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A note on a conjecture of Zhiqin Lu and Gang Tian

Abstract. The aim of this paper is to describe a particular family of metrics in $\mathbb{CP}^2$ that confirms a conjecture of Z. Lu and G. Tian given in [**18**].

Keywords. Szegö kernel; log term; Tian-Yau-Zeldich expansion.

Mathematics Subject Classification (2010): 53C55, 32Q20, 32A25.

1 - Introduction

Let (L, h) be a hermitian line bundle over a compact Kähler manifold (M, ω) of complex dimension n such that $\mathrm{Ric}(h) = \omega$, where $\mathrm{Ric}(h)$ is a two-form on M whose local expression is given by:

$$\mathrm{Ric}(h) = -\frac{i}{2}\partial\bar{\partial}\log h(\sigma(x), \sigma(x)) \tag{1}$$

for a trivializing holomorphic section $\sigma : U \to L \setminus \{0\}$. Let $(L^*, h^* = h^{-1})$ be the dual bundle of L and define the unit disk bundle of L^* as

$$D_h = \{v \in L^* | \rho(v, v) := 1 - h^*(v, v) > 0\},$$

ρ is called the *defining function* of D_h. It is well known that from the assumption $\mathrm{Ric}(h) = \omega$ it follows that D_h is a strictly pseudoconvex domain. Let dV be the natural measure on D_h defined by $dV = \frac{1}{n!}\pi^*(\omega^n) \wedge d\theta$, where $\frac{\partial}{\partial\theta}$ is the infinitesimal S^1-action on the unit circle bundle $X_h = \partial D_h$. The *Hardy space* $\mathcal{H}^2(X_h)$ is the separable Hilbert space defined as the closure in $L^2(X_h)$ of the set given by the

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restrictions to X_h of the continuous functions in $\bar{D}_h$ that are holomorphic in D_h (see [**5**] and also [**25**]).

Let $\{\varphi_0, \ldots, \varphi_n, \ldots\}$ be an orthonormal base of $\mathcal{H}^2(X_h)$, with respect to $\langle \cdot, \cdot \rangle$, then the Szegö kernel is

$$\mathcal{S}_h(v) = \sum_{j=0}^{\infty} \varphi_j(v)\overline{\varphi_j(v)} \qquad v \in D_h.$$

In general, the computation of Szegö kernel must be very complicated, but if the domain D_h is strictly pseudoconvex, from an important result by Boutet de Monvel and Sjöstrand ([**5**], Corollary 1.7), we know that there exist functions a and b continuous on $\bar{D}_h$ with $a \neq 0$ on X_h such that the Szegö kernel is given by

$$\mathcal{S}_h(v) = \frac{a(v)}{\rho(v)^{n+1}} + b(v) \log \rho(v) \qquad v \in D_h \tag{2}$$

where ρ is the defining function of D_h (see also [**4**] and [**3**]). The function $b(v)$ in (2) is called the *logarithmic term* (*log-term* from now on) of the Szegö kernel. One says that the log-term of the Szegö kernel of D_h vanishes if $b = 0$. The simplest example here is the complex projective space $\mathbb{CP}^n$ endowed with the Fubini-Study Kähler form ω_{FS}. Let $L = O(1)$ be the hyperplane bundle and $L^* = O(-1)$ its dual, namely the tautological bundle over $\mathbb{CP}^n$. Consider L equipped with the Hermitian metric h_{FS} such that $\mathrm{Ric}(h_{FS}) = \omega_{FS}$ and L^* endowed with h_{FS}^{-1}. A direct computation shows that the Szegö kernel of $D_{h_{FS}} \subset L^*$ has vanishing log-term (see [**8**]). The study of the log-term for the Bergman kernel and for the Szegö kernel has important analytic and geometric meanings. Among all there is the Ramadanov's conjecture [**20**] for the Bergman kernel which asserts that a strongly pseudoconvex bounded domain in $\mathbb{C}^n$ with smooth boundary and whose Bergman kernel has vanishing log-term, is biholomorphic to the unit ball in $\mathbb{C}^n$.

