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Sectional class of ample line bundles on smooth projective varieties

Abstract. Let X be an n-dimensional smooth projective variety defined over the field of complex numbers, let $L_1, \dots, L_{n-i}, A_1$ and A_2 be ample line bundles on X. In this paper, we will define the sectional class $\mathrm{cl}_i(X, L_1, \dots, L_{n-i}; A_1, A_2)$ for every integer i with $0 \le i \le n$, and we will investigate this invariant. In particular, for every integer i with $0 \le i \le n$, by setting $L_1 = \cdots = L_{n-i} = L$ and $A_1 = A_2 = L$, we give a classification of polarized manifolds (X, L) by the value of $\mathrm{cl}_i(X, L) := \mathrm{cl}_i(X, \underbrace{L, \cdots, L}_{n-i}; L, L)$.

Keywords. Ample vector bundle, (multi-)polarized manifold, class, sectional Euler number, sectional Betti number.

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1 - Introduction

Let X be a smooth projective variety of dimension n defined over the field of complex numbers, and let L be an ample line bundle on X. Then (X, L) is called a *polarized manifold*. Assume that L is very ample and let $\varphi : X \hookrightarrow \mathbb{P}^N$ be the morphism defined by $|L|$. Then φ is an embedding. In this situation, its dual variety $X^\vee \to (\mathbb{P}^N)^\vee$ is a hypersurface of N-dimensional projective space

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except some special types. Then the *class* $\mathrm{cl}(X, L)$ of (X, L) is defined by the following.

$$\mathrm{cl}(X,L)=\begin{cases}\deg(X^\vee), & \text{if } X^\vee \text{ is a hypersurface in } (\mathbb{P}^N)^\vee\\ 0, & \text{otherwise.}\end{cases}$$

A lot of investigations by using $\mathrm{cl}(X, L)$ have been obtained (for example [**23**], [**27**], [**32**], [**24**], [**28**], [**26**], [**1**], [**31**] and so on). In this paper, we are going to define a generalization of this invariant. Let X be a smooth projective variety of dimension n and let $L_1, \ldots, L_{n-i}, A_1$ and A_2 be ample (not necessarily very ample) line bundles on X. Then in Section 2 we will define the *sectional class* $\mathrm{cl}_i(X, L_1, \ldots, L_{n-i}; A_1, A_2)$ for every integer i with $0 \le i \le n$ (see Definition 2.8), and we will study some fundamental properties concerning this invariant. In Section 3, we consider the following special case: Let L be an ample (not necessarily very ample) line bundle on X and we set $L_1 = \cdots = L_{n-i} = L$ and $A_1 = A_2 = L$. Then we will define $\mathrm{cl}_i(X, L) := \mathrm{cl}_i(X, \underbrace{L, \cdots, L}_{n-i}; L, L)$. We will call this invariant the *ith sectional class of* (X, L). In Section 3, we study this invariant $\mathrm{cl}_i(X, L)$ for the case where L is not necessarily very ample and will get some results about $\mathrm{cl}_i(X, L)$.

Here we note the following: Assume that L is very ample. Then there exists a member $X_j \in |L_{j-1}|$ such that each X_j is a smooth projective manifold of dimension $n-j$ and $L_j := L|_{X_j}$ for every j with $1 \le j \le n-i$. In this case, we see that $\mathrm{cl}_i(X, L)$ is the class of the i dimensional polarized manifold (X_{n-i}, L_{n-i}). In particular, if $i = n$, then $\mathrm{cl}_n(X, L)$ is equal to the class $\mathrm{cl}(X, L)$ of (X, L) if L is very ample.

As we said above, there are a lot of works about the class $\mathrm{cl}(X, L)$ for very ample line bundles L, that is, the case where $i = n$ and L is very ample. Classifications of (X, L) concerning $\mathrm{cl}_i(X, L)$ are known for the following cases.

- The case where $i = n \le 3$ and L is very ample (see [**23**], [**27**], [**24**]).
- The case where $i = 2$, $n \ge 2$ and L is very ample (see [**32**], [**28**], [**26**]).
- The case where $i = n = 2$ and L is ample (see [**31**]).

In this paper, we give classifications of (X, L) by the value of $\mathrm{cl}_i(X, L)$ for the following cases, some of which are natural generalizations of the known results.

- The case where $i = 1$, $n \ge 3$, $\mathrm{cl}_1(X, L) \le 4$ and L is ample.
- The case where $i = 2$, $n \ge 3$, $\mathrm{cl}_2(X, L) \le 16$ and L is ample and spanned.
- The case where $i = 3$, $n \ge 3$, $\mathrm{cl}_3(X, L) \le 8$ and L is ample and spanned.
- The case where $i = 4$, $n \ge 5$, $\mathrm{cl}_4(X, L) \le 1$ (resp. $\mathrm{cl}_4(X, L) = 2$) and L is ample and spanned (resp. very ample).

In Subsection 3.1, we calculate $\mathrm{cl}_i(X,L)$ for some special cases. The results in Subsection 3.1 will be used in order to classify (X,L) by the value of $\mathrm{cl}_i(X,L)$. In Subsections 3.2, 3.3, 3.4 and 3.5 we obtain the classification of (X,L) by the value of $\mathrm{cl}_1(X,L)$, $\mathrm{cl}_2(X,L)$, $\mathrm{cl}_3(X,L)$ and $\mathrm{cl}_4(X,L)$.

We see from the definition of the ith sectional class that it is somewhat hard to calculate this invariant in general (see also [**18**]). But we expect that the ith sectional class has properties similar to those of the class of i-dimensional projective manifolds, and we believe that this invariant is useful for investigating polarized manifolds. We also hope that we can give a characterization of special polarized manifolds by the value of sectional classes. This is the reason why we define this invariant.

In our paper for the future, we will define and study the sectional class for the case of ample vector bundles.

2 - Definitions and fundamental results

Definition 2.1. Let $L_1,\dots,L_m$ be ample line bundles on a smooth projective variety X. Then $(X,L_1,\dots,L_m)$ is called a *multi-polarized manifold of type m*.

First we recall some invariants of polarized manifolds which are used later.

Definition 2.2 ([**16**, Definition 2.1.3]). Let X be a smooth projective variety of dimension n and let $\mathcal{E}$ be an ample vector bundle on X with rank $\mathcal{E}=r$. We assume that $r\le n$. For every integer p with $0\le p\le n-r$ we set

$$C_p^{n,r}(X,\mathcal{E}) := \sum_{k=0}^{p} c_k(X)s_{p-k}(\mathcal{E}^\vee).$$

Notation 2.1. (1) Let (X,L) be a polarized manifold of dimension n. Then the Euler-Poincaré characteristic $\chi(L^{\otimes t})$ of $L^{\otimes t}$ is a polynomial in t of degree n (see [**21**, chapter I, § 1]), and we put

$$\chi(L^{\otimes t}) = \sum_{j=0}^{n}\chi_j(X,L)\binom{t+j-1}{j}.$$

(2) Let Y be a smooth projective variety of dimension $i\ge 1$, let $\mathcal{T}_Y$ be the tangent bundle of Y and let Ω_Y be the dual bundle of $\mathcal{T}_Y$. For every integer j with $0\le j\le i$, we put

$$h_{i,j}(c_1(Y),\cdots,c_i(Y)) := \chi(\Omega_Y^j) = \int_Y \mathrm{ch}(\Omega_Y^j)\mathrm{Td}(\mathcal{T}_Y).$$

(Here $\mathrm{ch}(\Omega_Y^j)$ (resp. $\mathrm{Td}(\mathcal{T}_Y)$) denotes the Chern character of Ω_Y^j (resp. the Todd class of $\mathcal{T}_Y$). See [**20**, example 3.2.3 and example 3.2.4].)

(3) Let X be a smooth projective variety of dimension n. For every integers i and j with $0 \leq j \leq i \leq n$, we put

$$H_1(X;i,j) := \begin{cases} \sum_{s=0}^{i-j-1} (-1)^s h^s(\Omega_X^j) & \text{if } j \neq i, \\ 0 & \text{if } j = i, \end{cases}$$

$$H_2(X;i,j) := \begin{cases} \sum_{t=0}^{j-1} (-1)^{i-t} h^t(\Omega_X^{i-j}) & \text{if } j \neq 0, \\ 0 & \text{if } j = 0. \end{cases}$$

Definition 2.3 ([**9**], [**10**] and [**11**]). Let (X, L) be a polarized manifold of dimension n, and let i and j be integers with $0 \leq j \leq i \leq n$. (Here we use Notation 2.1.)

(i) The *ith sectional H-arithmetic genus* $\chi_i^H(X,L)$ *of* (X,L) is defined as follows:

$$\chi_i^H(X,L) := \chi_{n-i}(X,L).$$

(ii) The *ith sectional geometric genus* $g_i(X,L)$ *of* (X,L) is defined as follows:

$$g_i(X,L) := (-1)^i(\chi_{n-i}(X,L) - \chi(\mathcal{O}_X)) + \sum_{j=0}^{n-i} (-1)^{n-i-j} h^{n-j}(\mathcal{O}_X).$$

(iii) The *ith sectional Euler number* $e_i(X,L)$ *of* (X,L) is defined by the following:

$$e_i(X,L) := C_i^{n,n-i}(X, L^{\oplus n-i})L^{n-i}.$$

(iv) The *ith sectional Betti number* $b_i(X,L)$ *of* (X,L) is defined by the following:

$$b_i(X,L) := \begin{cases} e_0(X,L) & \text{if } i = 0, \\ (-1)^i \left(e_i(X,L) - \sum_{j=0}^{i-1} 2(-1)^j h^j(X,\mathbb{C}) \right) & \text{if } 1 \leq i \leq n. \end{cases}$$

(v) The *ith sectional Hodge number* $h_i^{j,i-j}(X,L)$ *of type* $(j, i-j)$ *of* (X,L) is defined by the following:

$$h_i^{j,i-j}(X,L) := (-1)^{i-j}\left\{ w_i^j(X,L) - H_1(X;i,j) - H_2(X;i,j) \right\},$$

where

$$w_i^j(X,L) := \begin{cases} h_{i,j}(C_1^{n,n-i}(X,L^{\oplus n-i}),\cdots,C_i^{n,n-i}(X,L^{\oplus n-i}))L^{n-i}, & \text{if } i>0,\\ L^n, & \text{if } i=0. \end{cases}$$

Next we recall some invariants of ample vector bundles.

