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Isomorphism for regular boundary value problems for elliptic differential-operator equations of the fourth order depending on a parameter

Abstract. We treat some fourth order elliptic differential-operator boundary value problems on a finite interval quadratically depending on a parameter. We prove an isomorphism result (which implies maximal L_p-regularity) in the corresponding abstract Sobolev spaces. The underlying space is a *UMD* Banach space. Then, for the corresponding homogeneous problems, we prove discreteness of the spectrum and two-fold completeness of a system of eigenvectors and associated vectors of the problem in the framework of Hilbert and *UMD* Banach spaces. We apply the obtained abstract results to non-local boundary value problems for elliptic and quasi-elliptic equations with a parameter in (bounded and unbounded) cylindrical domains.

Keywords. Abstract elliptic equation, quasi-elliptic equations, UMD Banach space, isomorphism, completeness of eigenfunctions, maximal L_p-regularity.

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1 - Introduction and basic notations

In our previous papers, joint with other authors ([5], [8], [2], [4]), we have studied various second order elliptic differential-operator boundary value problems on $[0,1]$ in *UMD* Banach spaces. When the problems do not contain the complex parameter λ, we prove Fredholmness of the problems. In order to obtain isomorphism theorems (or unique solvability theorems), we have considered the above problems depending on the parameter λ. For higher order elliptic differential-operator boundary value problems on a finite interval in *UMD* Banach spaces, we proved only Fredholmness of the problems (see [7] and [9]) and we succeeded to prove an isomorphism theorem for a very particular problem depending on the parameter and generated by one operator [9, Theorem 4]. The question was how to prove an isomorphism theorem (or a unique solvability theorem) for rather general higher order elliptic differential-operator boundary value problems on a finite interval in a Banach space. In the present paper, we consider some fourth order elliptic differential-operator boundary value problems on $[0,1]$ quadratically depending on the parameter, for which we prove an isomorphism result (which implies maximal L_p-regularity) in the corresponding abstract Sobolev spaces. The underlying space is a *UMD* Banach space. We also prove the corresponding estimate for the solution and its derivatives with respect to the right-hand sides of the equation and boundary conditions. The estimate is uniform with respect to the parameter λ.

Further, for the corresponding fourth order homogeneous elliptic problem, we prove discreteness of the spectrum and two-fold completeness of a system of eigenvectors and associated vectors (root vectors) of the problem in the framework of Hilbert and *UMD* Banach spaces. Discreteness of the spectrum and completeness of a system of root vectors for second order elliptic differential-operator boundary value problems on a finite interval have been previously studied in the framework of Hilbert spaces (see [13], [1], and [2] and the references therein) and in the framework of *UMD* Banach spaces (the only paper is [2] up to our best knowledge). Our mentioned above results for the fourth order abstract elliptic problems are new even in the framework of Hilbert spaces.

The obtained abstract results are illustrated by a number of applications to non-local boundary value problems for elliptic and quasi-elliptic equations with a parameter in (bounded and unbounded) cylindrical domains.

Let us give necessary definitions and notations.

If E and F are Banach spaces, $B(E,F)$ denotes the Banach space of all bounded, linear operators from E into F with the norm equal to the operator norm; moreover, $B(E) := B(E,E)$. The spectrum of a linear operator A in E is denoted by $\sigma(A)$, its resolvent set by $\rho(A)$. The domain and range of an operator A are denoted by $D(A)$ and $R(A)$, respectively. The resolvent of an operator A is denoted by $R(\lambda,A) := (\lambda I - A)^{-1}$.

A Banach space E is said to be of **class** HT, if the Hilbert transform is bounded on $L_p(\mathbb{R};E)$ for some (and then all) $p > 1$. Here the Hilbert transform H of a function $f \in S(\mathbb{R};E)$, the Schwartz space of rapidly decreasing E-valued functions, is defined by

$$Hf := \frac{1}{\pi} PV(\frac{1}{t}) * f,$$

i.e., $(Hf)(t) := \frac{1}{\pi} \lim_{\varepsilon \to 0} \int\limits_{|\tau|>\varepsilon} \frac{f(t-\tau)}{\tau} d\tau$. These spaces are often also called *UMD* Banach spaces, where the *UMD* stands for the property of *unconditional martingale differences.*

Definition 1.1. Let E be a complex Banach space, and let A be a closed linear operator in E. The operator A is called *sectorial* if the following conditions are satisfied:

(1) $\overline{D(A)} = E$, $\overline{R(A)} = E$, $(-\infty, 0) \subset \rho(A)$;

(2) $\|\lambda(\lambda I + A)^{-1}\| \leq M$ for all $\lambda > 0$, and some $M < \infty$.

Definition 1.2. Let E and F be Banach spaces. A family of operators $\mathcal{T} \subset B(E,F)$ is called *$\mathcal{R}$-bounded*, if there are a constant $C > 0$ and $p \geq 1$ such that for each natural number n, $T_1, T_2, \dots, T_n \in \mathcal{T}$, $u_1, u_2, \dots, u_n \in E$ and for all independent, symmetric, $\{-1,1\}$-valued random variables $\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n$ on $[0,1]$ (e.g., the Rademacher functions $\varepsilon_j(t) = \text{sign } \sin(2^j \pi t)$, $j = 1, \dots, n$) the inequality

$$\Big\| \sum_{j=1}^{n} \varepsilon_j T_j u_j \Big\|_{L_p((0,1);F)} \leq C \Big\| \sum_{j=1}^{n} \varepsilon_j u_j \Big\|_{L_p((0,1);E)}$$

is valid. The smallest such C is called $\mathcal{R}$-*bound* of $\mathcal{T}$ and is denoted by $\mathcal{R}\{\mathcal{T}\}_{E\to F}$. If $E = F$, the R-bound will be denoted by $R\{\mathcal{T}\}_E$.

Definition 1.3. A sectorial operator A in E is called $\mathcal{R}$-*sectorial in* $F \subset E$ (in particular, $F = E$), if

$$\mathcal{R}_A(0) := \mathcal{R}\{\lambda(\lambda I + A)^{-1} \ : \ \lambda > 0\}_F < \infty.$$

The number

$$\phi_A^{\mathcal{R}} := \inf\{\theta \in (0, \pi) \ : \{\mathcal{R}_A(\pi - \theta) < \infty\},$$

where $\mathcal{R}_A(\theta) := \mathcal{R}\{\lambda(\lambda I + A)^{-1} \ : \ |\arg \lambda| \le \theta\}_F$, is called the $\mathcal{R}$-*angle in* F of the operator A.

Generally, $\phi_A^{\mathcal{R}}$ may depend on F.

For the operator A closed in E, the domain of definition $D(A^n)$ of the operator A^n is turned into a Banach space $E(A^n)$ with respect to the norm

$$\|u\|_{E(A^n)} := \Big(\sum_{k=0}^{n} \|A^k u\|_E^2\Big)^{\frac{1}{2}}.$$

The operator A^n from $E(A^n)$ into E is bounded.

For the Banach spaces F and E, introduce the Banach space $W_p^n((0,1); F, E)$, $1 < p < \infty$, a natural number $n \ge 1$, of vector-valued functions with the finite norm

$$\|u\|_{W_p^n((0.1);F,E)} := \Big(\int_0^1 \|u(x)\|_F^p dx + \int_0^1 \|u^{(n)}(x)\|_E^p dx\Big)^{\frac{1}{p}}.$$

We write $W_p^n((0,1); E) := W_p^n((0,1); E, E)$.

2 - Isomorphism theorem for abstract fourth order elliptic boundary value problems quadratically depending on a parameter

Consider, in a *UMD* Banach space E, a boundary value problem in [0,1] for the fourth order abstract elliptic equation depending on a parameter

$$\begin{aligned}(L(\lambda)u)(x) \ &:= \ \lambda^2 u(x) - \lambda\big(2u''(x) + A_2 u(x)\big) \\ &\quad + u''''(x) + A_2 u''(x) + A_4 u(x) = f(x), \quad x \in (0,1),\end{aligned} \tag{2.1}$$

$$\begin{aligned}L_k u \ &:= \alpha_k u^{(m_k)}(0) + \beta_k u^{(m_k)}(1) = \varphi_k, \quad k = 1, 2, \\ L_k(\lambda)u \ &:= \alpha_k\big(u^{(m_k)}(0) - \lambda u^{(m_{k-2})}(0)\big) \\ &\quad + \beta_k\big(u^{(m_k)}(1) - \lambda u^{(m_{k-2})}(1)\big) = \varphi_k, \quad k = 3, 4,\end{aligned} \tag{2.2}$$

where $0 \le m_1, m_2 \le 1,\ m_3 = m_1 + 2,\ m_4 = m_2 + 2$; α_k and β_k are complex numbers; A_2 and A_4 are, generally speaking, unbounded operators in E. Let us formulate the main maximal L_p-regularity theorem.

Theorem 2.1. *Let the following conditions be satisfied:*

1. *the operator A_4 is closed, densely defined and invertible in a UMD Banach space E and $\mathcal{R}\{\mu R(\mu, A_4) : \arg \mu = \pi\}_E < \infty$;*[1]
2. *the operator A_2 is bounded from E_2 into E, where $E_2 := E(A_4^{\frac{1}{2}})$;*
3. *there exists $\psi \in [0, \pi)$ such that the operator pencil $L_0(\mu) := \mu^4 I + \mu^2 A_2 + A_4$ is invertible in E, for $\frac{\pi}{2} \ge |\arg \mu| \ge \frac{\pi - \psi}{2}$, and*

$$
\begin{aligned}
&\mathcal{R}\Big\{\mu^4 L_0(\mu)^{-1} \ : \ \frac{\pi}{2} \ge |\arg \mu| \ge \frac{\pi-\psi}{2}\Big\}_E < \infty;\\
&\mathcal{R}\Big\{A_4 L_0(\mu)^{-1} \ : \ \frac{\pi}{2} \ge |\arg \mu| \ge \frac{\pi-\psi}{2}\Big\}_E < \infty;\\
&\mathcal{R}\Big\{\mu^4 L_0(\mu)^{-1} \ : \ \frac{\pi}{2} \ge |\arg \mu| \ge \frac{\pi-\psi}{2}\Big\}_{E_2} < \infty;\\
&\mathcal{R}\Big\{A_4 L_0(\mu)^{-1} \ : \ \frac{\pi}{2} \ge |\arg \mu| \ge \frac{\pi-\psi}{2}\Big\}_{E_2} < \infty;
\end{aligned}
\tag{2.3}
$$

4. $(-1)^{m_1}\alpha_1\beta_2 - (-1)^{m_2}\alpha_2\beta_1 \ne 0$ *and* $(-1)^{m_1}\alpha_3\beta_4 - (-1)^{m_2}\alpha_4\beta_3 \ne 0$; *for $m_1 \ne m_2$, assume, in addition, that $\alpha_k = \alpha_{k+2},\ \beta_k = \beta_{k+2},\ k = 1, 2$.*

Then, for $|\arg \lambda| \le \psi$ and sufficiently large $|\lambda|$, the operator

$$\mathbb{L}(\lambda) : u \to \mathbb{L}(\lambda)u := \Big((L(\lambda)u)(x), L_1 u, L_2 u, L_3(\lambda)u, L_4(\lambda)u\Big),$$

is an isomorphism

$$\text{from } W_p^4((0,1); E(A_4), E) \text{ onto } L_p((0,1); E) \mathop{\times}_{k=1}^{4} (E(A_4), E)_{\frac{m_k}{4} + \frac{1}{4p}, p},$$

where $p \in (1, \infty)$, and, for these values of λ, the following estimate holds for the solution of the problem (2.1)–(2.2)[2]

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[1] In fact, this is equivalent to that A_4 is an invertible $\mathcal{R}$-sectorial operator in E with the $\mathcal{R}$-angle in E, $\phi_{A_4}^{\mathcal{R}} < \pi$ and, therefore, in particular, there exist fractional powers of A_4 (see, e.g., [**3**, Theorem 2.3]).