A corresponding conjecture for the Szegö kernel was formulated by M. Engliš and G. Zhang in [**8**], inspired by the paper [**18**] of G. Tian and Z. Lu. More precisely, they asked if the vanishing of log-term of the Szegö kernel of the disk bundle of a simply connected Kähler manifold implies that the circle bundle is diffeomorphic to the sphere or at least locally CR equivalent to the sphere. In [**8**], Engliš and Zhang showed a counterexample for both the diffeomorphic and locally CR equivalent cases. In the first case, they consider the tensor power of the tautological bundle L^* over the complex projective space, namely the line bundle $(L^*)^{\otimes m}$ over $\mathbb{CP}^n$: in this case the Szegö kernel of the disk bundle $D_{h_{FS}}$ of $(L^*)^{\otimes m}$ has no log-term, but $D_{h_{FS}}$, being the lens space $\mathbb{S}^{2n+1}/\mathbb{Z}_m$, is not diffeomorphic to $\mathbb{S}^{2n+1}$ for $m > 1$, (but CR equivalent to $\mathbb{S}^{2n+1}$),(see [**8**] for details). For the locally CR equivalent case, they consider compact symmetric spaces of higher rank, whose disk bundles have vanishing log-term, but they are not locally spherical at any point (neither diffeomorphic to $\mathbb{S}^{2n+1}$). A recent generalization of these results, can be found in [**3**], where the authors show that the disk bundles over homogeneous Hodge manifolds form a infinite family of strictly pseudoconvex domains (also smoothly bounded) for which the log-terms vanish but they are not locally CR equivalent to the sphere.

In paper [**18**] the authors analyse what happens to the log-term of the Szegö kernel of D_h, when one varies the metric h by preserving the corresponding cohomology class. They conjecture the following:

Conjecture (Z. Lu-G. Tian). *Let $\omega \in [\omega_{FS}]$ be a Kähler metric on $\mathbb{CP}^n$ in the same cohomology class of the Fubini-Study metric ω_{FS}. Let (L, h) be the hyperplane bundle whose curvature is ω, (i.e. $Ric(h) = \omega$). If the log-term of the Szegö kernel of the unit disk bundle $D_h \subset L^*$ vanish, then there is an automorphism $\varphi : \mathbb{CP}^n \to \mathbb{CP}^n$ such that $\varphi^*\omega = \omega_{FS}$.*

In the same paper the authors prove the validity of the conjecture for the case $n = 1$. However, the main result of Lu and Tian is the local version of the conjecture, in fact the conjecture above is true if the hermitian metric h is close to h_{FS} in the following sense:

Theorem (Z. Lu-G. Tian). *Let L be the hyperplane bundle of $\mathbb{CP}^n$ and let h a hermitian metric on L such that* $\mathrm{Ric}(h) = \omega$. *Assume that there exists $\varepsilon > 0$ (depending only on n) for which*

$$\left\| \frac{h}{h_{FS}} - 1 \right\|_{C^{2n+4}} < \varepsilon. \tag{3}$$

If the log-term of the Szegö kernel of the unit disk bundle D_h vanish, then there exists an automorphism φ of $\mathbb{CP}^n$ such that $\varphi^*(\omega) = \omega_{FS}$.

The aim of this paper is to show the validity of the Lu-Tian's Conjecture for a family of Kähler forms in $\mathbb{CP}^2$ cohomologous to $2\omega_{FS}$ and which does not satisfy condition (3). More precisely, for each $a \neq 0, a > 0$ we consider the one parameter family of Kähler forms on $\mathbb{CP}^2$given by

$$\omega_a = \Phi^*\omega_{FS} \tag{4}$$

where $a = |\alpha|^2, \alpha \in \mathbb{C}^*$ and Φ is a holomorphic Veronese-type embedding given by:

$$\mathbb{CP}^2 \xrightarrow{\Phi} \mathbb{CP}^5$$

$$[Z_0, Z_1, Z_2] \longmapsto [Z_0, Z_1, Z_2, \alpha Z_0 Z_1, \alpha Z_0 Z_2, \alpha Z_1 Z_2],$$

where Z_0, Z_1, Z_2 are homogeneous coordinates on $\mathbb{CP}^2$ (note that we are denoting with the same symbol the Fubiny-Study form of $\mathbb{CP}^2$ and of $\mathbb{CP}^5$). Our main result is:

