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Khovanov-Kauffman homology for embedded graphs

Abstract. We discuss the question of extending Khovanov homology from links to embedded graphs, by using the Kauffman topological invariant of embedded graphs by associating a family of links and knots to a such graph by using some local replacements at each vertex in the graph. This new concept of Khovanov-Kauffman homology of an embedded graph is constructed to be the sum of the Khovanov homologies of all the links and knots associated to this graph.

1 - Introduction

The idea of categorification the Jones polynomial is realized by Khovanov Homology for links which is a new link invariant introduced by Khovanov [**4**], [**1**]. For each link L in S^3 he defined a graded chain complex, with grading preserving differentials, whose graded Euler characteristic is equal to the Jones polynomial of the link L. The idea of Khovanov Homology for graphs arises from the same idea of Khovanov homology for links by the categorifications the chromatic polynomial of graphs. This was done by L. Helme-Guizon and Y. Rong [**2**], for each graph G, they defined a graded chain complex whose graded Euler characteristic is equal to the chromatic polynomial of G. In our work we try to recall the Khovanov homology for links.

We discuss the question of extending Khovanov homology from links to embedded graphs. This is based on a result of Kauffman that constructs a topological

invariant of embedded graphs in the 3-sphere by associating to such a graph a family of links and knots obtained using some local replacements at each vertex in the graph. He showed that it is a topological invariant by showing that the resulting knot and link types in the family thus constructed are invariant under a set of Reidemeister moves for embedded graphs that determine the ambient isotopy class of the embedded graphs. We build on this idea and simply define the Khovanov homology of an embedded graph to be the sum of the Khovanov homologies of all the links and knots in the Kauffman invariant associated to this graph. Since this family of links and knots is a topologically invariant, so is the Khovanov-Kauffman homology of embedded graphs defined in this manner. We close this paper by giving an example of computation of Khovanov-Kauffman homology for an embedded graph using this definition.

2 - Khovanov homology

In the following we recall a homology theory for knots and links embedded in the 3-sphere. We discuss later how to extend it to the case of embedded graphs.

2.1 - *Khovanov homology for links*

In recent years, many papers have appeared that discuss properties of Khovanov Homology theory, which was introduced in [4]. For each link $L \in \mathbf{S}^3$, Khovanov constructed a bi-graded chain complex associated with the diagram D for this link L and applied homology to get a group $Kh^{i,j}(L)$, whose Euler characteristic is the normalized Jones polynomial.

$$\sum_{i,j} (-1)^i q^j dim(Kh^{i,j}(L)) = J(L).$$

He also proved that, given two diagrams D and D' for the same link, the corresponding chain complexes are chain equivalent, hence, their homology groups are isomorphic. Thus, Khovanov homology is a link invariant.

2.2 - *The link cube*

Let L be a link with n crossings. At any small neighborhood of a crossing we can replace the crossing by a pair of parallel arcs and this operation is called a resolution. There are two types of these resolutions called 0-resolution (Horizontal resolution) and 1-resolution (Vertical resolution) as illustrated in Figure 1.
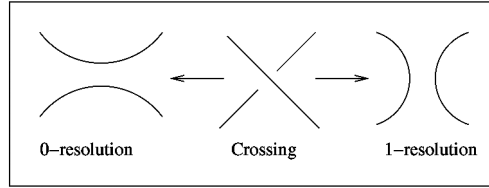
Fig. 1. 0 and 1- resolutions to each crossing.

We can construct a n-dimensional cube by applying the 0 and 1-resolutions n times to each crossing to get 2^n pictures called smoothings (which are one dimensional manifolds) S_α. Each of these can be indexed by a word α of n zeros and ones, i.e. $\alpha \in \{0,1\}^n$. Let ξ be an edge of the cube between two smoothings S_{α_1} and S_{α_2}, where S_{α_1} and S_{α_2} are identical smoothings except for a small neighborhood around the crossing that changes from 0 to 1-resolution. To each edge ξ we can assign a cobordism Σ_ξ (orientable surface whose boundary is the union of the circles in the smoothing at either end)

$$\Sigma_\xi : S_{\alpha_1} \longrightarrow S_{\alpha_2}.$$

This Σ_ξ is a product cobordism except in the neighborhood of the crossing, where it is the obvious saddle cobordism between the 0 and 1-resolutions. Khovanov constructed a complex by applying a $1+1$-dimensional TQFT (Topological Quantum Field Theory) which is a monoidal functor, by replacing each vertex S_α by a graded vector space V_α and each edge (cobordism) Σ_ξ by a linear map d_ξ. We set the group $CKh(D)$ to be the direct sum of the graded vector spaces for all the vertices and the differential on the summand $CKh(D)$ is a sum of the maps d_ξ for all edges ξ such that Tail(ξ)$= \alpha$ i.e.

$$(2.1) \qquad\qquad d^i(v) = \sum_\xi sign(-1)d_\xi(v)$$

where $v \in V_\alpha \subseteq CKh(D)$ and $sign(-1)$ is chosen such that $d^2 = 0$.