Definition 2.4 ([**16**, Definition 3.1.1]). Let X be a smooth projective variety of dimension n and let $\mathcal{E}$ be an ample vector bundle on X with rank $\mathcal{E}=r\leq n$. Then the *c_r-sectional H-arithmetic genus* $\chi^H_{n,r}(X,\mathcal{E})$ and the *c_r-sectional Euler number* $e_{n,r}(X,\mathcal{E})$ of $(X,\mathcal{E})$ are defined by the following[1]:

$$\chi^H_{n,r}(X,\mathcal{E}) := \mathrm{td}_{n-r}\big(C_1^{n,r}(X,\mathcal{E}),\cdots,C_{n-r}^{n,r}(X,\mathcal{E})\big)c_r(\mathcal{E}).$$

$$e_{n,r}(X,\mathcal{E}) := C_{n-r}^{n,r}(X,\mathcal{E})c_r(\mathcal{E}).$$

Remark 2.1. If $r=n$, then we see that $\chi^H_{n,r}(X,\mathcal{E})=c_n(\mathcal{E})$ and $e_{n,r}(X,\mathcal{E})=c_n(\mathcal{E})$.

Definition 2.5 ([**16**, Definition 3.2.1]). Let X be a smooth projective variety of dimension n and let $\mathcal{E}$ be an ample vector bundle on X with rank $\mathcal{E}=r\leq n$. Then the *c_r-sectional geometric genus* $g_{n,r}(X,\mathcal{E})$ and the *c_r-sectional Betti number* $b_{n,r}(X,\mathcal{E})$ *of* $(X,\mathcal{E})$ are defined by the following:

$$g_{n,r}(X,\mathcal{E}) := (-1)^{n-r}\chi^H_{n,r}(X,\mathcal{E}) + (-1)^{n-r+1}\chi(\mathcal{O}_X) + \sum_{k=0}^{r}(-1)^{r-k}h^{n-k}(\mathcal{O}_X).$$

$$b_{n,r}(X,\mathcal{E}) := \begin{cases} (-1)^{n-r}\Big(e_{n,r}(X,\mathcal{E}) - \sum_{j=0}^{n-r-1}2(-1)^j h^j(X,\mathbb{C})\Big), & \text{if } r<n,\\ e_{n,n}(X,\mathcal{E}), & \text{if } r=n. \end{cases}$$

Definition 2.6 ([**16**, Definition 3.3.1]). Let X be a smooth projective variety of dimension n and let $\mathcal{E}$ be an ample vector bundle on X with rank $\mathcal{E}=r\leq n$. Then the *c_r-sectional Hodge number* $h^{j,n-r-j}_{n,r}(X,\mathcal{E})$ *of type* $(j,n-r-j)$ *of* $(X,\mathcal{E})$ is defined by the following[2]:

$$h^{j,n-r-j}_{n,r}(X,\mathcal{E}) := (-1)^{n-r-j}\Big\{w^j_{n,r}(X,\mathcal{E}) - H_1(X;n-r,j) - H_2(X;n-r,j)\Big\}.$$

---

[1] Here td_{n-r} means the Todd polynomial of weight $n-r$.

[2] See Notation 2.1 (2).

Here we set

$$w^j_{n,r}(X,\mathcal{E}) := \begin{cases} h_{n-r,j}(C^{n,r}_1(X,\mathcal{E}),\cdots,C^{n,r}_{n-r}(X,\mathcal{E}))c_r(\mathcal{E}), & \text{if } r<n,\\ c_n(\mathcal{E}), & \text{if } r=n, \end{cases}$$

for every integer j with $0\le j\le n-r$.

There are the following relationships among these invariants.

Theorem 2.1. *Let X be a smooth projective variety of dimension n and let $\mathcal{E}$ be an ample vector bundle on X with* rank $\mathcal{E}=r$. *Assume that $r\le n-1$. For every integer j with $0\le j\le n-r$, we get the following.*

(i) $b_{n,r}(X,\mathcal{E})=\sum_{k=0}^{n-r}h^{k,n-r-k}_{n,r}(X,\mathcal{E})$.
(ii) $h^{j,n-r-j}_{n,r}(X,\mathcal{E})=h^{n-r-j,j}_{n,r}(X,\mathcal{E})$.
(iii) $h^{n-r,0}_{n,r}(X,\mathcal{E})=h^{0,n-r}_{n,r}(X,\mathcal{E})=g_{n,r}(X,\mathcal{E})$.
(iv) *If $n-r$ is odd, then $b_{n,r}(X,\mathcal{E})$ is even.*

Proof. See [**16**, Theorem 4.1]. □

By using above invariants, we define some invariants of multi-polarized manifolds.

Definition 2.7 ([**16**, Definition 5.1.1]) Let $(X,L_1,\dots,L_{n-i})$ be a multi-polarized manifold of type $n-i$ with $\dim X=n$, where i is an integer with $0\le i\le n-1$. Then we define the *ith sectional H-arithmetic genus* $\chi^H_i(X,L_1,\dots,L_{n-i})$, the *ith sectional Euler number* $e_i(X,L_1,\dots,L_{n-i})$, the *ith sectional geometric genus* $g_i(X,L_1,\dots,L_{n-i})$, the *ith sectional Betti number* $b_i(X,L_1,\dots,L_{n-i})$ and the *ith sectional Hodge number* $h^{j,i-j}_i(X,L_1,\dots,L_{n-i})$ *of type* $(j,i-j)$ for every integer j with $0\le j\le i$ are defined as follows.

$$\begin{aligned} \chi^H_i(X,L_1,\dots,L_{n-i}) &:= \chi^H_{n,n-i}(X,L_1\oplus\cdots\oplus L_{n-i}),\\ g_i(X,L_1,\dots,L_{n-i}) &:= g_{n,n-i}(X,L_1\oplus\cdots\oplus L_{n-i}),\\ e_i(X,L_1,\dots,L_{n-i}) &:= e_{n,n-i}(X,L_1\oplus\cdots\oplus L_{n-i}),\\ b_i(X,L_1,\dots,L_{n-i}) &:= b_{n,n-i}(X,L_1\oplus\cdots\oplus L_{n-i}),\\ h^{j,i-j}_i(X,L_1,\dots,L_{n-i}) &:= h^{j,i-j}_{n,n-i}(X,L_1\oplus\cdots\oplus L_{n-i}). \end{aligned}$$

Remark 2.2. (i) For the case of $i=n$, as a matter of convenience, we set

$$\begin{aligned} \chi^H_i(X,L_1,\dots,L_{n-i}) &:= \chi(\mathcal{O}_X), \quad g_i(X,L_1,\dots,L_{n-i}):=h^n(\mathcal{O}_X),\\ e_i(X,L_1,\dots,L_{n-i}) &:= e(X), \quad b_i(X,L_1,\dots,L_{n-i}):=b_n(X),\\ h^{j,i-j}_i(X,L_1,\dots,L_{n-i}) &:= h^{j,n-j}(X). \end{aligned}$$

Here $e(X):=\sum_{k=0}^{2n}(-1)^kh^k(X,\mathbb{C})$, $b_i(X):=h^i(X,\mathbb{C})$ and $h^{j,i-j}(X):=h^{i-j}(\Omega^j_X)$ for every integers i and j with $0\le j\le i\le n$.

(ii) Under the assumption that X is smooth, we see that $\chi_i^H(X, L_1, \dots, L_{n-i})$ (resp. $g_i(X, L_1, \dots, L_{n-i})$) in Definition 2.7 is equal to

$$\chi_i^H(X, L_1, \dots, L_{n-i}; \mathcal{O}_X) \quad (\text{resp. } g_i(X, L_1, \dots, L_{n-i}; \mathcal{O}_X))$$

in [**13**, Definition 2.1].

We also note that $\chi_i^H(X, L_1, \dots, L_{n-i})$ (resp. $g_i(X, L_1, \dots, L_{n-i})$) is defined for any *smooth* projective variety X in Definition 2.7, but $\chi_i^H(X, L_1, \dots, L_{n-i}; \mathcal{O}_X)$ (resp. $g_i(X, L_1, \dots, L_{n-i}; \mathcal{O}_X)$) is defined for any projective variety in [**13**, Definition 2.1].

Proposition 2.1. *Let $(X, L_1, \dots, L_{n-i})$ be a multi-polarized manifold of type $n-i$ with* $\dim X = n$, *where i is an integer with $0 \le i \le n-1$. Assume that a line bundle L is ample and $L_k = L$ for every integer k with $1 \le k \le n-i$. Then we have*

$$\begin{aligned}
\chi_i^H(X, L_1, \dots, L_{n-i}) &= \chi_i^H(X, L), \quad g_i(X, L_1, \dots, L_{n-i}) = g_i(X, L),\\
e_i(X, L_1, \dots, L_{n-i}) &= e_i(X, L), \quad b_i(X, L_1, \dots, L_{n-i}) = b_i(X, L),\\
h_i^{j,i-j}(X, L_1, \dots, L_{n-i}) &= h_i^{j,i-j}(X, L) \quad \textit{for every integer } j \textit{ with } 0 \le j \le i.
\end{aligned}$$

Here $\chi_i^H(X, L)$, $g_i(X, L)$, $e_i(X, L)$, $b_i(X, L)$ and $h_i^{j,i-j}(X, L)$ are sectional invariants defined in Definition 2.3.

