Riv. Mat. Univ. Parma, Vol. **5** (2014), 387-397

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A Whitney-type result about rectifiability of graphs

Abstract. Given $m > 0$ and a measurable set $E \subset \mathbb{R}^n$, $E^{(m)}$ denotes the set of m-density points of E, namely the points $x \in \mathbb{R}^n$ at which $\mathcal{L}^n(B(x,r)\backslash E)$ is an infinitesimal of order greater than r^m (as $r \to 0$). We investigate the size of $E^{(m)}$ in the particular case when E is a generalized Cantor set in $\mathbb{R}$. Moreover we prove the following result. Let $\varphi \in C^h(\Omega)$ and $\Phi \in C^h(\Omega; \mathbb{R}^n)$), where Ω is an open subset of $\mathbb{R}^n$ and $h \geq 1$. If $K := \{x \in \Omega \,|\, \nabla\varphi(x) = \Phi(x)\}$ then the graph of $\varphi_{|\Omega \cap K^{(n+h)}}$ is a n-dimensional C^{h+1}-rectifiable set.

Keywords. Rectifiable sets, geometric measure theory, Whitney extension theorem.

Mathematics Subject Classification (2010): Primary 49Q15, 28A75, 28A78; Secondary 28A12.

1 - Introduction

Definition 1.1. Let E be a measurable subset of $\mathbb{R}^n$ and $m > 0$. Then $x \in \mathbb{R}^n$ is said to be a "*m-density point of E*" if

$$\lim_{r \downarrow 0} \frac{\mathcal{L}^n(B(x,r)\backslash E)}{r^m} = 0.$$

The set of all m-density points of E will be denoted by $E^{(m)}$.

It is well-known that almost every point in a measurable subset E of $\mathbb{R}^n$ is a n-density point of E. Does such a statement hold with m in place of n, provided

Received: June 24, 2013; accepted: September 3, 2013.

$m > n$? Surprisingly enough the answer is "yes" when E is a locally finite perimeter set and $m = n + n/(n-1)$, as we proved in [**5**]. Such a fact supports the idea that locally finite perimeter sets are "closer" to open sets (for which one obviously has $E^{(m)} \supset E$ for all m) than generic sets with positive measure. For an arbitrary measurable set E of positive measure the answer is "no". As we will show below, one can even produce many examples of sets E of positive measure with $E^{(n+1)} = \emptyset$.

Just as one expects, for $m > n$, the notion of m-density reveals to be useful in extending arguments based on blow-up from open sets to more general situations. An example is given by the following result, proved in [**5**] and generalizing a well-known classical theorem.

Theorem 1.1. *Let λ and μ be differential forms of class C^1 in an open subset Ω of $\mathbb{R}^n$, respectively of degree h and $h+1$ (with $h \geq 1$). If define*

$$K := \{x \in \Omega \mid d\lambda(x) = \mu(x)\}$$

then $\Omega \cap K^{(n+1)} \subset K$ and $(d\mu)|_{\Omega\cap K^{(n+1)}} = 0$.

Corollary 1.1. *Let $\Phi \in C^1(\Omega; \mathbb{R}^n)$ and $\varphi \in C^1(\Omega)$, where Ω is an open subset of $\mathbb{R}^n$. If define*

$$K := \{x \in \Omega \mid \nabla\varphi(x) = \Phi(x)\}$$

then $\Omega \cap K^{(n+1)} \subset K$ and $(\mathrm{curl}\,\Phi)|_{\Omega\cap K^{(n+1)}} = 0$.

Another example is this theorem about C^2-rectifiability of graphs, which we obtained in [**5**] by combining Corollary 1.1 with results in [**2**] (largely based on [**3**]) and in [**4**].