Theorem 1.1. *Let ω_a be as above and let h_a be the hermitian product on $L \to \mathbb{CP}^2$ such that* $\mathrm{Ric}(h_a) = \omega_a$. *If the log-term of the Szegö kernel of D_{h_a} vanishes, then there is an automorphism $\varphi : \mathbb{CP}^2 \to \mathbb{CP}^2$ such that $\varphi^*\omega_a = \omega_{FS}$.*

The article is organized as follow: in the next Section we recall the result obtained by S. Zelditch [**25**] and by Z. Lu and G. Tian [**18**], needed in the proof of our main result. Section 3 is dedicated to the proof of Theorem 1.1.

2 - The work of S. Zelditch, Z. Lu and G.Tian

Let (L, h) be a Hermitian line bundle over a compact Kähler manifold (M, ω) of complex dimension n such that $\mathrm{Ric}(h) = \omega$. For all integer $m > 0$, consider the line bundle $(L^{\otimes m}, h_m)$ over (M, ω) with $\mathrm{Ric}(h_m) = m\omega$ and the space $H^0(L^{\otimes m})$ consisting of all holomorphic sections bounded with respect to the L^2-product

$$\langle s, t\rangle_m = \int_M h_m(s_j^m(x), s_j^m(x)) \frac{\omega^n}{n!}(x)$$

for $s, t \in H^0(L^{\otimes m})$. Compactness of M ensure that the dimension of $H^0_m = H^0(L^{\otimes m})$ is finite, say $\dim H^0_m = N_m + 1$. Given an orthonormal basis $s_0^m, \dots, s_{N_m}^m$ of H^0_m with respect to $\langle \cdot, \cdot \rangle_m$, define a smooth and positive real valued function $T_m(x)$ on M, called the *Kempf's distorsion function*:

$$T_m(x) := \sum_{j=0}^{N_m} h_m(s_j^m(x), s_j^m(x)). \tag{5}$$

It is not difficult to verify that this function depends only on the Kähler form $m\omega$ and not on the orthonormal basis chosen. The function T_m is known in literature with different names, for examples in Rawnsley [**21**] it's called $\eta-$function, and later re-named $\theta-$function in [**22**]. In [**13**] Kempf called T_m as *distorsion function* and it is also called distorsion function by Ji [**12**] for the abelian varieties and by Zhang in [**26**] for complex projective varieties. It coincides with the diagonal of the Bergman kernel on L^m associated to h_m and thus is also frequently called *Bergman kernel* in the literature (see, for example, [**19**]). An important result of S. Zelditch [**25**], that generalized Tian-Ruan's theorem [**23**], is the following:

Theorem 2.1 (Zelditch). *There exists a complete asymptotic expansion*

$$T_m(x) \sim a_0(x)m^n + a_1(x)m^{n-1} + a_2(x)m^{n-2}\dots$$

with $a_j(x)$ smooth and $a_0(x) = 1$. Moreover for $m \to +\infty$

$$\left\| T_m(x) - \sum_{j=0}^{k} a_j(x)m^{n-j} \right\|_{C^r} \leq C_{k,r}m^{n-k}.$$

This expansion is called *TYZ* (*Tian-Yau-Zelditch*) *expansion* and it is a key ingredient in the investigations of balanced metric, and an important tool for calculation of Szego kernel. In [**17**], Lu computes the first three coefficients a_1, a_2 and a_3 of this expansion and proves the following:

