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Cobordism group for embedded graphs

Abstract. We construct a cobordism group for embedded graphs using sequences of two basic operations, called "fusion" and "fission", which in terms of cobordisms correspond to the basic cobordisms obtained by attaching or removing a 1-handle.

Keywords. Graph cobordism group, embedded graphs, fusion and fission.

Mathematics Subject Classification (2010): 14L35.

1 - Introduction

The construction of the cobordism group for links and for knots and their relation is given in [**2**]. We then consider the question of constructing a similar cobordism group for embedded graphs in the 3-sphere. We show that this can actually be done in two different ways, both of which reduce to the same notion for links. The first one comes from the description of the cobordisms for links in terms of sequences of two basic operations, called "fusion" and "fission", which in terms of cobordisms correspond to the basic cobordisms obtained by attaching or removing a 1-handle. We define analogous operations of fusion and fission for embedded graphs and we introduce an equivalence relation of cobordism by iterated application of these two operations.

The second possible definition of cobordism of embedded graphs is a surface (meaning here 2-complex) in $S^3 \times [0,1]$ with boundary the union of the given graphs. While for links, where cobordisms are realized by smooth surfaces, these can always

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be decomposed into a sequence of handle attachments, hence into a sequence of fusions and fissions, in the case of graphs not all cobordisms realized by 2-complexes can be decomposed as fusions and fissions, hence the two notions are no longer equivalent.

2 - Knots and links cobordism groups

A notion of knot cobordism group and link cobordism group can be given by using cobordism classes of knots and links to form a group [**1**],[**2**]. The link cobordism group splits into the direct sum of the knot cobordism group and an infinite cyclic group which represents the linking number, which is invariant under link cobordism [**2**]. In this part we will give a survey about both knot and link cobordism groups. In a later part of this work we will show that the same idea can be adapted to construct a graph cobordism group.

2.1 - *Knot cobordism group*

We recall the concept of cobordism between knots introduced in [**1**]. Two knots K_1 and K_2 are called knot cobordic if there is a locally flat cylinder S in $\mathbf{S}^3 \times [0,1]$ with boundary $\partial S = K_1 \cup -K_2$ where $K_1 \subset \mathbf{S}^3 \times \{0\}$ and $K_2 \subset \mathbf{S}^3 \times \{1\}$. We then write $K_1 \sim K_2$. The critical points in the cylinder are assumed be minima (birth), maxima (death), and saddle points. In the birth point at some t_0 there is a sudden appearance of a point. The point becomes an unknotted circle in the level immediately above t_0. At a maxima or death point, a circle collapses to a point and disappears from higher levels.

For the saddle point, two curves touch and rejoin as illustrated in Figure 2.

These saddle points are of two types: *negative* if with increasing t the number of components of the cross sections decreases and *positive* if the number increases.

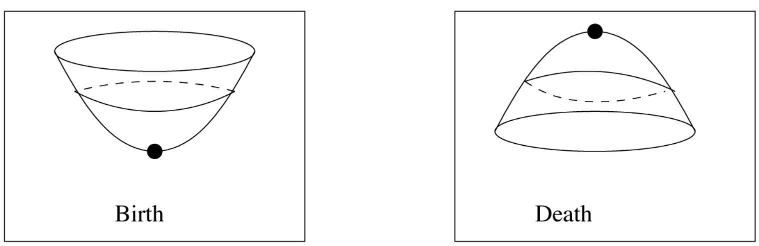

Fig. 1. Death and Birth Points.

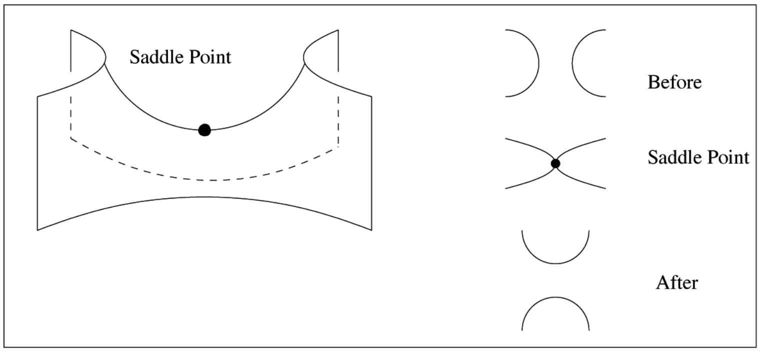

Fig. 2. Saddle Point.