Proof. See [**16**, Proposition 5.2.1]. □

Here we define the ith sectional class of multi-polarized manifolds.

Definition 2.8. Let X be a smooth projective variety of dimension $n \ge 1$, let i be an integer with $0 \le i \le n$ and let $L_1, \dots, L_{n-i}, A_1, A_2$ be ample line bundles on X. Then the *ith sectional class* of $(X, L_1, \dots, L_{n-i}; A_1, A_2)$ is defined by the following.

$$\mathrm{cl}_i(X, L_1, \dots, L_{n-i}; A_1, A_2)$$

$$:= \begin{cases}
e_0(X, L_1, \dots, L_n), & \text{if } i = 0,\\[1ex]
(-1)\{e_1(X, L_1, \dots, L_{n-1}) - e_0(X, L_1, \dots, L_{n-1}, A_1) \\ \qquad - e_0(X, L_1, \dots, L_{n-1}, A_2)\}, & \text{if } i = 1,\\[1ex]
(-1)^i\{e_i(X, L_1, \dots, L_{n-i}) - e_{i-1}(X, L_1, \dots, L_{n-i}, A_1) \\ \qquad - e_{i-1}(X, L_1, \dots, L_{n-i}, A_2) \\ \qquad + e_{i-2}(X, L_1, \dots, L_{n-i}, A_1, A_2)\}, & \text{if } 2 \le i \le n,
\end{cases}$$

where $e_k(X, L_1, \dots, L_{n-k})$ is the kth sectional Euler number of $(X, L_1, \dots, L_{n-k})$.

Remark 2.3. (i) If i is odd, then $e_i(X, L_1, \dots, L_{n-i})$ is even.

Proof. First we note that $1 \leq i$ because i is odd. Then by the definition of the ith sectional Betti number $b_i(X, L_1, \ldots, L_{n-i})$, we have

$$(1) \qquad e_i(X, L_1, \ldots, L_{n-i}) = 2\sum_{j=0}^{i-1}(-1)^j h^j(X, \mathbb{C}) + (-1)^i b_i(X, L_1, \ldots, L_{n-i}).$$

On the other hand, since i is odd, $b_i(X, L_1, \ldots, L_{n-i})$ is even by Definition 2.7 and Theorem 2.1. Hence $e_i(X, L_1, \ldots, L_{n-i})$ is even. □

So if i is odd and $A_1 = A_2 = A$, then we see that $\mathrm{cl}_i(X, L_1, \ldots, L_{n-i}; A, A)$ is even.

(ii) If $i = 0$, then $\mathrm{cl}_0(X, L_1, \ldots, L_n; A_1, A_2) = L_1 \cdots L_n > 0$.

Definition 2.9. Let (X, L) be a polarized manifold of dimension n and let i be an integer with $0 \leq i \leq n$. Then the *ith sectional class* of (X, L) is defined by the following.

$$\mathrm{cl}_i(X, L) := \mathrm{cl}_i(X, \underbrace{L, \ldots, L}_{n-i}; L, L).$$

Remark 2.4. Assume that L is very ample. Then there exists a sequence of smooth subvarieties $X \supset X_1 \supset \cdots \supset X_{n-i}$ such that $X_j \in |L_{j-1}|$ and $\dim X_j = n - j$ for every integer j with $1 \leq j \leq n - i$, where $L_j = L|_{X_j}$. In particular, X_{n-i} is a smooth projective variety of dimension i and L_{n-i} is a very ample line bundle on X_{n-i}. Then $\mathrm{cl}_i(X, L)$ is equal to the class of (X_{n-i}, L_{n-i}).

Remark 2.5. ([**22**, II-1]) Let X be an n-dimensional smooth projective variety and let L be a very ample line bundle on X. Let $X \hookrightarrow \mathbb{P}^N$ be the embedding defined by $|L|$. For every integer i with $0 \leq i \leq n$, Severi defined the notion of the ith *rank* $r_i(X)$ of X as follows.

$$r_i(X) = \int L^i (L^\vee)^{N-1-i}(CX).$$

Here CX denotes the conormal variety, $X^\vee$ denotes the dual variety of X and $L^\vee = \mathcal{O}_{X^\vee}(1)$. Then we see that $r_i(X) = \mathrm{cl}_{n-i}(X, L)$ (see [**22**, (6) Theorem in II]). We also note that if $i = 0$, then $r_0(X) = \mathrm{cl}_n(X, L)$ is called the *class* of X.

Remark 2.6. By Definitions 2.8 and 2.9 we see that

$$\mathrm{cl}_i(X, L) = \begin{cases} e_0(X, L), & \text{if } i = 0, \\ (-1)\{e_1(X, L) - 2e_0(X, L)\}, & \text{if } i = 1, \\ (-1)^i\{e_i(X, L) - 2e_{i-1}(X, L) + e_{i-2}(X, L)\}, & \text{if } 2 \leq i \leq n. \end{cases}$$

Here $e_i(X, L)$ is the ith sectional Euler number of (X, L).

Proposition 2.2. *Let X be a smooth projective variety of dimension n and let i be an integer with $0 \leq i \leq n$. Let $L_1, \dots, L_{n-i}, A_1, A_2$ be ample line bundles on X. Then the following holds.*

$$
\begin{aligned}
&\mathrm{cl}_i(X, L_1, \dots, L_{n-i}; A_1, A_2) \\
&= \begin{cases}
b_0(X, L_1, \dots, L_n), & \text{if } i = 0, \\[6pt]
b_1(X, L_1, \dots, L_{n-1}) \\
\quad + b_0(X, L_1, \dots, L_{n-1}, A_1) - b_0(X) \\
\quad + b_0(X, L_1, \dots, L_{n-1}, A_2) - b_0(X), & \text{if } i = 1, \\[6pt]
b_i(X, L_1, \dots, L_{n-i}) - b_{i-2}(X) \\
\quad + b_{i-1}(X, L_1, \dots, L_{n-i}, A_1) - b_{i-1}(X) \\
\quad + b_{i-1}(X, L_1, \dots, L_{n-i}, A_2) - b_{i-1}(X) \\
\quad + b_{i-2}(X, L_1, \dots, L_{n-i}, A_1, A_2) - b_{i-2}(X), & \text{if } 2 \leq i \leq n.
\end{cases}
\end{aligned}
$$

Proof. Since

$$
\begin{aligned}
&(-1)^i \left(2 \sum_{j=0}^{i-1} (-1)^j b_j(X) - 4 \sum_{j=0}^{i-2} (-1)^j b_j(X) + 2 \sum_{j=0}^{i-3} (-1)^j b_j(X) \right) \\
&= -2b_{i-1}(X) - 2b_{i-2}(X),
\end{aligned}
$$

the assertion holds by substituting the equality (1) in Remark 2.3 (i) for the formula in Definition 2.8. □

Corollary 2.1. *Let (X, L) be a polarized manifold of dimension n. For any integer i with $0 \leq i \leq n$, the following holds.*

$$
\begin{aligned}
&\mathrm{cl}_i(X, L) \\
&= \begin{cases}
b_0(X, L), & \text{if } i = 0, \\[6pt]
b_1(X, L) + 2b_0(X, L) - 2, & \text{if } i = 1, \\[6pt]
b_i(X, L) - b_{i-2}(X) + 2b_{i-1}(X, L) - 2b_{i-1}(X) \\
\quad + b_{i-2}(X, L) - b_{i-2}(X), & \text{if } 2 \leq i \leq n.
\end{cases}
\end{aligned}
$$

Next we study the non-negativity of the sectional class.

Theorem 2.2. *Let X be a smooth projective variety of dimension n and let i be an integer with $1 \leq i \leq n$. Let $L_1, \dots, L_{n-i}, A_1, A_2$ be ample and spanned line* *bundles on X. Then*

$$\mathrm{cl}_i(X, L_1, \dots, L_{n-i}; A_1, A_2) \geq 0.$$

Proof. (i) First we assume that $2 \leq i$. Then by Proposition 2.2, we get

$$\begin{aligned}
&\mathrm{cl}_i(X, L_1, \dots, L_{n-i}; A_1, A_2)\\
&= b_i(X, L_1, \dots, L_{n-i}) - b_{i-2}(X) + b_{i-1}(X, L_1, \dots, L_{n-i}, A_1) - b_{i-1}(X)\\
&\quad + b_{i-1}(X, L_1, \dots, L_{n-i}, A_2) - b_{i-1}(X)\\
&\quad + b_{i-2}(X, L_1, \dots, L_{n-i}, A_1, A_2) - b_{i-2}(X).
\end{aligned}$$

In general, for every ample and spanned line bundles $H_1, \dots, H_{n-j}$, by Definition 2.7 and [**16**, Proposition 4.1] we have $b_j(X, H_1, \dots, H_{n-j}) \geq b_j(X)$ for every integer j with $0 \leq j \leq n$. On the other hand, we obtain $b_i(X) \geq b_{i-2}(X)$ by the hard Lefschetz theorem [**29**, Corollary 3.1.40]. Therefore we get the assertion.

(ii) Next we assume that $i = 1$. Then by definition we have

$$\mathrm{cl}_1(X, L_1, \dots, L_{n-1}; A_1, A_2) = 2g_1(X, L_1, \dots, L_{n-1}) + L_1 \cdots L_{n-1}(A_1 + A_2) - 2,$$

where $g_1(X, L_1, \dots, L_{n-1})$ is the first sectional geometric genus of $(X, L_1, \dots, L_{n-1})$. We note that $g_1(X, L_1, \dots, L_{n-1}) \geq 0$ by [**15**, Theorem 6.1.1], and $L_1 \cdots L_{n-1}A_k \geq 1$ for $k = 1, 2$. So we have $\mathrm{cl}_1(X, L_1, \dots, L_{n-1}; A_1, A_2) \geq 0$. □

Remark 2.7. By (ii) in the proof of Theorem 2.2

$$\mathrm{cl}_1(X, L_1, \dots, L_{n-1}; A_1, A_2) \geq 0$$

holds for any merely ample line bundles $L_1, \dots, L_{n-i}, A_1, A_2$.

