2}+\frac{1}{2p}, p}$ into $(H(A), H)_{\frac{1}{2p}, p}$ and it holds the estimate

$$\left\| \left(A + \lambda^2 I\right)^{-1/2} \right\|_{B\left((H(A),H)_{\frac{1}{2}+\frac{1}{2p},p},\,(H(A),H)_{\frac{1}{2p},p}\right)} \leq C, \tag{2.16}$$

where $C > 0$ is some constant independent on λ;

b) the operator $\left(I+\alpha^{-1}\lambda\left(A+\lambda^2 I\right)^{-1/2}\right)^{-1}$, for λ from the sector $|\arg\lambda|\le\varphi<\frac{\pi}{2}$, is bounded from $(H(A),H)_{\frac{1}{2p},p}$ into $(H(A),H)_{\frac{1}{2p},p}$ and it holds the estimate

$$\left\|\left(I+\alpha^{-1}\lambda\left(A+\lambda^2 I\right)^{-1/2}\right)^{-1}\right\|_{B\left((H(A),H)_{\frac{1}{2p},p}\right)}\le C, \tag{2.17}$$

where $C>0$ is some constant independent on λ.

Note that a) was proved in [**6**]. Prove b). From the estimate (2.11) it follows that, for $|\arg\lambda|\le\varphi$, it also holds the estimate

$$\left\|\left(I+\alpha^{-1}\lambda\left(A+\lambda^2 I\right)^{-1/2}\right)^{-1}\right\|_{B(H(A))}\le C,\quad \exists C>0. \tag{2.18}$$

Then, according to the interpolation theorem [**20**, theorem 1.3.3/(a)] (see also [**22**, section 1.7.9]), from the estimates (2.11) and (2.18) it follows that, for λ from the sector $|\arg\lambda|\le\varphi<\frac{\pi}{2}$, the operator $\left[I+\alpha^{-1}\lambda\left(A+\lambda^2 I\right)^{-1/2}\right]^{-1}$ is bounded from $(H(A),H)_{\theta,p}$ into $(H(A),H)_{\theta,p}$, for any $\theta\in(0,1)$, and it holds the estimate

$$\left\|\left(I+\alpha^{-1}\lambda\left(A+\lambda^2 I\right)^{-1/2}\right)^{-1}\right\|_{B\left((H(A),H)_{\theta,p}\right)} \tag{2.19}$$

$$\le\left\|\left(I+\alpha^{-1}\lambda\left(A+\lambda^2 I\right)^{-1/2}\right)^{-1}\right\|_{B(H(A))}^{1-\theta}\left\|\left(I+\alpha^{-1}\lambda\left(A+\lambda^2 I\right)^{-1/2}\right)^{-1}\right\|_{B(H)}^{\theta}\le C.$$

Take $\theta=\frac{1}{2p}$ in (2.19). Then we get (2.17). So, from the representation (2.14), by virtue of the estimates (2.16) and (2.17), it follows that for λ from the sector $|\arg\lambda|\le\varphi<\frac{\pi}{2}$ the estimate (2.8) holds. Consequently, for λ from the sector $|\arg\lambda|\le\varphi$, the operator $A(\lambda)^{-1}$ is bounded from $\mathbb{H}$ into $(H(A),H)_{\frac{1}{2p},p}\dot{+}(H(A),H)_{\frac{1}{2p},p}$ and the estimate (2.7) holds. Then, from the equation (2.6), we have

$$\left(I+A(\lambda)^{-1}R(\lambda)\right)\begin{pmatrix}g_1\\ g_2\end{pmatrix}=A(\lambda)^{-1}\begin{pmatrix}f_1\\ f_2\end{pmatrix}. \tag{2.20}$$

We can now show that all the operators in the operator–matrix $A(\lambda)^{-1}R(\lambda)$, for sufficiently large $|\lambda|$ from the sector $|\arg\lambda|\le\varphi$, are bounded from $(H(A),H)_{\frac{1}{2p},p}$ into $(H(A),H)_{\frac{1}{2p},p}$. It is sufficient to show this for the operator

$$\left[\alpha\left(A+\lambda^2 I\right)^{1/2}+\lambda I\right]^{-1}\left[-\alpha\left(A+\lambda^2 I\right)^{1/2}+\lambda I\right]e^{-\left(A+\lambda^2 I\right)^{1/2}}.$$

From (2.15) it follows that for λ from the sector $|\arg\lambda| \le \varphi < \frac{\pi}{2}$ it holds the estimate

$$\left\| \left[\alpha\left(A + \lambda^2 I\right)^{1/2} + \lambda I \right]^{-1} \right\|_{B(H(A))} \le \frac{C}{1+|\lambda|}. \tag{2.21}$$

Then, by virtue of the interpolation theorem [**20**, theorem 1.3.3/(a)] (see also [**22**, section 1.7.9]), it follows from (2.15) and (2.21) that for λ from the sector $|\arg\lambda| \le \varphi < \frac{\pi}{2}$ it holds the estimate

$$\left\| \left[\alpha\left(A + \lambda^2 I\right)^{1/2} + \lambda I \right]^{-1} \right\|_{B\left((H(A),H)_{\frac{1}{2p},p}\right)} \le \frac{C}{1+|\lambda|}. \tag{2.22}$$

By virtue of [**22**, lemma 5.4.2/6], from the interpolation theorem [**20**, theorem 1.3.3/(a)] it also follows that, for $|\arg\lambda| \le \varphi < \frac{\pi}{2}$, the estimates

$$\left\| \left(A + \lambda^2 I\right)^{1/2} e^{-\left(A+\lambda^2 I\right)^{1/2}} \right\|_{B\left((H(A),H)_{\frac{1}{2p},p}\right)} \le Ce^{-\omega|\lambda|}, \quad \exists C, \omega > 0, \tag{2.23}$$

and

$$\left\| e^{-\left(A+\lambda^2 I\right)^{1/2}} \right\|_{B\left((H(A),H)_{\frac{1}{2p},p}\right)} \le Ce^{-\omega|\lambda|}, \quad \exists C, \omega > 0, \tag{2.24}$$

hold. Then, by virtue of the estimates (2.22)–(2.24), for λ from the sector $|\arg\lambda| \le \varphi < \frac{\pi}{2}$, we have

$$\left\| \left[\alpha\left(A + \lambda^2 I\right)^{1/2} + \lambda I \right]^{-1} \left[-\alpha\left(A + \lambda^2 I\right)^{1/2} + \lambda I \right] e^{-\left(A+\lambda^2 I\right)^{1/2}} \right\|_{B\left((H(A),H)_{\frac{1}{2p},p}\right)}$$
$$\le C\left((1+|\lambda|)^{-1} e^{-\omega|\lambda|} + e^{-\omega|\lambda|} \right) \le Ce^{-\omega|\lambda|}.$$