[2] By virtue of [**6**, Theorem 7 and Corollary 8], the embedding $W_p^4((0,1); E(A_4), E) \subset W_p^2((0,1); E(A_4^{\frac{1}{2}}))$ is continuous. Then, by virtue of condition (3), $A_2 u'' \in L_p((0,1); E)$.

$$
\begin{aligned}
&|\lambda|^2\|u\|_{L_p((0,1);E)} + |\lambda|\big(\|u''\|_{L_p((0,1);E)} + \|u\|_{L_p((0,1);E(A_4^{\frac{1}{2}}))}\big) \\
&\qquad + \|u''\|_{L_p((0,1);E(A_4^{\frac{1}{2}}))} + \|u''''\|_{L_p((0,1);E)} \\
&\le C\Big(\|f\|_{L_p((0,1);E)} + \sum_{k=1}^{4}\|\varphi_k\|_{(E(A_4),E)_{\frac{m_k}{4}+\frac{1}{4p},p}} \\
&\qquad + \sum_{k=1}^{2}|\lambda|^{1-\frac{m_k}{2}-\frac{1}{2p}}\big(\|\varphi_k\|_{E(A_4^{\frac{1}{2}})} + \|\varphi_{k+2}\|_E\big)\Big),
\end{aligned}
\tag{2.4}
$$

where the constant C does not depend on the parameter λ.

Proof. By the substitution

$$
v(x) := \begin{pmatrix} v_1(x) \\ v_2(x) \end{pmatrix} := \begin{pmatrix} u(x) \\ u''(x) - \lambda u(x) \end{pmatrix},
$$

problem (2.1)–(2.2) is reduced to the equivalent problem

$$
\begin{aligned}
&v''(x) = \mathbb{A}v(x) + \lambda v(x) + F(x), \quad x \in (0,1), \\
&a_k v^{(m_k)}(0) + b_k v^{(m_k)}(1) = \Phi_k, \quad k = 1,2,
\end{aligned}
\tag{2.5}
$$

where

$$
\mathbb{A} := \begin{pmatrix} 0 & I \\ -A_4 & -A_2 \end{pmatrix}, \quad a_k := \begin{pmatrix} \alpha_k I & 0 \\ 0 & \alpha_{k+2} I \end{pmatrix}, \quad b_k := \begin{pmatrix} \beta_k I & 0 \\ 0 & \beta_{k+2} I \end{pmatrix},
$$

$$
F(x) := \begin{pmatrix} 0 \\ f(x) \end{pmatrix}, \quad \Phi_k := \begin{pmatrix} \varphi_k \\ \varphi_{k+2} \end{pmatrix}.
$$

We consider the operator $\mathbb{A}$ in the space $\mathcal{E} := E_2 \times E$. Let $D(\mathbb{A}) := E(A_4) \times E_2$ and $F := (f_1, f_2) \in \mathcal{E} = E_2 \times E$. From the first equation of the system

$$
(\mu^2 I - \mathbb{A})v = F \tag{2.6}
$$

we find

$$
v_2 = \mu^2 v_1 - f_1.
$$

Substituting this expression into the second equation of system (2.6) we have

$$
\mu^2(\mu^2 v_1 - f_1) = -A_4 v_1 - A_2(\mu^2 v_1 - f_1) + f_2.
$$

Hence,

$$
L_0(\mu)v_1 = \mu^2 f_1 + A_2 f_1 + f_2,
$$

i.e., by condition (3), for $\frac{\pi}{2} \geq |\arg \mu| \geq \frac{\pi - \psi}{2}$,

$$v_1 = \mu^2 L_0(\mu)^{-1} f_1 + L_0(\mu)^{-1} A_2 f_1 + L_0(\mu)^{-1} f_2. \tag{2.7}$$

Consequently,

$$v_2 = \mu^4 L_0(\mu)^{-1} f_1 + \mu^2 L_0(\mu)^{-1} A_2 f_1 - f_1 + \mu^2 L_0(\mu)^{-1} f_2. \tag{2.8}$$

Since (2.7) and (2.8) define $(\mu^2 I - \mathbb{A})^{-1}$ then one can get that

$$\mathbb{A}(\mu^2 I - \mathbb{A})^{-1} = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}, \tag{2.9}$$

where

$$A_{11} = \mu^4 L_0(\mu)^{-1} + \mu^2 L_0(\mu)^{-1} A_2 - I,$$
$$A_{12} = \mu^2 L_0(\mu)^{-1},$$
$$A_{21} = -\mu^2 A_4 L_0(\mu)^{-1} - A_4 L_0(\mu)^{-1} A_2 - \mu^4 A_2 L_0(\mu)^{-1} - \mu^2 A_2 L_0(\mu)^{-1} A_2 + A_2,$$
$$A_{22} = -A_4 L_0(\mu)^{-1} - \mu^2 A_2 L_0(\mu)^{-1}.$$

From Venni's proposition (see [**6**, p. 500, with $X = Y = E$, $\alpha = 0$, $\beta = \frac{1}{2}$, $\gamma = 1$, $f(\mu) = \mu^2$, $B(\mu) = L_0(\mu)^{-1}$]) and the two first inequalities in (2.3), we get that

$$\mathcal{R}\Big\{\mu^2 A_4^{\frac{1}{2}} L_0(\mu)^{-1} \ : \ \frac{\pi}{2} \geq |\arg \mu| \geq \frac{\pi - \psi}{2}\Big\}_E < \infty. \tag{2.10}$$

Similarly, from the two last inequalities in (2.3), we get

$$\mathcal{R}\Big\{\mu^2 A_4^{\frac{1}{2}} L_0(\mu)^{-1} \ : \ \frac{\pi}{2} \geq |\arg \mu| \geq \frac{\pi - \psi}{2}\Big\}_{E_2} < \infty. \tag{2.11}$$

Using now the definition of $\mathcal{R}$-boundedness, conditions (2) and (3), and formulas (2.10) and (2.11), we obtain, from (2.9), that

$$\mathcal{R}\Big\{\mathbb{A}(\mu^2 I - \mathbb{A})^{-1} \ : \frac{\pi}{2} \geq |\arg \mu| \geq \frac{\pi - \psi}{2}\Big\}_{\mathcal{E}} < \infty.$$

From this and from the identity

$$\mu^2(\mu^2 I - \mathbb{A})^{-1} = \mathbb{A}(\mu^2 I - \mathbb{A})^{-1} + I,$$

we have, using, e.g., [**3**, Proposition 3.4],

$$\mathcal{R}\Big\{\mu^2(\mu^2 I - \mathbb{A})^{-1} \ : \ \frac{\pi}{2} \geq |\arg \mu| \geq \frac{\pi - \psi}{2}\Big\}_{\mathcal{E}} < \infty,$$

i.e.,

$$\mathcal{R}\Big\{\mu(\mu I-\mathbb{A})^{-1}\ :\ |\arg\mu|\geq\pi-\psi\Big\}_{\mathcal{E}}<\infty. \tag{2.12}$$

From condition (4), for $m_1\neq m_2$, it follows that $(-1)^{m_1}\alpha_1\beta_2-(-1)^{m_2}\alpha_2\beta_1\neq 0$ and $a_k v^{(m_k)}(0)+b_k v^{(m_k)}(1)=\alpha_k v^{(m_k)}(0)+\beta_k v^{(m_k)}(1)$ in (2.5). Then, by virtue of [**5**, Theorem 4 and Remark 3] (we use the remark only for the case $m_1=m_2$), the operator that corresponds to problem (2.5),

$$\mathbb{P}(\lambda): v\to\mathbb{P}(\lambda)v:=((D^2-\mathbb{A}-\lambda\mathbb{I})v(x), a_1v^{(m_1)}(0)+b_1v^{(m_1)}(1), a_2v^{(m_2)}(0)+b_2v^{(m_2)}(1)),$$

for $|\arg\lambda|\leq\psi$ and sufficiently large $|\lambda|$, is an isomorphism from $W_p^2((0,1);\mathcal{E}(\mathbb{A}),\mathcal{E})$ onto

$$L_p((0,1);\mathcal{E})\times(\mathcal{E}(\mathbb{A}),\mathcal{E})_{\frac{m_1}{2}+\frac{1}{2p},p}\times(\mathcal{E}(\mathbb{A}),\mathcal{E})_{\frac{m_2}{2}+\frac{1}{2p},p}$$

and, for these values of λ, the following estimate holds

$$\begin{aligned}&|\lambda|\|v\|_{L_p((0,1);\mathcal{E})}+\|v''\|_{L_p((0,1);\mathcal{E})}+\|\mathbb{A}v\|_{L_p((0,1);\mathcal{E})}\\ &\leq C\Big(\|F\|_{L_p((0,1);\mathcal{E})}+\sum_{k=1}^{2}\big(\|\Phi_k\|_{(\mathcal{E}(\mathbb{A}),\mathcal{E})_{\frac{m_k}{2}+\frac{1}{2p},p}}+|\lambda|^{1-\frac{m_k}{2}-\frac{1}{2p}}\|\Phi_k\|_{\mathcal{E}}\big)\Big).\end{aligned} \tag{2.13}$$

From (2.12), it follows that the operator $\mathbb{A}$ is closed. Consequently, $\mathcal{E}(\mathbb{A})=E(A_4)\times E_2$.

Further, we have $(\mathcal{E}(\mathbb{A}),\mathcal{E})_{\theta,p}=(E(A_4)\times E_2,E_2\times E)_{\theta,p}=(E(A_4),E_2)_{\theta,p}\times(E_2,E)_{\theta,p}$. Since $E_2:=E(A_4^{\frac{1}{2}})$ then, by virtue of [**12**, Theorem 1.3.3 and formula 1.15.4/(2)],

$$\begin{aligned}(E(A_4),E(A_4^{\frac{1}{2}}))_{\frac{m_k}{2}+\frac{1}{2p},p}&=(E(A_4^{\frac{1}{2}}),E(A_4))_{1-\frac{m_k}{2}-\frac{1}{2p},p}=(E,E(A_4))_{1-\frac{m_k}{4}-\frac{1}{4p},p}\\ &=(E(A_4),E)_{\frac{m_k}{4}+\frac{1}{4p},p},\qquad k=1,2.\end{aligned} \tag{2.14}$$

Since $m_{k+2}=m_k+2,\ k=1,2$, then, by calculations similar to the previous ones, using also, e.g., [**12**, Theorem 1.15.2], one can get

$$\begin{aligned}(E(A_4^{\frac{1}{2}}),E)_{\frac{m_k}{2}+\frac{1}{2p},p}&=(E,E(A_4^{\frac{1}{2}}))_{1-\frac{m_k}{2}-\frac{1}{2p},p}=(E,E(A_4))_{\frac{1}{2}-\frac{m_k}{4}-\frac{1}{4p},p}\\ &=(E(A_4),E)_{\frac{1}{2}+\frac{m_k}{4}+\frac{1}{4p},p}=(E(A_4),E)_{\frac{m_{k+2}}{4}+\frac{1}{4p},p},\quad k=1,2.\end{aligned} \tag{2.15}$$

Hence, the operator $\mathbb{L}(\lambda)$, for the same $|\arg\lambda|\leq\psi$ and sufficiently large $|\lambda|$, is an isomorphism from $W_p^4((0,1);E(A_4),E)$ onto $L_p((0,1);E)\ \dot{\times}_{k=1}^{4}\ (E(A_4),E)_{\frac{m_k}{4}+\frac{1}{4p},p}$.