By Definition 2.9, Remark 2.3 (ii) and Theorem 2.2 the following holds.

Corollary 2.2. *Let (X, L) be a polarized manifold of dimension n and let i be an integer with $0 \leq i \leq n$. Assume that L is base point free. Then $\mathrm{cl}_i(X, L) \geq 0$.*

Here we propose the following conjecture.

Conjecture 2.1. *Let X be a smooth projective variety of dimension n and let i be an integer with $0 \leq i \leq n$. Let $L_1, \dots, L_{n-i}, A_1, A_2$ be ample line bundles on X. Then*

$$\mathrm{cl}_i(X, L_1, \dots, L_{n-i}; A_1, A_2) \geq 0.$$

By Remark 2.3 (ii) (resp. Remark 2.7), this conjecture is true for the case where $i = 0$ (resp. $i = 1$).

If $i = 2$ and $\kappa(X) \geq 0$, then we can get the following lower bound.

Theorem 2.3. *Let X be a smooth projective variety of dimension n with $\kappa(X) \geq 0$ and let $L_1, \ldots, L_{n-2}, A_1, A_2$ be ample line bundles on X. Then the following inequality holds.*

$$\begin{aligned}
&\mathrm{cl}_2(X, L_1, \ldots, L_{n-2}; A_1, A_2) \\
&\geq \frac{1}{2n}\left(\sum_{j=1}^{n-2} L_j\right)^2 L_1 \cdots L_{n-2} + \frac{1}{2}\left(\sum_{j=1}^{n-2} L_j^2\right) L_1 \cdots L_{n-2} \\
&\quad + \sum_{j=1}^{2} (L_1 + \cdots + L_{n-2} + A_j) L_1 \cdots L_{n-2} A_j + L_1 \cdots L_{n-2} A_1 A_2.
\end{aligned}$$

Proof. First we note that

$$\begin{aligned}
&\mathrm{cl}_2(X, L_1, \ldots, L_{n-2}; A_1, A_2) \\
&= e_2(X, L_1, \ldots, L_{n-2}) + 2g_1(X, L_1, \ldots, L_{n-2}, A_1) - 2 \\
&\quad + 2g_1(X, L_1, \ldots, L_{n-2}, A_2) - 2 + L_1 \ldots L_{n-2} A_1 A_2.
\end{aligned}$$

From [**16**, Theorem 5.3.1], we have

$$e_2(X, L_1, \ldots, L_{n-2}) \geq \frac{1}{2n}\left(\sum_{j=1}^{n-2} L_j\right)^2 L_1 \cdots L_{n-2} + \frac{1}{2}\left(\sum_{j=1}^{n-2} L_j^2\right) L_1 \cdots L_{n-2}.$$

Moreover since $\kappa(X) \geq 0$ we have

$$2g_1(X, L_1, \ldots, L_{n-2}, A_k) - 2 \geq (L_1 + \cdots + L_{n-2} + A_k) L_1 \cdots L_{n-2} A_k$$

for $k = 1, 2$. So we get the assertion. $\square$

Next we consider the value of the sectional class of a reduction of multi-polarized manifolds.

Definition 2.10. ([**13**, Definition 1.5]) Let k be a positive integer.

(1) Let $(X, L_1, \cdots, L_k)$ and $(Y, A_1, \cdots, A_k)$ be n-dimensional multi-polarized manifolds of type k. Then $(X, L_1, \cdots, L_k)$ is called a *simple blowing up of a multi-polarized manifold* $(Y, A_1, \cdots, A_k)$ *of type* k if there exists a blowing up $\pi : X \to Y$ at a point $y \in Y$ such that $L_j = \pi^*(A_j) - E$ and $E|_E \cong \mathcal{O}_{\mathbb{P}^{n-1}}(-1)$ for every integer j with $1 \leq j \leq k$, where $E \cong \mathbb{P}^{n-1}$ is the exceptional effective divisor.

(2) A multi-polarized manifold $(\widetilde{X}, \widetilde{L_1}, \cdots, \widetilde{L_k})$ of type k is called a *reduction of* $(X, L_1, \cdots, L_k)$ if there exists a birational morphism

$$\pi : (X, L_1, \cdots, L_k) \to (\widetilde{X}, \widetilde{L_1}, \cdots, \widetilde{L_k})$$

such that π is a composite of simple blowing ups and $(\widetilde{X},\widetilde{L_1},\cdots,\widetilde{L_k})$ is not simple blowing up of another multi-polarized manifold of type k. This π is called the *reduction map*.

Proposition 2.3. *Let $(X,L_1,\cdots,L_{n-i},A_1,A_2)$ be a multi-polarized manifold of type $n-i+2$ with $\dim X=n\geq 2$, where i is an integer with $0\leq i\leq n$. Let $(Y,H_1,\cdots,H_{n-i},B_1,B_2)$ be a multi-polarized manifold of type $n-i+2$ such that $(X,L_1,\cdots,L_{n-i},A_1,A_2)$ is a composite of simple blowing ups of $(Y,H_1,\cdots,H_{n-i},B_1,B_2)$ and let γ be the number of its simple blowing ups. Then*

$$\begin{aligned}&\mathrm{cl}_i(X,L_1,\dots,L_{n-i};A_1,A_2)\\&=\begin{cases}\mathrm{cl}_0(Y,H_1,\dots,H_n;B_1,B_2)-\gamma, & \textit{if } i=0,\\ \mathrm{cl}_1(Y,H_1,\dots,H_{n-1};B_1,B_2)-2\gamma, & \textit{if } i=1,\\ \mathrm{cl}_i(Y,H_1,\dots,H_{n-i};B_1,B_2), & \textit{if } 2\leq i\leq n-1 \textit{ or } i=n\geq 2.\end{cases}\end{aligned}$$

Proof. By Definition 2.8, Remark 2.3 and [**16**, Proposition 5.3.1] and its proof, we get the assertion. □

Corollary 2.3. *Let (X,L) be a polarized manifold of dimension $n\geq 2$ and let (Y,H) be a polarized manifold such that (X,L) is a composite of simple blowing ups of (Y,H) and let γ be the number of its simple blowing ups. Then for every integer i with $0\leq i\leq n$, we have*

$$\mathrm{cl}_i(X,L)=\begin{cases}\mathrm{cl}_0(Y,H)-\gamma, & \textit{if } i=0,\\ \mathrm{cl}_1(Y,H)-2\gamma, & \textit{if } i=1,\\ \mathrm{cl}_i(Y,H), & \textit{if } 2\leq i\leq n-1 \textit{ or } i=n\geq 2.\end{cases}$$

Proof. By putting $L_1:=L,\cdots,L_{n-i}:=L$, $A_1:=L$, $A_2:=L$, $H_1:=H,\cdots,H_{n-i}:=H$, $B_1:=H$ and $B_2:=H$, we get the assertion by Proposition 2.3. □

3 - On classification of polarized manifolds (X,L) by the sectional class

In this section, we study a classification of polarized manifolds (X,L) by the ith sectional class $\mathrm{cl}_i(X,L)$.

Notation 3.1. (1) Let Y be a projective variety and let $\mathcal{E}$ be a vector bundle on Y. Then $\mathbb{P}_Y(\mathcal{E})$ denotes the projective bundle over Y associated with $\mathcal{E}$ and $H(\mathcal{E})$ denotes the tautological line bundle.

(2) Let (X, L) be a hyperquadric fibration over a smooth curve C. We put $\mathcal{E} := f_*(L)$. Then $\mathcal{E}$ is a locally free sheaf of rank $n+1$ on C. Let $\pi : \mathbb{P}_C(\mathcal{E}) \to C$ be the projective bundle. Then $X \in |2H(\mathcal{E}) + \pi^*(B)|$ for some $B \in \mathrm{Pic}(C)$ and $L = H(\mathcal{E})|_X$, where $H(\mathcal{E})$ is the tautological line bundle of $\mathbb{P}_C(\mathcal{E})$. We put $e := \deg \mathcal{E}$ and $b := \deg B$.

Definition 3.1. Let $\mathcal{F}$ be a vector bundle on a smooth projective variety X. Then for every integer j with $j \geq 0$, the *jth Segre class* $s_j(\mathcal{F})$ *of* $\mathcal{F}$ is defined by the following equation: $c_t(\mathcal{F}^\vee)s_t(\mathcal{F}) = 1$, where $\mathcal{F}^\vee := \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{O}_X)$, $c_t(\mathcal{F}^\vee)$ is the Chern polynomial of $\mathcal{F}^\vee$ and $s_t(\mathcal{F}) = \sum_{j \geq 0} s_j(\mathcal{F})t^j$.

Remark 3.1. (a) Let $\mathcal{F}$ be a vector bundle on a smooth projective variety X. Let $\tilde{s}_j(\mathcal{F})$ be the jth Segre class which is defined in [**20**, Chapter 3]. Then $s_j(\mathcal{F}) = \tilde{s}_j(\mathcal{F}^\vee)$.

(b) For every integer i with $1 \leq i$, $s_i(\mathcal{F})$ can be written by using the Chern classes $c_j(\mathcal{F})$ with $1 \leq j \leq i$. (For example, $s_1(\mathcal{F}) = c_1(\mathcal{F})$, $s_2(\mathcal{F}) = c_1(\mathcal{F})^2 - c_2(\mathcal{F})$, and so on.)

3.1 - *Calculations on the sectional class of some special polarized manifolds*

Here we calculate the sectional class of some special polarized manifolds which will be used in the following subsection. See also [**18**].

Example 3.1.1. Let (X, L) be a polarized manifold of dimension $n \geq 3$ and let $g(X, L)$ be the sectional genus. Assume that L is spanned and $g(X, L) \leq q(X) + 2$. Then (X, L) is one of the following types (see [**6**], [**7**] and [**8**]).