Theorem 1.2. *Let Φ, φ and K be as in Corollary* 1.1. *Then the graph of $\varphi|_{\Omega\cap K^{(n+1)}}$ is a n–dimensional C^2-rectifiable set, namely $\mathcal{H}^n$-almost all of it may be covered by countably many n-dimensional submanifolds of class C^2.*

Now we can show how to get examples of sets E of positive measure with $E^{(n+1)} = \emptyset$. First consider $\Phi \in C^1(\mathbb{R}^n; \mathbb{R}^n)$ such that $\mathrm{curl}\,\Phi$ is never vanishing. Then by [**1**, Theorem 1] there exist an open subset A of $\mathbb{R}^n$ and a function $\varphi \in C^1_c(\mathbb{R}^n)$ such that

$$K := \{x \in \mathbb{R}^n \mid \nabla\varphi(x) = \Phi(x)\} \supset \mathbb{R}^n \backslash A, \qquad \mathcal{L}^n(A) \leq 1.$$

Since $K^{(n+1)} = \emptyset$ by Corollary 1.1, one has $E^{(n+1)} = \emptyset$ for every measurable subset E of K.

In Section 2 of this paper we face the problem of computing the size of $E^{(m)}$ in the remarkable case when E is a generalized Cantor set in $\mathbb{R}$. Two main results concerning this problem will be presented. The first one states that for a wide family of "suitably thin" generalized Cantor sets E one has $\mathcal{L}^1(E^{(m)}) = 0$, provided $m \geq 2$ (Corollary 2.1). Then, for every fixed $m > 1$ and $\varepsilon > 0$, we show how to construct a generalized Cantor set E in $\mathbb{R}$ such that $\mathcal{L}^1(E^{(m)}) \geq 1 - \varepsilon$ (Proposition 2.2).

Section 3 is devoted to a generalization of Theorem 1.2. More specifically we prove that if Φ and φ are of class C^h with $h \geq 1$ and if K is defined as above, then the graph of $\varphi|_{K^{(n+h)}}$ is a $n-$dimensional C^{h+1}-rectifiable set (Theorem 3.1). The proof we present of this result is very short and relies on only [**8**, Theorem 3.6.2] (an L^p version of the Whitney extension theorem, due to Calderón and Zygmund [**3**]) and [**4**, Theorem 4.2], the latter stating that at the points in $K^{(n+h)}$ the function φ behaves nicely with respect to $(h+1)$-degree Taylor polynomial expansion in L^1_{loc}.

2 - m-density points and generalized Cantor sets in $\mathbb{R}$

Let $T = (\lambda_k)$ be a sequence of numbers in $(0, 1/2)$ and $C(T)$ denote the generalized Cantor set in $\mathbb{R}$ obtained according to the construction in [**7**, Section 4.11]. Then, for $k = 1, 2, \dots$, let $I_{k,1}, \dots, I_{k,2^k}$ and $A_{k,1}, \dots, A_{k,2^{k-1}}$ be, respectively, the equi-length closed intervals left and the equi-length open intervals removed at the k-th step of the construction, so that

$$I_{k,2j-1} \cup A_{k,j} \cup I_{k,2j} = I_{k-1,j}.$$

Also let

$$s_k := \mathcal{L}^1(I_{k,j}) = \lambda_1 \cdots \lambda_k, \qquad \rho_k := \frac{\mathcal{L}^1(A_{k,j})}{2} = \frac{s_{k-1} - 2s_k}{2} = \left(\frac{1}{2} - \lambda_k\right) s_{k-1}$$

and $\xi_{k,j}$ be the middle point of $I_{k,j}$. Now let $x_0 \in C(T)$ and observe that $x_0 \neq \xi_{k,j}$ for all k,j. For each $k = 1, 2, \dots$ there exists a unique $j(x_0, k) \in \{1, \dots, 2^k\}$ such that $x_0 \in I_{k,j(x_0,k)}$. Let $J_k(x_0)$ denote the closed interval whose endpoints are $\xi_{k,j(x_0,k)}$ and $\xi_{k,j(x_0,k)} + (\xi_{k,j(x_0,k)} - x_0)$.