Consequently, for sufficiently large $|\lambda|$ from the sector $|\arg\lambda| \le \varphi < \frac{\pi}{2}$, the operator $A(\lambda)^{-1}R(\lambda)$ is bounded from $(H(A),H)_{\frac{1}{2p},p} \dotplus (H(A),H)_{\frac{1}{2p},p}$ into $(H(A),H)_{\frac{1}{2p},p} \dotplus (H(A),H)_{\frac{1}{2p},p}$ and it holds the estimate

$$\left\| A(\lambda)^{-1} R(\lambda) \right\|_{B\left((H(A),H)_{\frac{1}{2p},p} \dotplus (H(A),H)_{\frac{1}{2p},p}\right)} \le Ce^{-\omega|\lambda|} < 1. \tag{2.25}$$

Hence, according to the Neumann identity, for $|\arg\lambda| \le \varphi < \frac{\pi}{2}$ and sufficiently large $|\lambda|$,

$$\left(I + A(\lambda)^{-1}R(\lambda)\right)^{-1} = I + \sum_{k=1}^{\infty} \left(-A(\lambda)^{-1}R(\lambda)\right)^k, \tag{2.26}$$

where the series converges in the norm of the space of bounded operators in $(H(A),H)_{\frac{1}{2p},p} \dotplus (H(A),H)_{\frac{1}{2p},p}$. Then, from (2.20), for sufficiently large $|\lambda|$ from the sector $|\arg\lambda| \le \varphi < \frac{\pi}{2}$, we have

$$\begin{pmatrix} g_1 \\ g_2 \end{pmatrix} = \left(I + A(\lambda)^{-1}R(\lambda)\right)^{-1} A(\lambda)^{-1} \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}.$$

Consequently, using the formulas of $A(\lambda)^{-1}$ and $R(\lambda)$ and (2.26), for sufficiently large $|\lambda|$ from the sector $|\arg\lambda| \le \varphi$, the elements g_1 and g_2 can be represented in the form

$$g_k = (C_{k1}(\lambda) + R_{k1}(\lambda))f_1 + (C_{k2}(\lambda) + R_{k2}(\lambda))f_2, \quad k = 1, 2, \tag{2.27}$$

where $C_{11}(\lambda) = 0$, $C_{12}(\lambda) = I$, $C_{21}(\lambda) = \left[\alpha\left(A + \lambda^2 I\right)^{1/2} + \lambda I\right]^{-1}$, $C_{22}(\lambda) = 0$, and $R_{kj}(\lambda)$ are some bounded operators acting from $(H(A),H)_{\frac{1}{2}+\frac{1}{2p},p}$ into $(H(A),H)_{\frac{1}{2p},p}$. Furthermore, from the estimaties (2.7) and (2.25) it follows that, for $|\arg\lambda| \le \varphi$ and $|\lambda| \to \infty$,

$$\left\|R_{kj}(\lambda)\right\|_{B\left((H(A),H)_{\frac{1}{2}+\frac{1}{2p},p},\,(H(A),H)_{\frac{1}{2p},p}\right)} \le Ce^{-\omega|\lambda|}, \quad \exists C, \omega > 0. \tag{2.28}$$

From the representations of $A(\lambda)^{-1}$ and $R(\lambda)$ it also follows that, for sufficiently large $|\lambda|$ from the sector $|\arg\lambda| \le \varphi < \frac{\pi}{2}$, for the operators $R_{kj}(\lambda)$ we have

$$\left\|R_{kj}(\lambda)\right\|_{B(H)} \le Ce^{-\omega|\lambda|}, \quad \exists C, \omega > 0. \tag{2.29}$$

Substituting (2.27) into (2.4), we get

$$u(x) = \sum_{k=1}^{2} \Big\{ e^{-x\left(A+\lambda^2 I\right)^{1/2}} (C_{1k}(\lambda) + R_{1k}(\lambda)) + e^{-(1-x)\left(A+\lambda^2 I\right)^{1/2}} (C_{2k}(\lambda) + R_{2k}(\lambda)) \Big\} f_k. \tag{2.30}$$

In order to show the estimate (2.3), it is necessary to estimate, for sufficiently large $|\lambda|$ from the sector $|\arg\lambda| \le \varphi < \frac{\pi}{2}$, some finite number of integrals in the space $L_p((0,1);H)$. The integrand expressions are obtained from the functions $u(x)$, $u''(x)$, $Au(x)$, where $u(x)$ is determined by the equality (2.30). Here, [**22**, theorem 5.4.2/1 and lemma 5.4.2/6] and the estimates (2.11), (2.12), (2.22), (2.28), and (2.29) are essentially used. Estimate one of these integrals, for example, the integral

$$|\lambda|^2 \left(\int_0^1 \left\| e^{-(1-x)\left(A+\lambda^2 I\right)^{1/2}} C_{21}(\lambda) f_1 \right\|_H^p dx \right)^{1/p}.$$

By virtue of [**22**, theorem 5.4.2/1 and lemma 5.4.2/6] and the estimates (2.11), (2.12), and (2.19), for sufficiently large $|\lambda|$ from the sector $|\arg\lambda| \le \varphi < \frac{\pi}{2}$, we have

$$|\lambda|^2 \left(\int_0^1 \left\| e^{-(1-x)\left(A+\lambda^2 I\right)^{1/2}} C_{21}(\lambda) f_1 \right\|_H^p dx \right)^{1/p}$$

$$= |\lambda|^2 \left(\int_0^1 \left\| e^{-(1-x)\left(A+\lambda^2 I\right)^{1/2}} \left[\alpha \left(A + \lambda^2 I \right)^{1/2} + \lambda I \right]^{-1} f_1 \right\|_H^p dx \right)^{1/p}$$

$$= |\lambda|^2 \left(\int_0^1 \left\| e^{-(1-x)\left(A+\lambda^2 I\right)^{1/2}} \frac{1}{\alpha} \left(A + \lambda^2 I \right)^{-1/2} \left[I + \alpha^{-1}\lambda \left(A + \lambda^2 I \right)^{-1/2} \right]^{-1} f_1 \right\|_H^p dx \right)^{1/p}$$