(a) $(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))$.
(b) $(\mathbb{Q}^n, \mathcal{O}_{\mathbb{Q}^n}(1))$.
(c) A scroll over a smooth curve.
(d) A Del Pezzo manifold with $L^n \geq 2$.[3]
(e) X is a double covering of $\mathbb{P}^n$ branched along a smooth hypersurface of degree 6, and L is the pull-back of $\mathcal{O}_{\mathbb{P}^n}(1)$.
(f) A scroll over a smooth surface S and (X, L) satisfies one of the types (2-1), (2-2) and (2-3) in [8, Theorem 3.3].
(g) A hyperquadric fibration over a smooth curve C and (X, L) satisfies one of the types (3-1) and (3-2) in [8, Theorem 3.3].

[3] Here we assume that L is spanned. So we see that $L^n \geq 2$.

Here we calculate the ith sectional class of the above (e), (f) and (g).

(I) If (X, L) is the type (e), then by [**18**, Proposition 2.2 in Example 2.1 (vii.7)], we have

i	0	$1 \leq i$
$\mathrm{cl}_i(X, L)$	2	$6 \cdot 5^{i-1}$

(II) Next we consider the case (f). Here we use the same notation as in [**8**, Theorem 3.3].

(II.1) First we assume that (X, L) is the type (2-1) in [**8**, Theorem 3.3]. Then we have $K_S = -2H_\alpha - 2H_\beta$, $c_1(\mathcal{E}) = 2H_\alpha + 3H_\beta$ and $c_2(\mathcal{E}) = (H_\alpha + 2H_\beta)(H_\alpha + H_\beta) = 3$. Hence $K_S^2 = 8$, $K_S c_1(\mathcal{E}) = -10$, $c_1(\mathcal{E})^2 = 12$ and $L^n = s_2(\mathcal{E}) = c_1(\mathcal{E})^2 - c_2(\mathcal{E}) = 9$. On the other hand since $c_2(S) = 12\chi(\mathcal{O}_S) - K_S^2 = 4$, by [**14**, Corollary 3.1 (3.1.2)] we have

i	0	1	2	3
$\mathrm{e}_i(X, L)$	9	-2	7	8

Therefore

i	0	1	2	3
$\mathrm{cl}_i(X, L)$	9	20	20	8

(II.2) Next we consider the type (2-2) in [8, Theorem 3.3]. Then $K_S = -3H + E$ and $\mathcal{E} = (2H - E)^{\oplus 2}$. Hence $K_S^2 = 8$, $c_1(\mathcal{E})^2 = (4H - 2E)^2 = 12$, $c_2(\mathcal{E}) = (2H - E)^2 = 3$, $K_S c_1(\mathcal{E}) = -10$ and $s_2(\mathcal{E}) = c_1(\mathcal{E})^2 - c_2(\mathcal{E}) = 9$. We also note that $c_2(S) = 12\chi(\mathcal{O}_S) - K_S^2 = 4$. Hence we have

i	0	1	2	3
$\mathrm{e}_i(X, L)$	9	-2	7	8

Therefore

i	0	1	2	3
$\mathrm{cl}_i(X, L)$	9	20	20	8

(II.3) Next we consider the type (2-3) in [8, Theorem 3.3]. Then $K_S = -2H(\mathcal{F}) +$

$c_1(\mathcal{F})F = -2H(\mathcal{F}) + F$, $\mathcal{E} = H(\mathcal{F}) \otimes p^*\mathcal{G}$, $\deg \mathcal{F} = 1$ and $\deg \mathcal{G} = 1$. Hence $K_S^2 = 4H(\mathcal{F})^2 - 4 = 0$, $c_1(\mathcal{E})^2 = (2H(\mathcal{F}) + F)^2 = 8$, $c_2(\mathcal{E}) = c_2(p^*\mathcal{G}) + H(\mathcal{G})c_1(p^*\mathcal{G}) + H(\mathcal{G})^2 = 2$, $K_S c_1(\mathcal{E}) = -4H(\mathcal{G})^2 = -4$ and $s_2(\mathcal{E}) = c_1(\mathcal{E})^2 - c_2(\mathcal{E}) = 6$. We also note that $c_2(S) = 12\chi(\mathcal{O}_S) - K_S^2 = 0$. Hence by [**14**, Corollary 3.1 (3.1.2)] we have

i	0	1	2	3
$\mathrm{e}_i(X, L)$	6	-4	2	0

Therefore

i	0	1	2	3
$\mathrm{cl}_i(X, L)$	6	16	16	8

(III) Finally we consider the case (g).

(III.1) First we assume that (X, L) is the type in the type (3-1) in [**8**, Theorem 3.3]. Then by [**18**, Example 2.1 (viii)] we have

i	0	1	2	3
$\mathrm{cl}_i(X, L)$	6	16	16	8

(III.2) Next we consider the type (3-2) in [**8**, Theorem 3.3]. Then $e = d - 3$ and $b = 6 - d$. So by [**18**, Example 2.1 (viii)] we have

i	0	1	$2 \leq i \leq n$
$\mathrm{cl}_i(X, L)$	d	$2d + 2$	$4(6 - d)(i - 1) + 4(d - 1)$

Here we note that $3 \leq d \leq 9$ holds in this case, and if $d = 8$ (resp. $d \neq 8$), then $3 \leq n \leq 4$ (resp. $n = 3$).

Example 3.1.2. Let (X, L) be a polarized manifold of dimension $n \geq 3$. Assume that $q(X) = 0$, L is spanned and $g(X, L) = 3$. Then (X, L) is one of (I-2), (III), (IV), (IV′) and (V) in [**19**, Theorem 2.1]. Here we calculate the second sectional class of (X, L), which will be used in Theorem 3.3.2.

(A) First we consider the case (I-2) in [**19**, Theorem 2.1]. Then by [**18**, Example 2.1 (viii)] we have $\mathrm{cl}_2(X, L) = 8e + 8b + 4(g(C) - 1) = 8e + 8b - 4 = 28$.

(B) Next we consider the case (III) in [**19**, Theorem 2.1].

(B.1a) If (X,L) is the type (III-1a), then $n=5$ and $\mathrm{cl}_2(X,L)=c_2(S)+3c_1(\mathcal{E})^2+2K_Sc_1(\mathcal{E})=27$ by [**18**, Example 2.1 (x)].

(B.1b) If (X,L) is the type (III-1b), then $n=4$. If $(S,\mathcal{E})=(\mathbb{P}^2,\mathcal{O}_{\mathbb{P}^2}(1)^{\oplus 2}\oplus\mathcal{O}_{\mathbb{P}^2}(2))$, then by [**18**, Example 2.1 (x)] we have $\mathrm{cl}_2(X,L)=c_2(S)+3c_1(\mathcal{E})^2+2K_Sc_1(\mathcal{E})=27$.

If $(S,\mathcal{E})=(\mathbb{P}^2,T_{\mathbb{P}^2}\oplus\mathcal{O}_{\mathbb{P}^2}(1))$, then by [**18**, Example 2.1 (x)] we have $\mathrm{cl}_2(X,L)=c_2(S)+3c_1(\mathcal{E})^2+2K_Sc_1(\mathcal{E})=27$.

(B.1c) If (X,L) is the type (III-1c), then $S\cong\mathbb{P}^2$, $\mathrm{rank}(\mathcal{E})=2$ and $c_1(\mathcal{E})=\mathcal{O}_{\mathbb{P}^2}(4)$. Hence $\mathrm{cl}_2(X,L)=c_2(S)+3c_1(\mathcal{E})^2+2K_Sc_1(\mathcal{E})=27$.

(B.2) If (X,L) is the type (III-2), then S is a Del Pezzo surface with $K_S^2=2$ and $\mathcal{E}$ is an ample vector bundle of rank two on S with $c_1(\mathcal{E})^2=8$ and $K_Sc_1(\mathcal{E})=-4$. Hence $\mathrm{cl}_2(X,L)=c_2(S)+3c_1(\mathcal{E})^2+2K_Sc_1(\mathcal{E})=26$.

(C) Next we consider the case (IV) in [**19**, Theorem 2.1]. By [**18**, Proposition 2.1 in Example 2.1 (vii.6)] we have $\mathrm{cl}_2(X,L)=4\cdot 3^2=36$.

(D) Next we consider the case (IV′) in [**19**, Theorem 2.1]. Since $\mathrm{cl}_2(X,L)$ and $\mathrm{cl}_3(X,L)$ are invariant under simple blowing ups by Corollary 2.3, we have $\mathrm{cl}_2(X,L)=4\cdot 3^2=36$.

(E) Next we consider the case (V) in [**19**, Theorem 2.1].

(E.1) If (X,L) is the type (V-1), then by [**18**, Proposition 2.2 in Example 2.1 (vii.7)] we have $\mathrm{cl}_2(X,L)=8\cdot 7^1=56$.

(E.2) If (X,L) is the type (V-2), then (X,L) is a Mukai manifold, that is, $\mathcal{O}_X(K_X+(n-2)L)=\mathcal{O}_X$. Hence by [**9**, Example 2.10 (7)] we have $g_2(X,L)=1$ and $\chi_2^H(X,L)=1-h^1(\mathcal{O}_X)+g_2(X,L)=2$, where $g_2(X,L)$ (resp. $\chi_2^H(X,L)$) is the second sectional geometric genus (resp. the second sectional H-arithmetic genus) of (X,L). Furthermore by [**12**, Proposition 3.1] we have

$$h_2^{1,1}(X,L)=10\chi_2^H(X,L)-(K_X+(n-2)L)^2L^{n-2}+2h^1(\mathcal{O}_X)=20.$$

Here $h_2^{1,1}(X,L)$ denotes the second sectional Hodge number of type $(1,1)$. Hence by [**11**, Theorem 3.1 (3.1.1), (3.1.3) and (3.1.4)] we get $b_2(X,L)=2g_2(X,L)+h_2^{1,1}(X,L)=22$. Since $b_1(X,L)=2g(X,L)=6$ (see [**11**, Remark 3.1 (2)]) and $b_0(X,L)=L^n$, we have

$$e_2(X,L)=2b_0(X)-2b_1(X)+b_2(X,L)=2-2\cdot 0+22=24,$$

$$e_1(X,L)=2b_0(X)-b_1(X,L)=2-6=-4,$$

$$e_0(X,L)=b_0(X,L)=4.$$

Therefore we get $\mathrm{cl}_2(X,L)=24-2(-4)+4=36$.