So, the first part of the statement of the theorem has been proved. Let us now obtain estimate (2.4). From (2.13), (2.14), and (2.15) it follows

$$
\begin{aligned}
&|\lambda|\big(\|u\|_{L_p((0,1);E_2)} + \|u'' - \lambda u\|_{L_p((0,1);E)}\big) \\
&\qquad + \|u''\|_{L_p((0,1);E_2)} + \|u'''' - \lambda u''\|_{L_p((0,1);E)} \\
&\qquad + \|u'' - \lambda u\|_{L_p((0,1);E_2)} + \|A_4 u + A_2(u'' - \lambda u)\|_{L_p((0,1);E)} \\
&\le C\Big(\|f\|_{L_p((0,1);E)} + \sum_{k=1}^{2}\Big[\|\varphi_k\|_{(E(A_4),E_2)_{\frac{m_k}{2}+\frac{1}{2p},p}} \\
&\qquad + \|\varphi_{k+2}\|_{(E_2,E)_{\frac{m_k}{2}+\frac{1}{2p},p}} + |\lambda|^{1-\frac{m_k}{2}-\frac{1}{2p}}(\|\varphi_k\|_{E_2} + \|\varphi_{k+2}\|_E)\Big]\Big) \\
&\le C\Big(\|f\|_{L_p((0,1);E)} + \sum_{k=1}^{4}\|\varphi_k\|_{(E(A_4),E)_{\frac{m_k}{4}+\frac{1}{4p},p}} \\
&\qquad + \sum_{k=1}^{2}|\lambda|^{1-\frac{m_k}{2}-\frac{1}{2p}}\big(\|\varphi_k\|_{E(A_4^{\frac{1}{2}})} + \|\varphi_{k+2}\|_E\big)\Big),
\end{aligned}
\tag{2.16}
$$

where the constant C does not depend on the parameter λ which is in $|\arg\lambda| \le \psi$ and $|\lambda|$ is sufficiently large. Moreover, here and throughout the paper, the constant C in estimates may change from line to line, but we keep the same notation C for all lines. Using the technique of the proof of [**13**, Theorem 3.2.1] together with the Fourier multiplier theorem in a *UMD* Banach space but with scalar multipliers (see, e.g., [**15**]), one can get that $\forall\varepsilon > 0$, $\exists C_\varepsilon > 0$ such that

$$\|u''\|_{L_p((0,1);E)} + |\lambda|\|u\|_{L_p((0,1);E)} \le C_\varepsilon\|u'' - \lambda u\|_{L_p((0,1);E)}, \quad |\arg\lambda| < \pi - \varepsilon,$$

where the constant C_ε also does not depend on λ. It means that the last inequality is also true in our angle $|\arg\lambda| \le \psi$ since $\psi < \pi$. Then, the left hand side of (2.16) is surely greater than $C_0\big(|\lambda|\|u\|_{L_p((0,1);E_2)} + |\lambda|\|u''\|_{L_p((0,1);E)} + |\lambda|^2\|u\|_{L_p((0,1);E)} + \|u''\|_{L_p((0,1);E_2)}\big)$, i.e.,

$$
\begin{aligned}
&|\lambda|\|u\|_{L_p((0,1);E_2)} + |\lambda|\|u''\|_{L_p((0,1);E)} + |\lambda|^2\|u\|_{L_p((0,1);E)} + \|u''\|_{L_p((0,1);E_2)} \\
&\qquad \le C\Big(\|f\|_{L_p((0,1);E)} + \sum_{k=1}^{4}\|\varphi_k\|_{(E(A_4),E)_{\frac{m_k}{4}+\frac{1}{4p},p}} \\
&\qquad\qquad + \sum_{k=1}^{2}|\lambda|^{1-\frac{m_k}{2}-\frac{1}{2p}}\big(\|\varphi_k\|_{E(A_4^{\frac{1}{2}})} + \|\varphi_{k+2}\|_E\big)\Big).
\end{aligned}
\tag{2.17}
$$

Then, from (2.16) and (2.17), we get

$$
\begin{aligned}
&\|u''''\|_{L_p((0,1);E)} \le \|u'''' - \lambda u''\|_{L_p((0,1);E)} + \|\lambda u''\|_{L_p((0,1);E)} \\
&\qquad \le C\Big(\|f\|_{L_p((0,1);E)} + \sum_{k=1}^{4}\|\varphi_k\|_{(E(A_4),E)_{\frac{m_k}{4}+\frac{1}{4p},p}} \\
&\qquad\qquad + \sum_{k=1}^{2}|\lambda|^{1-\frac{m_k}{2}-\frac{1}{2p}}\big(\|\varphi_k\|_{E(A_4^{\frac{1}{2}})} + \|\varphi_{k+2}\|_E\big)\Big).
\end{aligned}
\tag{2.18}
$$

In turn, (2.17) and (2.18) imply the desired estimate (2.4). □

3 - Application of the abstract isomorphism result to elliptic and quasi-elliptic equations with a quadratic parameter

In this section, we essentially use our previous paper [**7**], where the main calculations for this section have been done. Moreover, the calculations in [**7**] do not allow us to take odd order of derivatives in the operators A_2 and A_4 in application. That is why all our below examples contain only even order of derivatives in the equations.

In the domain $\Omega := [0,1] \times \mathbb{R}$, let us consider a non-local boundary value problem for elliptic equations of the fourth order with a parameter

$$
\begin{aligned}
(L(\lambda)u)(x,y) &:= \lambda^2 u(x,y) - \lambda\big(2D_x^2 u(x,y) + aD_y^2 u(x,y) - 2\gamma^2 u(x,y)\big) \\
&\quad + D_x^4 u(x,y) + (aD_y^2 - 2\gamma^2)D_x^2 u(x,y) + bD_y^4 u(x,y) - a\gamma^2 D_y^2 u(x,y) \\
&\quad + \gamma^4 u(x,y) = f(x,y), \quad (x,y) \in \Omega,
\end{aligned} \tag{3.1}
$$

$$
\begin{aligned}
(L_k u)(y) &:= \alpha_k D_x^{m_k} u(0,y) + \beta_k D_x^{m_k} u(1,y) = \varphi_k(y), \quad y \in \mathbb{R},\ k = 1,2, \\
(L_k(\lambda)u)(y) &:= \alpha_k\big(D_x^{m_k} u(0,y) - \lambda D_x^{m_k-2} u(0,y)\big) \\
&\quad + \beta_k\big(D_x^{m_k} u(1,y) - \lambda D_x^{m_k-2} u(1,y)\big) \\
&\quad = \varphi_k(y), \quad y \in \mathbb{R},\ k = 3,4,
\end{aligned} \tag{3.2}
$$

where $0 \le m_1, m_2 \le 1$, $m_3 = m_1 + 2$, $m_4 = m_2 + 2$; a, b, α_k, β_k are complex numbers; $\gamma \in \mathbb{R}$; f and φ_k are given functions; $D_x := \dfrac{\partial}{\partial x}$, $D_y := \dfrac{\partial}{\partial y}$. By $B^s_{q,p}(\mathbb{R}^n)$, n is natural, we denote the standard Besov space, see, e.g., [**12**, section 2.3.1].

Theorem 3.1. *Let the following conditions be satisfied*:

1. $0 \ne \gamma \in \mathbb{R}$, $0 \ne b \in \mathbb{C}$, $\arg b \ne \pi$;
2. *if* $\sigma := (\sigma_1, \sigma_2) \in \mathbb{R}^2$, $\sigma \ne 0$, *then* $\sigma_1^4 + a\sigma_1^2\sigma_2^2 + b\sigma_2^4 \ne 0$, $(\sigma_1;\sigma_2) \in \mathbb{R}^2$;
3. $(-1)^{m_1}\alpha_1\beta_2 - (-1)^{m_2}\alpha_2\beta_1 \ne 0$ *and* $(-1)^{m_1}\alpha_3\beta_4 - (-1)^{m_2}\alpha_4\beta_3 \ne 0$; *for* $m_1 \ne m_2$, *assume, in addition, that* $\alpha_k = \alpha_{k+2}$, $\beta_k = \beta_{k+2}$, $k = 1,2$.

Then, there exist $\delta > 0$ *sufficiently small and* $\lambda_0 \ge 0$ *such that, for* $|a| < \delta$ *and* $\lambda \ge \lambda_0$, *the operator*

$$
\mathbb{L}(\lambda) : u \to \mathbb{L}(\lambda)u := \Big((L(\lambda)u)(x,y), (L_1 u)(y), (L_2 u)(y), (L_3(\lambda)u)(y), (L_4(\lambda)u)(y)\Big),
$$

$$
\textit{from } W_p^4((0,1); W_q^4(\mathbb{R}), L_q(\mathbb{R})) \ \textit{ onto } \ L_p((0,1); L_q(\mathbb{R})) \mathop{\times}_{k=1}^{4} B_{q,p}^{4-m_k-\frac{1}{p}}(\mathbb{R}),
$$

where $q \in (1,\infty)$, $p \in (1,\infty)$, *is an isomorphism and, for these values of* λ, *the following estimate holds for the solution* $u(x,y)$ *of the problem* (3.1)–(3.2)

$$
\begin{aligned}
&|\lambda|^2\|u\|_{L_p((0,1);L_q(\mathbb{R}))} + |\lambda|\Big(\|\frac{\partial^2 u}{\partial x^2}\|_{L_p((0,1);L_q(\mathbb{R}))} + \|u\|_{L_p((0,1);W_q^2(\mathbb{R}))}\Big)\\
&\qquad + \|\frac{\partial^2 u}{\partial x^2}\|_{L_p((0,1);W_q^2(\mathbb{R}))} + \|\frac{\partial^4 u}{\partial x^4}\|_{L_p((0,1);L_q(\mathbb{R}))}\\
&\le C\Big(\|f\|_{L_p((0,1);L_q(\mathbb{R}))} + \sum_{k=1}^{4}\|\varphi_k\|_{B_{q,p}^{4-m_k-\frac{1}{p}}(\mathbb{R})}\\
&\qquad + \sum_{k=1}^{2}|\lambda|^{1-\frac{m_k}{2}-\frac{1}{2p}}\big(\|\varphi_k\|_{W_q^2(\mathbb{R})} + \|\varphi_{k+2}\|_{L_q(\mathbb{R})}\big)\Big),
\end{aligned}
$$

where the constant C does not depend on the parameter λ.