2.1 - *A family of "thin" generalized Cantor sets*

Going along the lines of [**2**, Appendix], we can easily obtain the following result (the proof of which is provided for the convenience of the reader).

Proposition 2.1. *Let $T = (\lambda_k)$ satisfy the condition*

$$C_k := \left(\frac{1}{2} - \lambda_k\right) 2^{k/3} \to +\infty \qquad (\text{as } k \to \infty). \tag{2.1}$$

and define

$$\varphi(x) := \int_0^x \operatorname{dist}(t, C(T))^{1/2} dt \qquad (x \in \mathbb{R}). \tag{2.2}$$

Then the graph of $\varphi|_{C(T)}$ intersects every C^2 graph in a zero-measure set.

Proof. Step 1. Let $x_0 \in C(T)$ and prove that

$$|\varphi(y) - \varphi(x_0)| \geq \frac{1}{4} C_{k+1}^{3/2} |y - x_0|^2 \qquad (k = 1, 2, \dots) \tag{2.3}$$

for all $y \in J_k(x_0)$.

To this aim, observe that, for each $k = 1, 2, \dots$, one has two possible cases (let $j(x_0, k)$ be indicated simply by j):

(a) $x_0 < \xi_{k,j}$, i.e. $x_0 \in I_{k+1,2j-1}$;

(b) $x_0 > \xi_{k,j}$, i.e. $x_0 \in I_{k+1,2j}$.

In the case (a) one has $J_k(x_0) = [\xi_{k,j}\, ,\, \xi_{k,j} + (\xi_{k,j} - x_0)]$ and

$$\begin{aligned}\varphi(\xi_{k,j}) - \varphi(x_0) &= \int_{x_0}^{\xi_{k,j}} \operatorname{dist}(t, C(T))^{1/2} dt \geq \int_{\xi_{k,j}-\rho_{k+1}}^{\xi_{k,j}} \operatorname{dist}(t, C(T))^{1/2} dt \\ &\geq \int_{\xi_{k,j}-\rho_{k+1}}^{\xi_{k,j}} [t - (\xi_{k,j} - \rho_{k+1})]^{1/2} dt = \int_0^{\rho_{k+1}} t^{1/2} dt = \frac{2}{3}\rho_{k+1}^{3/2} \\ &= \frac{2}{3}\left(\frac{1}{2} - \lambda_{k+1}\right)^{3/2} s_k^{3/2}.\end{aligned}$$

In the case (b), by an analogous computation, we get the same lower bound for $\varphi(x_0) - \varphi(\xi_{k,j})$. Putting together the two estimates, one finds

$$|\varphi(x_0) - \varphi(\xi_{k,j})| \geq \frac{2}{3}\left(\frac{1}{2} - \lambda_{k+1}\right)^{3/2} s_k^{3/2}.$$

Since

$$2|x_0 - \xi_{k,j}| \leq s_k \leq 2^{-k}$$

it follows that

$$\begin{aligned}|\varphi(x_0) - \varphi(\xi_{k,j})| &\geq \frac{2}{3}\left(\frac{1}{2} - \lambda_{k+1}\right)^{3/2} s_k^2\, s_k^{-1/2} \\ &\geq \frac{8}{3}\left(\frac{1}{2} - \lambda_{k+1}\right)^{3/2} 2^{k/2}|x_0 - \xi_{k,j}|^2 \\ &\geq C_{k+1}^{3/2}\, |x_0 - \xi_{k,j}|^2.\end{aligned}$$

We also observe that for all $y \in J_k(x_0)$ one has

$$|y - x_0| \leq |y - \xi_{k,j}| + |\xi_{k,j} - x_0| \leq 2|\xi_{k,j} - x_0|$$

and (by the monotonicity of φ)

$$|\varphi(y) - \varphi(x_0)| \geq |\varphi(\xi_{k,j}) - \varphi(x_0)|.$$

Hence (2.3) follows immediately.