Example 3.1.3. Let (X, L) be a polarized manifold of dimension $n \geq 3$ such that $h^0(L) \geq n+1$ and $L^n \leq 2$. Then we see that $\Delta(X, L) \leq 1$ and (X, L) is one of the following types.[4]

(i) $(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))$.

(ii) $(\mathbb{Q}^n, \mathcal{O}_{\mathbb{Q}^n}(1))$.

(iii) X is a double covering of $\mathbb{P}^n$ whose branch locus is of degree $2g(X, L)+2$ and L is the pull back of $\mathcal{O}_{\mathbb{P}^n}(1)$. In this case we see that $g(X, L) \geq 1$, and if $g(X, L) = 1$, then (X, L) is a Del Pezzo manifold.

If (X, L) is the type (iii), then by [**18**, Proposition 2.2 in Example 2.1 (vii.7)] we have $\mathrm{cl}_i(X, L) = (2g(X, L)+2)(2g(X, L)+1)^{i-1}$ for $i \geq 1$ and $\mathrm{cl}_0(X, L) = 2$.

Example 3.1.4. Let (X, L) be a polarized manifold of dimension $n \geq 3$ such that $b_2(X, L) = h^2(X, \mathbb{C})+1$. Here we calculate $\mathrm{cl}_i(X, L)$ if (X, L) is the type (e) in [**17**, Theorem 3.1].

(i) If $(S, \mathcal{E})$ is the type (e.1), then $c_1(\mathcal{E}) = \mathcal{O}_{\mathbb{P}^2}(3)$, $c_1(\mathcal{E})^2 = 9$, $c_2(S) = 3$, $c_2(\mathcal{E}) = 2$, $K_S^2 = 9$, $K_S c_1(\mathcal{E}) = -9$ and $s_2(\mathcal{E}) = c_1(\mathcal{E})^2 - c_2(\mathcal{E}) = 7$. Hence by [**18**, Example 2.1 (x)]

i	0	1	2	3
$\mathrm{cl}_i(X, L)$	7	14	12	4

In this case, $(X, L) = (\mathbb{P}_S(\mathcal{E}), H(\mathcal{E}))$ is a Del Pezzo 3-fold with $L^3 = 7$.

(ii) If $(S, \mathcal{E})$ is the type (e.2), then $c_1(\mathcal{E}) = \mathcal{O}_{\mathbb{Q}^2}(2)$, $c_1(\mathcal{E})^2 = 8$, $c_2(S) = 4$, $c_2(\mathcal{E}) = 2$, $K_S^2 = 8$, $K_S c_1(\mathcal{E}) = -8$ and $s_2(\mathcal{E}) = c_1(\mathcal{E})^2 - c_2(\mathcal{E}) = 6$. Hence by [**18**, Example 2.1 (x)]

i	0	1	2	3
$\mathrm{cl}_i(X, L)$	6	12	12	4

In this case, $(X, L) = (\mathbb{P}_S(\mathcal{E}), H(\mathcal{E}))$ is a Del Pezzo 3-fold with $L^3 = 6$.

(iii) If $(S, \mathcal{E})$ is the type (e.3), then $c_1(\mathcal{E}) = 2H(\mathcal{F}) + \pi^* c_1(\mathcal{G})$, $c_1(\mathcal{E})^2 = 8$, $c_2(S) = 0$, $c_2(\mathcal{E}) = 2$, $K_S^2 = 0$, $K_S c_1(\mathcal{E}) = -4$ and $s_2(\mathcal{E}) = c_1(\mathcal{E})^2 - c_2(\mathcal{E}) = 6$. Hence by [**18**, Example 2.1 (x)]

i	0	1	2	3
$\mathrm{cl}_i(X, L)$	6	16	16	8

[4] $\Delta(X, L)$ denotes the Δ-genus of (X, L) (see [**5**, (2.2)]).

(iv) If $(S, \mathcal{E})$ is the type (e.4), then there exists a line bundle $\mathcal{O}_{\mathbb{P}^2}(2b)$ such that the branch locus $C \in |\mathcal{O}_{\mathbb{P}^2}(2b)|$. In this case $c_1(\mathcal{E}) = f^*\mathcal{O}_{\mathbb{P}^2}(2)$, $c_1(\mathcal{E})^2 = 8$, $c_2(S) = 2c_2(\mathbb{P}^2) + 2g(C) - 2 = 4b^2 - 6b + 6$, $c_2(\mathcal{E}) = 2$, $K_S^2 = 2(b-3)^2$, $K_S c_1(\mathcal{E}) = 4(b-3)$ and $s_2(\mathcal{E}) = c_1(\mathcal{E})^2 - c_2(\mathcal{E}) = 6$. Hence by [**18**, Example 2.1 (x)]

i	0	1	2	3
$\mathrm{cl}_i(X, L)$	6	$4b+8$	$4b^2+2b+6$	$4b$

If $b = 1$, then $(X, L) = (\mathbb{P}_S(\mathcal{E}), H(\mathcal{E}))$ is a Del Pezzo 3-fold with $L^3 = 6$.

3.2 - *The case where* $i = 1$

In this subsection, we consider the case where $i = 1$. Here we assume that $n \geq 3$. In this case by [**11**, Remark 3.1 (2)] we have

(2) $$\mathrm{cl}_1(X, L) = -e_1(X, L) + 2e_0(X, L) = 2g(X, L) - 2 + 2L^n.$$

Since $g(X, L) \geq 0$ and $L^n \geq 1$, we see that $\mathrm{cl}_1(X, L) \geq 0$. We also note that $\mathrm{cl}_1(X, L)$ is even.

Next we consider a classification of (X, L) with small $\mathrm{cl}_1(X, L)$.

(I) First we consider the case where $\mathrm{cl}_1(X, L) = 0$.

Proposition 3.2.1. *Let (X, L) be a polarized manifold of dimension $n \geq 3$. If $\mathrm{cl}_1(X, L) = 0$, then (X, L) is isomorphic to $(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))$.*

Proof. If $\mathrm{cl}_1(X, L) = 0$, then we have $g(X, L) = 0$ and $L^n = 1$ from the equality (2). Therefore we see from [**5**, (12.1) Theorem and (5.10) Theorem] that (X, L) is isomorphic to $(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))$. □

(II) Next we consider the case where $\mathrm{cl}_1(X, L) = 2$.

Proposition 3.2.2. *Let (X, L) be a polarized manifold of dimension $n \geq 3$. If $\mathrm{cl}_1(X, L) = 2$, then (X, L) is one of the following types.*

(a) $(\mathbb{Q}^n, \mathcal{O}_{\mathbb{Q}^n}(1))$.

(b) *A Del Pezzo manifold and $L^n = 1$. In this case, X is a weighted hypersurface of degree 6 in the weighted projective space $\mathbb{P}(3, 2, 1, \ldots, 1)$.*

(c) *A scroll over an elliptic curve B and $L^n = 1$. In this case, $(X, L) = (\mathbb{P}_B(\mathcal{E}), H(\mathcal{E}))$, where $\mathcal{E}$ is an ample vector bundle of rank n on B with $c_1(\mathcal{E}) = 1$.*

Proof. Then by the equality (2) we have $(g(X,L),L^n)=(0,2)$ or $(1,1)$. If (X,L) is the first type, then by [**5**, (12.1) Theorem and (5.10) Theorem] (X,L) is the type (a) above. If (X,L) is the last type, then we see from [**5**, (12.3) Theorem] that (X,L) is either the type (b) or the type (c) above. $\square$

(III) Next we consider the case where $\mathrm{cl}_1(X,L)=4$.

Proposition 3.2.3. *Let (X,L) be a polarized manifold of dimension $n\geq 3$. If $\mathrm{cl}_1(X,L)=4$, then (X,L) is one of the following types.*

(a) $(\mathbb{P}_{\mathbb{P}^1}(\mathcal{E}),H(\mathcal{E}))$, where $\mathcal{E}\cong\mathcal{O}_{\mathbb{P}^1}(1)\oplus\mathcal{O}_{\mathbb{P}^1}(1)\oplus\mathcal{O}_{\mathbb{P}^1}(1)$.

(b) *A Del Pezzo manifold and $L^n=2$. In this case, X is a double covering of $\mathbb{P}^n$ branched along a smooth hypersurface of degree 4 and L is the pull-back of $\mathcal{O}_{\mathbb{P}^n}(1)$.*

(c) *A scroll over an elliptic curve B and $L^n=2$. In this case, $(X,L)=(\mathbb{P}_B(\mathcal{E}),H(\mathcal{E}))$, where $\mathcal{E}$ is an ample vector bundle of rank n on B with $c_1(\mathcal{E})=2$.*

(d) *$K_X=(3-n)L$ and $L^n=1$ hold.*[5]

(e) *(X,L) is a simple blowing up of (M,A), where M is a double covering of $\mathbb{P}^n$ with branch locus being a smooth hypersurface of degree 6 and $A=\pi^*(\mathcal{O}_{\mathbb{P}^n}(1))$, where $\pi:M\to\mathbb{P}^n$ is its double covering.*

(f) *$(\mathbb{P}_S(\mathcal{E}),H(\mathcal{E}))$, where $(S,\mathcal{E})$ is one of the types* 1), 2-i) *and* 4-b) *in* [**4**, (2.25) Theorem].

