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Involutions of graded central simple algebras

Abstract. Although much attention has already been paid to graded central simple algebras and their involutions, one piece of information is not yet well known: their involutions are classified by a cyclic group of order 8. The properties of this classification, and several problems in which it is useful, are here explained. Graded modules are the natural domain of application, but modules without gradation are also dealt with. A precise example (coming from quantum mechanics) shows which progresses this theory can achieve.

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Contents

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Much information about graded central simple algebras has been brought by many people, especially by C. T. C. Wall and H. Bass; the graded Brauer group (or Brauer-Wall group) $\mathrm{BW}(K)$ of a field K has become a usual tool in many works. Although much attention has already been paid to the involutions of these algebras, their study can still be improved. The benefit of this improvement is already noticeable with Clifford algebras, which were an important motivation of Wall's work in [**6**]; in many problems, a simplification of the arguments and a reduction of the calculations can be obtained by new general knowledge about graded central simple algebras with involution, rather than by new specialized knowledge about Clifford algebras. The concept of "class of involution" is a useful tool for this improvement; every involution of a graded central simple algebra has a class in the group R_8 of eighth roots of 1 in $\mathbb{C}$ (the field of complex numbers), as it is already pointed out in [**2**] and [**3**]. Here is a complete survey of the topic, with the most recent progresses.

Sections 1 and 2 rapidly recall the classical definitions and theorems needed in the study of graded central simple algebras. Classes of involutions are defined in Sections 3 and 4. In Section 5, there is a first application which involves the Brauer-Wall group $\mathrm{BW}(\mathbb{R})$ of the field of real numbers. Graded central simple algebras with involution were already classified in [**7**]; in Section 6, Wall's classification proves to gather two concepts which here are separated: the class of the algebra and the class of the involution. The main part of this article is the study of a graded module over a graded central simple algebra provided with an involution, in Sections 7 and 8; Theorems 8.3 and 8.4 show which efficient pieces of information can be derived from the class of the involution.

Sections 9 and 10 present an example: the study of a graded irreducible module M over a real Clifford algebra $C\ell(E)$ of type $(1,3)$; this example contains relevant problems because they are the algebraic preliminaries to quantum mechanics. The first difficulty is the selection of the problems that are meaningful for physicists; the present study relies on the selection proposed in [**5**], but applies quite different mathematical methods, which allow to settle all details from the beginning up to the Fiersz identities.

Finally, although the study of *graded* central simple algebras naturally involves *graded* modules (in agreement with [**1**]), it has remained usual to study irreducible modules over Clifford algebras (the so-called spinor spaces) without worrying about gradations; fortunately, the case of modules without gradation can be easily reduced to the case of graded modules, as it is explained in Sections 11 and 12.

1 - Parity gradations

In all this text, K is a field of characteristic $\neq 2$, and all spaces over K (therefore, all algebras too) are assumed to have a finite dimension. This basic field K is always referred to when no other field or ring is mentioned. Here every gradation is a parity gradation, in other words, a gradation over the group $\mathbb{Z}/2\mathbb{Z}$. If $M = M_0 \oplus M_1$ is a graded space, an element x of M is said to be *homogeneous* if it is even or odd, and its parity (an element of $\mathbb{Z}/2\mathbb{Z} = \{0, 1\}$) is denoted by ∂x. Whenever I use the symbol ∂, the following element is silently assumed to be homogeneous. The *grade automorphism* σ (or σ_M if more precision is needed) maps every homogeneous element x to $(-1)^{\partial x}x$. Since the characteristic of K is $\neq 2$, the gradation is determined by σ. A subspace P of M is said to be graded (by the gradation of M) if $P = (P \cap M_0) \oplus (P \cap M_1)$. The gradation of M is said to be *trivial* if $M_0 = M$ and $M_1 = 0$; it is said to be *balanced* if $\dim(M_0) = \dim(M_1)$.

When M and N are graded spaces, their tensor product $M \otimes N$, the space of all linear mappings $M \to N$, and the space of all bilinear mappings $M \times M \to N$ are also graded. The even linear mappings $M \to N$ are also called graded mappings because they are the morphisms in the category of graded spaces. All these definitions have become usual, and only some few pieces of information still deserve to be mentioned. The gradation of $M \otimes N$ is trivial if and only if both factors M and N are trivially graded; it is balanced if and only if at least one factor has a balanced gradation. Every space without gradation (for instance K itself) automatically receives the trivial gradation, and consequently a bilinear form $\beta : M \times M \to K$ is said to be even (relatively to the gradation of M) if $\beta(M_p, M_{1-p}) = 0$ for $p = 0, 1$, and it is said to be odd if $\beta(M_p, M_p) = 0$ for $p = 0, 1$.