A transformation from one cross section to another is called negative hyperbolic transformation if there is only one saddle point between the two cross sections and if the number of components decreases. We can define analogously a positive hyperbolic transformation.

Definition 2.1. [**2**] We say that two knots K_1 and K_2 are *related by an elementary cobordism* if the knot K_2 is obtained by $r-1$ negative hyperbolic transformations from a split link consisting of K_1 together with $r-1$ circles.

What we mean by split link is a link with n components ($K_i, i = 1 \dots n$) in $\mathbf{S}^3$ such that there are mutually disjoint n 3-cells ($D_i, i = 1 \dots n$) containing $K_i, i = 1, 2 \dots, n$.

Lemma 2.2. [**2**] *Two knots are called knot cobordic if and only if they are related by a sequence of elementary cobordisms.*

It is well known that the oriented knots form a commutative semigroup under the operation of composition $\#$. Given two knots K_1 and K_2, we can obtain a new knot by removing a small arc from each knot and then connecting the four endpoints by two new arcs. The resulting knot is called the composition of the two knots K_1 and K_2 and is denoted by $K_1 \# K_2$.

Notice that, if we take the composition of a knot K with the unknot $\bigcirc$ then the result is again K.

Lemma 2.3. *The set of oriented knots with the connecting operation $\#$ forms a semigroup with identity $\bigcirc$.*

Fox and Milnor [1] showed that composition of knots induces a composition on knot cobordism classes $[K] \# [K']$. This gives an abelian group G_K with $[\bigcirc]$ as identity and the negative is $-[K] = [-K]$, where the $-K$ denotes the reflected inverse of K.

Theorem 2.4. *The knot cobordism classes with the connected sum operation $\#$ form an abelian group, called the knot cobordism group and denoted by G_K.*

2.2 - *Link cobordism group*

For links, [2] the conjunction operation & between two links gives a commutative semigroup. $L_1 \& L_2$ is a link represented by the union of the two links $l_1 \cup l_2$ where l_1 represents L_1 and l_2 represents L_2 with mutually disjoint 3-cells D_1 and D_2 contain l_1 and l_2 respectively. Here "represents" means that we are working with ambient isotopy classes L_i of links (also called *link types*) and the l_i are chosen representatives of these classes. In the following we loosely refer to the classes L_i also simply as links, with the ambient isotopy equivalence implicitly understood. The zero of this semigroup is the link consisting of just the empty link. The link cobordism group is constructed using the conjunction operation and the cobordism classes. We recall below the definition of cobordism of links.

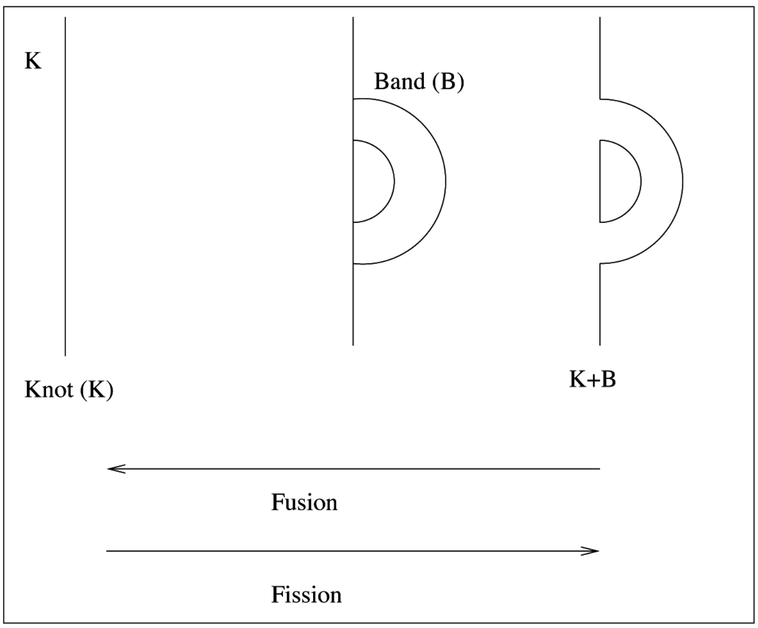

Fig. 3. Band.