(g) *A hyperquadric fibration over a smooth curve C. In this case C is one of the following types.*[6]

(g.1) *C is an elliptic curve, $b=1$ and $e=0$.*

(g.2) *$C\cong\mathbb{P}^1$, $\mathcal{E}\cong\mathcal{O}_{\mathbb{P}^1}(-1)\oplus\mathcal{O}_{\mathbb{P}^1}(-1)\oplus\mathcal{O}_{\mathbb{P}^1}\oplus\mathcal{O}_{\mathbb{P}^1}$ and $b=5$.*

(h) *$(\mathbb{P}_C(\mathcal{E}),H(\mathcal{E}))$, where C is a smooth curve of genus two and $\mathcal{E}$ is an ample vector bundle of rank n on C with $c_1(\mathcal{E})=1$.*

Proof. By the equality (2) in 3.2 we have $(g(X,L),L^n)=(0,3)$, $(1,2)$ or $(2,1)$. If $(g(X,L),L^n)=(0,3)$, then by [**5**, (12.1) Theorem and (5.10) Theorem] (X,L) is the type (a) above. If $(g(X,L),L^n)=(1,2)$, then we see from [**5**, (12.3) Theorem] that (X,L) is either the type (b) or (c) above. If $(g(X,L),L^n)=(2,1)$, then by using a list of

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[5] For some examples of this type, see [**3**, §2].

[6] We use Notation 3.1 (2).

a classification of polarized manifolds with $g(X,L)=2$ and $L^n=1$ (see [**3**, (1.10) Theorem, (3.7) and (3.30) Theorem]) we see that (X,L) is one of the types from (c) to (h) above. □

3.3 - *The case where* $i=2$

Here we classify polarized manifolds (X,L) such that L is spanned and $\mathrm{cl}_2(X,L)\leq 15$.

Theorem 3.3.1. *Let* (X,L) *be a polarized manifold* (X,L) *with* $\dim X=n\geq 3$. *Assume that L is spanned and* $\mathrm{cl}_2(X,L)\leq 15$. *Then* (X,L) *is one of the following.*

(a) $(\mathbb{P}^n,\mathcal{O}_{\mathbb{P}^n}(1))$. *In this case* $\mathrm{cl}_2(X,L)=0$.

(b) $(\mathbb{Q}^n,\mathcal{O}_{\mathbb{Q}^n}(1))$. *In this case* $\mathrm{cl}_2(X,L)=2$.

(c) *A scroll over a smooth curve. In this case* $3\leq \mathrm{cl}_2(X,L)\leq 15$.

(d) $(\mathbb{P}_S(\mathcal{E}),H(\mathcal{E}))$, *where* $(S,\mathcal{E})\cong(\mathbb{P}^2,\mathcal{O}_{\mathbb{P}^2}(1)\oplus\mathcal{O}_{\mathbb{P}^2}(1))$.
In this case $\mathrm{cl}_2(X,L)=3$.

(e) *A Del Pezzo manifold* (X,L) *with* $L^n\geq 2$. *In this case* $\mathrm{cl}_2(X,L)=12$.

Proof. We note that

$$\begin{aligned}\mathrm{cl}_2(X,L)&=b_2(X,L)-b_0(X)+2(b_1(X,L)-b_1(X))+b_0(X,L)-b_0(X)\\&=(b_2(X,L)-b_2(X))+(b_2(X)-b_0(X))+4(g(X,L)-h^1(\mathcal{O}_X))\\&\quad+(b_0(X,L)-b_0(X)).\end{aligned}$$

We also note that $b_0(X,L)\geq 1=b_0(X)$ and $b_2(X)\geq b_0(X)$. Since L is spanned, we have $b_2(X,L)\geq b_2(X)$ and $g(X,L)\geq h^1(\mathcal{O}_X)$ by [**11**, Proposition 3.3 (2)] and [**2**, Theorem 7.2.10]. Hence we get the following.

- If $0\leq \mathrm{cl}_2(X,L)\leq 3$, then $g(X,L)=h^1(\mathcal{O}_X)$ holds.
- If $4\leq \mathrm{cl}_2(X,L)\leq 7$, then $g(X,L)\leq h^1(\mathcal{O}_X)+1$ holds.
- If $8\leq \mathrm{cl}_2(X,L)\leq 11$, then $g(X,L)\leq h^1(\mathcal{O}_X)+2$ holds.
- If $\mathrm{cl}_2(X,L)=12$, then $g(X,L)\leq h^1(\mathcal{O}_X)+2$ or $L^n=1$ holds.
- If $\mathrm{cl}_2(X,L)=13$, then $g(X,L)\leq h^1(\mathcal{O}_X)+2$ or $L^n\leq 2$ holds.
- If $\mathrm{cl}_2(X,L)=14$, then $g(X,L)\leq h^1(\mathcal{O}_X)+2$ or $L^n\leq 2$ or $b_2(X,L)=b_2(X)$ holds.
- If $\mathrm{cl}_2(X,L)=15$, then $g(X,L)\leq h^1(\mathcal{O}_X)+2$ or $L^n\leq 2$ or $b_2(X,L)\leq b_2(X)+1$ holds.

Hence by [**12**, Theorem 4.1], [**17**, Theorem 3.1], Examples 3.1.1, 3.1.3 and 3.1.4 and [**18**, Example 2.1], we get the assertion. □

Next we consider the case where $\mathrm{cl}_2(X, L) = 16$.

Theorem 3.3.2. *Let* (X, L) *be a polarized manifold* (X, L) *with* $\dim X = n \geq 3$. *Assume that* L *is spanned and* $\mathrm{cl}_2(X, L) = 16$. *Then* (X, L) *is one of the following.*

- (a) $(\mathbb{P}_C(\mathcal{E}), H(\mathcal{E}))$, *where* C *is a smooth projective curve and* $\mathcal{E}$ *is an ample vector bundle of rank* n *on* C *with* $c_1(\mathcal{E}) = 16$.
- (b) *A hyperquadric fibration over an elliptic curve with* $e = 4$, $b = -2$ *and* $\mathcal{E}$ *is ample.*[7]
- (c) $(\mathbb{P}_S(\mathcal{E}), H(\mathcal{E}))$ *and* $(S, \mathcal{E}) \cong (\mathbb{P}_C(\mathcal{F}), \pi^*(\mathcal{G}) \otimes H(\mathcal{F}))$, *where* C *is an elliptic curve,* $\mathcal{F}$ *and* $\mathcal{G}$ *are indecomposable vector bundles of rank two on* C *with* $\deg \mathcal{F} = 1$ *and* $\deg \mathcal{G} = 1$, *and* $\pi : \mathbb{P}_C(\mathcal{F}) \to C$ *is the projection map.*

Proof. By the same argument as the proof of Theorem 3.3.1, either of the following 4 types occurs.

(i) $g(X, L) \leq h^1(\mathcal{O}_X) + 2$. (ii) $L^n \leq 2$. (iii) $b_2(X, L) \leq b_2(X) + 1$.
(iv) $g(X, L) = h^1(\mathcal{O}_X) + 3$, $L^n = 3$ and $b_2(X, L) = b_2(X) + 2$.

If (X, L) satisfies one of the cases (i), (ii), or (iii), then we see from [**12**, Theorem 4.1], [**17**, Theorem 3.1], Examples 3.1.1, 3.1.3 and 3.1.4, and [**18**, Example 2.1] that (X, L) is one of the types (a), (b) and (c) in Theorem 3.3.2. So we may assume that the case (iv) occurs. Then $\Delta(X, L) = n + L^n - h^0(L) \leq n + 3 - (n + 1) \leq 2$.

Claim 3.3.1. $h^1(\mathcal{O}_X) = 0$ *holds.*

Proof. If $\Delta(X, L) \leq 1$, then by [**5**, (5.10) Theorem and (6.7) Corollary] we get the assertion. So we may assume that $\Delta(X, L) = 2$. Then since L is spanned, $h^0(L) = n + 1$ and $L^n = 3$, the morphism $X \to \mathbb{P}^n$ defined by $|L|$ is a triple covering. So by [**29**, Theorem 7.1.15], we get the assertion. □

Hence $g(X, L) = 3$. Since $\mathrm{Bs}|L| = \emptyset$, we see from Example 3.1.2 that this case (iv) cannot occur. □

[7] We use Notation 3.1 (2).

3.4 - *The case where* $i = 3$

Here we consider a classification of (X, L) such that L is spanned and $\mathrm{cl}_3(X, L) \le 8$.

Theorem 3.4.1. *Let* (X, L) *be a polarized manifold with* $\dim X = n \ge 3$. *Assume that L is spanned and* $\mathrm{cl}_3(X, L) \le 8$. *Then* (X, L) *is one of the following.*

(a) $(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))$. *In this case* $\mathrm{cl}_3(X, L) = 0$.

(b) *A scroll over a smooth curve. In this case* $\mathrm{cl}_3(X, L) = 0$.

(c) $(\mathbb{Q}^n, \mathcal{O}_{\mathbb{Q}^n}(1))$. *In this case* $\mathrm{cl}_3(X, L) = 2$.

(d) $(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(2))$. *In this case* $\mathrm{cl}_3(X, L) = 4$.

(e) *A simple blowing up of* $(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(2))$. *In this case* $\mathrm{cl}_3(X, L) = 4$.

(f) $(\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1, \otimes_{i=1}^3 p_i^* \mathcal{O}_{\mathbb{P}^1}(1))$. *In this case* $\mathrm{cl}_3(X, L) = 4$.

(g) $(\mathbb{P}^2 \times \mathbb{P}^2, \otimes_{i=1}^2 p_i^* \mathcal{O}_{\mathbb{P}^2}(1))$. *In this case* $\mathrm{cl}_3(X, L) = 6$.