With every space N without gradation is associated a graded space $(N^2)^g$; it is the space $N \oplus N$ in which the even (resp. odd) elements are the couples (x, x) (resp. $(x, -x)$); the associated grade automorphism is the *swap automorphism* $(x, y) \longmapsto (y, x)$. Finally with every graded space $M = M_0 \oplus M_1$ is associated the space M^s with *shifted gradation*; it is made of elements x^s in bijection with the elements x of M, but $(M^s)_0 = (M_1)^s$ and $(M^s)_1 = (M_0)^s$, or equivalently, $\partial x^s = 1 - \partial x$.

Let A be an (associative and unital) algebra, and $A = A_0 \oplus A_1$ a gradation of the underlying space; A is said to be a graded algebra if the multiplication mapping $A \times A \to A$ is an even bilinear mapping. The gradations of the algebra A are in bijection with the involutive algebra automorphisms $\sigma : A \to A$. The center $\mathrm{Z}(A)$ of a graded algebra A is a graded subalgebra $\mathrm{Z}_0(A) \oplus \mathrm{Z}_1(A)$, and there is a canonical algebra morphism $K \to \mathrm{Z}_0(A)$ which allows to identify K with its image in $\mathrm{Z}_0(A)$ (provided that $A \neq 0$). We also consider $\mathrm{Z}^g(A)$ which has the same even component $\mathrm{Z}^g_0(A) = \mathrm{Z}_0(A)$; its odd component $\mathrm{Z}^g_1(A)$ is the subspace of all $b \in A_1$ such that $ba = \sigma(a)\, b$ for all $a \in A$. Therefore $2b^2 = 0$ for all $b \in \mathrm{Z}^g_1(A)$, whence $b^2 = 0$ in characteristic $\neq 2$.

With every algebra A is associated the opposite algebra A^o: it is the set of all elements a^o (with $a \in A$) provided with the multiplication $a^o b^o = (ba)^o$. When A is a graded algebra, we can also define the twisted algebra A^t in which the multiplication is defined by $a^t b^t = (-1)^{\partial a\, \partial b}(ab)^t$. Still more useful is the twisted opposite algebra A^{to} in which $a^{to} b^{to} = (-1)^{\partial a\, \partial b}(ba)^{to}$. Finally, when A and B are graded algebras, besides the ordinary $A \otimes B$ there is the twisted tensor product $A \hat{\otimes} B$ where $(a \otimes b)(a' \otimes b') = (-1)^{\partial b\, \partial a'}\, aa' \otimes bb'$.

2 - Graded central simple algebras

A graded algebra A is said to be a *graded central simple algebra* if $\mathrm{Z}_0(A) = K$, $\mathrm{Z}_1^g(A) = 0$, and the only graded (two-sided) ideals of A are 0 and A; since the characteristic is $\neq 2$, the condition $\mathrm{Z}_1^g(A) = 0$ is a consequence of the other conditions. Every central simple algebra C becomes a graded central simple algebra when it is provided with the trivial gradation; besides, with C is associated the graded central simple algebra $(C^2)^g$ provided with the gradation defined above, and the usual multiplication $(c, d)(c', d') = (cc', dd')$.

Let M be a graded vector space and let us set $n = \dim(M_0)$ and $n' = \dim(M_1)$; it is known that the graded algebra $\mathrm{End}(M)$ is a graded central simple algebra (if $n + n' > 0$). It is isomorphic to the graded matrix algebra $\mathrm{Mat}(n, n'; K)$ defined in this way: without its gradation it is the same object as $\mathrm{Mat}(n + n', K)$; and if all entries of a matrix vanish except one entry in the i-th row and the j-th column, this matrix is even (resp. odd) if i and j are on the same side (resp. on different sides) with respect to $n + \frac{1}{2}$. If B is a K-algebra without gradation (or equivalently, with the trivial gradation), in the same way we can define a graded algebra $\mathrm{Mat}(n, n'; B)$ isomorphic to $B \otimes \mathrm{Mat}(n, n'; K)$. If the gradation of the algebra B is not trivial, for every integer $n > 0$ there is a graded algebra $\mathrm{Mat}(n, B)$ isomorphic to $B \otimes \mathrm{Mat}(n, K)$. When B_1 contains invertible elements, it is not useful to define $\mathrm{Mat}(n, n'; B)$ because both $B \hat{\otimes} \mathrm{Mat}(n, n'; K)$ and $B \otimes \mathrm{Mat}(n, n'; K)$ prove to be isomorphic to $\mathrm{Mat}(n + n', B)$. Finally, a graded algebra B is called a *graded central division algebra* if $\mathrm{Z}_0(B) = K$ and all homogeneous elements of B are invertible except 0; such an algebra is graded central simple; if its gradation is trivial, it is the same thing as a central division algebra. I recall the fundamental theorems, and for the proofs I refer to [3], Section 6.6, where effective proofs have been selected among the works of previous authors.