Let L be a link in $\mathbf{S}^3$ containing r-components $k_1, ..., k_r$ with a split sublink $L_s = k_1 \cup k_2 \cup .... \cup k_t,\ t \leq r$ of L. Define a knot $\hat{\mathbf{K}}$ to be $k_1 + k_2 + ... + k_t + \partial B_{t+1} + \partial B_{t+2} + ... + \partial B_r$ where $\{B_{t+1}, B_{t+2}, B_{t+3} ...., B_r\}$ are disjoint bands in $\mathbf{S}^3$ spanning L_s [**2**]. The operation $+$ means additions in the homology sense. Put $L_1 = L_s \cup k_{t+1} \cup k_{t+2} .... \cup k_r$ and $L_2 = \hat{\mathbf{K}} \cup k_{t+1} \cup k_{t+2} .... \cup k_r$. Now, the operation of replacing L_1 by L_2 is called **fusion** and L_2 by L_1 is called **fission**.

Definition 2.5. [**2**] Two links will be called *link cobordic* if one can be obtained from the other by a sequence of fusions and fissions. This equivalence relation is denoted by $\simeq$. $[L]$ denotes the link cobordism class of L.

Theorem 2.6. [**2**] *The link cobordism classes with the conjunction operation form an abelian group, called the link cobordism group and denoted by* G_L.

Proof. For two cobordism classes $[L_1]$ and $[L_2]$ the multiplication between them is well defined and given by

$$[L_1] \,\&\, [L_2] = [L_1 \,\&\, L_2].$$

The zero of this operation is the class $[\bigcirc]$ which is the trivial link of a countable number of components. The negative of $[L]$ is $-[L] = [-L]$, where $-L$ denoted the reflected inverse of L. $\square$

Lemma 2.7. *For any link L, a conjunction* $L \,\&\, -L$ *is link cobordic to zero.*

To study the relation between the knot cobordism group G_K and link cobordism group G_L define a natural mapping $f : G_K \longrightarrow G_L$ which assigns to each knot cobordism class $[k]$ the corresponding link cobordism class $[L]$ where L is the knot k viewed as a one-component link. We claim that f is a homomorphism. f is well defined from the following lemma

Lemma 2.8. [**2**] *Two knots are link cobordic if and only if they are knot cobordic.*

Now, $K_1 \# K_2$ is a fusion of $K_1 \,\&\, K_2$ then $K_1 \# K_2$ is cobordic to $K_1 \,\&\, K_2$, therefore f is a homomorphism. Again by using the lemma 2.8, if a knot is link cobordic to zero then it is also knot cobordic to zero, and hence $ker(f)$ consists of just $\bigcirc$.

Lemma 2.9. *f is an isomorphism of* G_K *onto a subgroup of* G_L.

Theorem 2.10. [2] *$f(G_K)$ is a direct summand of G_L and it is a subgroup of G_L whose elements have total linking number zero. The other summand is isomorphic to the additive group of integers.*

3 - Graphs and cobordisms

3.1 - *Graph cobordism group*

In this section we construct cobordism groups for embedded graphs by extending the notions of cobordisms used in the case of links.

Definition 3.1. Two graphs E_1 and E_2 are called *cobordic* if there is a surface S have the boundary $\partial S = E_1 \cup -E_2$ with $E_1 = S \cap (\mathbf{S}^3 \times \{0\})$, $E_2 = S \cap (\mathbf{S}^3 \times \{1\})$ and we set $E_1 \sim E_2$. Here by "surfaces" we mean 2-dimensional simplicial complexes that are PL-embedded in $\mathbf{S}^3 \times [0,1]$. $[E]$ denotes the cobordism class of the graph E.

By using the graph cobordism classes and the conjunction operation &, we can induce a graph cobordism group. $E_1 \,\&\, E_2$ is a graph represented by the union of the two graphs $E_1 \cup E_2$ with mutually disjoint 3-cells D_1 and D_2 containing (representatives of) E_1 and E_2, respectively. Here again we do not distinguish in the notation between the ambient isotopy classes of embedded graphs (graph types) and a choice of representatives.

