Riv. Mat. Univ. Parma, Vol. **5** (2014), 311-333

TAKU KANAZAWA

Partial regularity for elliptic systems with VMO-coefficients

Abstract. We establish partial Hölder continuity for vector-valued solutions $u : \Omega \to \mathbb{R}^N$ to inhomogeneous elliptic systems of the type:

$$-\operatorname{div}(A(x,u,Du)) = f(x,u,Du) \qquad \text{in } \Omega,$$

where the coefficients $A : \Omega \times \mathbb{R}^N \times \operatorname{Hom}(\mathbb{R}^n, \mathbb{R}^N) \to \operatorname{Hom}(\mathbb{R}^n, \mathbb{R}^N)$ are possibly discontinuous with respect to x. More precisely, we assume a VMO-condition with respect to the x and continuity with respect to u and prove Hölder continuity of the solutions outside of singular sets.

Keywords. Nonlinear elliptic systems, Partial regularity, VMO-coefficients, $\mathcal{A}$-harmonic approximation.

Mathematics Subject Classification (2010): 35J60, 35B65.

1 - Introduction

In this paper, we consider the second order nonlinear elliptic systems in divergence form of the following type:

$$-\operatorname{div}(A(x,u,Du)) = f(x,u,Du) \qquad \text{in } \Omega. \tag{1.1}$$

Here Ω is bounded domain in $\mathbb{R}^n$, u takes values in $\mathbb{R}^N$ with coefficients $A : \Omega \times \mathbb{R}^N \times \operatorname{Hom}(\mathbb{R}^n, \mathbb{R}^N) \to \operatorname{Hom}(\mathbb{R}^n, \mathbb{R}^N)$.

The aim of this paper is to obtain a partial regularity result of weak solutions to (1.1) with discontinuous coefficients. More precisely, we assume that the partial mapping $x \mapsto A(x,u,\xi)/(1+|\xi|)^{p-1}$ has vanishing mean oscillation (VMO), uniformly

Received: February 15, 2013; accepted in revised form: July 30, 2013.

in (u, ξ). This means that A satisfies an estimate

$$|A(x,u,\xi) - (A(\cdot,u,\xi))_{x_0,\rho}| \le V_{x_0}(x,\rho)(1+|\xi|)^{p-1},$$

where $V_{x_0} : \mathbb{R}^n \times [0,\rho_0] \to [0,2L]$ are bounded functions with

$$\lim_{\rho \searrow 0} V(\rho) = 0, \quad V(\rho) := \sup_{x_0 \in \Omega} \sup_{0<r\le\rho} \fint_{B_r(x_0)\cap\Omega} V_{x_0}(x,r)dx.$$

We also assume that $u \mapsto A(x,u,\xi)/(1+|\xi|)^{p-1}$ is continuous, that is, there exists a modulus of continuity $\omega : [0,\infty) \to [0,\infty)$ such that an estimate

$$|A(x,u,\xi) - A(x,u_0,\xi)| \le L\omega(|u-u_0|^2)(1+|\xi|)^{p-1}$$

holds.

Regularity results under a VMO-condition have been established by Zheng [**15**] for quasi-linear elliptic systems or integral functionals. General functionals with VMO-coefficients were considered by Ragusa and Tachikawa [**14**], who generalized the low-dimensional results from problems with continuous coefficients to the case of VMO-coefficients. In particular, these results require that the dimension of domain is small, for example, $n \le p+2$ is required to obtain the Hölder continuity of the minimizers in [**14**]. In contrast, Bögelein, Duzaar, Habermann and Scheven [**1**] give the regularity result for homogeneous nonlinear elliptic system without dimension conditions.

Stronger assumptions such as the Hölder continuity with respect to (x, u) or a Dini-type condition lead to partial C^1-regularity with a quantitative modulus of continuity for Du; the modulus of continuity can be determined in dependence on the modulus of continuity of the coefficients (cf. Giaquinta and Modica [**10**], Duzaar and Grotowski [**7**], Duzaar and Gastel [**6**], Chen and Tan [**5**], Qiu [**13**] and the references therein).

As we knew, we could not expect continuity (and not even boundedness) of the gradient Du under continuous coefficients (or even more relaxed condition). The regularity result with continuous coefficients was already proved in eighties by Campanato [**3, 4**]. He proves that we could still expect local Hölder continuity of the solution u in special cases, for instance, in lower dimension $n \le p+2$. The result with arbitrary dimension was given by Foss and Mingione [**8**].

Our aim is to extend the homogeneous system result in [**1**] to inhomogeneous system. Therefore we assume the same structure conditions to coefficients A as in [**1**]. Under a suitable assumption to inhomogeneous term, we obtain Hölder continuity of weak solution (see Theorem 2.2).

Our proof is based on so-called $\mathcal{A}$-harmonic approximation (cf. [**7**, Lemma 2.1]; see also Lemma 3.2), introduced by Duzaar and Grotowski. They give a simplified (direct) proof of regularity results to the systems with Hölder continuous coefficients and a natural growth condition, without L^p-L^2-estimates for Du.

We close this section by briefly summarizing the notation used in this paper. As mentioned above, we consider a bounded domain $\Omega \subset \mathbb{R}^n$, and maps from Ω to $\mathbb{R}^N$, where we take $n \geq 2$, $N \geq 1$. For a given set X we denote by $\mathscr{L}^n(X)$ the n-dimensional Lebesgue measure. We write $B_\rho(x_0) := \{x \in \mathbb{R}^n : |x - x_0| < \rho\}$. For bounded set $X \subset \mathbb{R}^n$ with $\mathscr{L}^n(X) > 0$, we denote the average of a given function $g \in L^1(X, \mathbb{R}^N)$ by $\fint_X g dx$, that is, $\fint_X g dx = \frac{1}{\mathscr{L}^n(X)} \int_X g dx$. In particular, we write $g_{x_0,\rho} = \fint_{B_\rho(x_0)\cap\Omega} g dx$. We write $\mathrm{Bil}(\mathrm{Hom}(\mathbb{R}^n, \mathbb{R}^N))$ for the space of bilinear forms on the space $\mathrm{Hom}(\mathbb{R}^n, \mathbb{R}^N)$ of linear maps from $\mathbb{R}^n$ to $\mathbb{R}^N$. We denote c a positive constant, possibly varying from line by line. Special occurrences will be denoted by capital letters K, C_1, C_2 or the like.

2 - Statement of the results

Definition 2.1. We say $u \in W^{1,p}(\Omega, \mathbb{R}^N)$, $p \geq 2$ is a *weak solution* of (1.1) if u satisfies

$$\int_\Omega \langle A(x, u, Du), D\varphi \rangle dx = \int_\Omega \langle f, \varphi \rangle dx \tag{2.1}$$

for all $\varphi \in C_0^\infty(\Omega, \mathbb{R}^N)$, where $\langle \cdot, \cdot \rangle$ is the standard Euclidean inner product on $\mathbb{R}^N$ or $\mathbb{R}^{nN}$.

We assume following structure conditions.

(H1) $A(x, u, \xi)$ is differentiable in ξ with continuous derivatives, that is, there exists $L \geq 1$ such that

$$|A(x, u, \xi)| + (1 + |\xi|)|D_\xi A(x, u, \xi)| \leq L(1 + |\xi|)^{p-1} \tag{2.2}$$

for all $x \in \Omega$, $u \in \mathbb{R}^N$ and $\xi \in \mathrm{Hom}(\mathbb{R}^n, \mathbb{R}^N)$. Moreover, from this we deduce the modulus of continuity function $\mu : [0, \infty) \to [0, \infty)$ such that μ is bounded, concave, non-decreasing and we have

$$|D_\xi A(x, u, \xi) - D_\xi A(x, u, \xi_0)| \leq L\mu\left(\frac{|\xi - \xi_0|}{1 + |\xi| + |\xi_0|}\right)(1 + |\xi| + |\xi_0|)^{p-2} \tag{2.3}$$

for all $x \in \Omega$, $u \in \mathbb{R}^N$, $\xi, \xi_0 \in \mathrm{Hom}\ (\mathbb{R}^n, \mathbb{R}^N)$. Without loss of generality, we may assume $\mu \leq 1$.

(H2) $A(x,u,\xi)$ is uniformly strongly elliptic, that is, for some $\lambda > 0$ we have

$$\left\langle D_\xi A(x,u,\xi)v, v\right\rangle := \sum_{\substack{1\le i,\beta\le N\\1\le j,\alpha\le n}} D_{\xi^j_\beta} A^i_\alpha(x,u,\xi)v^\alpha_i v^\beta_j \ge \lambda |v|^2(1+|\xi|)^{p-2} \tag{2.4}$$

for all $x \in \Omega$, $u \in \mathbb{R}^N$, $\xi, v \in \mathrm{Hom}(\mathbb{R}^n, \mathbb{R}^N)$.