Theorem 2.1. *If A and B are graded algebras, then $A \hat{\otimes} B$ is graded central simple if and only if both A and B are graded central simple.*

Theorem 2.2. *If A is a graded algebra, there is a graded algebra morphism $A \hat{\otimes} A^{to} \to \mathrm{End}(A)$ which maps every $a \otimes b^{to}$ to the linear mapping $x \longmapsto (-1)^{\partial b\, \partial x}\, axb$. It is bijective if and only if A is graded central simple.*

Theorem 2.3. *The algebra A is graded central simple if and only if there is a field extension $K \to L$ such that the L-algebra $L \otimes A$ is isomorphic to $\mathrm{Mat}(n, n'; L)$ (for some suitable integers n and n') or to $(\mathrm{Mat}(n; L)^2)^g$ (for some suitable n). This is still true if we require L to be finite over K.*

Corollary 2.4. *When A is graded central simple, $\dim(A)$ is a square or the double of a square. In the first case, A is said to have the* even type, *and there are two integers n and n' such that $\dim(A_0) = n^2 + n'^2$ and $\dim(A_1) = 2nn'$. In the second case, A is said to have the* odd type, *and its gradation is balanced. When A has the even type and $A_1 \neq 0$, then $\mathrm{Z}(A_0)$ is a subalgebra of dimension 2 spanned by 1 and an even element ω such that $\omega^2 \in K$. When A has the odd type, $\mathrm{Z}(A)$ is a subalgebra of dimension 2 spanned by 1 and an odd element ω such that $\omega^2 \in K$. In both cases, ω is called an* Arf element, *and $a\omega = (-1)^{\partial a(1-\partial\omega)}\, \omega a$ for all $a \in A$.*

Corollary 2.5. *The subspace of A spanned by A_1 and all brackets $[a, b] = ab - ba$ is a hyperplane. If $\mathrm{Scal} : A \to K$ is a nonzero linear form that vanishes on this hyperplane, the bilinear form $(a, b) \longmapsto \mathrm{Scal}(ab)$ is symmetric and nondegenerate. Moreover, $\mathrm{Scal} \circ \varphi = \mathrm{Scal}$ whenever φ is an automorphism or an anti-automorphism of A.*

The linear form Scal in Corollary 2.5 is determined up to an invertible factor in K, but there are two usual precise determinations. One determination is called the *trace*, because it is related to the trace of matrices through the isomorphisms mentioned in Theorem 2.3; the trace of the unit element 1 of A is the image in K of the integer $\sqrt{\dim(A)}$ (if A has the even type) or $\sqrt{2\dim(A)}$ (if A has the odd type). The other determination is called the *scalar part* and satisfies the equality $\mathrm{Scal}(1) = 1$; it shall be preferred whenever it is available, in other words, whenever the image of $\dim(A)$ in K is invertible.

Theorem 2.6. *The algebra A is graded central simple if and only if it is isomorphic either to some $\mathrm{Mat}(n, n'; B)$ with B a central division algebra, or to some $\mathrm{Mat}(n, B)$ with B a graded central division algebra where $B_1 \neq 0$. The first case occurs when $\mathrm{Z}(A)$ is not a field (and then $nn' \neq 0$), and also when $A_1 = 0$ (and then $nn' = 0$); the second case occurs when $A_1 \neq 0$ and $\mathrm{Z}(A)$ is a field. In both cases, A determines B up to isomorphy.*

Corollary 2.7. *When A is a graded central simple algebra, A_1 contains invertible elements if and only if the gradation of A is balanced.*

Every graded central simple algebra A has a class $\mathrm{cl}(A)$ in the Brauer-Wall group $\mathrm{BW}(K)$ in such a way that these two conditions are satisfied: firstly, $\mathrm{cl}(A \hat{\otimes} A') = \mathrm{cl}(A)\,\mathrm{cl}(A')$ whenever A and A' are graded central simple; secondly, every algebra isomorphic to some $\mathrm{End}(M)$ (where M is a graded space), or equivalently to some $\mathrm{Mat}(n, n'; K)$, has a trivial class. Theorem 2.2 shows that $\mathrm{cl}(A)^{-1} = \mathrm{cl}(A^{to})$. The group $\mathrm{BW}(K)$ was first defined in [**6**]; the ordinary Brauer group $\mathrm{B}(K)$ is a subgroup of $\mathrm{BW}(K)$.