An element of $CKh^{i,j}(D)$ is said to have homological grading i and q-grading j where

$$(2.2) \qquad\qquad i = |\alpha| - n_-$$

$$(2.3) \qquad\qquad j = deg(v) + i + n_- + n_+$$

for all $v \in V_\alpha \subseteq CKh^{i,j}(D)$, $|\alpha|$ is the number of 1's in α, and n_-, n_+ represent the number of negative and positive crossings respectively in the diagram D.

2.3 - *Properties*

Here we recall some properties of Khovanov homology [**6**], [**5**].

Proposition 2.1.   (1) *If D' is a diagram obtained from D by the application of a Reidemeister moves then the complexes $(CKh^{*,*}(D))$ and $(CKh^{*,*}(D'))$ are homotopy equivalent.*

(2) *For an oriented link L with diagram D, the graded Euler characteristic satisfies*

$$(2.4) \qquad \sum (-1)^i qdim(Kh^{i,*}(L)) = J(L)$$

where $J(L)$ is the normalized Jones Polynomials for a link L and

$$\sum (-1)^i qdim(Kh^{i,*}(D)) = \sum (-1)^i qdim(CKh^{i,*}(D)).$$

(3) *Let L_{odd} and L_{even} be two links with odd and even number of components then $Kh^{*,even}(L_{odd}) = 0$ and $Kh^{*,odd}(L_{even}) = 0$.*

(4) *For two oriented link diagrams D and D', the chain complex of the disjoint union $D \sqcup D'$ is given by*

$$(2.5) \qquad CKh(D \sqcup D') = CKh(D) \otimes CKh(D').$$

(5) *For two oriented links L and L', the Khovanov homology of the disjoint union $L \sqcup L'$ satisfies*

$$Kh(L \sqcup L') = Kh(L) \otimes Kh(L').$$

(6) *Let D be an oriented link diagram of a link L with mirror image D^m diagram of the mirror link L^m. Then the chain complex $CKh(D^m)$ is isomorphic to the dual of $CKh(D)$ and*

$$Kh(L) \cong Kh(L^m).$$

3 - Khovanov-Kauffman homology for embedded graphs (KKh(G))

3.1 - *Kauffman's invariant of graphs*

We give now a survey of the Kauffman theory and show how to associate to an embedded graph in $\mathbf{S}^3$ a family of knots and links. We then use these results to give our definition of Khovanov homology for embedded graphs. In [**3**] Kauffman introduced a method for producing topological invariants of graphs embedded in $\mathbf{S}^3$.
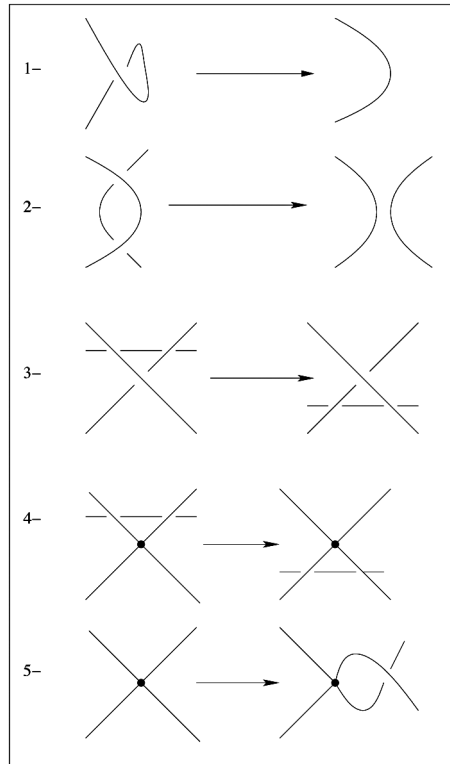
Fig. 2. Generalized Reidemeister moves by Kauffman.

The idea is to associate a collection of knots and links to a graph G so that this family is an invariant under the expanded Reidemeister moves defined by Kauffman and reported here in Figure 2.

He defined in his work an ambient isotopy for non-rigid (topological) vertices. (Physically, the rigid vertex concept corresponds to a network of rigid disks each with (four) flexible tubes or strings emanating from it.) Kauffman proved that piecewise linear ambient isotopies of embedded graphs in $\mathbf{S}^3$ correspond to a sequence of generalized Reidemeister moves for planar diagrams of the embedded graphs.