(H3) $A(x,u,\xi)$ is continuous with respect to u. There exists a bounded, concave and non-decreasing function $\omega : [0,\infty) \to [0,\infty)$ satisfying

$$|A(x,u,\xi) - A(x,u_0,\xi)| \le L\omega\Big(|u-u_0|^2\Big)(1+|\xi|)^{p-1} \tag{2.5}$$

for all $x \in \Omega$, $u, u_0 \in \mathbb{R}^N$, $\xi \in \mathrm{Hom}(\mathbb{R}^n, \mathbb{R}^N)$. Without loss of generality, we may assume $\omega \le 1$.

(H4) $x \mapsto A(x,u,\xi)/(1+|\xi|)^{p-1}$ fulfils the following VMO-conditions uniformly in u and ξ:

$$|A(x,u,\xi) - (A(\cdot,u,\xi))_{x_0,\rho}| \le V_{x_0}(x,\rho)(1+|\xi|)^{p-1}, \qquad \text{for all } x \in B_\rho(x_0)$$

whenever $x_0 \in \Omega$, $0 < \rho < \rho_0$, $u \in \mathbb{R}^N$ and $\xi \in \mathrm{Hom}\,(\mathbb{R}^n, \mathbb{R}^N)$, where $\rho_0 > 0$ and $V_{x_0} : \mathbb{R}^n \times [0,\rho_0] \to [0, 2L]$ are bounded functions satisfying

$$\lim_{\rho\searrow 0} V(\rho) = 0, \qquad V(\rho) := \sup_{x_0\in\Omega}\ \sup_{0<r\le\rho} \fint_{B_r(x_0)\cap\Omega} V_{x_0}(x,r)dx. \tag{2.6}$$

(H5) $f(x,u,\xi)$ has p-growth, that is, there exist constants $a, b \ge 0$, with a possibly depending on $M > 0$, such that

$$|f(x,u,\xi)| \le a(M)|\xi|^p + b \tag{2.7}$$

for all $x \in \Omega$, $u \in \mathbb{R}^N$ with $|u| \le M$ and $\xi \in \mathrm{Hom}(\mathbb{R}^n, \mathbb{R}^N)$.

Now, we are ready to state our main theorem.

Theorem 2.2. *Let $u \in W^{1,p}(\Omega,\mathbb{R}^N) \cap L^\infty(\Omega,\mathbb{R}^N)$ be a bounded weak solution of* (1.1) *under the structure conditions* **(H1)**, **(H2)**, **(H3)**, **(H4)** *and* **(H5)** *satisfying $\|u\|_\infty \le M$ and $2^{(10-9p)/2}\lambda > a(M)M$. Then there exists an open set $\Omega_u \subseteq \Omega$ with $\mathscr{L}^n(\Omega\setminus\Omega_u) = 0$ such that $u \in C^{0,\alpha}_{\mathrm{loc}}(\Omega_u,\mathbb{R}^N)$ for every $\alpha \in (0,1)$. Moreover, we have $\Omega\setminus\Omega_u \subseteq \Sigma_1 \cup \Sigma_2$, where*

$$\Sigma_1 := \left\{ x_0 \in \Omega : \liminf_{\rho\searrow 0} \fint_{B_\rho(x_0)} |Du - (Du)_{x_0,\rho}|^p dx > 0 \right\},$$

$$\Sigma_2 := \left\{ x_0 \in \Omega : \limsup_{\rho\searrow 0} |(Du)_{x_0,\rho}| = \infty \right\}.$$

3 - Preliminaries

In this section we present $\mathcal{A}$-harmonic approximation lemma and some standard estimates for the proof of the regularity theorem.

First we state the definition of $\mathcal{A}$-harmonic function and recall $\mathcal{A}$-harmonic approximation lemma as below.

Definition 3.1 ([**7**, Section 1]). For a given $\mathcal{A} \in \mathrm{Bil}(\mathrm{Hom}(\mathbb{R}^n, \mathbb{R}^N))$, we say that $h \in W^{1,p}(\Omega, \mathbb{R}^N)$ is an *$\mathcal{A}$-harmonic function*, if h satisfies

$$\int_{\Omega} \mathcal{A}(Dh, D\varphi)dx = 0$$

for all $\varphi \in C_0^\infty(\Omega, \mathbb{R}^N)$.

Lemma 3.2 ([**1**, Lemma 2.3]). *Let $\lambda > 0$, $L > 0$, $p \geq 2$ and $n, N \in \mathbb{N}$ with $n \geq 2$ given. For every $\varepsilon > 0$, there exists a constant $\delta = \delta(n, N, L, \lambda, \varepsilon) \in (0, 1]$ such that the following holds: assume that $\gamma \in [0, 1]$ and $\mathcal{A} \in \mathrm{Bil}(\mathrm{Hom}(\mathbb{R}^n, \mathbb{R}^N))$ with the property*

$$\mathcal{A}(v, v) \geq \lambda |v|^2, \qquad \textit{for all } v \in \mathrm{Hom}(\mathbb{R}^n, \mathbb{R}^N), \tag{3.1}$$

$$\mathcal{A}(v, \tilde{v}) \leq L|v||\tilde{v}|, \quad \textit{for all } v, \tilde{v} \in \mathrm{Hom}(\mathbb{R}^n, \mathbb{R}^N). \tag{3.2}$$

Furthermore, let $g \in W^{1,2}(B_\rho(x_0), \mathbb{R}^N)$ be an approximately $\mathcal{A}$-harmonic map in sense that there holds

$$⨍_{B_\rho(x_0)} \{|Dg|^2 + \gamma^{p-2}|Dg|^p\}dx \leq 1, \tag{3.3}$$

$$\left| ⨍_{B_\rho(x_0)} \mathcal{A}(Dg, D\varphi)dx \right| \leq \delta \sup_{B_\rho(x_0)} |D\varphi|, \quad \textit{for all } \varphi \in C_c^1(B_\rho(x_0), \mathbb{R}^N). \tag{3.4}$$

Then there exists an $\mathcal{A}$-harmonic function h that satisfies

$$⨍_{B_\rho(x_0)} \left\{ \left|\frac{h-g}{\rho}\right|^2 + \gamma^{p-2}\left|\frac{h-g}{\rho}\right|^p \right\}dx \leq \varepsilon, \tag{3.5}$$

$$⨍_{B_\rho(x_0)} \left\{ |Dh|^2 + \gamma^{p-2}|Dh|^p \right\}dx \leq c(n, p). \tag{3.6}$$

Next is a standard estimates for the solutions to homogeneous second order elliptic systems with constant coefficients, due originally to Campanato [**2**, Teorema 9.2]. For convenience, we state the estimate in a slightly general form than the original one.

Theorem 3.3 ([**7**, Teorema 2.3]). *Consider $\mathcal{A}$, λ and L as in Lemma 3.2. Then there exists $C_0 \geq 1$ depending only on n, N, λ and L such that any $\mathcal{A}$-harmonic function h on $B_{\rho/2}(x_0)$ satisfies*

$$\left(\frac{\rho}{2}\right)^2 \sup_{B_{\rho/4}(x_0)} |Dh|^2 + \left(\frac{\rho}{2}\right)^4 \sup_{B_{\rho/4}(x_0)} |D^2h|^2 \leq C_0 \left(\frac{\rho}{2}\right)^2 ⨍_{B_{\rho/2}(x_0)} |Dh|^2 dx. \tag{3.7}$$

We state the Poincaré inequality (Lemma 3.4) in a convenient form. The proof can be founded in several literature, for example [**9**, Proposition 3.10].

Lemma 3.4. *There exists $C_P \geq 1$ depending only on n such that every $u \in W^{1,p}(B_\rho(x_0), \mathbb{R}^N)$ satisfies*

$$\int_{B_\rho(x_0)} |u - u_{x_0,\rho}|^p dx \leq C_P \rho^p \int_{B_\rho(x_0)} |Du|^p dx. \tag{3.8}$$

Given a function $u \in L^2(B_\rho(x_0), \mathbb{R}^N)$, where $x_0 \in \mathbb{R}^n$ and $\rho > 0$. We write $\ell_{x_0,\rho}$ for the minimizer of the functional

$$\ell \mapsto ⨍_{B_\rho(x_0)} |u - \ell|^2 dx \tag{3.9}$$

among all affine functions $\ell : \mathbb{R}^n \to \mathbb{R}^N$. Let write $\ell_{x_0,\rho}(x) := \ell_{x_0,\rho}(x_0) + D\ell_{x_0,\rho}(x - x_0)$. It is easy to check that $\ell_{x_0,\rho}(x_0) = u_{x_0,\rho}$ and

$$D\ell_{x_0,\rho} = \frac{n+2}{\rho^2} ⨍_{B_\rho(x_0)} u \otimes (x - x_0) dx, \tag{3.10}$$

where $\xi \otimes \zeta = \xi_i \zeta^\alpha$. Based on this formula, elementary calculations yield the following estimates.