Theorem 2.6 is a natural generalization of a well known theorem about central simple algebras. It shows that every Brauer-Wall class contains a graded central division algebra which is unique up to isomorphy; thus $\mathrm{BW}(K)$ *is also the set of isomorphy classes of graded central division algebras.*

3 - A calculation of dimensions

Let A be a graded vector space and τ a graded involutive endomorphism of A; let τ_0 and τ_1 be the endomorphisms of A_0 and A_1 induced by τ; the dimensions of the eigenspaces of τ_0 and τ_1 are assumed to be known. Similar hypotheses also hold for (A', τ'), and we consider the graded involutive endomorphism $\tau \otimes \tau'$ of $A \otimes A'$. The calculation of the dimensions of the eigenspaces of $(\tau \otimes \tau')_0$ and $(\tau \otimes \tau')_1$ is an easy problem. I propose a solution that later will enable us to settle a more difficult problem. Since τ commutes with the grade automorphism σ, there is a couple $(\tau, \sigma\tau)$ of graded involutive endomorphisms on A; with τ, I associate this element $D'(\tau)$ of $\mathbb{R}^2$:

$$(3.1)\quad \big(\dim(\ker(\tau - \mathrm{id})) - \dim(\ker(\tau + \mathrm{id})), \dim(\ker(\sigma\tau - \mathrm{id})) - \dim(\ker(\sigma\tau + \mathrm{id}))\big),$$

and the solution to the problem is given by the equality

$$(3.2)\qquad D'(\tau \otimes \tau') = D'(\tau)\, D'(\tau') .$$

When K is a field of characteristic 0, the equality (3.2) is a consequence of this fact: if f and f' are endomorphisms of A and A', the trace of $f \otimes f'$ is the product of the traces of f and f'. The algebra $\mathbb{R}^2$ is isomorphic to the algebra $\mathbb{R} \oplus i'\mathbb{R}$ generated by an element i' such that $i'^2 = 1$; when $D'(\tau)$ is treated as an element of $\mathbb{R} \oplus i'\mathbb{R}$, it becomes

$$(3.3)\quad \dim(\ker((\tau_0 - \mathrm{id})) - \dim(\ker((\tau_0 + \mathrm{id})) \\ + i' \dim(\ker(\tau_1 - \mathrm{id})) - i' \dim(\ker(\tau_1 + \mathrm{id}));$$

this expression confirms that the knowledge of the dimensions of the eigenspaces of τ_0 and τ_1 is equivalent to the knowledge of $D'(\tau)$ and the dimensions of A_0 and A_1. Therefore, $D'(\tau \otimes \tau')$ gives the four wanted dimensions.

Nevertheless, in the following sections I am not interested in $\tau \otimes \tau'$, but in $\tau \tilde{\otimes} \tau'$ which is the graded involutive endomorphism of $A \otimes A'$ defined by

$$(\tau \tilde{\otimes} \tau')(a \otimes a') = (-1)^{\partial a \, \partial a'} \, \tau(a) \otimes \tau'(a') . \tag{3.4}$$

The symbol $\tilde{\otimes}$ has not the same meaning as $\hat{\otimes}$; indeed, when f and f' are homogeneous endomorphisms of A and A', then

$$(f \hat{\otimes} f')(a \otimes a') = (-1)^{\partial a \, \partial f'} f(a) \otimes f'(a') .$$

Whereas $\tau \otimes \tau'$ is the direct sum of the four components $\tau_p \otimes \tau'_q$ (with $p,\, q \in \mathbb{Z}/2\mathbb{Z}$), in $\tau \tilde{\otimes} \tau'$ the component $\tau_1 \otimes \tau'_1$ has been replaced with $-\tau_1 \otimes \tau'_1$; to compensate this modification, it suffices to replace the element i' in the formula (3.3) with an element i such that $i^2 = -1$. In other words, with τ I now associate the element $D(\tau)$ of $\mathbb{C}$ equal to

$$\begin{aligned} &\dim(\ker(\tau_0 - \mathrm{id})) - \dim(\ker(\tau_0 + \mathrm{id})) \\ &\qquad\qquad + i \dim(\ker(\tau_1 - \mathrm{id})) - i \dim(\ker(\tau_1 + \mathrm{id})), \end{aligned} \tag{3.5}$$

so that I can write the equality which solves the problem for $\tau \tilde{\otimes} \tau'$:

$$D(\tau \tilde{\otimes} \tau') = D(\tau)\, D(\tau') . \tag{3.6}$$

At first sight, the information given by $D(\tau)$ is a couple of integers; nevertheless, in the next sections it will only be an integer k modulo 8, because this property shall always hold true:

$$\exists k \in \mathbb{Z}, \qquad D(\tau) = \sqrt{\dim(A)} \exp\left(\frac{ik\pi}{4}\right) . \tag{3.7}$$