Proof. Let us denote $E := L_q(\mathbb{R})$. Consider in $L_q(\mathbb{R})$ operators A_2 and A_4 which are defined by the equalities

$$
\begin{aligned}
&D(A_2) := W_q^2(\mathbb{R}), \quad (A_2u)(y) := au''(y) - 2\gamma^2 u(y),\\
&D(A_4) := W_q^4(\mathbb{R}), \quad (A_4u)(y) := bu''''(y) - a\gamma^2 u''(y) + \gamma^4 u(y).
\end{aligned}
$$

Then, problem (3.1)–(3.2) can be rewritten in the operator form

$$
\begin{aligned}
&\lambda^2 u(x) - \lambda\big(2u''(x) + A_2u(x)\big) + u''''(x) + A_2u''(x) + A_4u(x) = f(x), \quad x \in (0,1),\\
&\alpha_k u^{(m_k)}(0) + \beta_k u^{(m_k)}(1) = \varphi_k, \quad k = 1,2, \qquad\qquad (3.3)\\
&\alpha_k\big(u^{(m_k)}(0) - \lambda u^{(m_{k-2})}(0)\big) + \beta_k\big(u^{(m_k)}(1) - \lambda u^{(m_{k-2})}(1)\big) = \varphi_k, \quad k = 3,4,
\end{aligned}
$$

where $u(x) := u(x,\cdot)$, $f(x) := f(x,\cdot)$ are functions with values in the Banach space $E = L_q(\mathbb{R})$ and $\varphi_k := \varphi_k(\,\cdot\,)$.

We now apply Theorem 2.1, for $\psi = 0$, to problem (3.3). In fact, we have to check conditions (1)–(3) of Theorem 2.1, for $\psi = 0$, and they have been checked in the proof of Theorem 5 in our paper [7]. We also refer the reader to [**12**, section 2.4.1] for the characterization of the Besov space $B_{q,p}^{4-m_k-\frac{1}{p}}(\mathbb{R})$ as a corresponding interpolation space $(E(A_4),E)_{\frac{m_k}{4}+\frac{1}{4p},p}$, where $E(A_4) = W_q^4(\mathbb{R})$ and $E = L_q(\mathbb{R})$. □

The next application is in the domain $\Omega := [0,1] \times \mathbb{R}^n$, $n \ge 1$. Consider a non-local boundary value problem for elliptic ($m = 2$) and quasi-elliptic ($m \ne 2$ is natural) equations with a parameter

$$
\begin{aligned}
(L(\lambda)u)(x,y) &:= \lambda^2 u(x,y) - 2\lambda D_x^2 u(x,y) + D_x^4 u(x,y)\\
&\quad + \sum_{|\alpha|=2m} a_\alpha(y) D_y^\alpha u(x,y) + \nu u(x,y) = f(x,y), \quad (x,y) \in \Omega, \qquad (3.4)
\end{aligned}
$$

$$
\begin{aligned}
(L_k u)(y) &:= \alpha_k D_x^{m_k} u(0,y) + \beta_k D_x^{m_k} u(1,y) = \varphi_k(y), \quad y \in \mathbb{R}^n,\ k = 1,2,\\
(L_k(\lambda)u)(y) &:= \alpha_k \big(D_x^{m_k} u(0,y) - \lambda D_x^{m_{k-2}} u(0,y)\big)\\
&\quad + \beta_k \big(D_x^{m_k} u(1,y) - \lambda D_x^{m_{k-2}} u(1,y)\big)\\
&\quad = \varphi_k(y), \quad y \in \mathbb{R}^n,\ k = 3,4,
\end{aligned} \tag{3.5}
$$

where $0 \le m_1, m_2 \le 1$, $m_3 = m_1 + 2$, $m_4 = m_2 + 2$; $\nu > 0$, α_k and β_k are complex numbers, f and φ_k are given functions, $D_x := \dfrac{\partial}{\partial x}$, $D_y^\alpha := D^\alpha := D_1^{\alpha_1} \cdots D_n^{\alpha_n}$, $D_j := -i\dfrac{\partial}{\partial y_j}$, $|\alpha| = \alpha_1 + \cdots + \alpha_n$.

Recall (see, e.g., [**10**, p. 790]) that, for $M > 0, \omega_0 \in [0,\pi)$, an operator of the form $(Cu)(x) := \sum_{|\alpha| \le 2m} c_\alpha(x) D^\alpha u(x)$ with complex-valued $c_\alpha \in L_\infty(\mathbb{R}^n)$, $|\alpha| \le 2m$, is called (M,ω_0)-elliptic if $\sum_{|\alpha|=2m} \|c_\alpha\|_\infty \le M$ and the principal symbol

$$
C_\pi(x,\xi) := \sum_{|\alpha|=2m} c_\alpha(x)\xi^\alpha, \quad x,\xi \in \mathbb{R}^n,
$$

of the operator C satisfies, for all $x,\xi \in \mathbb{R}^n$, that the spectrum $\sigma(C_\pi(x,\xi)) \subset \overline{\Sigma_{\omega_0}}$ and $|C_\pi(x,\xi)| \ge M^{-1}|\xi|^{2m}$, where $\Sigma_{\omega_0} := \{\lambda \in \mathbb{C}\backslash\{0\} : |\arg \lambda| < \omega_0\}$. If a bounded domain $G \subset \mathbb{R}^n$ with C^{2m} boundary is considered instead of $\mathbb{R}^n$, then a similar definition is given with $c_\alpha \in C(\overline{G})$.

Let $(Au)(y) := \sum_{|\alpha|=2m} a_\alpha(y) D^\alpha u(y)$ be an (M,ω_0)-elliptic operator, for some $M > 0$, $\omega_0 \in [0,\pi)$, such that the complex-valued coefficients a_α are Hölder continuous, i.e., $a_\alpha \in C^\gamma(\mathbb{R}^n)$, $|\alpha| = 2m$, for some $\gamma > 0$.

Theorem 3.2. *Let, in addition to the above,* $(-1)^{m_1}\alpha_1\beta_2 - (-1)^{m_2}\alpha_2\beta_1 \ne 0$ *and* $(-1)^{m_1}\alpha_3\beta_4 - (-1)^{m_2}\alpha_4\beta_3 \ne 0$; *for* $m_1 \ne m_2$, *assume also that* $\alpha_k = \alpha_{k+2}$, $\beta_k = \beta_{k+2}$, $k = 1,2$.

Then, there exist $\nu > 0$ *sufficiently large and* $\lambda_0 \ge 0$ *such that, for* $\lambda \ge \lambda_0$, *the operator*

$$
\mathbb{L}(\lambda) : u \to \mathbb{L}(\lambda)u := \Big((L(\lambda)u)(x,y), (L_1 u)(y), (L_2 u)(y), (L_3(\lambda)u)(y), (L_4(\lambda)u)(y)\Big),
$$

from $W_p^4((0,1); W_q^{2m}(\mathbb{R}^n), L_q(\mathbb{R}^n))$ *onto* $L_p((0,1); L_q(\mathbb{R}^n)) \times_{k=1}^{4} B_{q,p}^{2m - \frac{mm_k}{2} - \frac{m}{2p}}(\mathbb{R}^n)$, *where* $q \in (1,\infty)$, $p \in (1,\infty)$, *is an isomorphism and, for these values of* λ, *the following estimate holds for the solution* $u(x,y)$ *of the problem* (3.4)–(3.5)

$$
\begin{aligned}
&|\lambda|^2\|u\|_{L_p((0,1);L_q(\mathbb{R}^n))} + |\lambda|\Big(\|\frac{\partial^2 u}{\partial x^2}\|_{L_p((0,1);L_q(\mathbb{R}^n))} + \|u\|_{L_p((0,1);W_q^m(\mathbb{R}^n))}\Big) \\
&\qquad + \|\frac{\partial^2 u}{\partial x^2}\|_{L_p((0,1);W_q^m(\mathbb{R}^n))} + \|\frac{\partial^4 u}{\partial x^4}\|_{L_p((0,1);L_q(\mathbb{R}^n))} \\
&\le C\Big(\|f\|_{L_p((0,1);L_q(\mathbb{R}^n))} + \sum_{k=1}^{4}\|\varphi_k\|_{B_{q,p}^{2m-\frac{mm_k}{2}-\frac{m}{2p}}(\mathbb{R}^n)} \\
&\qquad + \sum_{k=1}^{2}|\lambda|^{1-\frac{m_k}{2}-\frac{1}{2p}}\big(\|\varphi_k\|_{W_q^m(\mathbb{R}^n)} + \|\varphi_{k+2}\|_{L_q(\mathbb{R}^n)}\big)\Big),
\end{aligned}
$$

where the constant C does not depend on the parameter λ.

Proof. Let us denote $E := L_q(\mathbb{R}^n)$. Consider in E the operator A_4 which is defined by the equalities

$$D(A_4) := W_q^{2m}(\mathbb{R}^n), \quad (A_4u)(y) := Au(y) + \nu u(y),$$

where $\nu > 0$ is sufficiently large. Then, problem (3.4)–(3.5) can be rewritten in the operator form

$$
\begin{aligned}
&\lambda^2 u(x) - 2\lambda u''(x) + u''''(x) + A_4 u(x) = f(x), \quad x \in (0,1), \\
&\alpha_k u^{(m_k)}(0) + \beta_k u^{(m_k)}(1) = \varphi_k, \quad k = 1,2, \qquad (3.6)\\
&\alpha_k\big(u^{(m_k)}(0) - \lambda u^{(m_{k-2})}(0)\big) + \beta_k\big(u^{(m_k)}(1) - \lambda u^{(m_{k-2})}(1)\big) = \varphi_k, \quad k = 3,4,
\end{aligned}
$$

where $u(x) := u(x,\cdot)$, $f(x) := f(x,\cdot)$ are functions with values in the Banach space $E = L_q(\mathbb{R}^n)$ and $\varphi_k := \varphi_k(\,\cdot\,)$.

We apply Theorem 2.1, with $\psi = 0$, to problem (3.6) the conditions of which, with $\psi = 0$, have been checked in the proof of Theorem 6 in [**7**]. We also refer the reader to [**12**, section 2.4.1] for the characterization of the Besov space $B_{q,p}^{2m-\frac{mm_k}{2}-\frac{m}{2p}}(\mathbb{R}^n)$ as a corresponding interpolation space $(E(A_4),E)_{\frac{m_k}{4}+\frac{1}{4p},p}$, where $E(A_4) = W_q^{2m}(\mathbb{R}^n)$ and $E = L_q(\mathbb{R}^n)$. □

Remark 3.1. *Using the technique of the proof of Theorem* 5.1, *we can get the isomorphism theorem even for* $|\arg\lambda| \le \psi$ *and sufficiently large* $|\lambda|$, *for any* $0 \le \psi < \dfrac{\pi - \omega_0}{2}$.

Finally, in the cylindrical domain $\Omega := [0,1] \times G$, where $G \subset \mathbb{R}^n$, $n \ge 2$, is a bounded domain with an $(n-1)$-dimensional boundary $\partial G \in C^{2m}$, which locally admits rectification, let us consider a non-local boundary value problem for elliptic ($m = 2$) and quasi-elliptic ($m \ne 2$ is natural) equations with a parameter

$$(3.7)\quad \begin{aligned}(L(\lambda)u)(x,y) :=&\lambda^2 u(x,y) - 2\lambda D_x^2 u(x,y) + D_x^4 u(x,y)\\ &+\sum_{|\alpha|=2m} a_\alpha(y) D_y^\alpha u(x,y) + vu(x,y) = f(x,y), \quad (x,y)\in\Omega,\end{aligned}$$

$$(3.8)\quad \begin{aligned}(L_k u)(y) &:= \alpha_k D_x^{m_k} u(0,y) + \beta_k D_x^{m_k} u(1,y) = \varphi_k(y), \quad y\in G,\ k=1,2,\\ (L_k(\lambda)u)(y) &:= \alpha_k\big(D_x^{m_k}u(0,y) - \lambda D_x^{m_{k-2}}u(0,y)\big)\\ &\quad+\beta_k\big(D_x^{m_k}u(1,y) - \lambda D_x^{m_{k-2}}u(1,y)\big)\\ &\quad= \varphi_k(y), \quad y\in G,\ k=3,4,\end{aligned}$$

$$(3.9)\quad (B_\ell u)(x,y') := \sum_{|\beta|\le p_\ell} b_{\ell\beta}(y') D_y^\beta u(x,y') = 0, \quad (x,y')\in[0,1]\times\partial G,\ \ell=1,\dots,m,$$

where $0\le m_1, m_2\le 1,\ m_3 = m_1+2,\ m_4 = m_2+2,\ p_\ell \le 2m-1;\ \alpha_k$ and β_k are complex numbers; f and φ_k are given functions; $D_x := \frac{\partial}{\partial x}$, $D_y^\alpha := D^\alpha := D_1^{\alpha_1}\cdots D_n^{\alpha_n}$, $D_j := -i\frac{\partial}{\partial y_j}$, $|\alpha| = \alpha_1+\cdots+\alpha_n$.