$$\le \frac{1}{|\alpha|} |\lambda|^2 \left\| \left(A + \lambda^2 I \right)^{-1} \right\|_{B(H)}$$

$$\times \left(\int_0^1 \left\| \left(A + \lambda^2 I \right)^{1/2} e^{-(1-x)\left(A+\lambda^2 I\right)^{1/2}} \left[I + \alpha^{-1}\lambda \left(A + \lambda^2 I \right)^{-1/2} \right]^{-1} f_1 \right\|_H^p dx \right)^{1/p}$$

$$\le \frac{1}{|\alpha|} |\lambda|^2 \frac{1}{1+|\lambda|^2} \left(\left\| \left[I + \alpha^{-1}\lambda \left(A + \lambda^2 I \right)^{-1/2} \right]^{-1} f_1 \right\|_{(H(A),H)_{\frac{1}{2}+\frac{1}{2p},p}} \right.$$

$$\left. + |\lambda|^{2\left(\frac{1}{2}-\frac{1}{2p}\right)} \left\| \left[I + \alpha^{-1}\lambda \left(A + \lambda^2 I \right)^{-1/2} \right]^{-1} f_1 \right\|_H \right)$$

$$\le C \left(\|f_1\|_{(H(A),H)_{\frac{1}{2}+\frac{1}{2p},p}} + |\lambda|^{1-\frac{1}{p}} \|f_1\|_H \right).$$

□

3 - Nonhomogeneous equations

Let us now consider our full boundary value problem in a separable Hilbert space H, i.e.,

$$L(\lambda, D)u := \lambda^2 u(x) - u''(x) + Au(x) = f(x), \quad x \in (0,1), \tag{3.1}$$

$$\begin{aligned} L_1(\lambda)u &:= \alpha u'(1) + \lambda u(1) = f_1, \\ L_2 u &:= u(0) = f_2. \end{aligned} \tag{3.2}$$

Theorem 2. *Let conditions of Theorem* 1 *be satisfied.*

Then the operator $\mathbb{L}(\lambda) : u \to \mathbb{L}(\lambda)u := (L(\lambda, D)u, L_1(\lambda)u,\ L_2 u)$, *for sufficiently large* $|\lambda|$ *from the sector* $|\arg\lambda| \leq \varphi < \frac{\pi}{2}$, *is an isomorphism from* $W_p^2((0,1); H(A), H)$ *onto* $L_p((0,1); H) \dotplus (H(A), H)_{\theta_1,p} \dotplus (H(A), H)_{\theta_2,p}$, *where* $\theta_1 = \frac{1}{2} + \frac{1}{2p}$, $\theta_2 = \frac{1}{2p}$, $p \in (1, \infty)$, *and, for these* λ, *the following estimate is valid for the solution of the problem* (3.1), (3.2)

$$\begin{aligned} &|\lambda|^2 \|u\|_{L_p((0,1);H)} + \|u''\|_{L_p((0,1);H)} + \|Au\|_{L_p((0,1);H)} \\ &\leq C\left[|\lambda| \|f\|_{L_p((0,1);H)} + \sum_{k=1}^{2}\left(\|f_k\|_{(H(A),H)_{\theta_k,p}} + |\lambda|^{2(1-\theta_k)}\|f_k\|_H\right)\right]. \end{aligned} \tag{3.3}$$

Proof. The injectivity of the mapping $\mathbb{L}(\lambda)$ follows from theorem 1, since the homogeneous boundary value problem corresponding to the boundary value problem (3.1), (3.2), for sufficiently large $|\lambda|$ from the sector $|\arg\lambda| \leq \varphi < \frac{\pi}{2}$, has only a trivial solution. Thus, it is sufficient to show that $\mathbb{L}(\lambda)$ is surjective, i.e., for any $f \in L_p((0,1); H)$ and any $f_1 \in (H(A), H)_{\theta_1,p}$, $f_2 \in (H(A), H)_{\theta_2,p}$, there exists a solution of problem (3.1), (3.2) belonging to $W_p^2((0,1); H(A), H)$. Define $\tilde{f}(x) := f(x)$ if $x \in (0,1)$ and $\tilde{f}(x) = 0$ if $x \notin (0,1)$.

A solution of the problem (3.1), (3.2) can be represented in the form of the sum $u(x) = u_1(x) + u_2(x)$, where $u_1(x)$ is the restriction on $(0,1)$ of the solution $\tilde{u}_1(x)$ of the equation

$$L(\lambda, D)\tilde{u}_1(x) = \tilde{f}(x), \quad x \in \mathbb{R} = (-\infty, +\infty), \tag{3.4}$$

and $u_2(x)$ is a solution of the problem

$$L(\lambda, D)u_2(x) = 0,\ x \in (0,1),\ L_1(\lambda)u_2 = f_1 - L_1(\lambda)u_1,\ L_2u_2 = f_2 - L_2u_1. \tag{3.5}$$

It is obvious that a solution of the equation (3.4) is given by the following formula

$$\tilde{u}_1(x) = \frac{1}{2\pi}\int_R e^{i\mu x} L(\lambda, i\mu)^{-1} F\tilde{f}(\mu)d\mu$$

where $F\tilde{f}$ is the Fourier transform of the function $\tilde{f}(x)$ and $L(\lambda, \sigma) = -\sigma^2 I + A + \lambda^2 I$. Further, it is proved in [**22**, theorem 5.4.4] that the solution $\tilde{u}_1$ belongs to $W_p^2(R; H(A), H)$ and, for the solution, it holds the estimate

$$|\lambda|^2 \|\tilde{u}_1\|_{L_p(\mathbb{R};H)} + \|\tilde{u}_1\|_{W_p^2(\mathbb{R};H(A),H)} \leq C\left\|\tilde{f}\right\|_{L_p(\mathbb{R};H)}, \quad |\arg\lambda| \leq \varphi. \tag{3.6}$$

Therefore, $u_1 \in W_p^2((0,1); H(A), H)$ and, from (3.6), for $|\arg\lambda| \leq \varphi$, we have

$$|\lambda|^2 \|u_1\|_{L_p((0,1);H)} + \|u_1\|_{W_p^2((0,1);H(A),H)} \leq C\|f\|_{L_p((0,1);H)}. \tag{3.7}$$