Step 2. Let $f \in C^2(\mathbb{R})$ and prove that

$$\mathcal{L}^1(F) = 0, \qquad F := \{x \in C(T)\,|\,\varphi(x) = f(x)\}.$$

It will be enough to prove that F does not contain points of density. To this aim, assume (by absurd) the existence of a point x_0 of density of F. Since F is closed, one has

$$x_0 \in F \subset C(T).$$

If define

$$U_k := (x_0 - 2\rho\,,\, x_0 + 2\rho); \qquad \rho := |\xi_{k,j(x_0,k)} - x_0|$$

then

$$\frac{\mathcal{L}^1(F \cap U_k)}{\mathcal{L}^1(U_k)} > \frac{3}{4} \quad \text{i.e.} \quad \mathcal{L}^1(F \cap U_k) > 3\rho$$

provided k is sufficiently large. It follows that (for k large enough and denoting $J_k(x_0)$ simply by J_k)

$$\mathcal{L}^1(F \cap J_k) = \mathcal{L}^1(F \cap U_k) - \mathcal{L}^1(F \cap (U_k \backslash J_k)) > 3\rho - \mathcal{L}^1(U_k \backslash J_k) = 0$$

hence there exists $y_k \in F \cap J_k$. Observing that $f'(x_0) = 0$ and recalling (2.3), we obtain

$$\frac{|f(y_k) - f(x_0) - f'(x_0)\,(y_k - x_0)|}{|y_k - x_0|^2} = \frac{|\varphi(y_k) - \varphi(x_0)|}{|y_k - x_0|^2} \to +\infty$$

as $k \to +\infty$. This result contradicts the assumption that f is of class C^2. □

Remark 2.1. In [**2**, Appendix] it is considered the case of $T = (\lambda_k)$ with

$$\lambda_k := \frac{1}{2} - \frac{1}{2(k+i)^2}$$

which obviously satisfies (2.1).

Corollary 2.1. *If* $E := C(T)$ *with* T *given as in Proposition* 2.1, *then* $\mathcal{L}^1(E^{(m)}) = 0$ *for all* $m \geq 2$.

Proof. Since E is closed, one has $E^{(2)} \subset E$. It follows from Proposition 2.1 and Theorem 1.2 (with $n := 1$, $\Omega := \mathbb{R}$, φ given by (2.2) and $\Phi := 0$) that the graph of $\varphi|_{E^{(2)}}$ has measure zero, hence $\mathcal{L}^1(E^{(2)}) = 0$. The conclusion follows from the obvious inclusion $E^{(m)} \subset E^{(2)}$, for all $m \geq 2$. □

2.2 - *A family of "fat" generalized Cantor sets*

If E is an open set, one obviously has $E^{(m)} \supset E$. Hence we expect that $E^{(m)}$ has positive measure whenever E is a "fat enough" generalized Cantor set. This section is devoted to illustrating how to get, given any $m > 1$, examples of generalized Cantor sets with such a property.

If σ denotes a given positive number, we can chose $T = (\lambda_k)$ in such a way that

$$\mathcal{L}^1(A_{k,j}) = \sigma^{km} \qquad (k = 1, 2, \ldots; j = 1, \ldots, 2^{k-1}) \tag{2.4}$$

provided σ is small enough. Indeed, if (2.4) holds, then the following condition

$$\begin{cases} \mathcal{L}^1(A_{1,1}) < 1 \\ \sum_{i=1}^{2^k} \mathcal{L}^1(A_{k+1,i}) < 1 - \sum_{j=1}^{k} \sum_{i=1}^{2^{j-1}} \mathcal{L}^1(A_{j,i}) \quad \text{for } k = 1, 2, \ldots \end{cases}$$

i.e.

$$\begin{cases} \sigma^m < 1 \\ 2^k \sigma^{(k+1)m} < 1 - \sum_{j=1}^{k} 2^{j-1} \sigma^{jm} \quad \text{for } k = 1, 2, \ldots \end{cases}$$

is easily seen to be verified when σ is small enough.