Let $(Au)(y) := \sum_{|\alpha|=2m} a_\alpha(y) D^\alpha u(y)$ be an (M,ω_0)-elliptic operator, for some $M>0$, $\omega_0\in[0,\pi)$. Complex-valued coefficients $a_\alpha\in C^\gamma(\overline{G})$, $|\alpha|=2m$. Complex-valued coefficients $b_{\ell\beta}$ of the boundary conditions B_ℓ belong to $C^{2m-p_\ell+\gamma}(\overline{G})$, where $\gamma\in(0,1)$ (the continuation of the coefficients from ∂G into G is possible without loss of generality). We assume that $(A, B_1,\dots,B_m)$ satisfies the Lopatinskii-Shapiro condition (see, e.g., [**3**, p. 100]) at every point $y'\in\partial G$.

By $B^s_{q,p}(G)$ we denote the standard Besov space, see, e.g., [**12**, section 4.2.1]. Before the formulation of the next theorem, let us make the following remark.

Remark 3.2. *By $W_q^{2m}(G; B_\ell u = 0, \ell = 1,\dots,m)$ we denote the space of functions u from $W_q^{2m}(G)$ which satisfy all boundary conditions $B_\ell u = 0$, $\ell = 1,\dots,m$. By $B_{q,p}^{2m-\frac{m_k m}{2}-\frac{m}{2p}}\big(G; B_\ell u = 0, p_\ell < 2m - \frac{m_k m}{2} - \frac{m}{2p} - \frac{1}{q}\big)$ we denote the space of functions u from the Besov space $B_{q,p}^{2m-\frac{m_k m}{2}-\frac{m}{2p}}(G)$ which satisfy only boundary conditions $B_\ell u = 0$ with the order $p_\ell < 2m - \frac{m_k m}{2} - \frac{m}{2p} - \frac{1}{q}$. Moreover, we refer the reader to* [**12**, *section* 4.3.3] *for the characterization of the space $B_{q,p}^{2m-\frac{m_k m}{2}-\frac{m}{2p}}\big(G; B_\ell u = 0, p_\ell < 2m - \frac{m_k m}{2} - \frac{m}{2p} - \frac{1}{q}\big)$ as a corresponding interpolation space $(E(A_4), E)_{\frac{m_k}{4}+\frac{1}{4p},p}$, where $E(A_4) = W_q^{2m}(G; B_\ell u = 0, \ell = 1,\dots,m)$ and $E = L_q(G)$.*

Theorem 3.3. *Let, in addition to the above,* $(-1)^{m_1}\alpha_1\beta_2-(-1)^{m_2}\alpha_2\beta_1\neq 0$ *and* $(-1)^{m_1}\alpha_3\beta_4-(-1)^{m_2}\alpha_4\beta_3\neq 0$; *for* $m_1\neq m_2$, *assume also that* $\alpha_k=\alpha_{k+2}$, $\beta_k=\beta_{k+2}$, $k=1,2$.

Then, there exist $\nu>0$ *sufficiently large and* $\lambda_0\geq 0$ *such that, for* $\lambda\geq\lambda_0$, *the operator*

$$\mathbb{L}(\lambda):u\to\mathbb{L}(\lambda)u:=\Big((L(\lambda)u)(x,y),(L_1u)(y),(L_2u)(y),(L_3(\lambda)u)(y),(L_4(\lambda)u)(y)\Big)$$

from $W_p^4((0,1);W_q^{2m}(G;B_\ell u=0,\ell=1,\dots,m),L_q(G))$ *onto*

$$L_p((0,1);L_q(G))\mathop{\times}_{k=1}^{4}B_{q,p}^{2m-\frac{m_km}{2}-\frac{m}{2p}}\Big(G;B_\ell u=0,p_\ell<2m-\frac{m_km}{2}-\frac{m}{2p}-\frac{1}{q}\Big),$$

where $q\in(1,\infty)$, $p\in(1,\infty)$, *is an isomorphism and, for these values of* λ, *the following estimate holds for the solution* $u(x,y)$ *of the problem* (3.7)–(3.9)

$$\begin{aligned}|\lambda|^2\|u\|_{L_p((0,1);L_q(G))}&+|\lambda|\Big(\Big\|\frac{\partial^2u}{\partial x^2}\Big\|_{L_p((0,1);L_q(G))}+\|u\|_{L_p((0,1);W_q^m(G))}\Big)\\&+\Big\|\frac{\partial^2u}{\partial x^2}\Big\|_{L_p((0,1);W_q^m(G))}+\Big\|\frac{\partial^4u}{\partial x^4}\Big\|_{L_p((0,1);L_q(G))}\\\leq C\Big(&\|f\|_{L_p((0,1);L_q(G))}+\sum_{k=1}^{4}\|\varphi_k\|_{B_{q,p}^{2m-\frac{mm_k}{2}-\frac{m}{2p}}(G)}\\&+\sum_{k=1}^{2}|\lambda|^{1-\frac{m_k}{2}-\frac{1}{2p}}\big(\|\varphi_k\|_{W_q^m(G)}+\|\varphi_{k+2}\|_{L_q(G)}\big)\Big),\end{aligned}$$

where the constant C *does not depend on the parameter* λ.

Proof. Let us denote $E:=L_q(G)$. Consider in E the operator A_4 which is defined by the equalities

$$D(A_4):=W_q^{2m}(G;B_\ell u=0,\ell=1,\dots,m),\quad (A_4u)(y):=Au(y)+\nu u(y),$$

where $\nu>0$ is sufficiently large. Then, problem (3.7)–(3.9) can be rewritten in the operator form

$$\begin{aligned}&\lambda^2u(x)-2\lambda u''(x)+u''''(x)+A_4u(x)=f(x),\quad x\in(0,1),\\&\alpha_ku^{(m_k)}(0)+\beta_ku^{(m_k)}(1)=\varphi_k,\quad k=1,2,\\&\alpha_k\big(u^{(m_k)}(0)-\lambda u^{(m_{k-2})}(0)\big)+\beta_k\big(u^{(m_k)}(1)-\lambda u^{(m_{k-2})}(1)\big)=\varphi_k,\quad k=3,4,\end{aligned}\tag{3.10}$$

$u(x):=u(x,\cdot),f(x):=f(x,\cdot)$ are functions with values in the Banach space $E=L_q(G)$ and $\varphi_k:=\varphi_k(\cdot)$.

Apply Theorem 2.1, for $\psi=0$, to problem (3.10). Conditions (1)–(3) of Theorem 2.1, for $\psi=0$, have been checked in the proof of Theorem 7 in [7]. □

Remark 3.3. *The same remark as Remark* 3.1 *can be done also here.*

4 - Two-fold completeness theorem for abstract fourth order elliptic boundary value problems quadratically depending on a parameter

We start from the necessary definitions.

Let X and X^ν, $\nu = 1, \dots, m$, be Banach spaces. Consider a problem for a system of polynomial operator pencils in X,

$$
\begin{aligned}
L(\lambda)u &:= \lambda^n u + \lambda^{n-1} B_1 u + \cdots + B_n u = 0, \\
L_\nu(\lambda)u &:= \lambda^{n_\nu} A_{\nu 0} u + \lambda^{n_\nu - 1} A_{\nu 1} u + \cdots + A_{\nu n_\nu} u = 0, \quad \nu = 1, \dots, m,
\end{aligned}
\tag{4.1}
$$

where $n \geq 1$, $0 \leq n_\nu \leq n-1$, $m \geq 0$; B_k are, generally speaking, unbounded operators in X; and $A_{\nu k}$, $k = 0, \dots, n_\nu$ are, generally speaking, unbounded operators from X into X^ν. Let there exist a Banach space $X_n \subset X$, such that operators B_k, $k = 1, \dots, n$, from X_n into X, act boundedly, and operators $A_{\nu k}$, $k = 0, \dots, n_\nu$, $\nu = 1, \dots, m$, from X_n into X^ν, act boundedly.

A number λ_0 is called an **eigenvalue** of problem (4.1) if the problem

$$L(\lambda_0)u = 0, \quad L_\nu(\lambda_0)u = 0, \quad \nu = 1, \dots, m$$

has a nontrivial solution belonging to X_n. The nontrivial solution $u_0 \in X_n$ is called an **eigenvector** of problem (4.1) corresponding to the eigenvalue λ_0. A solution $u_p \in X_n$, $p = 1, 2, \dots$, of the problem

$$
\begin{aligned}
L(\lambda_0)u_p + \frac{1}{1!} L'(\lambda_0) u_{p-1} + \cdots + \frac{1}{p!} L^{(p)}(\lambda_0) u_0 &= 0, \\
L_\nu(\lambda_0)u_p + \frac{1}{1!} L_\nu'(\lambda_0) u_{p-1} + \cdots + \frac{1}{p!} L_\nu^{(p)}(\lambda_0) u_0 &= 0, \quad \nu = 1, \dots, m,
\end{aligned}
$$

is called an **associated vector of rank** p to the eigenvector u_0 of problem (4.1).

Eigenvectors and associated vectors of problem (4.1) are combined under the general name **root vectors** of problem (4.1).

A complex number λ is called a **regular point** of problem (4.1) or of the operator pencil $\mathbb{L}(\lambda) : u \to \mathbb{L}(\lambda)u := (L(\lambda)u, L_1(\lambda)u, \dots, L_m(\lambda)u)$, which acts boundedly from X_n into $X \times X^1 \times \cdots \times X^m$, if the problem

$$L(\lambda)u = f, \quad L_\nu(\lambda)u = f_\nu, \quad \nu = 1, \dots, m$$

has a unique solution $u \in X_n$, for any $f \in X, f_\nu \in X^\nu$, and the estimate

$$\|u\|_{X_n} \leq C(\lambda) \Big(\|f\|_X + \sum_{\nu=1}^m \|f_\nu\|_{X^\nu} \Big)$$

is satisfied. The set of all regular points (the **resolvent set**) of problem (4.1) is denoted by $\rho(\mathbb{L}(\lambda))$.

The complement of the regular point set in the complex plane is called the **spectrum** of problem (4.1) or of the operator pencil $\mathbb{L}(\lambda)$ and is denoted by $\sigma(\mathbb{L}(\lambda))$.

As usual, the spectrum of problem (4.1) is called **discrete** if $\sigma(\mathbb{L}(\lambda))$ consists of isolated eigenvalues with finite algebraic multiplicities and infinity is the only limit point of $\sigma(\mathbb{L}(\lambda))$.