Theorem 2.2 (Lu). *Each of the coefficients a_j of the Zelditch expansion is a polynomial of the curvature and its covariant derivatives at x of the metric g of the manifold. In particular we have*

$$a_0 = 1,$$

$$a_1 = \frac{1}{2}\mathrm{Scal},$$

$$a_2 = \frac{1}{3}\Delta\,\mathrm{Scal} + \frac{1}{24}(|R|^2 - 4|\mathrm{Ric}|^2 + 3\mathrm{Scal}^2)$$

$$a_3 = \frac{1}{8}\Delta\Delta\,\mathrm{Scal} + \frac{1}{24}\mathit{divdiv}(R,\mathrm{Ric}) - \frac{1}{6}\mathit{divdiv}(\mathrm{ScalRic}) + \frac{1}{48}\Delta(|R|^2 - 4|\mathrm{Ric}|^2 + 8\mathrm{Scal}^2)$$
$$+\frac{1}{48}\mathrm{Scal}(\mathrm{Scal}^2 - 4|\mathrm{Ric}|^2 + |R|^2) + \frac{1}{24}(\sigma_3(\mathrm{Ric}) - \mathrm{Ric}(R,R) - R(\mathrm{Ric},\mathrm{Ric}))$$

where R, Ric, *and* Scal *represent the curvature tensor, the Ricci curvature and the scalar curvature of g respectively, and Δ represents the Laplacian of M.*

For more details and more precisely definition of each element in the previous expressions see Appendix A.

From Theorem 2.1 and Theorem 2.2 above follows that the coefficients a_k can be found by finitely many algebraic operations (see also [**14**] and [**15**] for the computation of the coefficients a_k's through Calabi's Diastasis function).

The main results obtained by Z. Lu and G. Tian in [**18**] is the close relation between the vanishing of the log-term of the Szegö Kernel constructed on the disk bundle $D_h \subset L^*$ and the vanishing of coefficients a_k's of the TYZ expansion of (M,ω) for $k > n$. We summarize them in the following:

Theorem 2.3. *Let (L,h) be a positive line bundle over a complex compact manifold (M,ω) of dimension n such that $\mathrm{Ric}(h) = \omega$. If the log-term of the Szegö kernel of $D_h \subset L^*$ vanishes then the coefficients a_k of TYZ in Theorem* 2.1 *vanish for $k > n$.*

3 - Proof of Theorem 1.1

In order to prove Theorem 1.1, we consider standard affine coordinates in $\mathbb{CP}^2$ in the chart $U_0 = \{Z_0 \neq 0\}$. Then the Kähler form ω_a in (4) is given in this coordinates by:

$$\omega_a = \frac{i}{2}\partial\bar{\partial}\log(1 + |z_1|^4 + |z_2|^4 + a|z_1|^2 + a|z_2|^2 + a|z_1|^2|z_2|^2)$$

with $a = |\alpha|^2$.

Suppose that the log term of the Szegö kernel of

$$D_{h_a} = \{v \in L^* | \rho(v, v) := 1 - h_a^*(v, v) > 0\} \subset L^*,$$

with L^* dual of the universal line bundle of $\mathbb{CP}^2$, vanishes. Then, by Theorem 2.3, the coefficients $a_k = 0$, for $k > 2$. In particular $a_3 = 0$, that combined with Theorem 2.2, gives the following equation

$$a_3 = \frac{1}{8}\Delta\Delta\,\mathrm{Scal} + \frac{1}{24}divdiv(R, Ric) - \frac{1}{6}divdiv(\mathrm{Scal}Ric) + \frac{1}{48}\Delta(|R|^2 - 4|Ric|^2 + 8\mathrm{Scal}^2)$$
$$+\frac{1}{48}\mathrm{Scal}(\mathrm{Scal}^2 - 4|Ric|^2 + |R|^2) + \frac{1}{24}(\sigma_3(Ric) - Ric(R, R) - R(Ric, Ric)) = 0.$$