Lemma 3.2. *The graph cobordism classes in the sense of Definition* 3.1 *with the conjunction operation form an abelian group called the graph cobordism group and denoted by G_E.*

Proof. For two cobordism classes $[E_1]$ and $[E_2]$ the operation between them is given by

$$[E_1] \,\&\, [E_2] = [E_1 \,\&\, E_2].$$

This operation is well defined. To show that : Suppose $E_1 \sim F_1$, for two graphs E_1 and F_1. Then there exists a surface S_1 with boundary $\partial S_1 = E_1 \cup -F_1$. Suppose also, $E_2 \sim F_2$, for two graphs E_2 and F_2. Then there exists a surface S_2 with boundary $\partial S_2 = E_2 \cup -F_2$. We want to show that $E_1 \,\&\, E_2 \sim F_1 \,\&\, F_2$, *i.e.* we want to find a surface S with boundary $\partial S = (E_1 \,\&\, E_2) \cup -(F_1 \,\&\, F_2)$.

Define the cobordism S to be $S_1 \,\&\, S_2$ where $S_1 \,\&\, S_2$ represents $S_1 \cup S_2$ with mutually disjoint 4-cells $D_1 \times [0,1]$ and $D_2 \times [0,1]$, containing S_1 and S_2 respectively with $D_1 \times \{0\}$ containing E_1, $D_2 \times \{0\}$ containing F_1, $D_1 \times \{1\}$ containing E_2 and $D_2 \times \{1\}$ containing F_2. The boundary of S is given by,

$$\partial S = \partial(S_1 \,\&\, S_2) = \partial S_1 \,\&\, \partial S_2 = \partial S_1 \cup \partial S_2 = (E_1 \cup -F_1) \cup (E_2 \cup -F_2) = (E_1 \,\&\, E_2) \cup -(F_1 \,\&\, F_2).$$

Then the operation is well defined. The zero of this operation is the class [○] which is the trivial graph of a countable number of components. The negative of $[E]$ is $-[E] = [-E]$, where $-E$ denoted the reflected inverse of E. □

3.2 - *Fusion and fission for embedded graphs*

We now describe a special kind of cobordisms between embedded graphs, namely the basic cobordisms that correspond to attaching a 1-handle and that give rise to the analog in the context of graphs of the operations of fusion and fission described already in the case of links. Let E be a graph containing n-components with a split subgraph $E_s = G_1 \cup G_2 \cup G_3 ... \cup G_t$. We can define a new graph $\hat{E}$ to be $G_1 + G_2 + G_3 ... + G_t + \partial B_{t+1} + \partial B_{t+2} + ... + \partial B_n$ where $\{B_{t+1}, B_{t+2}, B_{t+3} ...., B_n\}$ are disjoint bands in $\mathbf{S}^3$ spanning E_s. The graph $\hat{E}$ depends on E and on a choice of a vertex in each G_i, see Figure 4. The operation $+$ means addition in the homology sense. Put $E_1 = E_s \cup G_{t+1} \cup G_{t+2} .... \cup G_n$ and $E_2 = \hat{E} + G_{t+1} + G_{t+2} .... + G_n$. Now, the operation of replacing E_1 by E_2 is called **fusion** and E_2 by E_1 is called **fission**.

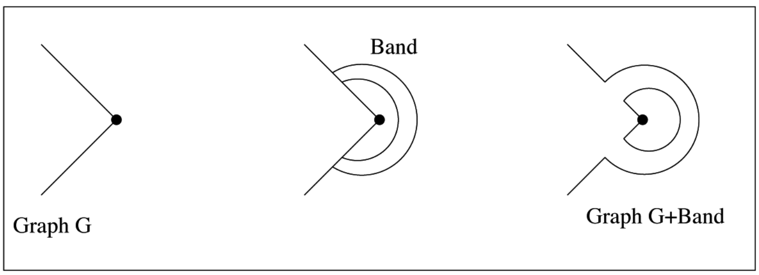

Fig. 4. Attaching a Band to a Graph.

Notice that, in order to make sure that all resulting graphs will still have at least one vertex, one needs to assume that the 1-handle is attached in such a way that there is at least an intermediate vertex in between the two segments where the 1-handle is attached, as the Figure 4, hence the dependence on the choice of vertices.