Now let $B_{k,j}$ be the open interval of length $(2\sigma)^k$ centered at the middle point of $A_{k,j}$, define

$$E_k := [0,1] \backslash \bigcup_{j=1}^{k} \bigcup_{i=1}^{2^{j-1}} A_{j,i}, \qquad E := \bigcap_{k=1}^{\infty} E_k = C(T)$$

and

$$F := [0,1] \backslash \bigcup_{j=1}^{\infty} \bigcup_{i=1}^{2^{j-1}} B_{j,i}.$$

The measure of the generalized Cantor set E can be computed as the limit of $\mathcal{L}(E_k)$ for $k \to \infty$. Since

$$\mathcal{L}^1(E_k) = 1 - \sum_{j=1}^{k} 2^{j-1} \sigma^{jm} = 1 - \sigma^m \sum_{j=0}^{k-1} (2\sigma^m)^j = 1 - \sigma^m \frac{1 - (2\sigma^m)^k}{1 - 2\sigma^m} \tag{2.5}$$

we find

$$\mathcal{L}^1(E) = 1 - \frac{\sigma^m}{1 - 2\sigma^m}. \tag{2.6}$$

Observe that, for σ small enough

$$\frac{\mathcal{L}^1(A_{j,i})}{\mathcal{L}^1(B_{j,i})} = \frac{\sigma^{jm}}{2^j \sigma^j} = \left(\frac{\sigma^{m-1}}{2}\right)^j < 1 \qquad (j = 1, 2, \ldots) \tag{2.7}$$

hence

$$F \subset E.$$

Moreover, from

$$E \backslash F \subset [0,1] \backslash F = \bigcup_{j=1}^{\infty} \bigcup_{i=1}^{2^{j-1}} B_{j,i}$$

it follows that

$$\mathcal{L}^1(E \backslash F) \le \sum_{j=1}^{\infty} 2^{j-1} (2\sigma)^j = \frac{2\sigma}{1 - 4\sigma} \to 0 \qquad (\text{as } \sigma \downarrow 0). \tag{2.8}$$

This result holds.

Proposition 2.2. *For σ small enough, one has $F \backslash \{0,1\} \subset E^{(m)}$. Moreover*

$$\mathcal{L}^1(E^{(m)}) \ge \mathcal{L}^1(E) - \frac{2\sigma}{1 - 4\sigma}. \tag{2.9}$$

Proof. Consider $x \in F \backslash \{0,1\}$ and define

$$\delta_j := \frac{\mathcal{L}^1(B_{j,i}) - \mathcal{L}^1(A_{j,i})}{2}$$

which is positive for all $j = 1, 2, \ldots$, by (2.7). A trivial computation shows that the inequality $\delta_{j+1} < \delta_j$ is equivalent to

$$\left(\frac{\sigma^{m-1}}{2}\right)^j < \frac{1-2\sigma}{1-\sigma^m}$$

which is clearly verified for all j, provided σ is small enough. Thus the δ_j form a decreasing infinitesimal sequence.

Now let r be positive with

$$r < \min\{\delta_1, x, 1-x\} \tag{2.10}$$

and denote with $\kappa(r)$ the maximum of the k such that $r < \delta_k$, hence

$$\delta_{\kappa(r)+1} \leq r < \delta_{\kappa(r)} < \ldots < \delta_1. \tag{2.11}$$