A long but straightforward computation obtained also with the use of a computer program and expressions in Appendix A below, gives that function a_3 evaluated at the origin reads

$$\begin{aligned} a_3(0,0) &= \frac{1}{6}\frac{3a^6 - 30a^5 - 67a^4 + 278a^3 + 904a^2 - 704a - 2592}{a^6} \\ &= \frac{1}{6}\frac{(3a^5 - 24a^4 - 115a^3 + 48a^2 + 1000a + 1296)(a-2)}{a^6} \end{aligned} \tag{6}$$

while evaluating a_3 at the point $(1, 1)$ it reads

$$\begin{aligned} a_3(1,1) = &-\frac{1}{3}\,\frac{28139a^8 - 526469a^7 - 57190a^6 + 6561820a^5 + 2946788a^4 +}{(1+a)} \\ &\frac{-22781096a^3 - 16867840a^2 + 19757632a + 16922624}{(a^2+8a+16)^4(a+4)}. \\ = &-\frac{1}{3}\,\frac{(28139a^7 - 470191a^6 - 997572a^5 + 4566676a^4 + 12080140a^3}{(1+a)} \\ &\frac{+1379184a^2 - 14109472a - 8461312)(a-2)}{(a^2+8a+16)^4(a+4)}. \end{aligned} \tag{7}$$

With a bit of calculation and using Descartes' rule of signs and the intermediate value theorem, we found out that the positive zeros of (6) are x_1, x_2, x_3 with $x_1 = 2$,

$x_2 \in \left]\frac{31}{10}, \frac{32}{10}\right[$ and $x_3 \in]11, 12[$ while the positive solutions of (7) are y_1, y_2, y_3, y_4 with $y_1 = 2$, $y_2 \in]1, 2[$, $y_3 \in \left]\frac{34}{10}, \frac{35}{10}\right[$ and $y_4 \in]18, 19[$ so we can conclude that the only value of a for which the coefficient a_3 is zero for all points is $a = 2$ that is the only Fubini-Study metric of the family and this ends the proof of Theorem 1.1.

Let us finally point out that the proof of Theorem 1.1 can not be achieve by Lu-Tian's Theorem, since h_a doesn't satisfy condition (3). Indeed, let $\sigma_0 : U_0 \to L \setminus \{0\}$ be the trivializing section given by

$$\sigma_0([Z_0, Z_1, Z_2]) = ([1, z_1, z_2], (1, z_1, z_2)).$$

Then the local expression of the hermitian metric h_a and of the hermitian metric h^2_{FS} such that $\mathrm{Ric}(h^2_{FS}) = 2\omega_{FS}$ are given respectively by

$$h_a(\sigma_0([Z_0, Z_1, Z_2]), \sigma_0([Z_0, Z_1, Z_2])) = \frac{1}{(1 + |z_1|^4 + |z_2|^4 + a|z_1|^2 + a|z_2|^2 + a|z_1|^2|z_2|^2)},$$

and

$$h^2_{FS}(\sigma_0([Z_0, Z_1, Z_2]), \sigma_0([Z_0, Z_1, Z_2])) = \frac{1}{(1 + |z_1|^2 + |z_2|^2)^2}.$$

If condition (3) were satisfied then the quantity

$$\left\| \frac{(1 + |z_1|^4 + |z_2|^4 + 2|z_1|^2 + 2|z_2|^2 + 2|z_1|^2|z_2|^2)}{(1 + |z_1|^4 + |z_2|^4 + a|z_1|^2 + a|z_2|^2 + a|z_1|^2|z_2|^2)} - 1 \right\|$$

would be bounded. By passing to polar coordinates $(z_1, z_2) = \rho(\cos\vartheta, \sin\vartheta)$ one gets

$$\lim_{\cos\vartheta\sin\vartheta \to -\frac{1}{a}} \ \lim_{\rho \to +\infty} \left\| \frac{(2 - a)[\rho(\cos\vartheta + \sin\vartheta) + \rho^2 \cos\vartheta \sin\vartheta]}{(1 + \rho^2 + a[\rho(\cos\vartheta + \sin\vartheta) + \rho^2 \cos\vartheta \sin\vartheta]} \right\| = +\infty,$$

which yields the desired contradiction.

A - Appendix

In this Appendix we recall notations used in Theorem 2.2.

Let M be a n-dimensional complex manifold endowed with a Kähler metric g whose local expression is $g = \sum_{j\bar{k}}^{n} g_{j\bar{k}} dz_j d\bar{z}_k$.