Lemma 3.5 ([**12**, Lemma 2]). *Assume $u \in L^2(B_\rho(x_0), \mathbb{R}^N)$, $x_0 \in \mathbb{R}^n$, $\rho > 0$ and $0 < \theta \leq 1$. With $\ell_{x_0,\rho}$ and $\ell_{x_0,\theta\rho}$, we denote the affine functions from $\mathbb{R}^n$ to $\mathbb{R}^N$ defined* *as above for the radii* ρ *and* $\theta\rho$ *respectively. Then we have*

$$|D\ell_{x_0,\rho} - D\ell_{x_0,\theta\rho}|^2 \leq \frac{n(n+2)}{(\theta\rho)^2} \fint_{B_{\theta\rho}(x_0)} |u - \ell_{x_0,\rho}|^2 dx, \tag{3.11}$$

and more generally,

$$|D\ell_{x_0,\rho} - D\ell|^2 \leq \frac{n(n+2)}{\rho^2} \fint_{B_\rho(x_0)} |u - \ell|^2 dx, \tag{3.12}$$

for all affine functions $\ell : \mathbb{R}^n \to \mathbb{R}^N$.

The estimate (3.12) implies, in particular, that $\ell_{x_0,\rho}$ has the following quasi-minimizing property for the L^p-norm. The proof can be founded in [**1**, Section 2].

Lemma 3.6. *Consider the minimizer of* (3.9), *that is,* $\ell_{x_0,\rho}$. *For any affine functions* $\ell : \mathbb{R}^n \to \mathbb{R}^N$ *and* $p \geq 2$ *we have*

$$\fint_{B_\rho(x_0)} |u - \ell_{x_0,\rho}|^p dx \leq c(n,p) \fint_{B_\rho(x_0)} |u - \ell|^p dx. \tag{3.13}$$

Using Young's inequality, we obtain the following lemma.

Lemma 3.7. *Consider fixed* $a, b \geq 0$, $p \geq 1$. *Then for any* $\varepsilon > 0$, *there exists* $K = K(p,\varepsilon) \geq 0$ *satisfying*

$$(a+b)^p \leq (1+\varepsilon)a^p + Kb^p. \tag{3.14}$$

Proof. We first consider the case $p = 2k-1$ for $k \in \mathbb{N}$. By binomial theorem, we have

$$\begin{aligned}(a+b)^{2k-1} &= \sum_{m=0}^{2k-1} \binom{2k-1}{m} a^{2k-1-m} b^m \\ &= a^{2k-1} + b^{2k-1} + \sum_{m=1}^{k-1} \binom{2k-1}{m} (a^{2k-1-m} b^m + a^m b^{2k-1-m}).\end{aligned}$$

Using Young's inequality, we obtain

$$\sum_{m=1}^{k-1} \binom{2k-1}{m} (a^{2k-1-m} b^m + a^m b^{2k-1-m}) \leq \sum_{m=1}^{k-1} \binom{2k-1}{m} (\varepsilon' a^{2k-1} + c(k,m,\varepsilon') b^{2k-1}),$$

where $\varepsilon' > 0$ will be fixed later. Thus, we get

$$
\begin{aligned}
(a+b)^{2k-1} &\le a^{2k-1} + b^{2k-1} + \sum_{m=1}^{k-1} \binom{2k-1}{m} (\varepsilon' a^{2k-1} + c(k,m,\varepsilon') b^{2k-1}) \\
&= \left\{ 1 + \varepsilon' \sum_{m=1}^{k-1} \binom{2k-1}{m} \right\} a^{2k-1} + \left\{ 1 + \sum_{m=1}^{k-1} \binom{2k-1}{m} c(k,m,\varepsilon') \right\} b^{2k-1}.
\end{aligned}
$$

For any $\varepsilon > 0$ we conclude (3.7) by taking ε' as $\varepsilon = \varepsilon' \sum_{m=1}^{k-1} \binom{2k-1}{m}$.

In case of $p = 2k$, we may estimate similarly as above, hence we get

$$
\begin{aligned}
(a+b)^{2k} &= \sum_{m=0}^{2k} \binom{2k}{m} a^{2k-m} b^m \\
&= a^{2k} + b^{2k} + \sum_{m=1}^{k-1} \binom{2k}{m} (a^{2k-m} b^m + a^m b^{2k-m}) + \binom{2k}{k} a^k b^k \\
&\le a^{2k} + b^{2k} + \sum_{m=1}^{k-1} \binom{2k}{m} (\varepsilon' a^{2k} + c(k,m,\varepsilon') b^{2k}) + \binom{2k}{k} \left(\varepsilon' a^{2k} + \frac{1}{\varepsilon'} b^{2k} \right) \\
&= \left\{ 1 + \varepsilon' \sum_{m=1}^{k} \binom{2k}{m} \right\} a^{2k} + \left\{ 1 + \sum_{m=1}^{k-1} \binom{2k}{m} c(k,m,\varepsilon') + \frac{1}{\varepsilon'} \right\} b^{2k}.
\end{aligned}
$$

This conclude that we have (3.7) for $p \in \mathbb{N}$.

For general $p \ge 1$, let $[p]$ be the greatest integer not greater then p. We write

$$
(a+b)^p = (a+b)^{[p]} (a+b)^{p-[p]}.
$$

By $0 \le p - [p] < 1$, we have

$$
(a+b)^{p-[p]} \le a^{p-[p]} + b^{p-[p]}.
$$

For $\varepsilon' > 0$ to be fixed later, we get

$$
(a+b)^{[p]} \le (1+\varepsilon') a^{[p]} + K(p,\varepsilon') b^{[p]},
$$

since $[p] \in \mathbb{N}$. Combining two estimates, we obtain

$$
\begin{aligned}
(a+b)^p &\le \left\{ (1+\varepsilon') a^{[p]} + K(p,\varepsilon') b^{[p]} \right\} (a^{p-[p]} + b^{p-[p]}) \\
&= (1+\varepsilon') a^p + K(p,\varepsilon') b^p + (1+\varepsilon') a^{[p]} b^{p-[p]} + K(p,\varepsilon') a^{p-[p]} b^{[p]} \\
&\le (1+\varepsilon') a^p + K(p,\varepsilon') b^p + (1+\varepsilon' + K(p,\varepsilon'))(a^{[p]} b^{p-[p]} + a^{p-[p]} b^{[p]}).
\end{aligned}
$$

Again for $\varepsilon'' > 0$ to be fixed later, by using Young's inequality, we conclude

$$(a+b)^p \le (1+\varepsilon')a^p + K(p,\varepsilon')b^p + (1+\varepsilon' + K(p,\varepsilon'))(\varepsilon'' a^p + c(p,\varepsilon'')b^p).$$

Take $\varepsilon' = \varepsilon/2$ and $\varepsilon'' = \varepsilon'/(1+\varepsilon'+K(p,\varepsilon'))$, and this complete the proof. □

Lemma 3.8 ([**11**, Lemma 2.1]). *For $\delta \ge 0$, and for all $a, b \in \mathbb{R}^k$ we have*

$$4^{-(1+2\delta)} \le \frac{\int_0^1 (1+|sa+(1-s)b|^2)^{\delta/2} ds}{(1+|a|^2+|b-a|^2)^{\delta/2}} \le 4^{\delta}. \tag{3.15}$$

4 - Proof of the main theorem

To obtain the regularity result (Theorem 2.2), we first prove Caccioppoli-type inequality. In the followings, we define $q > 0$ as the dual exponent of $p \ge 2$, that is, $q = p/(p-1)$. Here we note that $q \le 2$.

Lemma 4.1. *Let $u \in W^{1,p}(\Omega, \mathbb{R}^N) \cap L^\infty(\Omega, \mathbb{R}^N)$ be a bounded weak solution of the elliptic system* (1.1) *under the structure condition* (**H1**),(**H2**),(**H3**),(**H4**) *and* (**H5**) *with satisfying $\|u\|_\infty \le M$ and $2^{(10-9p)/2}\lambda > a(M)M$. For any $x_0 \in \Omega$ and $\rho \le 1$ with $B_\rho(x_0) \Subset \Omega$, and any affine functions $\ell : \mathbb{R}^n \to \mathbb{R}^N$ with $|\ell(x_0)| \le M$, we have the estimate*

$$\begin{aligned} &\fint_{B_{\frac{\rho}{2}}(x_0)} \left\{ \frac{|Du - D\ell|^2}{(1+|D\ell|)^2} + \frac{|Du - D\ell|^p}{(1+|D\ell|)^p} \right\} dx \\ &\le C_1 \Bigg[\fint_{B_\rho(x_0)} \left\{ \frac{|u-\ell|^2}{\rho^2(1+|D\ell|)^2} + \frac{|u-\ell|^p}{\rho^p(1+|D\ell|)^p} \right\} dx \\ &\quad + \omega\left(\fint_{B_\rho(x_0)} |u - \ell(x_0)|^2 dx \right) + V(\rho) + (a^q |D\ell|^q + b^q)\rho^q \Bigg], \end{aligned} \tag{4.1}$$

with the constant $C_1 = C_1(\lambda, p, L, a(M), M) \ge 1$.