In order to give a definition of n-fold completeness, let us consider a system of differential-operator equations corresponding to (4.1)

$$\begin{aligned} L(D)u &:= u^{(n)}(t) + B_1 u^{(n-1)}(t) + \cdots + B_n u(t) = 0, \quad t > 0, \\ L_\nu(D)u &:= A_{\nu 0} u^{(n_\nu)}(t) + \cdots + A_{\nu n_\nu} u(t) = 0, \quad \nu = 1, \dots, m, \ t > 0, \end{aligned} \tag{4.2}$$

$$u^{(k)}(0) = v_{k+1}, \quad k = 0, \dots, n-1, \tag{4.3}$$

where v_k, $k = 1, \dots, n$, are given elements of X, $D := \dfrac{d}{dt}$. Derivatives are understood in an (abstract) strong sense, if one considers (4.2) in abstract n-times continuously differentiable functions spaces $C^n([0,T];X)$, or in an (abstract) generalized sense, if one considers (4.2) in abstract Sobolev spaces $W_p^n((0,T);X)$.

By virtue of [**13**, Lemma 2.2.1/1], a function of the form

$$u(t) := e^{\lambda_0 t}\Big(\frac{t^k}{k!}u_0 + \frac{t^{k-1}}{(k-1)!}u_1 + \cdots + u_k\Big) \tag{4.4}$$

is a solution of system (4.2), if and only if the system of vectors $u_0, u_1, \cdots, u_k$ is a chain of root vectors of problem (4.1), corresponding to the eigenvalue λ_0.

A solution of the form (4.4) is called an **elementary solution** of system (4.2).

The possibility to approximate a solution to the Cauchy problem (4.2)–(4.3) by linear combinations of the elementary solutions suggests that the vector $(v_1, v_2, \dots, v_n)$ should be approximated by linear combinations of vectors of the form

$$(u(0), u'(0), \dots, u^{(n-1)}(0)), \tag{4.5}$$

where $u(t)$ is an elementary solution.

Let $\mathcal{X}$ be a Banach space, continuously embedded into $\overset{n}{\times} X$. In particular, $\mathcal{X} = \overset{n}{\times} X$.

A system of root vectors of problem (4.1) is called **n-fold complete in the space $\mathcal{X}$** if the system of vectors (4.5) is complete in $\mathcal{X}$.

Let A be a bounded operator from a Banach space E into a Banach space F. Then, numbers

$$\tilde{s}_j(A;E,F) := \inf_{\substack{\dim R(K) < j \\ K \in B(E,F)}} \|A - K\|_{B(E,F)}, \quad j = 1, 2, \dots$$

are said to be the **approximation numbers** of A.

Let an operator A from a Hilbert space H into a Hilbert space H_1 be bounded. From $(A^*Au,u)_H = (Au,Au)_H \geq 0$ it follows that the operator A^*A in H is non-negative. In turn, it implies that there exists a unique non-negative selfadjoint operator $T := (A^*A)^{\frac{1}{2}}$ in H. If A from a Hilbert space H into a Hilbert space H_1 is compact, then, in addition to the above, the operator $T = (A^*A)^{\frac{1}{2}}$ in H is compact. The eigenvalues of the operator T are called **singular numbers** of the compact operator A and are denoted by $s_j(A;H,H_1)$. Enumerate the singular numbers in decreasing order, taking into account their multiplicities, so that

$$s_j(A;H,H_1) := \lambda_j(T), \quad j = 1,\dots,\infty.$$

In the framework of Hilbert spaces, the approximation numbers of a compact operator A coincide with its singular numbers (see, e.g., [**13**, Theorem 1.2.10/2]).

Consider now a homogeneous problem corresponding to problem (2.1)–(2.2), in order to investigate two-fold completeness for a system of its root vectors

$$\begin{aligned}(L(\lambda)u)(x) &:= \lambda^2 u(x) - \lambda\big(2u''(x) + A_2u(x)\big) + u''''(x)\\ &\quad + A_2u''(x) + A_4u(x) = 0, \quad x \in (0,1),\\ L_k u &:= \alpha_k u^{(m_k)}(0) + \beta_k u^{(m_k)}(1) = 0, \quad k = 1,2,\\ L_k(\lambda)u &:= \alpha_k\big(u^{(m_k)}(0) - \lambda u^{(m_{k-2})}(0)\big)\\ &\quad + \beta_k\big(u^{(m_k)}(1) - \lambda u^{(m_{k-2})}(1)\big) = 0, \quad k = 3,4.\end{aligned} \tag{4.6}$$

Denote

$$\begin{aligned}\mathcal{E}_1 := \Big\{v \mid v &:= (v_1,v_2) \in W_p^4((0,1);E(A_4),E) \times W_p^2((0,1);E(A_4^{\frac{1}{2}}),E),\\ &\alpha_1 v_1^{(m_1)}(0) + \beta_1 v_1^{(m_1)}(1) = 0, \quad \alpha_2 v_1^{(m_2)}(0) + \beta_2 v_1^{(m_2)}(1) = 0,\\ &\alpha_1 v_2^{(m_1)}(0) + \beta_1 v_2^{(m_1)}(1) = 0, \quad \alpha_2 v_2^{(m_2)}(0) + \beta_2 v_2^{(m_2)}(1) = 0,\\ &-\alpha_3 v_2^{(m_1)}(0) - \beta_3 v_2^{(m_1)}(1) + \alpha_3 v_1^{(m_3)}(0) + \beta_3 v_1^{(m_3)}(1) = 0,\\ &-\alpha_4 v_2^{(m_2)}(0) - \beta_4 v_2^{(m_2)}(1) + \alpha_4 v_1^{(m_4)}(0) + \beta_4 v_1^{(m_4)}(1) = 0\Big\},\\ \mathcal{E} := \Big\{v \mid v &:= (v_1,v_2) \in W_p^2((0,1);E(A_4^{\frac{1}{2}}),E) \times L_p((0,1);E),\\ &\alpha_1 v_1^{(m_1)}(0) + \beta_1 v_1^{(m_1)}(1) = 0, \quad \alpha_2 v_1^{(m_2)}(0) + \beta_2 v_1^{(m_2)}(1) = 0\Big\}.\end{aligned} \tag{4.7}$$

Theorem 4.1. *Let E be a separable, reflexive UMD Banach space and*

1. *the embedding $E(A_4) \subset E$ is compact and, for some $s > \frac{1}{2}$, for the embedding operators J_1 from $W_p^2((0,1);E(A_4^{\frac{1}{2}}),E)$ into $L_p((0,1);E)$ and J_2 from $W_p^4((0,1);E(A_4),E)$ into $W_p^2((0,1);E(A_4^{\frac{1}{2}}),E)$ it holds that the approximation numbers*

$$\tilde{s}_j\big(J_1; W_p^2((0,1); E(A_4^{\frac{1}{2}}), E), L_p((0,1); E)\big) \le Cj^{-s}, \quad j = 1, 2, \dots,$$
$$\tilde{s}_j\big(J_2; W_p^4((0,1); E(A_4), E), W_p^2((0,1); E(A_4^{\frac{1}{2}}), E)\big) \le Cj^{-s}, \quad j = 1, 2, \dots;$$

2. *all conditions of Theorem* 2.1 *are satisfied* (*condition* (3) *with some* $\frac{\frac{5}{2}-s}{2}\pi < \psi < \pi$ *if* $\frac{1}{2} < s \le \frac{5}{2}$ *and with some* $0 \le \psi < \pi$ *if* $s > \frac{5}{2}$);

3. *the spectrum of problem* (4.6) *is not empty.*

Then, the spectrum of problem (4.6) *is discrete and a system of root vectors of problem* (4.6) *is two-fold complete in the spaces* $\mathcal{E}$ *and* $\mathcal{E}_1$ *and, therefore, in* $L_p((0,1); E) \times L_p((0,1); E)$.

Proof. We are going to use Theorem 6.1 from the Appendix to problem (4.6). To this end, we introduce the corresponding spaces and operators and rewrite problem (4.6) in the form of (6.1).

Denote the Banach spaces

$$X_0 := X := L_p((0,1); E),$$
$$X_1 := W_p^2((0,1); E(A_4^{\frac{1}{2}}), E),$$
$$X_2 := W_p^4((0,1); E(A_4), E),$$
$$X^k := (E(A_4), E)_{\frac{m_k}{4}+\frac{1}{4p}, p}, \quad k = 1, \dots, 4$$

and consider, in X, operators B_1 and B_2 which are defined by the equalities

$$(B_1 u)(x) := -2u''(x) - A_2 u(x), \quad D(B_1) := X_1,$$
$$(B_2 u)(x) := u''''(x) + A_2 u''(x) + A_4 u(x), \quad D(B_2) := X_2.$$

Introduce also the boundary operators

$$A_{10}u := \alpha_1 u^{(m_1)}(0) + \beta_1 u^{(m_1)}(1), \qquad A_{20}u := \alpha_2 u^{(m_2)}(0) + \beta_2 u^{(m_2)}(1),$$
$$A_{30}u := -\alpha_3 u^{(m_1)}(0) - \beta_3 u^{(m_1)}(1), \quad A_{31}u := \alpha_3 u^{(m_3)}(0) + \beta_3 u^{(m_3)}(1),$$
$$A_{40}u := -\alpha_4 u^{(m_2)}(0) - \beta_4 u^{(m_2)}(1), \quad A_{41}u := \alpha_4 u^{(m_4)}(0) + \beta_4 u^{(m_4)}(1).$$

Finally, choose

$$X_0^1 := X_0^3 := X^3, \quad X_0^2 := X_0^4 := X^4.$$

Then, problem (4.6) has form (6.1), with $n = 2$, $n_1 = n_2 = 0$, $n_3 = n_4 = 1$, $m = 4$,

$$\begin{aligned} L(\lambda)u &= \lambda^2 u + \lambda B_1 u + B_2 u = 0, \\ L_\nu(\lambda)u &:= L_\nu u = A_{\nu 0}u = 0, \quad \nu = 1, 2, \\ L_\nu(\lambda)u &= \lambda A_{\nu 0}u + A_{\nu 1}u = 0, \quad \nu = 3, 4, \end{aligned} \tag{4.8}$$

to which we apply Theorem 6.1 from the Appendix. Let us start to check all conditions of Theorem 6.1.

Since the embedding $E(A_4) \subset E$ is compact, we have, by [**12**, Theorem 1.16.4/2] (see also [**13**, Lemma 1.7.3/9]) and condition (1) of Theorem 2.1, that the embeddings $E(A_4^\alpha) \subset E(A_4^\beta)$, $1 \geq \alpha > \beta \geq 0$, are also compact. The embedding $X_1 \subset W_p^1((0,1); E(A_4^{\frac{1}{4}}), E)$ is bounded (see [**6**, Theorem 7 and Corollary 8]). Then, by [**13**, Theorem 5.2.1/1], the embedding $X_1 \subset X_0 = X$ is compact. In a similar way, as in the proof of [**13**, Theorem 5.2.1/1], one can conclude that the embedding $X_2 \subset X_1$ is also compact. Moreover, since E is a separable then $X_0 = L_p((0,1); E)$ is also separable and, therefore, X_1 and X_2 are also separable (as dense subspaces of X_0). Further, X_0 is reflexive (see [**11**, Theorem 5.7]). Since the operator $A_4^{\frac{1}{2}}$ is closed then the graph $\{(u, A_4^{\frac{1}{2}}u),\ u \in D(A_4^{\frac{1}{2}})\}$ is a closed subspace of the reflexive space $E \times E$, i.e., $E(A_4^{\frac{1}{2}})$ is reflexive. Then, again, by [**11**, Theorem 5.7], the space $L_p((0,1); E(A_4^{\frac{1}{2}}))$ is reflexive. By the mapping $u \to (u, u'')$, the space $X_1 = W_p^2((0,1); E(A_4^{\frac{1}{2}}), E)$ becomes a closed subspace of the reflexive space $L_p((0,1); E(A_4^{\frac{1}{2}})) \times L_p((0,1); E)$. Then, X_1 is also reflexive. Similarly, X_2 is reflexive. So, the first condition of Theorem 6.1 has been checked.