(h) *A hyperquadric fibration over a smooth curve C, and one of the following holds.*[8]

(h.1) $g(C) = 1$, $n = 3$, $L^3 = 6$, $e = 4$, $b = -2$, *and* $\mathcal{E}$ *is ample. In this case* $\mathrm{cl}_3(X, L) = 8$.

(h.2) $g(C) = 0$, $n = 3$, $L^3 = 9$, $e = 6$, $b = -3$ *and* $\mathcal{E} \cong \mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1}(2)$ (*see* [3, (3.30) Theorem 9)]). *In this case* $\mathrm{cl}_3(X, L) = 8$.

(i) $(\mathbb{P}_S(\mathcal{E}), H(\mathcal{E}))$ *and* $(S, \mathcal{E})$ *is one of the following.*

(i.1) $(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1) \oplus \mathcal{O}_{\mathbb{P}^2}(1))$. *In this case* $\mathrm{cl}_3(X, L) = 0$.

(i.2) $(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1) \oplus \mathcal{O}_{\mathbb{P}^2}(2))$. *In this case* $\mathrm{cl}_3(X, L) = 4$.

(i.3) $(\mathbb{Q}^2, \mathcal{O}_{\mathbb{Q}^2}(1) \oplus \mathcal{O}_{\mathbb{Q}^2}(1))$. *In this case* $\mathrm{cl}_3(X, L) = 4$.

(i.4) *S is a double covering* $f : S \to \mathbb{P}^2$ *branched along a smooth hypersurface of degree* 2, *and* $\mathcal{E} = f^*(\mathcal{O}_{\mathbb{P}^2}(1)) \oplus f^*(\mathcal{O}_{\mathbb{P}^2}(1))$. *In this case* $\mathrm{cl}_3(X, L) = 4$.

(i.5) $(\mathbb{P}^2, T_{\mathbb{P}^2})$. *In this case* $\mathrm{cl}_3(X, L) = 6$.

(i.6) $(\mathbb{P}_C(\mathcal{F}), \pi^*(\mathcal{G}) \otimes H(\mathcal{F}))$, *where C is an elliptic curve,* $\mathcal{F}$ *and* $\mathcal{G}$ *are indecomposable vector bundles of rank two on C with* $\deg \mathcal{F} = 1$ *and* $\deg \mathcal{G} = 1$, *and* $\pi : \mathbb{P}_C(\mathcal{F}) \to C$ *is the projection map. In this case* $\mathrm{cl}_3(X, L) = 8$.

[8] We use Notation 3.1 (2).

(i.7) *S is a double covering $f : S \to \mathbb{P}^2$ branched along a smooth hypersurface of degree 4, and $\mathcal{E} \cong f^*(\mathcal{O}_{\mathbb{P}^2}(1)) \oplus f^*(\mathcal{O}_{\mathbb{P}^2}(1))$. In this case* $\mathrm{cl}_3(X,L) = 8$.

(i.8) $(\mathbb{P}^1_\alpha \times \mathbb{P}^1_\beta, [H_\alpha + 2H_\beta] \oplus [H_\alpha + H_\beta])$ *and H_α (resp. H_β) is the ample generator of* $\mathrm{Pic}(\mathbb{P}_\alpha)$ *(resp.* $\mathrm{Pic}(\mathbb{P}_\beta)$*). In this case* $\mathrm{cl}_3(X,L) = 8$.

(i.9) *S is the blowing up of $\mathbb{P}^2$ at a point and $\mathcal{E} \cong [2H - E]^{\oplus 2}$, where H is the pull-back of $\mathcal{O}_{\mathbb{P}^2}(1)$ and E is the exceptional divisor. In this case* $\mathrm{cl}_3(X,L) = 8$.

Proof. We note that

$$\begin{aligned}\mathrm{cl}_3(X,L) &= b_3(X,L) - b_1(X) + 2(b_2(X,L) - b_2(X)) + b_1(X,L) - b_1(X)\\ &= (b_3(X,L) - b_3(X)) + (b_3(X) - b_1(X)) + 2(b_2(X,L) - b_2(X))\\ &\quad + 2(g(X,L) - h^1(\mathcal{O}_X)).\end{aligned}$$

We also note that $\mathrm{cl}_3(X,L)$ is even and $b_3(X) \geq b_1(X)$. Since L is spanned, we have $b_3(X,L) \geq b_3(X)$, $b_2(X,L) \geq b_2(X)$ and $g(X,L) \geq h^1(\mathcal{O}_X)$ by [**11**, Proposition 3.3 (2)] and [**2**, Theorem 7.2.10]. Hence we get the following.

- If $0 \leq \mathrm{cl}_3(X,L) \leq 2$, then $b_2(X,L) \leq b_2(X) + 1$ holds.
- If $\mathrm{cl}_3(X,L) = 4$, then $b_2(X,L) \leq b_2(X) + 1$ or $g(X,L) = h^1(\mathcal{O}_X)$ holds.
- If $\mathrm{cl}_3(X,L) = 6$, then $b_2(X,L) \leq b_2(X) + 1$ or $g(X,L) \leq h^1(\mathcal{O}_X) + 1$ holds.
- If $\mathrm{cl}_3(X,L) = 8$, then $b_2(X,L) \leq b_2(X) + 1$ or $g(X,L) \leq h^1(\mathcal{O}_X) + 2$ holds.

By [**12**, Theorem 4.1], [**17**, Theorem 3.1], Examples 3.1.1 and 3.1.4, and [**18**, Example 2.1] we get the assertion.[9] □

3.5 - *The case where $i = 4$*

Here we consider a classification of (X,L) such that L is spanned (resp. very ample) and $\mathrm{cl}_4(X,L) \leq 1$ (resp. $\mathrm{cl}_4(X,L) = 2$).

Theorem 3.5.1. *Let (X,L) be a polarized manifold (X,L) with* $\dim X = n \geq 4$. *Assume that L is spanned and* $\mathrm{cl}_4(X,L) \leq 1$. *Then (X,L) is one of the following.*

(a) $(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))$. *In this case* $\mathrm{cl}_4(X,L) = 0$.

(b) *A scroll over a smooth curve. In this case* $\mathrm{cl}_4(X,L) = 0$.

[9] We note that the type (2-1) (resp. (2-2), (2-3), (3-1) and (3-2)) in [8, Theorem 3.3] corresponds to (i.8) (resp. (i.9), (i.6), (h.1) and (h.2)).

Proof. We note that the following equality holds.

$$\begin{aligned}\mathrm{cl}_4(X,L) &= b_4(X,L) - b_2(X) + 2(b_3(X,L) - b_3(X)) + b_2(X,L) - b_2(X)\\ &= b_4(X,L) - b_4(X) + (b_4(X) - b_2(X)) + 2(b_3(X,L) - b_3(X))\\ &\quad + b_2(X,L) - b_2(X).\end{aligned}$$

Since L is spanned, we see from [**11**, Proposition 3.3 (2)] that $b_4(X,L) \geq b_4(X)$, $b_3(X,L) \geq b_3(X)$ and $b_2(X,L) \geq b_2(X)$ hold. Furthermore by the strong Lefschetz theorem, we have $b_4(X) \geq b_2(X)$. Hence if $\mathrm{cl}_4(X,L) \leq 1$, then $b_2(X,L) \leq b_2(X) + 1$. Since $n \geq 4$, we can easily check that (X,L) is one of the above types by [**12**, Theorem 4.1], [**17**, Theorem 3.1] and [**18**, Example 2.1]. □

Remark 3.5.1. If L is spanned, then there does not exist (X,L) with $\mathrm{cl}_4(X,L) = 1$.

Theorem 3.5.2. *Let (X,L) be a polarized manifold (X,L) with $\dim X = n \geq 5$. Assume that L is very ample and $\mathrm{cl}_4(X,L) = 2$. Then (X,L) is $(\mathbb{Q}^n, \mathcal{O}_{\mathbb{Q}^n}(1))$.*

Proof. By the same argument as above, (X,L) with $\mathrm{cl}_4(X,L) = 2$ satisfies one of the following.

(I) $b_2(X,L) \leq b_2(X) + 1$. (II) $b_4(X,L) = b_4(X)$.

(I) If $b_2(X,L) \leq b_2(X) + 1$ holds, then by [**12**, Theorem 4.1], [**17**, Theorem 3.1] and [**18**, Example 2.1] we see that (X,L) with $\mathrm{cl}_4(X,L) = 2$ is $(\mathbb{Q}^n, \mathcal{O}_{\mathbb{Q}^n}(1))$.

(II) Next we assume that $b_4(X,L) = b_4(X)$ holds. Then by [**12**, Theorem 4.2], we see that (X,L) is one of the following types since we assume that $n \geq 5$.

(II.1) $(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))$.

(II.2) A scroll over a smooth projective curve.

(II.3) $(\mathbb{P}_S(\mathcal{E}), H(\mathcal{E}))$, where S is a smooth projective surface and $\mathcal{E}$ is an ample vector bundle of rank $n-1$ on S.

(II.4) X is the Plücker embedding of $G(2,5)$ and $L = \mathcal{O}_X(1)$. In this case $n = 6$.

(II.5) X is a nonsingular hyperplane section of the Plücker embedding of $G(2,5)$ in $\mathbb{P}^9$ and $L = \mathcal{O}_X(1)$. In this case $n = 5$.

Then by calculating $\mathrm{cl}_4(X,L)$, we see from [**18**, Example 2.1] that $\mathrm{cl}_4(X,L) = 0$ (resp. 0, $c_2(\mathcal{E})$, 5 and 5) if (X,L) is the type (II.1) (resp. (II.2), (II.3), (II.4) and (II.5)). Hence we find that the type (II.3) is possible and in this case $c_2(\mathcal{E}) = 2$.

But by [**30**, Theorem 6.1] and [**25**], the rank of $\mathcal{E}$ is two and so we have $n = 3$. This contradicts the assumption that $n \geq 5$. □

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