When this property is true, then

$$\dim(\ker(\tau_0 - \mathrm{id})) = \frac{1}{2} \dim(A_0) + \frac{1}{2} \sqrt{\dim(A)} \cos\left(\frac{k\pi}{4}\right) , \tag{3.8}$$

$$\dim(\ker(\tau_0 + \mathrm{id})) = \frac{1}{2} \dim(A_0) - \frac{1}{2} \sqrt{\dim(A)} \cos\left(\frac{k\pi}{4}\right) , \tag{3.9}$$

$$\dim(\ker(\tau_1 - \mathrm{id})) = \frac{1}{2} \dim(A_1) + \frac{1}{2} \sqrt{\dim(A)} \sin\left(\frac{k\pi}{4}\right) , \tag{3.10}$$

$$\dim(\ker(\tau_1 + \mathrm{id})) = \frac{1}{2} \dim(A_1) - \frac{1}{2} \sqrt{\dim(A)} \sin\left(\frac{k\pi}{4}\right) . \tag{3.11}$$

The property (3.7) suggests to divide $D(\tau)$ by the square root of $\dim(A)$. The quotient $D(\tau)/\sqrt{\dim(A)}$ is called the *class of the involutive endomorphism* τ, and denoted by $\mathrm{cl}(\tau)$. In [**2**] and [**3**], it was called the *complex divided trace* of τ; this name is quite suggestive when K is a field of characteristic 0, but it may be misleading when K is a field of characteristic ≥ 3, because the words "complex trace" suggest an element of an extension of K. Unless a better name is proposed, I will say "the class of τ", although this terminology lets the word "class" become ambivalent: it means either a family of involutive endomorphisms that have a common property, or the complex number that serves to describe this common property.

4 - The involutions under consideration

Let A be a graded algebra over K. An involutive K-linear mapping $\tau : A \to A$ is said to be an involution of A if $\tau(a)\,\tau(b) = \tau(ba)$ for all $a,\ b \in A$ (whence $\tau(1) = 1$), and $\tau(A_p) = A_p$ for $p = 0, 1$. In other words, the involutions of A are the graded involutive anti-automorphisms of A.

The proof of the next theorem is very easy.

Theorem 4.1. *When τ and τ' are involutions of the graded algebras A and A', there is a unique involution τ'' on $A \hat{\otimes} A'$ that extends τ and τ', and it is equal to $\tau \tilde{\otimes} \tau'$ (defined in* (3.4)*).*

Consequently, the class $\mathrm{cl}(\tau)$, defined at the end of Section 3, becomes very useful because of this multiplicative property:

$$\mathrm{cl}(\tau \tilde{\otimes} \tau') = \mathrm{cl}(\tau)\,\mathrm{cl}(\tau')\,. \tag{4.1}$$

Let us calculate some classes. The quadratic extension $A = K \oplus K\eta$, generated by an odd element η such that η^2 is an invertible element of K, has two involutions: the identity mapping, and the grade automorphism; their classes are respectively $\exp(i\pi/4)$ and $\exp(-i\pi/4)$. When $\eta^2 = 1$, then $A \cong (K^2)^g$. Now let us consider $\mathrm{Mat}(n, n'; K)$ and the transposition of matrices which is an involution T of $\mathrm{Mat}(n, n'; K)$; the dimensions of the kernels of $(T_0 - \mathrm{id})$ and $(T_0 + \mathrm{id})$ are $\frac{1}{2}n(n+1) + \frac{1}{2}n'(n'+1)$ and $\frac{1}{2}n(n-1) + \frac{1}{2}n'(n'-1)$; the kernels of $(T_1 - \mathrm{id})$ and $(T_1 + \mathrm{id})$ have the same dimension nn'; the resulting equality $\mathrm{cl}(T) = 1$ deserves to be emphasized in the next lemma.

Lemma 4.2. *The class of the transposition of matrices in* $\mathrm{Mat}(n, n'; K)$ *is* 1.

Corollary 4.3. *Let M be a graded vector space, $\mathcal{B}(M)$ the graded space of all bilinear forms $\beta : M \times M \to K$, and $\theta^\dagger$ the graded involutive endormorphism of $\mathcal{B}(M)$ defined by $\theta^\dagger(\beta)(x,y) = \beta(y,x)$. The class of $\theta^\dagger$ is* 1.

The next theorem is already proved in [3], Section 6.8. But below I propose a different proof which, in spite of being a little longer, has two advantages: it can be given immediately, unlike the other proof which needs Theorem 8.4 and must be postponed into a later step; secondly, it shall soon help us in Section 6.

Theorem 4.4. *The class of an involution τ of a graded central simple algebra A is an eighth root of* 1 *in* $\mathbb{C}$. *When A has the even type, then* $\mathrm{cl}(\tau)^4 = 1$, *and* $\mathrm{cl}(\tau)^2$ *is equal to* 1 *or to* -1 *according as τ operates trivially or not on* $\mathrm{Z}(A_0)$. *When A has the odd type, then* $\mathrm{cl}(\tau)^4 = -1$, *and* $\mathrm{cl}(\tau)^2$ *is equal to i or to $-i$ according as τ operates trivially or not on* $\mathrm{Z}(A)$.