Proof. Assume $x_0 \in \Omega$ and $\rho \le 1$ satisfy $B_\rho(x_0) \Subset \Omega$. We take a standard cut-off function $\eta \in C_0^\infty(B_\rho(x_0))$ satisfying $0 \le \eta \le 1$, $|D\eta| \le 4/\rho$, $\eta \equiv 1$ on $B_{\rho/2}(x_0)$. Then

$\varphi := \eta^p(u-\ell)$ is admissible as a test function in (2.1), and we obtain

$$\fint_{B_\rho(x_0)} \eta^p \langle A(x,u,Du), Du - D\ell\rangle dx = -\fint_{B_\rho(x_0)} \langle A(x,u,Du), p\eta^{p-1} D\eta \otimes (u-\ell)\rangle dx + \fint_{B_\rho(x_0)} \langle f, \varphi\rangle dx. \tag{4.2}$$

Furthermore, we have

$$-\fint_{B_\rho(x_0)} \eta^p \langle A(x,u,D\ell), Du - D\ell\rangle dx = \fint_{B_\rho(x_0)} \langle A(x,u,D\ell), p\eta^{p-1} D\eta \otimes (u-\ell)\rangle dx - \fint_{B_\rho(x_0)} \langle A(x,u,D\ell), D\varphi\rangle dx, \tag{4.3}$$

and

$$\fint_{B_\rho(x_0)} \langle (A(\cdot, \ell(x_0), D\ell))_{x_0,\rho}, D\varphi\rangle dx = 0. \tag{4.4}$$

Adding (4.2), (4.3) and (4.4), we obtain

$$\begin{aligned} &\fint_{B_\rho(x_0)} \eta^p \langle A(x,u,Du) - A(x,u,D\ell), Du - D\ell\rangle dx \\ &= -\fint_{B_\rho(x_0)} \langle A(x,u,Du) - A(x,u,D\ell), p\eta^{p-1} D\eta \otimes (u-\ell)\rangle dx \\ &\quad - \fint_{B_\rho(x_0)} \langle A(x,u,D\ell) - A(x,\ell(x_0),D\ell), D\varphi\rangle dx \\ &\quad - \fint_{B_\rho(x_0)} \langle A(x,\ell(x_0),D\ell) - (A(\cdot,\ell(x_0),D\ell))_{x_0,\rho}, D\varphi\rangle dx \\ &\quad + \fint_{B_\rho(x_0)} \langle f, \varphi\rangle dx \\ &=: \mathrm{I} + \mathrm{II} + \mathrm{III} + \mathrm{IV}. \end{aligned} \tag{4.5}$$

The terms I, II, III, IV are defined above. Using the ellipticity condition (**H2**) to the left-hand side of (4.5), we get

$$\begin{aligned}
&\langle A(x,u,Du)-A(x,u,D\ell), Du-D\ell\rangle \\
&= \int_0^1 \langle D_\xi A(x,u,sDu+(1-s)D\ell)(Du-D\ell), Du-D\ell\rangle ds \\
&\geq \lambda |Du-D\ell|^2 \int_0^1 (1+|sDu+(1-s)D\ell|)^{p-2} ds.
\end{aligned} \tag{4.6}$$

Then by using (3.15) in Lemma 3.8, we obtain

$$\begin{aligned}
&\langle A(x,u,Du)-A(x,u,D\ell), Du-D\ell\rangle \\
&\geq \lambda |Du-D\ell|^2 \int_0^1 (1+|sDu+(1-s)D\ell|^2)^{(p-2)/2} ds \\
&\geq 2^{(12-9p)/2}\lambda \Big\{(1+|D\ell|)^{p-2}|Du-D\ell|^2+|Du-D\ell|^p\Big\}.
\end{aligned} \tag{4.7}$$

For $\varepsilon>0$ to be fixed later, using (**H1**) and Young's inequality, we have

$$\begin{aligned}
|\,\mathrm{I}\,| &\leq \fint_{B_\rho(x_0)} p\eta^{p-1}\left|\int_0^1 D_\xi A(x,u,D\ell+s(Du-D\ell))(Du-D\ell)ds\right||D\eta||u-\ell|dx \\
&\leq \fint_{B_\rho(x_0)} c(p,L)\eta^{p-1}\Big\{(1+|D\ell|)^{p-2}+|Du-D\ell|^{p-2}\Big\}|Du-D\ell||D\eta||u-\ell|dx \\
&\leq \varepsilon \fint_{B_\rho(x_0)} \eta^p\Big\{(1+|D\ell|)^{p-2}|Du-D\ell|^2+|Du-D\ell|^p\Big\}dx \\
&\quad + c(p,L,\varepsilon)\fint_{B_\rho(x_0)}\left\{(1+|D\ell|)^{p-2}\left|\frac{u-\ell}{\rho}\right|^2+\left|\frac{u-\ell}{\rho}\right|^p\right\}dx.
\end{aligned} \tag{4.8}$$

In order to estimate II, we use (**H3**), $D\varphi=\eta^p(Du-D\ell)+p\eta^{p-1}D\eta\otimes(u-\ell)$, and again Young's inequality, we get

$$
\begin{aligned}
|\,\mathrm{II}\,| \le\ & \varepsilon \fint_{B_\rho(x_0)} \eta^p |Du - D\ell|^p dx + \varepsilon^{-q/p} \fint_{B_\rho(x_0)} L^q \omega^q \Big(|u-\ell(x_0)|^2\Big)(1+|D\ell|)^p dx \\
& + \varepsilon \fint_{B_\rho(x_0)} \left|\frac{u-\ell}{\rho}\right|^p dx + \varepsilon^{-q/p} \fint_{B_\rho(x_0)} (4Lp)^q \omega^q \Big(|u-\ell(x_0)|^2\Big)(1+|D\ell|)^p dx \\
\le\ & \varepsilon \fint_{B_\rho(x_0)} \eta^p |Du - D\ell|^p dx + \varepsilon \fint_{B_\rho(x_0)} \left|\frac{u-\ell}{\rho}\right|^p dx \\
& + c(p,L,\varepsilon)(1+|D\ell|)^p \omega \left(\fint_{B_\rho(x_0)} |u-\ell(x_0)|^2 dx \right),
\end{aligned} \tag{4.9}
$$

where we use Jensen's inequality in the last inequality. We next estimate III by using the VMO-condition (**H4**) and Young's inequality, we have

$$
\begin{aligned}
|\,\mathrm{III}\,| \le & \frac{\varepsilon}{2^{p-1}} \fint_{B_\rho(x_0)} \left\{ \eta^p |Du - D\ell| + \frac{4p|u-\ell|}{\rho} \right\}^p dx \\
& + \left(\frac{2^{p-1}}{\varepsilon}\right)^{q/p} \fint_{B_\rho(x_0)} {V_{x_0}}^q(x,\rho)(1+|D\ell|)^p dx.
\end{aligned}
$$

Then using the fact that ${V_{x_0}}^q = {V_{x_0}}^{q-1} \cdot V_{x_0} \le (2L)^{q-1} V_{x_0} \le 2L V_{x_0}$, we infer

$$
|\mathrm{III}| \le \varepsilon \fint_{B_\rho(x_0)} \eta^p |Du - D\ell|^p dx + c(p,\varepsilon) \fint_{B_\rho(x_0)} \left|\frac{u-\ell}{\rho}\right|^p dx + c(p,L,\varepsilon)(1+|D\ell|)^p V(\rho). \tag{4.10}
$$