By virtue of [**22**, theorem 1.7.7/1] (see also [**20**, theorem 1.8.2]) and the inequality (3.7), we have

$$u_1^{(s)}(x_0) \in (H(A),H)_{\frac{s}{2}+\frac{1}{2p},p}, \quad \forall x_0 \in [0,1],\ s=0,1.$$

Hence, $L_1(\lambda)u_1 \in (H(A),H)_{\theta_1,p}$ since $(H(A),H)_{\frac{1}{2p},p} \subset (H(A),H)_{\frac{1}{2}+\frac{1}{2p},p}$, and $L_2u \in (H(A),H)_{\theta_2,p}$. Thus, by virtue of theorem 1, the problem (3.5) has a unique solution $u_2(x)$ that belongs to $W_p^2((0,1);H(A),H)$, for sufficiently large $|\lambda|$ from the sector $|\arg\lambda| \le \varphi$. Furthermore, for the solution of the problem (3.5), for $|\arg\lambda| \le \varphi$, $|\lambda| \to \infty$, we have

$$\begin{aligned}(3.8)\qquad & |\lambda|^2\|u_2\|_{L_p((0,1);H)}+\|u_2''\|_{L_p((0,1);H)}+\|Au_2\|_{L_p((0,1);H)} \\ &\le C\Big(\|f_1-L_1(\lambda)u_1\|_{(H(A),H)_{\frac{1}{2}+\frac{1}{2p},p}}+\|f_2-L_2u_1\|_{(H(A),H)_{\frac{1}{2p},p}} \\ &\quad +|\lambda|^{2(1-\theta_1)}\|f_1-L_1(\lambda)u_1\|_H+|\lambda|^{2(1-\theta_2)}\|f_2-L_2u_1\|_H\Big) \\ &\le C\Big(\|f_1\|_{(H(A),H)_{\frac{1}{2}+\frac{1}{2p},p}}+\|u_1'(1)\|_{(H(A),H)_{\frac{1}{2}+\frac{1}{2p},p}} \\ &\quad +|\lambda|\|u_1(1)\|_{(H(A),H)_{\frac{1}{2}+\frac{1}{2p},p}}+\|f_2\|_{(H(A),H)_{\frac{1}{2p},p}}+\|u_1(0)\|_{(H(A),H)_{\frac{1}{2p},p}} \\ &\quad +|\lambda|^{2\left(\frac{1}{2}-\frac{1}{2p}\right)}\Big(\|f_1\|_H+\|u_1'(1)\|_H+|\lambda|\|u_1(1)\|_H\Big) \\ &\quad +|\lambda|^{2\left(1-\frac{1}{2p}\right)}\Big(\|f_2\|_H+\|u_1(0)\|_H\Big)\Big).\end{aligned}$$

By virtue of [**22**, theorem 1.7.7/1] (see also [**20**, theorem 1.8.2]) and (3.7), for any $x_0 \in [0,1]$, we have

$$(3.9)\qquad \left\|u_1^{(s)}(x_0)\right\|_{(H(A),H)_{\frac{s}{2}+\frac{1}{2p},p}} \le C\|u_1\|_{W_p^2((0,1);H(A),H)} \le C\|f\|_{L_p((0,1);H)}, \quad s=0,1.$$

By virtue of [**22**, theorem 1.7.7/2], for any complex number λ and any $u \in W_p^2((0,1);H)$, $s=0,1$,

$$(3.10)\qquad |\lambda|^{2-s}\|u^{(s)}(x_0)\|_H \le C\Big(|\lambda|^{\frac{1}{p}}\|u\|_{W_p^2((0,1);H)}+|\lambda|^{2+\frac{1}{p}}\|u\|_{L_p((0,1);H)}\Big).$$

Dividing (3.10) by $|\lambda|^{\frac{1}{p}}$, for $\lambda \in \mathbb{C}$, $u \in W_p^2((0,1);H)$, $s=0,1$, we have

$$(3.11)\qquad |\lambda|^{2-s-\frac{1}{p}}\|u^{(s)}(x_0)\|_H \le C\Big(\|u\|_{W_p^2((0,1);H)}+|\lambda|^2\|u\|_{L_p((0,1);H)}\Big),\ s=0,1.$$

Then, from (3.7) and (3.11), for $|\arg\lambda| \le \varphi < \dfrac{\pi}{2}$, we have

$$\begin{aligned}(3.12)\qquad & |\lambda|^{2\left(1-\frac{s}{2}-\frac{1}{2p}\right)}\left\|u_1^{(s)}(x_0)\right\|_H \le C\Big(\|u_1\|_{W_p^2((0,1);H)}+|\lambda|^2\|u_1\|_{L_p((0,1);H)}\Big) \\ &\le C\Big(\|u_1\|_{W_p^2((0,1);H(A),H)}+|\lambda|^2\|u_1\|_{L_p((0,1);H)}\Big) \le C\|f\|_{L_p((0,1);H)}, \quad s=0,1.\end{aligned}$$

According to the estimates (3.9) and (3.12), from (3.8), for $|\arg \lambda| \leq \varphi < \frac{\pi}{2}$, $|\lambda| \to \infty$, we have

$$\text{(3.13)} \qquad |\lambda|^2 \|u_2\|_{L_p((0,1);H)} + \|u_2''\|_{L_p((0,1);H)} + \|Au_2\|_{L_p((0,1);H)}$$
$$\leq C\left[|\lambda| \|f\|_{L_p((0,1);H)} + \sum_{k=1}^{2} \left(\|f_k\|_{(H(A),H)_{\theta_k,p}} + |\lambda|^{2(1-\theta_k)} \|f_k\|_H \right) \right],$$

where $\theta_1 = \frac{1}{2} + \frac{1}{2p}$, $\theta_2 = \frac{1}{2p}$.

Then, from (3.7) and (3.13) it follows (3.3) since $u = u_1 + u_2$. The theorem is proved. □

4 - Asymptotic formulae for eigenvalues of the homogeneous problem (1.1), (1.2) for some special case

In a separable Hilbert space H, consider the following boundary value problem on $[0, 1]$ for a second order elliptic differential-operator equation

$$\text{(4.1)} \qquad -u''(x) + Au(x) = \lambda^2 u(x), \quad x \in (0, 1),$$

$$\text{(4.2)} \qquad \begin{aligned} &u'(1) + \lambda u(1) = 0, \\ &u(0) = 0, \end{aligned}$$

where λ is the spectral parameter, A is a linear unbounded, selfadjoint, positive-definite operator in H and A^{-1} is a compact operator in H. This problem is a homogeneous problem corresponding to (1.1), (1.2) if one replaces λ by $i\lambda$ and takes $\alpha = i$ in (1.1), (1.2).