Proof. Since an extension of the field K does not affect $\mathrm{cl}(\tau)$, it follows from Theorem 2.3 that it suffices to prove Theorem 4.4 when A is equal either to $\mathrm{Mat}(n, n'; K)$ or to $(\mathrm{Mat}(n,K)^2)^g$.

First we suppose $A = \mathrm{Mat}(n, n'; K)$. Let us compare τ with the transposition T of matrices. Because of the Skolem-Noether theorem, there is an invertible matrix $c \in A$ such that $\tau(z) = c\,T(z)\,c^{-1}$ for all $z \in A$. Since τ is involutive, $T(c)\,c^{-1}$ belongs to the center of A which is K, and it rapidly follows that $T(c) = \pm\, c$. Since τ is graded, it operates on $\mathrm{Z}(A_0)$, where its operation is either trivial or not; in the first case, it easily follows that c is even; in the second case (which may occur only if $n = n'$), c is odd. Thus we meet 4 cases, according as c is symmetric or skew symmetric, and according as c is even or odd. When c is symmetric, $\ker(\tau - \mathrm{id})$ (resp. $\ker(\tau + \mathrm{id})$) is the subspace of all yc^{-1} such that $T(y) = y$ (resp. $T(y) = -y$). When c is skew-symmetric, the condition $T(y) = y$ must be replaced with $T(y) = -y$, and conversely. According to the parity of c, y has the same parity as yc^{-1} or the other parity. When c is even, then $\mathrm{cl}(\tau) = \pm\,1$ (after a calculation similar to the proof of Lemma 4.1), and when c is odd, then $\mathrm{cl}(\tau) = \pm\, i$.

Now we suppose that $A = (\mathrm{Mat}(n,K)^2)^g$. The operation of τ in $\mathrm{Z}(A)$ may be trivial or not. If it is trivial, there is matrix $c \in \mathrm{Mat}(n,K)$ such that $\tau(x,0) = (cT(x)c^{-1},\ 0)$ for all $x \in \mathrm{Mat}(n,K)$. Since τ is graded, the equality $\tau(0,y) = (0,\ cT(y)c^{-1})$ also holds true. Since τ is involutive, c must be symmetric or skew symmetric, and after similar calculations it follows that $\mathrm{cl}(\tau) = \pm\,(1+i)/\sqrt{2}$. When the action of τ in $\mathrm{Z}(A)$ is not trivial, a similar argument shows that $\tau(x,y) = (cT(y)c^{-1},\ cT(x)c^{-1})$ for some matrix c that is symmetric or skew symmetric, whence $\mathrm{cl}(\tau) = \pm\,(1-i)/\sqrt{2}$. □

With every involution τ on a graded algebra A is associated another involution $\sigma\tau$ which operates as τ (resp. $-\tau$) on A_0 (resp. A_1); the classes of τ and $\sigma\tau$ are complex conjugate; when A is graded central simple, it follows from Theorem 4.4 that $\mathrm{cl}(\sigma\tau) = \mathrm{cl}(\tau)^{-1}$.

Let R_8 be the group of eighth roots of 1 in $\mathbb{C}$. Now a class in $\mathrm{BW}(K) \times R_8$ is associated with every couple (A, τ) where τ is an involution of a graded central simple algebra A. Since the type of the algebra A is given both by the class of A and the class of τ, it is false that every element of $\mathrm{BW}(K) \times R_8$ is the class of some (A, τ), but it is easy to prove that this assertion is true for every element of $\mathrm{BW}(K) \times_2 R_8$ which is the subgroup of all $(w, \gamma) \in \mathrm{BW}(K) \times R_8$ such that the type of w (even or odd) agrees with the sign of $\gamma^4 = \pm 1$.

Clifford algebras were a strong motivation for the study of graded central simple algebras. If q is a nondegenerate quadratic form on a vector space E, the Clifford algebra $\mathrm{C}\ell(E, q)$ (that is the algebra generated by all $x \in E$ with the relations $x^2 = q(x)$) is a graded central simple algebra. The reversion ρ is the involution on $\mathrm{C}\ell(E, q)$ that induces the identity mapping on E, and the cliffordian conjugation $\rho\sigma$ is the involution that induces $-\,\mathrm{id}$ on E.