For $\varepsilon' > 0$ to be fixed later, using (**H5**), Lemma 3.7 and Young's inequality, we have

$$
\begin{aligned}
|\,\mathrm{IV}\,| \le & \fint_{B_\rho(x_0)} a(|Du - D\ell| + |D\ell|)^p \eta^p |u-\ell| dx + \fint_{B_\rho(x_0)} (b\eta\rho) \left|\frac{u-\ell}{\rho}\right| dx \\
\le & \fint_{B_\rho(x_0)} a\eta^p \{(1+\varepsilon')|Du - D\ell|^p + K(p,\varepsilon')|D\ell|^p\} |u-\ell| dx + \varepsilon b^q \rho^q \\
& + \varepsilon^{-p/q} \fint_{B_\rho(x_0)} \left|\frac{u-\ell}{\rho}\right|^p dx \\
\le\ & a(1+\varepsilon')(2M + |D\ell|\rho) \fint_{B_\rho(x_0)} \eta^p |Du - D\ell|^p dx + c(p,\varepsilon) \fint_{B_\rho(x_0)} \left|\frac{u-\ell}{\rho}\right|^p dx \\
& + \varepsilon(1+|D\ell|)^p \rho^q \{a^q K^q |D\ell|^q + b^q\}.
\end{aligned} \tag{4.11}
$$

Combining (4.5), (4.7), (4.9), (4.10) and (4.11), and set $\lambda' = 2^{(12-9p)/2}\lambda$ $\Lambda := \lambda' - 3\varepsilon - a(1+\varepsilon')(2M + |D\ell|\rho)$, this gives

$$
\begin{aligned}
&\Lambda \fint_{B_\rho(x_0)} \eta^p \left\{ \frac{|Du - D\ell|^2}{(1+|D\ell|)^2} + \frac{|Du - D\ell|^p}{(1+|D\ell|)^p} \right\} dx \\
&\leq c(p, L, \varepsilon) \left[\fint_{B_\rho(x_0)} \left\{ \left| \frac{u-\ell}{\rho(1+|D\ell|)} \right|^2 + \left| \frac{u-\ell}{\rho(1+|D\ell|)} \right|^p \right\} dx \right. \\
&\quad \left. + \omega\left(\fint_{B_\rho(x_0)} |u - \ell(x_0)|^2 dx \right) + V(\rho) \right] \\
&\quad + \varepsilon\{ a^q (1 + K(p, \varepsilon'))^q |D\ell|^q + b^q \} \rho^q.
\end{aligned}
\tag{4.12}
$$

Now choose $\varepsilon = \varepsilon(\lambda, p, a(M), M) > 0$ and $\varepsilon' = \varepsilon'(\lambda, p, a(M), M) > 0$ in a right way (for more precise way of choosing ε and ε', we refer to [7, Lemma 4.1]), we obtain (4.1). □

Remark 4.2. *If we insert $p = 2$ to the "smallness condition" $2^{(10-9p)/2}\lambda > a(M)M$, we obtain $\lambda/16 > a(M)M$. On the other hand, we only need $\lambda/2 > a(M)M$ to prove the Caccioppoli-type inequality* (*Lemma* 4.1) *since the term $(1 + |sDu + (1-s)v|)^{p-2}$ in* (4.6) *vanishes when $p = 2$. This gap happens because the left-hand side inequality of* (3.15) *in Lemma* 3.8, *which we used to estimate $(1 + |sDu + (1-s)v|)^{p-2}$ from below, could not take equal when $\delta = p - 2 = 0$.*

To use the $\mathcal{A}$-harmonic approximation lemma, we need to estimate $\fint_{B_\rho(x_0)} \mathcal{A}(D(u-\ell), D\varphi)dx$.

Lemma 4.3. *Assume the same assumptions in Lemma* 4.1. *Then for any $x_0 \in \Omega$ and $\rho \leq \rho_0$ satisfy $B_{2\rho}(x_0) \Subset \Omega$, and any affine functions $\ell : \mathbb{R}^n \to \mathbb{R}^N$ with $|\ell(x_0)| \leq M$, the inequality*

$$
\begin{aligned}
\fint_{B_\rho(x_0)} \mathcal{A}(Dv, D\varphi)dx &\leq C_2(1 + |D\ell|) \Big[\mu^{1/2}\Big(\sqrt{\Psi_*(x_0, 2\rho, \ell)}\Big)\sqrt{\Psi_*(x_0, 2\rho, \ell)} \\
&\quad + \Psi_*(x_0, 2\rho, \ell) + \rho(a|D\ell|^p + b) \Big] \sup_{B_\rho(x_0)} |D\varphi|
\end{aligned}
\tag{4.13}
$$

holds for all $\varphi \in C_0^\infty(B_\rho(x_0), \mathbb{R}^N)$ and a constant $C_2 = C_2(n, \lambda, L, p, a(M)) \geq 1$, where

$$\mathcal{A}(Dv, D\varphi) := \frac{1}{(1+|D\ell|)^{p-1}} \Big\langle \left(D_\xi A(\cdot, \ell(x_0), D\ell)\right)_{x_0,\rho} Dv, D\varphi \Big\rangle,$$

$$\Phi(x_0, \rho, \ell) := \fint_{B_\rho(x_0)} \left\{ \frac{|Du - D\ell|^2}{(1+|D\ell|)^2} + \frac{|Du - D\ell|^p}{(1+|D\ell|)^p} \right\} dx,$$

$$\Psi(x_0, \rho, \ell) := \fint_{B_\rho(x_0)} \left\{ \frac{|u-\ell|^2}{\rho^2(1+|D\ell|)^2} + \frac{|u-\ell|^p}{\rho^p(1+|D\ell|)^p} \right\} dx,$$

$$\Psi_*(x_0, \rho, \ell) := \Psi(x_0, \rho, \ell) + \omega\left(\fint_{B_\rho(x_0)} |u - \ell(x_0)|^2 dx \right) + V(\rho) + (a^q |D\ell|^q + b^q)\rho^q,$$

$$v := u - \ell = u - \ell(x_0) - D\ell(x - x_0).$$

Proof. Assume $x_0 \in \Omega$ and $\rho \le 1$ satisfy $B_{2\rho}(x_0) \Subset \Omega$. Without loss of generality we may assume $\sup_{B_\rho(x_0)} |D\varphi| \le 1$. Note $\sup_{B_\rho(x_0)} |\varphi| \le \rho \le 1$. Using the fact that $\int_{B_\rho(x_0)} A(x_0, \xi, v) D\varphi dx = 0$, we deduce

$$\begin{aligned}
&(1+|D\ell|)^{p-1} \fint_{B_\rho(x_0)} \mathcal{A}(Dv, D\varphi) dx \\
&= \fint_{B_\rho(x_0)} \int_0^1 \Big\langle \Big[\left(D_\xi A(\cdot, \ell(x_0), D\ell)\right)_{x_0,\rho} - \left(D_\xi A(\cdot, \ell(x_0), D\ell + sDv)\right)_{x_0,\rho} \Big] Dv, D\varphi \Big\rangle ds dx \\
&\quad + \fint_{B_\rho(x_0)} \Big\langle \left(A(\cdot, \ell(x_0), Du)\right)_{x_0,\rho} - A(x, \ell(x_0), Du), D\varphi \Big\rangle dx \\
&\quad + \fint_{B_\rho(x_0)} \langle A(x, \ell(x_0), Du) - A(x, u, Du), D\varphi \rangle dx \\
&\quad + \fint_{B_\rho(x_0)} \langle f, \varphi \rangle dx \\
&=: \mathrm{I} + \mathrm{II} + \mathrm{III} + \mathrm{IV}
\end{aligned} \tag{4.14}$$

where terms $\mathrm{I}, \mathrm{II}, \mathrm{III}, \mathrm{IV}$ are define above.

Using the modulus of continuity μ from (**H1**), Jensen's inequality and Hölder's inequality, we estimate

$$
\begin{aligned}
|\,\mathrm{I}\,| &\le c(p,L)(1+|D\ell|)^{p-1} \fint_{B_\rho(x_0)} \mu\Big(\frac{|Du-D\ell|}{1+|D\ell|}\Big)\Big\{\frac{|Du-D\ell|}{1+|D\ell|}+\frac{|Du-D\ell|^{p-1}}{(1+|D\ell|)^{p-1}}\Big\}dx \\
&\le c\,(1+|D\ell|)^{p-1}\Big[\mu^{1/2}\Big(\sqrt{\Phi(x_0,\rho,\ell)}\Big)\sqrt{\Phi(x_0,\rho,\ell)} \\
&\qquad + \mu^{1/p}\Big(\Phi^{1/2}(x_0,\rho,\ell)\Big)\Phi^{1/q}(x_0,\rho,\ell)\Big] \\
&\le c\,(1+|D\ell|)^{p-1}\Big[\mu^{1/2}\Big(\sqrt{\Phi(x_0,\rho,\ell)}\Big)\sqrt{\Phi(x_0,\rho,\ell)}+\Phi(x_0,\rho,\ell)\Big].
\end{aligned}
\tag{4.15}
$$

The last inequality follows from the fact that $a^{1/p}b^{1/q}=a^{1/p}b^{1/p}b^{(p-2)/p}\le a^{1/2}b^{1/2}+b$ holds by Young's inequality.