Lemma 1. *The eigenvalues of the boundary value problem* (4.1), (4.2) *are real.*

Proof. Denote the eigenvectors of the operator A, corresponding to the eigenvalues $\mu_k \to +\infty$, by φ_k, where $k = 1, 2, 3, ...$ It is known that $\{\varphi_k\}$ forms an orthonormal basis in H. Then, from the expansion $u(x) = \sum_{k=1}^{\infty} (u(x), \varphi_k)_H \varphi_k$, we get, for the Fourier coefficients $u_k(x) = (u(x), \varphi_k)_H$, the following spectral problem

$$\text{(4.3)} \qquad -u_k''(x) + \mu_k u_k(x) = \lambda^2 u_k(x), \quad x \in (0, 1),$$

$$\text{(4.4)} \qquad \begin{aligned} &u_k'(1) + \lambda u_k(1) = 0, \\ &u_k(0) = 0. \end{aligned}$$

Thus, the study of the eigenvalues of the boundary value problem (4.1), (4.2) is reduced to the study of the eigenvalues of the boundary value problem (4.3), (4.4) for different natural k. The spectrum of the boundary value problem (4.3), (4.4) consists of those λ for which the problem (4.3), (4.4) has a non-trivial solution $u_k(x)$, at least for one k. The number $\lambda = \pm\sqrt{\mu_k}$ cannot be an eigenvalue of the problem (4.3), (4.4), for sufficiently large k, since if $\lambda = \pm\sqrt{\mu_k}, \mu_k \neq 1$ then the problem (4.3), (4.4) has only a trivial solution.

Let λ be an eigenvalue of the boundary value problem (4.3), (4.4) and let $u_k(x, \lambda)$ be the corresponding eigenfunction. Multiply the both sides of equality (4.3) by the function $\overline{u_k(x, \lambda)}$ and integrate the obtained identity with respect to x from 0 to 1:

$$-\int_0^1 u_k''(x,\lambda)\overline{u_k(x,\lambda)}dx + \mu_k \int_0^1 |u_k(x,\lambda)|^2 dx = \lambda^2 \int_0^1 |u_k(x,\lambda)|^2 dx. \tag{4.5}$$

Using the formula of integration by parts and the boundary conditions (4.4), we get

$$\begin{aligned}\int_0^1 u_k''(x,\lambda)\overline{u_k(x,\lambda)}dx &= \overline{u_k(x,\lambda)}u_k'(x,\lambda)\Big|_0^1 - \int_0^1 u_k'(x,\lambda)\overline{u_k'(x,\lambda)}dx \\ &= \overline{u_k(1,\lambda)}u_k'(1,\lambda) - \overline{u_k(0,\lambda)}u_k'(0,\lambda) - \int_0^1 \left|u_k'(x,\lambda)\right|^2 dx \\ &= -\lambda|u_k(1,\lambda)|^2 - \int_0^1 \left|u_k'(x,\lambda)\right|^2 dx.\end{aligned}$$

From here and (4.5), it follows that

$$\lambda^2 \int_0^1 |u_k(x,\lambda)|^2 dx - \lambda|u_k(1,\lambda)|^2 - \mu_k \int_0^1 |u_k(x,\lambda)|^2 dx - \int_0^1 \left|u_k'(x,\lambda)\right|^2 dx = 0. \tag{4.6}$$

Denote

$$a_k(\lambda) = \int_0^1 |u_k(x,\lambda)|^2 dx, \qquad b_k(\lambda) = -|u_k(1,\lambda)|^2,$$

$$c_k(\lambda) = -\mu_k \int_0^1 |u_k(x,\lambda)|^2 dx - \int_0^1 \left|u_k'(x,\lambda)\right|^2 dx.$$

Then, we can rewrite the equation (4.6) in the form

$$a_k(\lambda)\lambda^2 + b_k(\lambda)\lambda + c_k(\lambda) = 0. \tag{4.7}$$

Since, for each k, $a_k(\lambda) > 0$, $b_k(\lambda) \leq 0$, $c_k(\lambda) < 0$ then $b_k^2(\lambda) - 4a_k(\lambda)c_k(\lambda) > 0$.

Consequently, for each k, the equation (4.7) has only real roots λ. Lemma 1 is proved. □

Theorem 3. *Let A be a selfadjoint positive-definite operator in a separable Hilbert space H and let A^{-1} be a compact operator in H.*

Then the boundary value problem (4.1), (4.2) *has the following three series of eigenvalues:*[1]

$$\lambda_k^{(1)} \sim -\sqrt{\frac{\mu_k}{2}}, \quad k \to +\infty,$$

and

$$\lambda_n^{(2,k)} = \sqrt{\mu_k + \gamma_n}\ , \quad \lambda_n^{(3,k)} = -\sqrt{\mu_k + \delta_n},$$

where $\mu_k \to +\infty$ are the eigenvalues of the operator A; $\gamma_n \sim n^2\pi^2$, $\delta_n \sim n^2\pi^2$ when $n \to +\infty$.

Proof. The general solution of the ordinary differential equation (4.3) has the form

$$u_k(x,\lambda) = C_1 e^{-x\sqrt{\mu_k-\lambda^2}} + C_2 e^{-(1-x)\sqrt{\mu_k-\lambda^2}}, \tag{4.8}$$

where C_i, $i = 1, 2$ are arbitrary constants. Substituting (4.8) into (4.4), we get a system with respect to C_i, $i = 1, 2$, whose determinant is of the form

$$K(\lambda) = \left(\lambda - \sqrt{\mu_k - \lambda^2}\right) e^{-2\sqrt{\mu_k-\lambda^2}} - \left(\lambda + \sqrt{\mu_k - \lambda^2}\right).$$

Consequently, the eigenvalues of the boundary value problem (4.3), (4.4) consist of those real $\lambda \neq \pm\sqrt{\mu_k}$ which satisfy the following equation, at least for one μ_k,

$$\left(\lambda - \sqrt{\mu_k - \lambda^2}\right) e^{-2\sqrt{\mu_k-\lambda^2}} - \left(\lambda + \sqrt{\mu_k - \lambda^2}\right) = 0. \tag{4.9}$$

Therefore, the eigenvalues of the boundary value problem (4.3), (4.4) or, the same, of (4.1), (4.2), are zeros of the function (with respect to λ, $\lambda \neq \pm\sqrt{\mu_k}$), standing at the left-hand side of the equation (4.9), for each k. Rewrite the equation (4.9) in the form

$$\lambda \sinh\left(\sqrt{\mu_k - \lambda^2}\right) + \sqrt{\mu_k - \lambda^2} \cosh\left(\sqrt{\mu_k - \lambda^2}\right) = 0. \tag{4.10}$$

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[1] By asymptotic behavior $\lambda_n \sim f(n), n \to +\infty$, we mean the standard concept, i.e., $\lim\limits_{n\to+\infty} \dfrac{\lambda_n}{f(n)} = 1$.