The second condition of Theorem 6.1 is just our condition (1).

The third condition of Theorem 6.1 is obvious if we take into account condition (2) of Theorem 2.1 and the second footnote of Theorem 2.1.

The fourth condition of Theorem 6.1 follows from [**12**, Theorem 1.8.2], which is written in a more convenient form in [**13**, Theorem 1.7.7/1], and (2.15).

It can be observed, that $\mathcal{X}_1$ and $\mathcal{X}$ in condition (5) of Theorem 6.1 are exactly given as $\mathcal{E}_1$ and $\mathcal{E}$, respectively, in (4.7). Then, the denseness of $\mathcal{E}_1$ in $\mathcal{E}$, i.e., condition (5) of Theorem 6.1, can be proved using ideas of the proofs of [**13**, Theorem 3.4.2/1 and Lemma 5.4.7/1].

Obviously, $(E(A_4), E)_{\frac{m_k}{4}+\frac{1}{4p},p} \subset E$, $k = 3, 4$. From (2.14) it follows that $(E(A_4), E)_{\frac{m_k}{4}+\frac{1}{4p},p} = (E(A_4), E(A_4^{\frac{1}{2}}))_{\frac{m_k}{2}+\frac{1}{2p},p} \subset E(A_4^{\frac{1}{2}})$, $k = 1, 2$. Then, for the nonhomogeneous problem associated with (4.8), i.e., for problem (2.1)–(2.2), from estimate (2.4), we get

$$|\lambda|\big(\|u''\|_{L_p((0,1);E)} + \|u\|_{L_p((0,1);E(A_4^{\frac{1}{2}}))}\big) \leq C|\lambda|^{1-\frac{1}{2p}}\Big(\|f\|_{L_p((0,1);E)} + \sum_{k=1}^{4} \|\varphi_k\|_{(E(A_4),E)_{\frac{m_k}{4}+\frac{1}{4p},p}}\Big),$$

for $|\arg\lambda| \leq \psi$ and sufficiently large $|\lambda|$, i.e.,

$$\|u\|_{X_1} \leq C|\lambda|^{-\frac{1}{2p}}\|(f, \varphi_1, \varphi_2, \varphi_3, \varphi_4)\|_{X\times X^1\times X^2\times X^3\times X^4},$$

for $|\arg \lambda| \leq \psi$ and sufficiently large $|\lambda|$. From this inequality, using our condition (2), we get condition (6) of Theorem 6.1 with $\eta = -\dfrac{1}{2p}$.

Condition (7) of Theorem 6.1 follows from our condition (3). □

In the framework of Hilbert spaces and $p = 2$ in Theorem 2.1, we get the following two-fold completeness theorem for problem (4.6) (we write everywhere H instead of E, in order to distinguish between Hilbert and Banach spaces cases). Note that we write singular numbers s_j instead of approximation numbers $\tilde{s}_j$, since, as it was mentioned above, they coincide in the Hilbert spaces settings.

Theorem 4.2. *Let H be a Hilbert space and*

1. *the embedding $H(A_4) \subset H$ is compact and, for some $t > 0$, for the embedding operator J from $H(A_4^{\frac{1}{2}})$ into H it holds that the singular numbers*
$$s_j\big(J; H(A_4^{\frac{1}{2}}), H\big) \leq Cj^{-t}, \quad j = 1, 2, \ldots;$$
2. *all conditions of Theorem 2.1 are satisfied (condition (3) with some $\dfrac{2}{2+t}\pi < \psi < \pi$).*

Then, the spectrum of problem (4.6) is discrete and a system of root vectors of problem (4.6) is two-fold complete in the spaces $\mathcal{H}$ and $\mathcal{H}_1$ ($\mathcal{H} := \mathcal{E}$ and $\mathcal{H}_1 := \mathcal{E}_1$ from (4.7) with $E = H$ and $p = 2$) and, therefore, in $L_2((0,1); H) \times L_2((0,1); H)$.

Proof. Observe that from condition (1), by [**13**, Lemma 1.7.8/6], we get
$$s_j\big(J_1; W_2^2((0,1); H(A_4^{\frac{1}{2}}), H), L_2((0,1); H)\big) \leq Cj^{-\frac{2t}{2+t}}, \quad j = 1, 2, \ldots.$$
On the other hand, from condition (1), using [**13**, Lemmas 1.2.10/3 and 4], we get, for the embedding operator J from $H(A_4)$ into $H(A_4^{\frac{1}{2}})$,
$$s_j\big(J; H(A_4), H(A_4^{\frac{1}{2}})\big) \leq Cs_j\big(J; H(A_4^{\frac{1}{2}}), H\big) \leq Cj^{-t}, \quad j = 1, 2, \ldots.$$
This implies, in a similar way as [**13**, Lemma 1.7.8/6], that
$$s_j\big(J_2; W_2^4((0,1); H(A_4), H), W_2^2((0,1); H(A_4^{\frac{1}{2}}), H)\big) \leq Cj^{-\frac{2t}{2+t}}, \quad j = 1, 2, \ldots.$$
Then, the proof is the same as of that of Theorem 4.1. We only use [**13**, Theorem 2.3.2/1] instead of Theorem 6.1 from the Appendix, taking into account Remark 6.1 from the Appendix. □

5 - Application of the abstract completeness result to quasi-elliptic equations with a quadratic parameter

We show an application of Theorem 4.2.

In the cylindrical domain $\Omega := [0,1] \times G$, where $G \subset \mathbb{R}^n$, $n \geq 2$, is a bounded domain with an $(n-1)$-dimensional boundary $\partial G \in C^{2m}$, which locally admits rectification, let us consider a homogeneous non-local boundary value problem with a parameter

$$\begin{aligned}(L(\lambda)u)(x,y) :=& \lambda^2 u(x,y) - 2\lambda D_x^2 u(x,y) + D_x^4 u(x,y)\\ &+ \sum_{|\alpha|=2m} a_\alpha(y) D_y^\alpha u(x,y) + vu(x,y) = 0, \quad (x,y) \in \Omega,\end{aligned} \tag{5.1}$$

$$\begin{aligned}(L_k u)(y) &:= \alpha_k D_x^{m_k} u(0,y) + \beta_k D_x^{m_k} u(1,y) = 0, \quad y \in G,\ k = 1,2,\\ (L_k(\lambda)u)(y) &:= \alpha_k \big(D_x^{m_k} u(0,y) - \lambda D_x^{m_k-2} u(0,y)\big)\\ &\quad + \beta_k \big(D_x^{m_k} u(1,y) - \lambda D_x^{m_k-2} u(1,y)\big)\\ &\quad = 0, \quad y \in G,\ k = 3,4,\end{aligned} \tag{5.2}$$

$$(B_\ell u)(x,y') := \sum_{|\beta| \leq p_\ell} b_{\ell\beta}(y') D_y^\beta u(x,y') = 0, \quad (x,y') \in [0,1] \times \partial G,\ \ell = 1,\dots,m, \tag{5.3}$$

where $0 \leq m_1, m_2 \leq 1$, $m_3 = m_1 + 2$, $m_4 = m_2 + 2$, $p_\ell \leq 2m-1$; α_k and β_k are complex numbers; f and φ_k are given functions; $D_x := \dfrac{\partial}{\partial x}$, $D_y^\alpha := D^\alpha := D_1^{\alpha_1} \cdots D_n^{\alpha_n}$, $D_j := -i\dfrac{\partial}{\partial y_j}$, $|\alpha| = \alpha_1 + \cdots + \alpha_n$.

Here, we consider the same operator $(Au)(y) := \sum_{|\alpha|=2m} a_\alpha(y) D^\alpha u(y)$ and the same restrictions on $(A, B_1, \dots, B_m)$ as just before Remark 3.2.

Denote

$$\begin{aligned}\mathcal{H}_1 := \Big\{ v | v :=& (v_1,v_2) \in W_2^4((0,1); W_2^{2m}(G), L_2(G)) \times W_2^2((0,1); W_2^m(G), L_2(G)),\\ &\alpha_1 v_1^{(m_1)}(0,y) + \beta_1 v_1^{(m_1)}(1,y) = 0, \quad \alpha_2 v_1^{(m_2)}(0,y) + \beta_2 v_1^{(m_2)}(1,y) = 0,\\ &\alpha_1 v_2^{(m_1)}(0,y) + \beta_1 v_2^{(m_1)}(1,y) = 0, \quad \alpha_2 v_2^{(m_2)}(0,y) + \beta_2 v_2^{(m_2)}(1,y) = 0,\\ &-\alpha_3 v_2^{(m_1)}(0,y) - \beta_3 v_2^{(m_1)}(1,y) + \alpha_3 v_1^{(m_3)}(0,y) + \beta_3 v_1^{(m_3)}(1,y) = 0,\\ &-\alpha_4 v_2^{(m_2)}(0,y) - \beta_4 v_2^{(m_2)}(1,y) + \alpha_4 v_1^{(m_4)}(0,y) + \beta_4 v_1^{(m_4)}(1,y) = 0,\ y \in G \Big\},\\ \mathcal{H} := \Big\{ v \mid v :=& (v_1,v_2) \in W_2^2((0,1); W_2^m(G), L_2(G)) \times L_2((0,1); L_2(G)),\\ &\alpha_1 v_1^{(m_1)}(0,y) + \beta_1 v_1^{(m_1)}(1,y) = 0, \quad \alpha_2 v_1^{(m_2)}(0,y) + \beta_2 v_1^{(m_2)}(1,y) = 0,\ y \in G \Big\}.\end{aligned} \tag{5.4}$$

Theorem 5.1. *Let, in addition to the above,*

1. $\dfrac{m}{n} > \dfrac{4\pi}{\pi-\omega_0} - 2$;
2. $(-1)^{m_1}\alpha_1\beta_2 - (-1)^{m_2}\alpha_2\beta_1 \neq 0$ *and* $(-1)^{m_1}\alpha_3\beta_4 - (-1)^{m_2}\alpha_4\beta_3 \neq 0$; *for* $m_1 \neq m_2$, *assume also that* $\alpha_k = \alpha_{k+2},\ \beta_k = \beta_{k+2},\ k = 1,2$.

Then, there exists $\nu > 0$ *sufficiently large such that the spectrum of problem* (5.1)–(5.3) *is discrete and a system of root vectors of problem* (5.1)–(5.3) *is two-fold complete in the spaces* $\mathcal{H}_1$ *and* $\mathcal{H}$ *and, therefore, in* $L_2(\Omega) \times L_2(\Omega)$.