By using the VMO-condition, Young's inequality and the bound $V_{x_0}(x,\rho)\le 2L$, the term II can be estimated as

$$
\begin{aligned}
|\,\mathrm{II}\,| &\le c(p)(1+|D\ell|)^{p-1} \fint_{B_\rho(x_0)} \Big\{V_{x_0}(x,\rho)+V_{x_0}(x,\rho)\frac{|Du-D\ell|^{p-1}}{(1+|D\ell|)^{p-1}}\Big\}dx \\
&\le c\,(1+|D\ell|)^{p-1}\Big[\Big(1+(2L)^{p-1}\Big)V(\rho)+\Phi(x_0,\rho,\ell)\Big].
\end{aligned}
\tag{4.16}
$$

Similarly, we estimate the term III by using the continuity condition (**H3**), Young's inequality, the bound $\omega\le 1$ and Jensen's inequality. This leads us to

$$
\begin{aligned}
|\,\mathrm{III}\,| &\le L \fint_{B_\rho(x_0)} (1+|D\ell|+|Du-D\ell|)^{p-1}\omega\Big(|u-\ell(x_0)|^2\Big)dx \\
&\le c(p,L)(1+|D\ell|)^{p-1}\left[\omega\left(\fint_{B_\rho(x_0)} |u-\ell(x_0)|^2dx\right)+\Phi(x_0,\rho,\ell)\right].
\end{aligned}
\tag{4.17}
$$

By using the growth condition (**H5**) and $\sup_{B_\rho(x_0)}|\varphi|\le\rho\le 1$, we have

$$
\begin{aligned}
|\,\mathrm{IV}\,| &\le \fint_{B_\rho(x_0)} \rho(a|Du|^p+b)dx \\
&\le 2^{p-1}a(1+|D\ell|)^p\Phi(x_0,\rho,\ell)+2^{p-1}\rho(1+|D\ell|)^{p-1}(a|D\ell|^p+b).
\end{aligned}
\tag{4.18}
$$

Combining (4.14) with the estimates (4.15), (4.16), (4.17) and (4.18), we finally arrive at

$$\begin{aligned}
&\fint_{B_\rho(x_0)} \mathcal{A}(Dv, D\varphi)dx \\
&\qquad \le c(p, L, a(M))(1+|D\ell|) \\
&\qquad \times \left[\mu^{1/2}\left(\sqrt{\Phi(x_0,\rho,\ell)}\right)\sqrt{\Phi(x_0,\rho,\ell)} + \Phi(x_0,\rho,\ell) + \Psi_*(x_0,\rho,\ell) + \rho(a|D\ell|^p + b)\right] \\
&\qquad \le C_2(1+|D\ell|)\left[\mu^{1/2}\left(\sqrt{\Psi_*(x_0,2\rho,\ell)}\right)\sqrt{\Psi_*(x_0,2\rho,\ell)}\right. \\
&\qquad\qquad\qquad\qquad \left. + \Psi_*(x_0,2\rho,\ell) + \rho(a|D\ell|^p + b)\right],
\end{aligned}$$

where we use Caccioppoli-type inequality (Lemma 4.1), $\Phi(x_0,\rho,\ell) \le C_1\Psi_*(x_0,2\rho,\ell)$ and the concavity of μ to have $\mu(cs) \le c\mu(s)$ for $c \ge 1$ at the last step. □

From now on, we write $\Phi(\rho) = \Phi(x_0,\rho,\ell_{x_0,\rho})$, $\Psi(\rho) = \Psi(x_0,\rho,\ell_{x_0,\rho})$, $\Psi_*(\rho) = \Psi_*(x_0,\rho,\ell_{x_0,\rho})$ for $x_0 \in \Omega$ and $0 < \rho \le 1$. Here $\ell_{x_0,\rho}$ is a minimizer of (3.9).

Now we are in the position to establish the excess improvement.

Lemma 4.4. *Assume the same assumptions with Lemma 4.3. Let $\theta \in (0, 1/4]$ be arbitrary and impose the following smallness conditions on the excess:*

(i) $\mu^{1/2}\left(\sqrt{\Psi_*(\rho)}\right) + \sqrt{\Psi_*(\rho)} \le \dfrac{\delta}{2}$ *with the constant* $\delta = \delta(n, N, p, \lambda, L, \theta^{n+p+2})$ *from Lemma 3.2,*

(ii) $\Psi(\rho) \le \dfrac{\theta^{n+2}}{4n(n+2)}$,

(iii) $\gamma(\rho) := \left[{\Psi_*}^{q/2}(\rho) + \delta^{-q}\rho^q(a|D\ell_{x_0,\rho}| + b)^q\right]^{1/q} \le 1.$

Then there holds the excess improvement estimate

$$\Psi(\theta\rho) \le C_3\theta^2\Psi_*(\rho) \tag{4.19}$$

with a constant $C_3 \ge 1$ that depends only on n, N, λ, L, p, $a(M)$, M and θ.

Proof. We first rescale u and set

$$w := \frac{u - \ell_{x_0,\rho}}{C_2(1 + |D\ell_{x_0,\rho}|)\gamma}.$$

We claim that w satisfies the assumptions of Lemma 3.2. By Lemma 4.3, with

$\rho/2$ and $\ell_{x_0,\rho}$ instead of ρ and ℓ, and assumption (i), the map w is approximately $\mathcal{A}$-harmonic in the sense that

$$\begin{aligned}\fint_{B_{\rho/2}(x_0)} \mathcal{A}(Dw, D\varphi)dx &\le \left[\mu^{1/2}\Big(\sqrt{\Psi_*(\rho)}\Big) + \sqrt{\Psi_*(\rho)} + \frac{\delta}{2}\right] \sup_{B_{\rho/2}(x_0)} |D\varphi| \\ &\le \delta \sup_{B_{\rho/2}(x_0)} |D\varphi|,\end{aligned}$$

for all $\varphi \in C_0^\infty(B_{\rho/2}(x_0), \mathbb{R}^N)$, with the constant δ determined by Lemma 3.2 for the choice $\varepsilon = \theta^{n+p+2}$. Moreover, the choice of C_2, which implies $C_2 \ge C_1$, and the Caccioppoli-type inequality (Lemma 4.1) infer

$$\fint_{B_{\rho/2}(x_0)} \Big\{|Dw|^2 + \gamma^{p-2}|Dw|^p\Big\}dx \le \frac{C_1\Psi_*(\rho)}{{C_2}^2\gamma^2} \le \frac{C_1}{{C_2}^2} \le 1.$$

Thus, Lemma 3.2 ensures the existence of an $\mathcal{A}$-harmonic map h with the properties

$$\fint_{B_{\rho/2}(x_0)} \left\{\left|\frac{w-h}{\rho/2}\right|^2 + \gamma^{p-2}\left|\frac{w-h}{\rho/2}\right|^p\right\}dx \le \theta^{n+p+2}, \tag{4.20}$$

$$\fint_{B_{\rho/2}(x_0)} \Big\{|Dh|^2 + \gamma^{p-2}|Dh|^p\Big\}dx \le c(n,p). \tag{4.21}$$

Since h is $\mathcal{A}$-harmonic, Theorem 3.3 yields the estimate for $s = 2$ as well as for $s = p$

$$\sup_{B_{\rho/4}(x_0)} |D^2 h|^s \le c(s,n,N,p,\lambda,L)\Big(\frac{\rho}{2}\Big)^{-s}.$$

Therefore, using Taylor's theorem, we have the decay estimate, where $\theta \in (0, 1/4]$ can be chosen arbitrarily:

$$\begin{aligned}&\gamma^{s-2}(\theta\rho)^{-s} \fint_{B_{\theta\rho}(x_0)} |w - h(x_0) - Dh(x_0)(x-x_0)|^s dx \\ &\le 2^{s-1}\gamma^{s-2}(\theta\rho)^{-s}\left[\fint_{B_{\theta\rho}(x_0)} |w-h|^s dx + \fint_{B_{\theta\rho}(x_0)} |h(x) - h(x_0) - Dh(x_0)(x-x_0)|^s dx\right] \\ &\le c(s,n,N,p,\lambda,L)\theta^2.\end{aligned}$$