Theorem 3.1. *Any piecewise linear (PL) ambient isotopy of embedded graphs is generated by the moves of Figure 2, that is, if two embedded graphs are ambient isotopic, then any two diagrams of them are related by a finite sequence of the moves of Figure 2* [**3**].
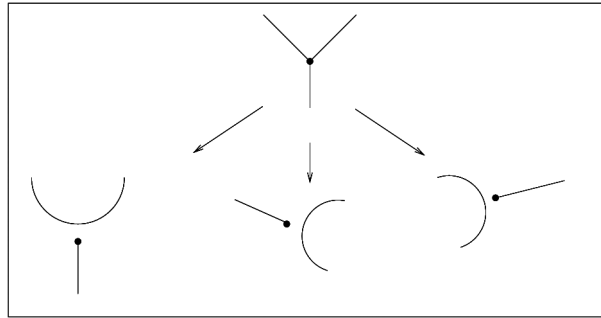
Fig. 3. Local replacement to a vertex in the graph G.

Let G be a graph embedded in $\mathbf{S}^3$. The procedure described by Kauffman of how to associate to G a family of knots and links prescribes that we should make a local replacement as in Figure 3 to each vertex in G. Such a replacement at a vertex v connects two edges and isolates all other edges at that vertex, leaving them as free ends. Let $r(G, v)$ denote the link formed by the closed curves formed by this process at a vertex v. One retains the link $r(G, v)$, while eliminating all the remaining unknotted arcs. Define then $T(G)$ to be the family of the links $r(G, v)$ for all possible replacement choices,

$$T(G) = \cup_{v \in V(G)} r(G, v).$$

For example see Figure 4.

Theorem 3.2. *Let G be any graph embedded in $\mathbf{S}^3$, and presented diagrammatically. Then the family of knots and links $T(G)$, taken up to ambient isotopy, is a topological invariant of G* [3].

For example, in the Figure 4 the graph G_2 is not ambient isotopic to the graph G_1, since $T(G_2)$ contains a non-trivial link.
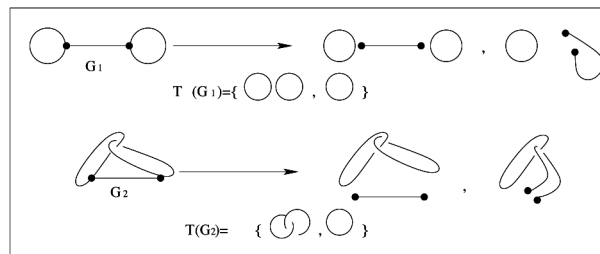

Fig. 4. Family of links associated to a graph.

3.2 - *Definition of Khovanov-Kauffman homology for embedded graphs KKh(G)*

Now we are ready to speak about a new concept of Khovanov-Kauffman homology for embedded graphs by using Khovanov homology for the links (knots) and Kauffman theory of associate a family of links to an embedded graph G, as described above.

Definition 3.3.   Let G be an embedded graph with $T(G) = \{L_1, L_2, ..., L_n\}$ the family of links associated to G by the Kauffman procedure. Let $Kh(L_i)$ be the usual Khovanov homology of the link L_i in this family. Then the *Khovanov-Kauffman homology for the embedded graph G* (*KKh(G)*) is given by

$$KKh(G) = Kh(L_1) \oplus Kh(L_2) \oplus .... \oplus Kh(L_n).$$

Its graded Euler characteristic is the sum of the graded Euler characteristics of the Khovanov homology of each link, *i.e.* the sum of the Jones polynomials,

(3.1) $$\sum_{i,j,k} (-1)^i q^j dim(Kh^{i,j}(L_k)) = \sum_k J(L_k).$$

We show some simple explicit examples.

Example 3.4.   In Figure 4 $T(G_1) = \{\bigcirc\bigcirc, \bigcirc\}$ then for $Kh(\bigcirc) = \mathbb{Q}$

$$KKh(G_1) = Kh(\bigcirc\bigcirc) \oplus Kh(\bigcirc).$$

Now, from Proposition 2.1 no. 5

$$KKh(G_1) = Kh(\bigcirc) \otimes Kh(\bigcirc) \oplus Kh(\bigcirc)$$

$$KKh(G_1) = \mathbb{Q} \otimes \mathbb{Q} \oplus \mathbb{Q} = \mathbb{Q} \oplus \mathbb{Q}.$$

Another example comes from $T(G_2) = \{\overset{\frown}{\mathcal{C}\mathcal{O}}, \bigcirc\}$ then

$$KKh(G_2) = Kh(\overset{\frown}{\mathcal{C}\mathcal{O}}) \oplus Kh(\bigcirc).$$

Since $Kh^{0,0}(\bigcirc) = \mathbb{Q}$, and from [6]

$$Kh(\overset{\frown}{\mathcal{C}\mathcal{O}}) =$$

j \ i	-2	-1	0
0			$\mathbb{Q}$
-1			
-2			$\mathbb{Q}$
-3			
-4	$\mathbb{Q}$		
-5			
-6	$\mathbb{Q}$		

Then,

$$KKh(G_2) =$$

i \ j	-2	-1	0
0			$\mathbb{Q} \oplus \mathbb{Q}$
-1			
-2			$\mathbb{Q}$
-3			
-4	$\mathbb{Q}$		
-5			
-6	$\mathbb{Q}$		

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