Thus, the eigenvalues of the problem (4.1), (4.2) consist of those real $\lambda \neq \pm\sqrt{\mu_k}$ which satisfy the equation (4.10), at least for one μ_k.

Find the eigenvalues λ for which $\lambda^2 < \mu_k$. Assume $\sqrt{\mu_k - \lambda^2} = y$ $(0 < y \leq \sqrt{\mu_k})$. Then $\lambda = \pm\sqrt{\mu_k - y^2}$. First, take $\lambda = \sqrt{\mu_k - y^2}$ in the equation (4.10). Then, the equation (4.10) takes the form

$$\sqrt{\mu_k - y^2}\sinh(y) + y\cosh(y) = 0, \qquad 0 < y \leq \sqrt{\mu_k}. \tag{4.11}$$

Consider the function $f_k(y) = \sqrt{\mu_k - y^2}\sinh(y) + y\cosh(y)$, $y \in (0, \sqrt{\mu_k}\,]$. Obviously, for each fixed k and for all $y \in (0, \sqrt{\mu_k}\,]$, $f_k(y) > 0$. Therefore, the equation (4.11) has no solutions on the interval $(0, \sqrt{\mu_k}\,]$, for any k.

Now, in the equation (4.10), take $\lambda = -\sqrt{\mu_k - y^2}$. In this case, the equation (4.10) is equivalent to the equation

$$\sqrt{\mu_k - y^2} - y\coth(y) = 0, \quad y \in (0, \sqrt{\mu_k}\,]. \tag{4.12}$$

Consider the function $\varphi_k(y) = \sqrt{\mu_k - y^2} - y\coth(y)$, $y \in (0, \sqrt{\mu_k}\,]$. The derivative $\varphi_k'(y) = -\dfrac{y}{\sqrt{\mu_k - y^2}} - \dfrac{\sinh(2y) - 2y}{2\sinh^2(y)} < 0$, for $y \in (0, \sqrt{\mu_k}\,]$, since $\sinh(2y) > 2y$, for $y > 0$. This means that $\varphi_k(y)$ strongly monotonically decreases on $(0, \sqrt{\mu_k}\,]$, for each k. Obviously, $\varphi_k\left(\sqrt{\dfrac{\mu_k}{2}}\right) = \sqrt{\dfrac{\mu_k}{2}}\left(1 - \coth\left(\sqrt{\dfrac{\mu_k}{2}}\right)\right) < 0$, for each k. On the other hand, some simple calculations show that

$$\varphi_k\left(\sqrt{\frac{\mu_k}{2} - \frac{1}{\mu_k}}\right) = \sqrt{\frac{\mu_k}{2} - \frac{1}{\mu_k}}\left(\sqrt{1 + \frac{4}{\mu_k^2 - 2}} - \left(1 + \frac{2}{e^{2\sqrt{\frac{\mu_k}{2} - \frac{1}{\mu_k}}} - 1}\right)\right) > 0,$$

starting with some k, since, for any fixed number m, $\lim\limits_{x \to +\infty} \dfrac{e^x}{x^m} = +\infty$. Therefore, the equation (4.12), starting with some k, has exactly one zero y_k and it belongs to the interval $\left(\sqrt{\dfrac{\mu_k}{2} - \dfrac{1}{\mu_k}}, \sqrt{\dfrac{\mu_k}{2}}\right)$, i.e., $y_k \sim \sqrt{\dfrac{\mu_k}{2}}$. Hence, for the first series of the eigenvalues $\lambda = -\sqrt{\mu_k - y^2}$ of the problem (4.1), (4.2), for $\lambda^2 < \mu_k$, we get the asymptotic formula

$$\lambda_k^{(1)} \sim -\sqrt{\frac{\mu_k}{2}}, \quad k \to +\infty.$$

Now, let us study the eigenvalues λ of the problem (4.1), (4.2) for which $\lambda^2 > \mu_k$. Set $z = \sqrt{\lambda^2 - \mu_k}$ $(0 < z < +\infty)$. Then $\sqrt{\mu_k - \lambda^2} = iz$, $\sinh(\sqrt{\mu_k - \lambda^2}) = \sinh(iz) = i\sin z$, $\cosh(\sqrt{\mu_k - \lambda^2}) = \cosh(iz) = \cos z$, $\lambda^2 = z^2 + \mu_k$, $\lambda = \pm\sqrt{z^2 + \mu_k}$.

First, we take $\lambda = \sqrt{z^2 + \mu_k}$ in the equation (4.10). Then the equation (4.10) takes the form

$$\sqrt{z^2 + \mu_k}\sin z + z\cos z = 0, \quad z \in (0, +\infty). \tag{4.13}$$

Obviously, $z \neq n\pi,\ n = 1, 2, \ldots$. Then the equation (4.13) is equivalent to the equation

$$1 + \frac{z}{\sqrt{z^2 + \mu_k}}\cot z = 0, \quad z \in (0, +\infty), \quad z \neq n\pi, \quad n = 1, 2, \ldots. \tag{4.14}$$

Consider the function $F_k(z) = 1 + \dfrac{z}{\sqrt{z^2+\mu_k}}\cot z$. Since, at each interval $((n-1)\pi, n\pi)$, $n = 1, 2, \ldots$, the function $F_k(z)$ gets the values from $-\infty$ to $+\infty$ and its derivative

$$F_k'(z) = \frac{\mu_k(\sin 2z - 2z) - 2z^3}{2(z^2 + \mu_k)^{3/2}\sin^2 z} < 0$$

then, therein, for each k, the function $F_k(z)$ has only one zero $z_n^{(k)}$: $(n-1)\pi < z_n^{(k)} < n\pi,\ n = 1, 2, \ldots$. Find the asymptotic formula for $z_n^{(k)}$, for each k, when $n \to +\infty$.