Here we applied the energy bound (4.20) for the last estimate. Scaling back to u and using Lemma 3.6, we conclude

$$
\begin{aligned}
&(\theta\rho)^{-s} \fint_{B_{\theta\rho}(x_0)} |u - \ell_{x_0,\theta\rho}|^s dx \\
&\le c(n,s)(\theta\rho)^{-s} \fint_{B_{\theta\rho}(x_0)} |u - \ell_{x_0,\rho} - C_2\gamma(1+|D\ell_{x_0,\rho}|)(h(x_0)+Dh(x_0)(x-x_0))|^s dx \\
&= c(s,n,N,p,\lambda,L,a(M))(\theta\rho)^{-s}\gamma^s(1+|D\ell_{x_0,\rho}|)^s \\
&\qquad\qquad \fint_{B_{\theta\rho}(x_0)} |w - h(x_0) - Dh(x_0)(x-x_0)|^s dx \qquad (4.22)\\
&\le c\,\gamma^2(1+|D\ell_{x_0,\rho}|)^s\theta^2 \\
&\le c\,(1+|D\ell_{x_0,\rho}|)^s\theta^2\Big[\Psi_*{}^{q/2}(\rho) + 2^{q/p}\delta^{-q}\Psi_*(\rho)\Big]^{2/q} \\
&\le c\,(1+|D\ell_{x_0,\rho}|)^s\theta^2\Psi_*(\rho).
\end{aligned}
$$

Here we would like to replace the term $|D\ell_{x_0,\rho}|$ on the right-hand side by $|D\ell_{x_0,\theta\rho}|$. For this, we use (3.11) and the assumption (ii) in order to estimate

$$
\begin{aligned}
|D\ell_{x_0,\rho} - D\ell_{x_0,\theta\rho}|^2 &\le \frac{n(n+2)}{(\theta\rho)^2} \fint_{B_{\theta\rho}(x_0)} |u - \ell_{x_0,\rho}|^2 dx \\
&\le \frac{n(n+2)}{\theta^{n+2}\rho^2} \fint_{B_{\rho}(x_0)} |u - \ell_{x_0,\rho}|^2 dx \\
&\le \frac{n(n+2)}{\theta^{n+2}}(1+|D\ell_{x_0,\rho}|)^2\Psi(\rho) \le \frac{1}{4}(1+|D\ell_{x_0,\rho}|)^2.
\end{aligned}
$$

This yields

$$
1+|D\ell_{x_0,\rho}| \le 1+|D\ell_{x_0,\theta\rho}| + |D\ell_{x_0,\rho} - D\ell_{x_0,\theta\rho}| \le 1+|D\ell_{x_0,\theta\rho}| + \frac{1}{2}(1+|D\ell_{x_0,\rho}|),
$$

and after reabsorbing the last term from the right-hand side on the left, we also obtain

$$
1+|D\ell_{x_0,\rho}| \le 2(1+|D\ell_{x_0,\theta\rho}|).
$$

Plugging this into (4.22), we deduce

$$
(\theta\rho)^{-s} \fint_{B_{\theta\rho}(x_0)} |u - \ell_{x_0,\theta\rho}|^s dx \le c(s,n,N,p,\lambda,L,a(M))(1+|D\ell_{x_0,\theta\rho}|)^s\theta^2\Psi_*(\rho)
$$

for $s = 2$ and $s = p$. Dividing by $(1 + |D\ell_{x_0,\theta\rho}|)^s$, then adding the corresponding terms for $s = 2$ and $s = p$, we deduce the claim. □

We fix an arbitrarily Hölder exponent $\alpha \in (0, 1)$ and define the Campanato-type excess

$$C_\alpha(x_0, \rho) := \rho^{-2\alpha} \fint_{B_\rho(x_0)} |u - u_{x_0,\rho}|^2 dx.$$

In the following lemma, we iterate the excess improvement estimate (4.19) from Lemma 4.4 and obtain the bounededness of the two excess functionals, C_α and Ψ.

Lemma 4.5. *Under the same assumptions with Lemma 4.4, for every $\alpha \in (0, 1)$, there exist constants $\varepsilon_*, \kappa_*, \rho_* > 0$ and $\theta \in (0, 1/8]$, all depending at most on n, N, λ, p, L, α, ρ_0, $\mu(\,\cdot\,)$, $\omega(\,\cdot\,)$, $V(\,\cdot\,)$, $a(M)$, b and M, such that the conditions*

$$(A_0) \qquad \Psi(\rho) < \varepsilon_*, \quad \textit{and} \quad C_\alpha(x_0, \rho) < \kappa_*$$

for all $\rho \in (0, \rho_*)$ with $B_\rho(x_0) \Subset \Omega$, imply

$$(A_k) \qquad \Psi(\theta^k \rho) < \varepsilon_*, \quad \textit{and} \quad C_\alpha(x_0, \theta^k \rho) < \kappa_*$$

respectively, for every $k \in \mathbb{N}$.

Proof. We begin by choosing the constants. First, let

$$\theta := \min\left\{ \left(\frac{1}{16n(n+2)} \right)^{1/(2-2\alpha)}, \frac{1}{\sqrt{4C_3}} \right\} \le \frac{1}{8},$$

with the constant C_3 determined in Lemma 4.3. In particular, the choice of $\theta = \theta(n, N, \lambda, L, a, M, \alpha) > 0$ fixes the constant $\delta = \delta(n, N, \lambda, L, a, M, \alpha) > 0$ from Lemma 3.2. Next, we fix an $\varepsilon_* = \varepsilon_*(n, N, \lambda, L, a, M, \alpha, \mu(\,\cdot\,)) > 0$ sufficiently small to ensure

$$\varepsilon_* \le \frac{\theta^{n+2}}{16n(n+2)} \quad \text{and} \quad \mu^{1/2}\left(\sqrt{4\varepsilon_*}\right) + \sqrt{4\varepsilon_*} \le \frac{\delta}{2}.$$

Then, we choose $\kappa_* = \kappa_*(n, N, \lambda, L, a, M, \alpha, \mu(\,\cdot\,), \omega(\,\cdot\,)) > 0$ so small that

$$\omega(\kappa_*) < \varepsilon_*.$$

Finally, we fix $\rho_* = \rho_*(n, N, \lambda, p, L, \alpha, \rho_0, \mu(\,\cdot\,), \omega(\,\cdot\,), V(\,\cdot\,), a, b, M) > 0$ small enough to guarantee

$$\rho_* \le \min\{\rho_0, {\kappa_*}^{1/(2-2\alpha)}, 1\}, \;\; V(\rho_*) < \varepsilon_* \text{ and } \left\{ \left(a\sqrt{n(n+2)\kappa_*} \right)^q + b^q \right\} {\rho_*}^{q\alpha} < \varepsilon_*.$$

Now we prove the assertion (A_k) by induction. We assume that we have already established (A_k) up to some $k \in \mathbb{N} \cup \{0\}$. We begin with proving the first part of the assertion (A_{k+1}), that is, the one concerning $\Psi(\theta^{k+1}\rho)$. First, using (3.12) with $\ell \equiv u_{x_0,\theta^k\rho}$, we obtain

$$\begin{aligned} |D\ell_{x_0,\theta^k\rho}|^2 &\le \frac{n(n+2)}{(\theta^k\rho)^2} \fint_{B_{\theta^k\rho}(x_0)} |u - u_{x_0,\theta^k\rho}|^2 dx \\ &= n(n+2)(\theta^k\rho)^{2\alpha-2} C_\alpha(x_0, \theta^k\rho) \\ &\le n(n+2){\rho_*}^{2\alpha-2}\kappa_*. \end{aligned} \tag{4.23}$$

Thus, the assumption (A_k), the choice of κ_* and ρ_*, and the above estimate infer

$$\begin{aligned} \Psi_*(\theta^k\rho) &\le \Psi(\theta^k\rho) + \omega(C_\alpha(x_0,\theta^k\rho)) + V(\theta^k\rho) + (a^q|D\ell_{x_0,\theta^k\rho}|^q + b^q)(\theta^k\rho)^q \\ &\le \varepsilon_* + \omega(\kappa_*) + V(\rho_*) + \left(\left(a\sqrt{n(n+2)\kappa_*}\right)^q + b^q\right){\rho_*}^{q\alpha} < 4\varepsilon_*. \end{aligned} \tag{4.24}$$

Now it is easy to check that our choice of ε_* implies that the smallness condition assumptions (i) and (ii) in Lemma 4.4 are satisfied on the level $\theta^k\rho$, that is, we have

$$\mu^{1/2}\left(\sqrt{\Psi_*(\theta^k\rho)}\right) + \sqrt{\Psi_*(\theta^k\rho)} < \mu^{1/2}\left(\sqrt{4\varepsilon_*}\right) + \sqrt{4\varepsilon_*} \le \frac{\delta}{2}, \tag{4.25}$$

and

$$\Psi(\theta^k\rho) < \varepsilon_* < \frac{\theta^{n+2}}{4n(n+2)}. \tag{4.26}$$

Furthermore, we have the smallness condition assumption (iii), that is,

$$\gamma(\theta^k\rho) = \left[{\Psi_*}^{q/2}(\theta^k\rho) + \delta^{-q}(\theta^k\rho)^q(a|D\ell_{x_0,\theta^k\rho}| + b)^q\right]^{1/q} \le 1. \tag{4.27}$$