Theorem 4.5. *If q is a nondegenerate quadratic form on a space E of dimension n, the classes of the reversion and conjugation in $\mathrm{C}\ell(E, q)$ are*

$$\mathrm{cl}(\rho) = \exp\Big(\frac{in\pi}{4}\Big), \qquad \mathrm{cl}(\rho\sigma) = \exp\Big(\frac{-in\pi}{4}\Big).$$

Proof. If E is the orthogonal direct sum of two subspaces E' and E'', and if q' and q'' are the restrictions of q to E' and E'', there is a canonical isomorphism from $\mathrm{C}\ell(E, q)$ onto $\mathrm{C}\ell(E', q') \hat{\otimes} \mathrm{C}\ell(E'', q'')$ that extends the linear mapping $x' + x'' \longmapsto x' \otimes 1 + 1 \otimes x''$ (where x' runs through E' and x'' through E''). Because of (4.1), Theorem 4.5 is true for (E, q) if it is true for (E', q') and (E'', q''). By means of an orthogonal basis of E, the problem is reduced to the trivial case of a space E of dimension 1. □

Remark. The case of a field K of characteristic 2 is treated in [3]. Over such a field, there are only graded central simple algebras of the even type, and their involutions are classified by the group R_2 of square roots of 1: if τ is an involution of a graded central simple algebra A, its class is 1 or -1 according as τ operates on $\mathrm{Z}(A_0)$ in a trivial way or not; therefore, this class gives the same information as the square of the class mentioned in Theorem 4.4. More generally, let K be a commutative, associative and unital ring, $\mathrm{Spec}(K)$ the set of its prime ideals, and $\mathrm{Spec}_2(K)$ the open subset of prime ideals that do not contain the image of 2 in K; let A be a graded

Azumaya algebra over K; this means that A is a finitely generated projective module, and that $A/\mathfrak{m}A$ is a graded central simple algebra over $K/\mathfrak{m}$ for every maximal ideal $\mathfrak{m}$ of K; when $\mathrm{Spec}_2(K) \neq \emptyset$, $\mathrm{cl}(A)$ is a continuous (locally constant) mapping $\mathrm{Spec}_2(K) \to R_8$; and $\mathrm{cl}(A)^2$ is a continuous mapping $\mathrm{Spec}(K) \to R_4$ which can only take the values ± 1 outside $\mathrm{Spec}_2(K)$.

5 - Positive involutions

In Section 5, K is the field $\mathbb{R}$ of real numbers. Every nondegenerate quadratic form, or nondegenerate symmetric bilinear form, has a *type* (m, n) which gives the maximal dimension m (resp. n) of the subspaces on which it is positive definite (resp. negative definite); its *signature* is $m - n$.

Wall proved in [**6**] that $\mathrm{BW}(\mathbb{R})$ was a cyclic group of order 8; his argument was generalized in [**1**]. Here I propose a quite different argument in three steps. Firstly, it is easy to find 8 different graded central division algebras over $\mathbb{R}$. The algebras $\mathbb{R}$ and $\mathbb{H}$ give the ordinary Brauer group $\mathrm{B}(\mathbb{R})$. Then in dimension 2 we find $(\mathbb{R}^2)^g$ (defined in Section 2) and the algebra $\mathbb{C}^g$ generated over $\mathbb{R}$ by an *odd* element i such that $i^2 = -1$. In dimension 4 we find $\mathbb{H}^g$ and $\mathbb{H}'^g$; the former is generated by an *even* i and an *odd* j such that $i^2 = j^2 = -1$ and $ji = -ij$; the latter is generated by an *even* i' and an *odd* j' such that $-i'^2 = j'^2 = 1$ and $j'i' = -i'j'$. In dimension 8 we find $(\mathbb{H}^2)^g$ and $\mathbb{H} \otimes \mathbb{C}^g$; both contain two *even* elements i' and j', or i and j, generating a subalgebra isomorphic to $\mathbb{H}$; the former (resp. the latter) still contains a *central odd* element k' (resp. k) such that $k'^2 = 1$ (resp. $k^2 = -1$). These descriptions show that $\mathbb{C}^g \cong ((\mathbb{R}^2)^g)^{to}$, $\mathbb{H}'^g \cong (\mathbb{H}^g)^{to}$ and $\mathbb{H} \otimes \mathbb{C}^g \cong ((\mathbb{H}^2)^g)^{to}$. Secondly, since $\mathbb{R}$, $\mathbb{C}$ and $\mathbb{H}$ are the only division algebras over $\mathbb{R}$, it is easy to deduce from Theorem 2.3 that we have found all graded division algebras over $\mathbb{R}$ (up to isomorphy). Thirdly, the multiplication in the group $\mathrm{BW}(\mathbb{R})$ shall be revealed by Theorems 5.1, 5.2 and 5.4 which I am now going to present.

Every graded central simple algebra A over $\mathbb{R}$ is provided with a linear form Scal that satisfies the properties mentioned in Corollary 2.5, and the property $\mathrm{Scal}(1) = 1$. An involution τ on A is said to be a *positive involution* if the symmetric bilinear form $(a, b) \longmapsto \mathrm{Scal}(a\tau(b))$ is positive definite on A. If τ is a positive involution, then $\varphi\tau\varphi^{-1}$ is a positive involution for every graded automorphism or anti-automorphism φ of A.