From (4.14), we have

$$\cot z = -\frac{\sqrt{z^2 + \mu_k}}{z}, \quad z \in (0, +\infty), \quad z \neq n\pi,\ n = 1, 2, \ldots.$$

Denote $q_k(z) = -\dfrac{\sqrt{z^2 + \mu_k}}{z}$, $z \in (0, +\infty)$. Obviously, for each k, $q_k(z) < 0$, $q_k'(z) = \dfrac{\mu_k}{z^2\sqrt{z^2 + \mu_k}} > 0$, and

$$q_k''(z) = -\mu_k\frac{2\mu_k + 3z^2}{z^3(z^2 + \mu_k)^{3/2}} < 0,$$

i.e., $q_k(z)$, for each k, is a negative, increasing, concave up function. Moreover, $\lim\limits_{z\to+\infty} q_k(z) = -1$, i.e., the straight line $z = -1$ is a horizontal asymptote of the function $q_k(z)$, for each k, and $\lim\limits_{z\to 0+} q_k(z) = -\infty$. On the other hand, the points $z_n^{(k)}$, for each k, are the abscissas of the intersection points of $q_k(z)$ and the branches of the function $\cot z$, $z > 0$. Then, $z_n^{(k)}$, for each k, approach the abscissas of the intersection points of the branches of the function $\cot z$ and the straight line $z = -1$ when natural $n \to +\infty$, i.e., $z_n^{(k)}$, for each k, are the approximate solutions of the equation $\cot z = -1$ on $((n-1)\pi, n\pi)$. So, for each k,

$$z_n^{(k)} \sim \operatorname{arccot}(-1) + (n-1)\pi = \frac{3\pi}{4} + (n-1)\pi = \left(n - \frac{1}{4}\right)\pi, \ \ n \to +\infty.$$

Since $\lambda = \sqrt{z^2 + \mu_k}$, we get that

$$\lambda_n^{(2,k)} = \sqrt{\mu_k + (z_n^{(k)})^2} = \sqrt{\mu_k + \gamma_n},$$

where $\gamma_n \sim (n - \frac{1}{4})^2 \pi^2$, i.e., $\gamma_n \sim n^2\pi^2$, $n \to +\infty$.

Now, take $\lambda = -\sqrt{z^2 + \mu_k}$ in the equation (4.10). Then the equation (4.10) takes the form

$$\sqrt{z^2 + \mu_k} \sin z - z \cos z = 0, \quad z \in (0, +\infty). \tag{4.15}$$

Obviuosly $z \neq n\pi,\ n = 1, 2, \ldots$. Then the equation (4.15) is equivalent to the equation

$$1 - \frac{z}{\sqrt{z^2 + \mu_k}} \cot z = 0, \quad z \in (0, +\infty), z \neq n\pi,\ n = 1, 2, \ldots . \tag{4.16}$$

In the same way, as the equation (4.14), investigate the equation (4.16) and show that the last series of the eigenvalues of the boundary value problem (4.1), (4.2) has the following representation

$$\lambda_n^{(3,k)} = -\sqrt{\mu_k + \delta_n}\ ,$$

where $\delta_n \sim n^2\pi^2$, $n \to +\infty$. The theorem is proved. □

5 - Application of abstract results to elliptic partial differential equations

Let us consider a boundary value problem with a parameter for an elliptic partial differential equation of the second order in the square $[0,1] \times [0,1]$

$$L(\lambda, D_x, D_y)u := \lambda^2 u(x,y) - D_x^2 u(x,y) - D_y\big(a(y)D_y u(x,y)\big) = f(x,y), \tag{5.1}$$

$$\begin{aligned} L_1(\lambda)u &:= \alpha D_x u(1,y) + \lambda u(1,y) = f_1(y), \quad y \in [0,1], \\ L_2 u &:= u(0,y) = f_2(y), \quad y \in [0,1]\,, \end{aligned} \tag{5.2}$$

$$u(x,0) = u(x,1) = 0, \ \ x \in [0,1], \tag{5.3}$$

where $D_x := \dfrac{\partial}{\partial x}$, $D_y := \dfrac{\partial}{\partial y}$.

Denote the interpolation space of Sobolev spaces by

$$B^s_{q,p}(0,1) := \Big(W_q^{s_0}(0,1), W_q^{s_1}(0,1)\Big)_{\theta,p},$$

where $0 \leq s_0, s_1$ are integers, $0 < \theta < 1$, $1 < q < \infty$, $1 \leq p \leq \infty$ and

$s = (1-\theta)s_0 + \theta s_1$. Set

$$W_p^s(0,1) := B_{p,p}^s(0,1) := \Big(W_p^{s_0}(0,1), W_p^{s_1}(0,1)\Big)_{\theta,p},$$

if $0 < s \neq$ integer.

Theorem 4. *Let the following conditions be fulfilled:*

1. $a(\cdot) \in C^1[0,1]$, $a(y) > 0$ *for* $y \in [0,1]$;
2. $\alpha \neq 0$ *is a complex number with* $|\arg \alpha| \leq \dfrac{\pi}{2}$.

Then the operator $\mathbb{L}(\lambda) : u \to \mathbb{L}(\lambda)u := \big(L\big(\lambda, D_x, D_y\big)u, L_1(\lambda)u, L_2u\big)$, *for* $|\arg\lambda| \leq \varphi < \dfrac{\pi}{2}$ *and sufficiently large* $|\lambda|$, *is an isomorphism from* $W_p^2\big((0,1); W_2^2(0,1), L_2(0,1)\big)$, $1 < p < \infty$, *onto*

$$L_p((0,1); L_2(0,1)) \dotplus B_{2,p,*}^{1-\frac{1}{p}}(0,1) \dotplus B_{2,p,*}^{2-\frac{1}{p}}(0,1),$$

where

$$B_{2,p,_*}^{1-\frac{1}{p}}(0,1) := \begin{cases} B_{2,p}^{1-\frac{1}{p}}(0,1), \;\; 1 < p < 2, \\ W_2^{\frac{1}{2}}\left((0,1);\; \int\limits_0^1 (\min\{x, 1-x\})^{-1}|u(x)|^2 dx < \infty\right), \;\; p = 2, \\ B_{2,p}^{1-\frac{1}{p}}((0,1); u(0) = u(1) = 0), \;\; p > 2, \end{cases}$$