To check (4.27), first, note that $\Psi_*(\theta^k\rho) < 1$ holds by the estimate (4.24) and the choice of ε_*. This implies

$${\Psi_*}^{q/2}(\theta^k\rho) \le {\Psi_*}^{1/2}(\theta^k\rho) < \sqrt{4\varepsilon_*} \le \frac{\delta}{4}. \tag{4.28}$$

Next, using (4.23) and ${\rho_*}^{\alpha-1} \ge 1$, we obtain

$$\begin{aligned} \delta^{-q}(\theta^k\rho)^q(a|D\ell|_{x_0,\theta^k\rho} + b)^q &\le \delta^{-q}{\rho_*}^q\left(a\sqrt{n(n+2)\kappa_*}{\rho_*}^{\alpha-1} + b\right)^q \\ &\le \delta^{-q}{\rho_*}^{q\alpha}\left(a\sqrt{n(n+2)\kappa_*} + b\right)^q \\ &\le \delta^{-q}{\rho_*}^{q\alpha}2^{q/p}\left\{\left(a\sqrt{n(n+2)\kappa_*}\right)^q + b^q\right\}. \end{aligned}$$

Then the choice of ρ_* and ε_* imply

$$\delta^{-q}(\theta^k\rho)^q(a|D\ell|_{x_0,\theta^k\rho}+b)^q \le \delta^{-q}2^{q/p}\varepsilon_* \le 2^{-4+q/p}\delta^{2-q} \le \frac{\delta}{8}. \tag{4.29}$$

Therefore combining (4.28) and (4.29), we have (4.27). We may thus apply Lemma 4.4 with the radius $\theta^k\rho$ instead of ρ, which yields

$$\Psi(\theta^{k+1}\rho) \le C_3\theta^2\Psi_*(\theta^k\rho) < 4C_3\theta^2\varepsilon_* \le \varepsilon_*,$$

by the choice of θ. We have thus established the first part of the assertion (A_{k+1}) and it remains to prove the second one, that is, the one concerning $C_\alpha(x_0,\theta^{k+1}\rho)$. For this aim, we first compute

$$\frac{1}{(\theta^k\rho)^2}\fint_{B_{\theta^k\rho}(x_0)}|u-\ell_{x_0,\theta^k\rho}|^2dx \le (1+|D\ell_{x_0,\theta^k\rho}|)^2\Psi(\theta^k\rho) \le 2\varepsilon_*+2\varepsilon_*|D\ell_{x_0,\theta^k\rho}|^2$$

where we used the assumption (A_k) in the last step. Since $\ell_{x_0,\theta^k\rho}(x)=u_{x_0,\theta^k\rho}+D\ell_{x_0,\theta^k\rho}(x-x_0)$, we can estimate

$$\begin{aligned}
C_\alpha(x_0,\theta^{k+1}\rho) &\le (\theta^{k+1}\rho)^{-2\alpha}\fint_{B_{\theta^{k+1}\rho}(x_0)}|u-u_{x_0,\theta^k\rho}|^2dx\\
&\le 2(\theta^{k+1}\rho)^{-2\alpha}\left[\fint_{B_{\theta^{k+1}\rho}(x_0)}|u-\ell_{x_0,\theta^k\rho}|^2dx+|D\ell_{x_0,\theta^k\rho}|^2(\theta^{k+1}\rho)^2\right]\\
&\le 2(\theta^{k+1}\rho)^{-2\alpha}\left[\theta^{-n}\fint_{B_{\theta^k\rho}(x_0)}|u-\ell_{x_0,\theta^k\rho}|^2dx+|D\ell_{x_0,\theta^k\rho}|^2(\theta^{k+1}\rho)^2\right]\\
&\le 4(\theta^k\rho)^{2-2\alpha}\big[\varepsilon_*\theta^{-n-2\alpha}+|D\ell_{x_0,\theta^k\rho}|^2(\varepsilon_*\theta^{-n-2\alpha}+\theta^{2-2\alpha})\big].
\end{aligned}$$

Using (4.23) and recalling the choice of ρ_*, ε_* and θ, we deduce

$$\begin{aligned}
C_\alpha(x_0,\theta^{k+1}\rho) &\le 4{\rho_*}^{2-2\alpha}\big[\varepsilon_*\theta^{-n-2\alpha}+n(n+2)\kappa_*{\rho_*}^{2-2\alpha}(\varepsilon_*\theta^{-n-2\alpha}+\theta^{2-2\alpha})\big]\\
&\le \frac{1}{4}{\rho_*}^{2-2\alpha}\theta^{2-2\alpha}+8n(n+2)\kappa_*\theta^{2-2\alpha}\\
&\le \frac{1}{4}\kappa_*+\frac{1}{2}\kappa_*<\kappa_*.
\end{aligned}$$

This proves the second part of the assertion (A_{k+1}) and finally we conclude the proof of the lemma. □

Now, to obtain the regularity result (Theorem 2.2), it is similar arguments as in [**1**, Section 3.5] by using Lemma 4.5.

Acknowledgments. This results are obtained as the Master thesis in Tokyo University of Science under advising by Professor Atsushi Tachikawa. The author thanks Professor Atsushi Tachikawa for helpful discussions.

References

[**1**] V. Bögelein, F. Duzaar, J. Habermann and C. Scheven, *Partial Hölder continuity for discontinuous elliptic problems with VMO-coefficients*, Proc. Lond. Math. Soc. (3) **103** (2011), 371-404.

[**2**] S. Campanato, *Equazioni ellittiche del* II° *ordine e spazi* $\mathfrak{L}^{(2,\lambda)}$, Ann. Mat. Pura Appl. (4) **69** (1965), 321-381.

[**3**] S. Campanato, *Hölder continuity and partial Hölder continuity results for $H^{1,q}$-solutions of nonlinear elliptic systems with controlled growth*, Rend. Sem. Mat. Fis. Milano **52** (1982), 435-472.

[**4**] S. Campanato, *Hölder continuity of the solutions of some nonlinear elliptic systems*, Adv. in Math. **48** (1983), 16-43.

[**5**] S. Chen and Z. Tan, *Optimal interior partial regularity for nonlinear elliptic systems under the natural growth condition: the method of A-harmonic approximation*, Acta Math. Sci. Ser. B Engl. Ed. **27** (2007), 491-508.

[**6**] F. Duzaar and A. Gastel, *Nonlinear elliptic systems with Dini continuous coefficients*, Arch. Math. (Basel) **78** (2002), 58-73.

[**7**] F. Duzaar and J. F. Grotowski, *Optimal interior partial regularity for nonlinear elliptic systems: the method of A-harmonic approximation*, Manuscripta Math. **103** (2000), 267-298.

[**8**] M. Foss and G. Mingione, *Partial continuity for elliptic problems*, Ann. Inst. H. Poincaré Anal. Non Linéaire **25** (2008), 471-503.

[**9**] M. Giaquinta and L. Martinazzi, *An introduction to the regularity theory for elliptic systems, harmonic maps and minimal graphs*, Appunti. Scuola Normale Superiore di Pisa (Nuova Serie) [Lecture Notes. Scuola Normale Superiore di Pisa (New Series)], vol. 2, Edizioni della Normale, Pisa 2005.

[**10**] M. Giaquinta and G. Modica, *Almost-everywhere regularity results for solutions of nonlinear elliptic systems*, Manuscripta Math. **28** (1979), 109-158.

[**11**] M. Giaquinta and G. Modica, *Partial regularity of minimizers of quasiconvex integrals*, Ann. Inst. H. Poincaré Anal. Non Linéaire **3** (1986), 185-208.

[**12**] M. Kronz, *Partial regularity results for minimizers of quasiconvex functionals of higher order*, Ann. Inst. H. Poincaré Anal. Non Linéaire **19** (2002), 81-112.

[13] Y. Qiu, *Optimal partial regularity of second order nonlinear elliptic systems with Dini continuous coefficients for the superquadratic case*, Nonlinear Anal. **75** (2012), 3574-3590.

[14] M. A. Ragusa and A. Tachikawa, *Regularity of minimizers of some variational integrals with discontinuity*, Z. Anal. Anwend. **27** (2008), 469-482.

[15] S. Z. Zheng, *Partial regularity for quasi-linear elliptic systems with VMO coefficients under the natural growth*, Chinese Ann. Math. Ser. A **29** (2008), 49-58.

Taku Kanazawa
Graduate School of Mathematics
Nagoya University
Chikusa-ku, Nagoya, 464-8602
JAPAN
e-mail: taku.kanazawa@math.nagoya-u.ac.jp