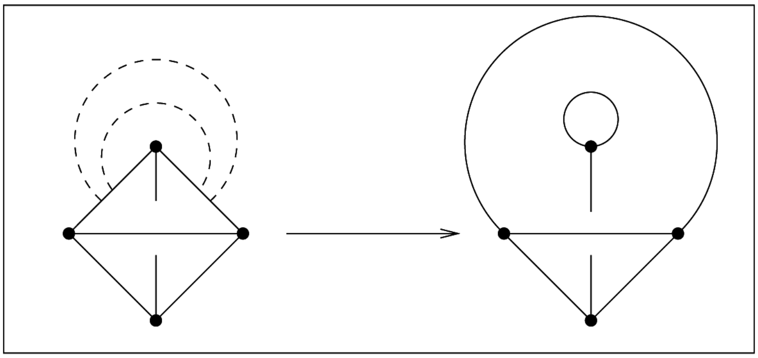

Fig. 5. Attaching a Band does not change the number of graph components.

Remark 3.3. Unlike the case of links, a fusion and fission for graphs does not necessarily change the number of components. For example see the Figure 5.

We can use the operations of fusion and fission described above to give another possible definition of cobordism of embedded graphs.

Definition 3.4. Two graphs will be called *graph cobordic* if one can be obtained from the other by a sequence of fusions and fissions. We denote this equivalence relation by $\simeq$, and by $[E]$ the graph cobordism class of E.

Thus we have two corresponding definitions for the graph cobordism group. One can check from the definition of fusion and fission that they give the existence of a cobordism (surface) between two graphs E_1 and E_2.

Lemma 3.5. *Two graphs E_1 and E_2 that are cobordant in the sense of Definition* 3.4 *are also cobordant in the sense of Definition* 3.1. *The converse, however, is not necessary true.*

Proof. As we have seen, a fusion/fission operation is equivalent to adding or removing a band to a graph and this implies the existence of a saddle cobordism given by the attached 1-handle, as illustrated in Figure 2. By combining this cobordism with the identity cobordism in the region outside where the 1-handle is attached, one obtains a PL-cobordism between E_1 and E_2. This shows that cobordism in the sense of Definition 3.4 implies cobordism in the sense of Definition 3.1. The reason why the converse need not be true is that, unlike what happens with the cobordisms given by embedded smooth surfaces used in the case of links, the cobordisms of graphs given by PL-embedded 2-complexes are not always decomposable as a finite set of fundamental saddle cobordism given by a 1-handle. Thus, having a PL-cobordism (surface in the sense of a 2-complex) between two embedded graphs E_1 and E_2 does not necessarily imply the existence of a finite sequence of fusions and fissions. In fact, observe that the number of vertices is preserved under fusions and fissions and the number of edges is also preserved, so the Euler characteristic is also preserved. Then one obtains an example by considering a cobordism by a PL embedded 2-complex that does not preserve the euler characteristic such as the figure below.

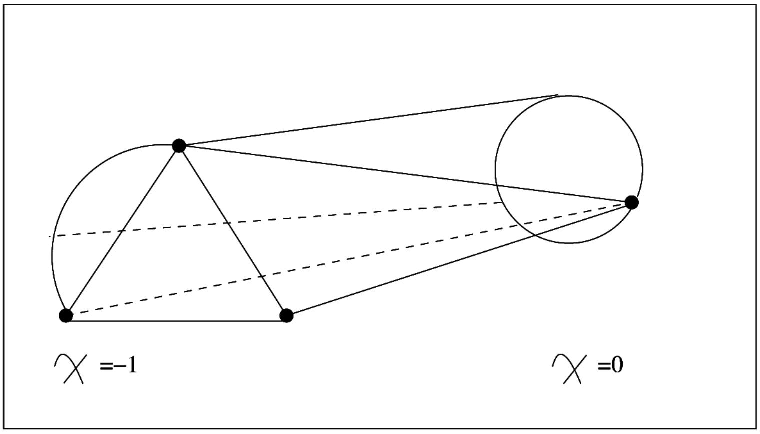

Fig. 6.

□

Lemma 3.6. *The graph cobordism classes in the sense of Definition* 3.4 *with the conjunction operation form an abelian group called the graph cobordism group and denoted by* G_F.

Proof. The proof is the same as the proof on lemma 2 since fusion and fission are a special case of cobordisms. □

The result of Lemma 3.5 shows that there are different equivalence classes $[E_1] \neq [E_2]$ in G_F that are identified $[E_1] = [E_2]$ in G_E. Thus, the number of cobordism classes when using Definition 3.1 is smaller that the number of classes by the fusion/fission method of Definition 3.4.

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