$$B_{2,p,_*}^{2-\frac{1}{p}}(0,1) := B_{2,p}^{2-\frac{1}{p}}((0,1); u(0) = u(1) = 0),$$

and, for these λ, *for the solution of the problem* (5.1)–(5.3) *it holds the following estimate*

$$\begin{aligned} (5.4) \qquad & |\lambda|^2 \|u(x,y)\|_{L_p((0,1),L_2(0,1))} + \big\|D_x^2 u(x,y)\big\|_{L_p((0,1);L_2(0,1))} \\ & + \big\|D_y\big(a(y) D_y u(x,y)\big)\big\|_{L_p((0,1);L_2(0,1))} \\ & \leq C\Bigg[|\lambda| \|f(x,y)\|_{L_p((0,1);L_2(0,1))} + \|f_1(y)\|_{B_{2,p}^{1-\frac{1}{p}}(0,1)} \\ & + \|f_2(y)\|_{B_{2,p}^{2-\frac{1}{p}}(0,1)} + |\lambda|^{1-\frac{1}{p}} \|f_1(y)\|_{L_2(0,1)} + |\lambda|^{2-\frac{1}{p}} \|f_2(y)\|_{L_2(0,1)}\Bigg]. \end{aligned}$$

Proof. In the space $H = L_2(0,1)$, consider an operator A which is defined by the following equalities

$$D(A) := W_2^2((0,1), u(0) = u(1) = 0), \;\; Au := (-a(y)u'(y))'.$$

Then, we can rewrite the problem (5.1)–(5.3) in the operator form (3.1), (3.2) and apply Theorem 2. From condition 1 it follows that the operator A is selfadjoint, positive-definite in $H = L_2(0,1)$, i.e., the only thing remains is to write down explicitly the interpolation spaces $(H(A),H)_{\theta_k,p}$, where $\theta_1 = \dfrac{1}{2} + \dfrac{1}{2p}$ and $\theta_2 = \dfrac{1}{2p}$.

By virtue of [**20**, theorem 4.3.3],

$$(H(A),H)_{\frac{1}{2}+\frac{1}{2p},p} = \left(W_2^2((0,1);u(0)=u(1)=0),\ L_2(0,1)\right)_{\frac{1}{2}+\frac{1}{2p},p}$$

$$= \begin{cases} B_{2,p}^{1-\frac{1}{p}}(0,1), & 1<p<2, \\ W_2^{\frac{1}{2}}\left((0,1);\ \int\limits_0^1 (\min\{x,1-x\})^{-1}|u(x)|^2 dx < \infty\right), & p=2, \\ B_{2,p}^{1-\frac{1}{p}}((0,1);u(0)=u(1)=0), & p>2, \end{cases}$$

i.e., $(H(A),H)_{\frac{1}{2}+\frac{1}{2p},p} = B_{2,p,*}^{1-\frac{1}{p}}(0,1)$ and

$$(H(A),H)_{\frac{1}{2p},p} = \left(W_2^2((0,1);u(0)=u(1)=0),\ L_2(0,1)\right)_{\frac{1}{2p},p}$$
$$= B_{2,p}^{2-\frac{1}{p}}((0,1);u(0)=u(1)=0) = B_{2,p,*}^{2-\frac{1}{p}}(0,1).$$

The theorem is proved. □

Consider now, in the square $[0,1]\times[0,1]$, the eigenvalue problem

(5.5) $$-\frac{\partial^2 v(x,y)}{\partial x^2} + \frac{\partial^4 v(x,y)}{\partial y^4} + \omega v(x,y) = \lambda^2 v(x,y),$$

(5.6) $$\frac{\partial v(1,y)}{\partial x} + \lambda v(1,y) = 0,\qquad v(0,y) = 0,$$

(5.7) $$\frac{\partial^j v(x,0)}{\partial y^j} = \frac{\partial^j v(x,1)}{\partial y^j},\quad j=0,1,2,3,$$

where $\omega > 0$ is some number. Rewrite the problem (5.5)–(5.7) in the operator form

$$-u''(x) + Au(x) = \lambda^2 u(x),\quad x\in(0,1),$$

$$u'(1) + \lambda u(1) = 0,$$
$$u(0) = 0,$$

where $u(x) = v(x,\cdot)$ is a vector-function with values in the Hilbert space

$H = L_2(0,1)$ and the operator A is defined as follows

$$D(A) = W_2^4\Big((0,1);\, u^{(j)}(0) = u^{(j)}(1),\ j = 0,1,2,3\Big), \quad Au = \frac{d^4u}{dy^4} + \omega u. \tag{5.8}$$

Obviously, the operator A, defined by the formula (5.8), is selfadjoint and positive-definite and A^{-1} is a compact operator in $L_2(0,1)$ since the embedding $D(A) \subset L_2(0,1)$ is compact. Some standard calculations show that the eigenvalues of the operator A are equal to $\mu_k(A) = 16k^4\pi^4 + \omega$, $k = 0,1,2,\dots$. Then, by virtue of Theorem 3, for the eigenvalues of the boundary value problem (5.5)–(5.7) the following three series are obtained:

$$\lambda_k^{(1)} \sim -2\sqrt{2}k^2\pi^2, \quad k \to +\infty,$$

and

$$\lambda_n^{(2,k)} = \sqrt{16k^4\pi^4 + \gamma_n}\ , \quad \lambda_n^{(3,k)} = -\sqrt{16k^4\pi^4 + \delta_n}\ ,$$

where $\gamma_n \sim n^2\pi^2$, $\delta_n \sim n^2\pi^2$ when $n \to +\infty$. Using, e.g., [**16**, Lemma], one can write the last two series as asymptotic formulae (with respect to one index instead of two indexes of the series):

$$\lambda_m^{(2)} \sim \left(\frac{2\pi^2}{\gamma}\right)^{\frac{2}{3}} m^{\frac{2}{3}}, \quad \lambda_m^{(3)} \sim -\left(\frac{2\pi^2}{\gamma}\right)^{\frac{2}{3}} m^{\frac{2}{3}}, \quad m \to +\infty,$$

where $\gamma = \int_0^{\frac{\pi}{2}} (\cos t)^{\frac{3}{2}}\, dt$.

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