This implies in particular that $B(x, r)$ cannot intersect the $A_{j,i}$ with $j = 1, \ldots, \kappa(r)$, namely

$$B(x,r) \cap \left(\bigcup_{j=1}^{\kappa(r)} \bigcup_{i=1}^{2^{j-1}} A_{j,i} \right) = \emptyset$$

which, also by (2.10), can be written as

$$B(x,r) \backslash E_{\kappa(r)} = \emptyset. \tag{2.12}$$

Since $E \subset E_{\kappa(r)}$, one has

$$E = E_{\kappa(r)} \backslash (E_{\kappa(r)} \backslash E)$$

and then

$$\begin{aligned} B(x,r) \backslash E &= B(x,r) \cap \left(E_{\kappa(r)} \cap (E_{\kappa(r)} \backslash E)^c\right)^c \\ &= B(x,r) \cap \left(E^c_{\kappa(r)} \cup (E_{\kappa(r)} \backslash E)\right) \\ &= (B(x,r) \backslash E_{\kappa(r)}) \cup (B(x,r) \cap (E_{\kappa(r)} \backslash E)). \end{aligned}$$

From this equality and (2.12) it follows that

$$\mathcal{L}^1(B(x,r) \backslash E) \leq \mathcal{L}^1(E_{\kappa(r)} \backslash E).$$

Recalling again (2.11), we obtain

$$\frac{\mathcal{L}^1(B(x,r) \backslash E)}{r^m} \leq \frac{\mathcal{L}^1(E_{\kappa(r)} \backslash E)}{\delta^m_{\kappa(r)+1}} \tag{2.13}$$

for all positive r. But (2.5) and (2.6) yield

$$\begin{aligned}\frac{\mathcal{L}^1(E_k\backslash E)}{\delta_{k+1}^m} &= \frac{\mathcal{L}^1(E_k) - \mathcal{L}^1(E)}{\delta_{k+1}^m} \\ &= \frac{\sigma^m(2\sigma^m)^k}{1-2\sigma^m} \times \frac{2^m}{\left[(2\sigma)^{k+1} - \sigma^{(k+1)m}\right]^m} \\ &= \frac{1}{1-2\sigma^m} \times \left[1 - \left(\frac{\sigma^{m-1}}{2}\right)^{k+1}\right]^{-m} \times \frac{1}{2^{k(m-1)}}\end{aligned}$$

hence

$$\lim_{k\to\infty} \frac{\mathcal{L}^1(E_k\backslash E)}{\delta_{k+1}^m} = 0.$$

Now (2.13) yields $x \in E^{(m)}$, which proves the first claim in the statement. Finally, the inequality (2.9) follows from (2.8):

$$\mathcal{L}^1(E^{(m)}) \geq \mathcal{L}^1(F) = \mathcal{L}^1(E) - \mathcal{L}^1(E\backslash F) = \mathcal{L}^1(E) - \frac{2\sigma}{1-4\sigma}.$$

□

3 - C^{h+1}-rectifiability via L^p Whitney extension theorem

Let $\varphi \in C^h(\Omega)$ and $\Phi \in C^h(\Omega; \mathbb{R}^n)$, where Ω is an open subset of $\mathbb{R}^n$ and $h \geq 1$. Define

$$K := \{x \in \Omega \,|\, \nabla\varphi(x) = \Phi(x)\}.$$

Consider the Taylor-type polynomial

$$P_x(y) := \sum_{j=1}^{h} \frac{\langle D^j\varphi(x)|(y-x)^j\rangle}{j!} + \frac{\langle D^h\Phi(x)|(y-x)^h\rangle \cdot (y-x)}{(h+1)!} \qquad (x \in \Omega, y \in \mathbb{R}^n)$$

where

$$\langle D^j\varphi(x)|(y-x)^j\rangle := \sum_{\lambda\in\{1,\dots,n\}^j} (y_{\lambda_1} - x_{\lambda_1})\cdots(y_{\lambda_j} - x_{\lambda_j}) \frac{\partial^j\varphi}{\partial x_{\lambda_1}\cdots\partial x_{\lambda_j}}(x)$$

and

$$\langle D^h\Phi(x)|(y-x)^h\rangle := \sum_{\lambda\in\{1,\dots,n\}^h} (y_{\lambda_1} - x_{\lambda_1})\cdots(y_{\lambda_h} - x_{\lambda_h}) \frac{\partial^h\Phi}{\partial x_{\lambda_1}\cdots\partial x_{\lambda_h}}(x).$$

From the proof of [4, Theorem 4.2] we can easily extrapolate the following result.