The curvature and the Ricci tensors are defined locally by

$$R_{i\bar{j}k\bar{l}} = \frac{\partial^2 g_{i\bar{j}}}{\partial z_k \partial \bar{z}_l} - \sum_{p=1}^{n}\sum_{q=1}^{n} g^{p\bar{q}} \frac{\partial g_{i\bar{q}}}{\partial z_k} \frac{\partial g_{p\bar{j}}}{\partial \bar{z}_l}, \qquad Ric_{i\bar{j}} = -\sum_{k,l=1}^{n} g^{k\bar{l}} R_{i\bar{j}k\bar{l}}$$

and the scalar curvature Scal as the trace of the Ricci curvature reads

$$\mathrm{Scal} = \sum_{i,j=1}^{n} g^{i\bar{j}} R_{i\bar{j}}.$$

Furthermore, by the usual definition of the Riemannian norm we have:

$$|R|^2 = \sum_{i,j,k,l,p,q,r,s=1}^{n} \overline{g^{i\bar{p}}} g^{j\bar{q}} \overline{g^{k\bar{r}}} g^{l\bar{s}} R_{i\bar{j}k\bar{l}} \overline{R_{p\bar{q}r\bar{s}}} \text{ and } |Ric|^2 = \sum_{i,j,k,l=1}^{n} \overline{g^{i\bar{k}}} g^{j\bar{l}} R_{i\bar{j}} \overline{R_{k\bar{l}}}.$$

Finally recall that $\varDelta = \sum_{i=1}^{n} \sum_{j=1}^{n} g^{i\bar{j}} \frac{\partial^2}{\partial z_i \partial \bar{z}_j}$ and we define

$$divdiv(\mathrm{Scal} Ric) = 2|D' \,\mathrm{Scal}|^2 + \sum_{i,j=1}^{n} R_{i\bar{j}} \frac{\partial^2 \,\mathrm{Scal}}{\partial z_i \partial \bar{z}_j} + \mathrm{Scal} \varDelta \,\mathrm{Scal}$$

$$divdiv(R, Ric) = -\sum_{i,j=1}^{n} R_{i\bar{j}} \frac{\partial^2 \,\mathrm{Scal}}{\partial z_i \partial \bar{z}_j} - 2|D' Ric|^2 + \\ + \sum_{i,j,k,l,p,q,r=1}^{n} g^{i\bar{p}} R_{p\bar{i}k\bar{q}} g^{q\bar{k}} g^{i\bar{r}} R_{r\bar{j},k\bar{l}} - R(Ric, Ric) - \sigma_3(Ric)$$

where

$$|D' \,\mathrm{Scal}|^2 = \sum_{i,j=1}^{n} g^{i\bar{j}} \frac{\partial \,\mathrm{Scal}}{\partial z_i} \frac{\partial \,\mathrm{Scal}}{\partial \bar{z}_j}, \qquad |D' Ric|^2 = \sum_{i,j,k,l,m=1}^{n} \overline{g^{i\bar{k}}} g^{j\bar{l}} R_{i\bar{j},m} \overline{R_{k\bar{l},m}}$$

with $R_{i\bar{j},k} = \frac{\partial R_{i\bar{j}}}{\partial z_k} - \sum_{s=1}^{n} \varGamma_{ik}^{s} R_{s\bar{j}}$,

$$\sigma_3(Ric) = \sum_{a,b,c,i,j,k=1}^{n} g^{i\bar{a}} R_{a\bar{j}} g^{j\bar{b}} R_{b\bar{k}} g^{k\bar{c}} R_{c\bar{i}},$$

$$Ric(R, R) = \sum_{i,j,k,l,p,q,r,s,t,u=1}^{n} g^{il} Ric_{lj} g^{jr} R_{rkps} g^{sq} g^{kt} R_{tiqu} g^{up}$$

and

$$R(Ric, Ric) = \sum_{i,j,k,l,p,q,r,s=1}^{n} g^{ip} R_{pjkq} g^{ql} g^{jr} Ric_{ri} g^{sk} Ric_{ls}.$$

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