Proof. Let us denote $H := L_2(G)$. Consider in H the operator A_4 which is defined by the equalities

$$D(A_4) := W_2^{2m}(G; B_\ell u = 0, \ell = 1,\dots,m), \quad (A_4u)(y) := Au(y) + \nu u(y),$$

where $\nu > 0$ is sufficiently large. Then, problem (5.1)–(5.3) can be rewritten in the operator form (4.6):

$$(5.5)\qquad \begin{aligned} (L(\lambda)u)(x) &:= \lambda^2 u(x) - 2\lambda u''(x) + u''''(x) + A_4u(x) = 0, \quad x \in (0,1),\\ L_k u &:= \alpha_k u^{(m_k)}(0) + \beta_k u^{(m_k)}(1) = 0, \quad k = 1,2,\\ L_k(\lambda)u &:= \alpha_k\big(u^{(m_k)}(0) - \lambda u^{(m_{k-2})}(0)\big)\\ &\quad + \beta_k\big(u^{(m_k)}(1) - \lambda u^{(m_{k-2})}(1)\big) = 0, \quad k = 3,4, \end{aligned}$$

Apply Theorem 4.2 to problem (5.5). From [**10**, Proposition 9.8] it follows that the operator A_4, for sufficiently large $\nu > 0$, has a bounded H^∞-calculus in $L_2(G)$, therefore, A_4 has *BIP* in $L_2(G)$. Then, by [**12**, Theorem 1.15.3], $H(A_4^{1-\frac{k}{2m}}) = [L_2(G), W_2^{2m}(G; B_\ell u = 0, \ell = 1,\dots,m)]_{1-\frac{k}{2m}}$, $k = 1,\dots,2m-1$. On the other hand, by virtue of [**12**, Theorem 4.3.3],

$$[L_2(G), W_2^{2m}(G; B_\ell u = 0, \ell = 1,\dots,m)]_{1-\frac{k}{2m}} \\ = W_2^{2m-k}(G; B_\ell u = 0, p_\ell < 2m-k), k = 1,\dots,2m-1.$$

Hence, $H(A_4^{1-\frac{k}{2m}}) = W_2^{2m-k}(G; B_\ell u = 0,\ p_\ell < 2m-k)$, $k = 0,\dots,2m-1$. In particular, $H_2 := H(A_4^{\frac{1}{2}}) = W_2^m(G; B_\ell u = 0, p_\ell < m)$. Therefore, condition (1) of Theorem 4.2, with $t = \dfrac{m}{n}$, follows, e.g., from [**12**, formula 4.10.2/(14)] (remind that, in Hilbert spaces, approximation numbers and singular numbers are the same).

We have now to check the conditions of Theorem 2.1 (see condition (2) of Theorem 4.2). Condition (1) of Theorem 2.1 have been checked in the proof of Theorem 7 in [**7**]. Condition (2) of Theorem 2.1 is obvious since $A_2 = 0$. The only condition remains to be checked is condition (3) of Theorem 2.1 for some $\psi \in \big(\dfrac{2}{2+t}\pi, \pi\big)$. Take any ω,

$\omega_0 < \omega < \pi$. Then, there exists $\nu > 0$ such that $\mathcal{R}$-sectoriality of A_4, with the $\mathcal{R}$-angle in $L_2(G)$, $\phi^{\mathcal{R}}_{A_4} \le \omega$, i.e.,

$$\mathcal{R}\{\mu R(\mu, A_4) \ : \ |\arg \mu| > \omega\}_{L_2(G)} < \infty, \tag{5.6}$$

follows from [3, Theorem 8.2]. Since $A_2 = 0$ then $L_0(\mu) = \mu^4 I + A_4$, i.e., $L_0(\mu)^{-1} = -R(-\mu^4, A_4)$. Therefore, $\forall \varepsilon > 0$ there exists $\nu > 0$ such that, from (5.6), for $0 \le \psi < \dfrac{\pi - \omega_0}{2} - \varepsilon$, we get

$$\mathcal{R}\Big\{\mu^4 L_0(\mu)^{-1} \ : \ \frac{\pi}{2} \ge |\arg \mu| \ge \frac{\pi - \psi}{2}\Big\}_{L_2(G)} < \infty. \tag{5.7}$$

Since $A_4 L_0(\mu)^{-1} = I - \mu^4 L_0(\mu)^{-1}$ then, using, e.g., [3, Proposition 3.4], we get

$$\mathcal{R}\Big\{A_4 L_0(\mu)^{-1} \ : \ \frac{\pi}{2} \ge |\arg \mu| \ge \frac{\pi - \psi}{2}\Big\}_{L_2(G)} < \infty. \tag{5.8}$$

So, (5.7) and (5.8) are two first inequalities in condition (3) of Theorem 2.1.

On the other hand, it can be easily seen from Definitions 1.2 and 1.3 that A_4 is an R-sectorial operator in $H_2 = H(A_4^{\frac{1}{2}}) = W_2^m(G; B_\ell u = 0, p_\ell < m)$ with the same $\mathcal{R}$-angle in $L_2(G)$ of that of A_4, $\phi^{\mathcal{R}}_{A_4} \le \omega$. In other words,

$$\mathcal{R}\{\mu R(\mu, A_4) \ : \ |\arg \mu| > \omega\}_{W_2^m(G)} < \infty.$$

So, again, taking into account that $L_0(\mu)^{-1} = -R(-\mu^4, A_4)$, we get, for the same above ψ,

$$\mathcal{R}\Big\{\mu^4 L_0(\mu)^{-1} \ : \ \frac{\pi}{2} \ge |\arg \mu| \ge \frac{\pi - \psi}{2}\Big\}_{W_2^m(G)} < \infty \tag{5.9}$$

and

$$\mathcal{R}\Big\{A_4 L_0(\mu)^{-1} \ : \ \frac{\pi}{2} \ge |\arg \mu| \ge \frac{\pi - \psi}{2}\Big\}_{W_2^m(G)} < \infty. \tag{5.10}$$

Inequalities (5.9) and (5.10) are the two last inequalities in condition (3) of Theorem 2.1.

Note that all the above inequalities (5.7)–(5.10) are true for any $0 \le \psi < \dfrac{\pi - \omega_0}{2} - \varepsilon$, but condition (2) of Theorem 4.2, in fact, condition (3) of Theorem 2.1 is claimed for $\dfrac{2}{2+t}\pi < \psi < \pi$. Therefore, $\dfrac{2}{2+t}\pi < \psi < \dfrac{\pi - \omega_0}{2} - \varepsilon$. Since $t = \dfrac{m}{n}$ then it should be $t = \dfrac{m}{n} > \dfrac{4\pi}{\pi - \omega_0} - 2$, which is our condition (1). □

Note that if A_4 is a selfadjoint, positive definite operator, then $\omega_0 = 0$ and then the theorem is true for $m > 2n$. So, even if $n = 2$ then $m > 4$, i.e., equation (5.1) is quasi-elliptic and not elliptic.

6 - Appendix

Let X and X^ν, $\nu = 1, \ldots, m$, be Banach spaces. Consider a problem for a system of polynomial operator pencils in X,

$$\begin{aligned} &L(\lambda)u := \lambda^n u + \lambda^{n-1} B_1 u + \cdots + B_n u = 0, \\ &L_\nu(\lambda)u := \lambda^{n_\nu} A_{\nu 0} u + \lambda^{n_\nu - 1} A_{\nu 1} u + \cdots + A_{\nu n_\nu} u = 0, \quad \nu = 1, \ldots, m, \end{aligned} \tag{6.1}$$

where $n \geq 1$, $0 \leq n_\nu \leq n-1$, $m \geq 0$. By $\tilde{s}_j$ we denote the approximation numbers (see section 4 for the definition).

Theorem 6.1 ([**14**]). *Let the following conditions be satisfied:*

1. *there exist separable, reflexive Banach spaces* X_k, $k = 0, \ldots, n$, *for which the compact embeddings* $X_n \subset X_{n-1} \subset \cdots \subset X_0 = X$ *take place*;
2. *for some* $s > \dfrac{1}{2}$, $\tilde{s}_j(J_k; X_k, X_{k-1}) \leq Cj^{-s}$, $j = 1, 2, \ldots$, $k = 1, \ldots, n$ *hold, where* J_k *denotes the embedding operator from* X_k *into* X_{k-1};
3. *the operators* B_k, $k = 1, \ldots, n$, *from* X_k *into* X, *act boundedly*;
4. *the operators* $A_{\nu k}$, $k = 0, \ldots, n_\nu$, $\nu = 1, \ldots, m$, *from* $X_{n-n_\nu+k}$ *into* X^ν, *act boundedly*;
5. *there exist Banach spaces* X_0^ν *such that continuous embeddings* $X^\nu \subset X_0^\nu$, $\nu = 1, \ldots, m$, *hold, and the linear manifold*

$$\mathcal{X}_1 := \Big\{ v \mid v := (v_1, \ldots, v_n) \in \mathop{\times}_{k=0}^{n-1} X_{n-k}, \quad \sum_{k=0}^{n_\nu} A_{\nu k} v_{n_\nu - k + s} = 0,$$
$$\text{for such integers } \nu \in [1, m] \text{ and } s \in [1, n - n_\nu] \text{ for which}$$
$$A_{\nu k} (\text{for all } k = 0, \ldots, n_\nu) \text{ from } X_{n+1-n_\nu+k-s} \text{ into } X_0^\nu \text{ are bounded} \Big\}$$

is dense in the Banach space

$$\mathcal{X} := \Big\{ v \mid v := (v_1, \ldots, v_n) \in \mathop{\times}_{k=0}^{n-1} X_{n-k-1}, \quad \sum_{k=0}^{n_\nu} A_{\nu k} v_{n_\nu - k + s} = 0,$$
$$\text{for such integers } \nu \in [1, m] \text{ and } s \in [1, n - n_\nu - 1] \text{ for which}$$
$$A_{\nu k} \ (\text{for all } k = 0, \ldots, n_\nu) \text{ from } X_{n-n_\nu+k-s} \text{ into } X_0^\nu \text{ are bounded} \Big\};$$

6. *there exist*[3] *rays* ℓ_k *with angles between neighboring rays less than* $(s - \frac{1}{2})\pi$ *and a number* η *such that all numbers* λ *on* ℓ_k, *with sufficiently*

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[3] For $s > \dfrac{5}{2}$ the existence of one such ray is enough.

large moduli, are regular points of the operator pencil $\mathbb{L}(\lambda): u \to \mathbb{L}(\lambda)u := (L(\lambda)u, L_1(\lambda)u, \ldots, L_m(\lambda)u)$, *which acts boundedly from* X_n *into* $X \times X^1 \times \cdots \times X^m$, *and*

$$\|\mathbb{L}(\lambda)^{-1}\|_{B(X \times X^1 \times \cdots \times X^m, X_{n-1})} \leq C|\lambda|^{\eta}, \quad \lambda \in \ell_k, \ |\lambda| \to \infty;$$

7. *the spectrum of problem* (6.1) (*or of the operator pencil* $\mathbb{L}(\lambda)$) *is not empty.*

Then, the spectrum of problem (6.1) *is discrete and a system of root vectors of problem* (6.1) *is n-fold complete in the spaces* $\mathcal{X}$ *and* $\mathcal{X}_1$.

Remark 6.1. *In the framework of Hilbert spaces, this theorem is presented in* [**13**, Theorem 2.3.2/1] *and there are a few generalizations in the conditions of the theorem:*

(a) *it should be* $s > 0$ *instead of* $s > \frac{1}{2}$ *and the approximation numbers* $\tilde{s}_j$ *are just singular numbers* s_j *in condition* (2)*;*
(b) *it should be* $s\pi$ *instead of* $\big(s - \frac{1}{2}\big)\pi$ *in condition* (6) *and, then, if* $s > 2$ *the existence of one ray is enough;*
(c) *condition* (7) *should be omitted.*

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