Proposition 3.1. *One has*

$$\lim_{r\downarrow 0}\frac{1}{r^{n+h+1}}\int\limits_{B(x,r)}|\varphi(y)-P_x(y)|\,dy=0$$

for all $x\in\Omega\cap K^{(n+h)}$.

With these preliminary remarks in mind and through results by Calderón and Zygmund [3], remarkably presented in [8, Section 3.6], we are in position to get an easy proof of the following theorem generalizing Theorem 1.2.

Theorem 3.1. *Let* Ω *be an open subset of* $\mathbb{R}^n$ *and* $h\geq 1$. *Consider* $\varphi\in C^h(\Omega)$, $\Phi\in C^h(\Omega;\mathbb{R}^n)$ *and define*

$$K:=\{x\in\Omega\,|\,\nabla\varphi(x)=\Phi(x)\}.$$

Then the graph of $\varphi|_{\Omega\cap K^{(n+h)}}$ *is a* n*–dimensional* C^{h+1}*-rectifiable set, namely* $\mathcal{H}^n$*-almost all of it may be covered by countably many* n*-dimensional submanifolds of class* C^{h+1}.

Proof. Proposition 3.1 implies

$$\Omega\cap K^{(n+h)}=\bigcup_{j=1}^{\infty}E_j$$

where E_j denotes the set of points $x\in K^{(n+h)}$ such that

$$\operatorname{dist}(x,\mathbb{R}^n\backslash\Omega)>\frac{1}{j}\qquad(\text{hence } x\in\Omega\ \text{ too})$$

and

$$\frac{1}{r^{h+1}}\;⨍_{B(x,r)}|\varphi(y)-P_x(y)|\,dy\leq j,\qquad\text{for all } r\in\left(0,\frac{1}{2j}\right).$$

Hence it will be enough to show that the graph of $\varphi|_{E_j}$ is a n-dimensional C^{h+1}-rectifiable set (for all j). But the E_j are measurable and $E_j\cap B(0,m)$ can be approximated in measure by compact subsets (for all j,m), thus we are reduced to prove the following claim.

Claim. Let j be fixed arbitrarily and let C be a compact subset of E_j. Then the graph Γ of $\varphi|_C$ is a n-dimensional C^{h+1}-rectifiable set.

To this aim, we begin by applying [8, Theorem 3.6.2]. It implies the existence of a function f of class $C^{h,1}$ in an open neighbourhood of C such that $f|_C=\varphi|_C$. By

[**6**, Theorem 3.1.15], for each $m = 1, 2, \dots$ there exists $g_m \in C^{h+1}(\mathbb{R}^n)$ such that

$$\mathcal{L}^n(C \backslash C_m) \leq \frac{1}{m}$$

where

$$C_m := \{x \in C \,|\, g_m(x) = f(x)\} = \{x \in C \,|\, g_m(x) = \varphi(x)\}.$$

Denoting by Γ_m the graph of g_m, one has

$$\mathcal{H}^n(\Gamma \backslash \cup_j \Gamma_j) \leq \mathcal{H}^n(\Gamma \backslash \Gamma_m) \leq \int_{C \backslash C_m} (1 + \|\nabla \varphi\|^2)^{1/2} \leq \frac{1 + \max_C \|\nabla \varphi\|}{m}$$

for all m. Hence

$$\mathcal{H}^n(\Gamma \backslash \cup_j \Gamma_j) = 0$$

and this concludes the proof of Claim. □

Remark 3.1. Under the assumptions of Theorem 3.1, in general, we cannot expect the graph of $\varphi|_{\Omega \cap K}$ to be a $n-$dimensional C^{h+1}-rectifiable set. An example is provided by Proposition 2.1 (where $n := 1$, $h := 1$, $\Omega := \mathbb{R}$ and $\Phi := 0$).

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