Theorem 5.1. *Every graded central division algebra over $\mathbb{R}$ admits a unique positive involution. The classes of the positive involutions of the algebras*

$$\mathbb{R}, \ (\mathbb{R}^2)^g, \ \mathbb{H}'^g, \ \mathbb{H} \otimes \mathbb{C}^g, \ \mathbb{H}, \ (\mathbb{H}^2)^g, \ \mathbb{H}^g, \ \mathbb{C}^g$$

are respectively equal to $\exp(ik\pi/4)$ *with* $k = 0, 1, 2, 3, 4, 5, 6, 7.$

Proof. If τ is a positive involution of A, then $\sigma\tau$ is a positive involution on A^{to}. If C is a central simple algebra (without gradation), then every positive involution τ of C gives a positive involution of $(C^2)^g$, which is $(c,d) \longmapsto (\tau(c),\tau(d))$, because the scalar part of each (c,d) is $\mathrm{Scal}(c+d)/2$. These evident statements allow us to find a positive involution on each of the 8 algebras listed in Theorem 5.1. Then it is easy to verify the uniqueness of each positive involution and to calculate its class. □

Theorem 5.2. *If τ and τ' are positive involutions respectively on A and A', then $\tau \tilde{\otimes} \tau'$ is a positive involution on $A \hat{\otimes} A'$.*

Proof. Let β be the bilinear form on A defined by $\beta(a,b) = \mathrm{Scal}(a\tau(b))$; similarly β' (resp. β'') is the bilinear form on A' (resp. $A \hat{\otimes} A'$) involving τ' (resp. $\tau'' = \tau \tilde{\otimes} \tau'$). Since a tensor product of positive definite symmetric bilinear forms is positive definite, it suffices to prove that $\beta'' = \beta \otimes \beta'$. It is easy to verify that the linear form $\mathrm{Scal}'' : A \hat{\otimes} A' \to K$ is the tensor product of the linear forms $\mathrm{Scal} : A \to K$ and $\mathrm{Scal}' : A' \to K$. A straightforward calculation shows that (for all $a,\ b \in A$ and all $a',\ b' \in A'$)

$$\beta''(a \otimes a',\ b \otimes b') \;=\; (-1)^{\partial b(\partial a' + \partial b')}\, \beta(a,b)\, \beta'(a',b')\,.$$

The twisting exponent $\partial b(\partial a' + \partial b')$ has no effect; this is clear if $\partial a' + \partial b' = 0$; and when $\partial a' + \partial b' = 1$, the equality $\beta'(a',b') = 0$ prevents it from having any effect. Therefore $\beta'' = \beta \otimes \beta'$. □

Lemma 5.3. *If a graded central simple algebra A admits several positive involutions, their classes in R_8 are all equal.*

Proof. Let τ be a positive involution, and let us set $\beta(a,b) = \mathrm{Scal}(a\tau(b))$ as previously. It is clear that A_0 and A_1 are orthogonal for the symmetric bilinear form χ defined by $\chi(a,b) = \mathrm{Scal}(ab)$. Also $\ker(\tau - \mathrm{id})$ and $\ker(\tau + \mathrm{id})$ are orthogonal for χ; indeed if $\tau(a) = a$ and $\tau(b) = -b$, then

$$\chi(a,b) = \mathrm{Scal}(ab) = \mathrm{Scal}(\tau(ab)) = \mathrm{Scal}(\tau(b)\tau(a)) = -\mathrm{Scal}(ba) = -\chi(a,b)\,.$$

The restriction of χ to $\ker(\tau - \mathrm{id})$ (resp. $\ker(\tau + \mathrm{id})$) is equal to the restriction of β (resp. $-\beta$). Since β is positive definite, the signatures of the restrictions χ_0 and χ_1 of χ to A_0 and A_1 are

$$\begin{aligned} \mathrm{sgn}(\chi_0) &= \dim(\ker(\tau_0 - \mathrm{id})) - \dim(\ker(\tau_0 + \mathrm{id}))\,, \\ \mathrm{sgn}(\chi_1) &= \dim(\ker(\tau_1 - \mathrm{id})) - \dim(\ker(\tau_1 + \mathrm{id}))\,. \end{aligned}$$

Consequently $\mathrm{cl}(\tau)$ is equal to $\mathrm{sgn}(\chi_0) + i\,\mathrm{sgn}(\chi_1)$ divided by $\sqrt{\dim(A)}$. This equality holds for all positive involutions on A. □

Theorem 5.4. *Every graded central simple algebra A admits positive involutions, and their class in R_8 only depends on the class of A in* BW($\mathbb{R}$).

Proof. The transposition T of matrices in $\mathrm{Mat}(n, n'; \mathbb{R})$ is a positive involution, because for every $c \in \mathrm{Mat}(n, n'; \mathbb{R})$ the trace of $c\,T(c)$ is the sum of the squares of all entries of c. Because of Theorem 2.6, we consider a graded central division algebra B, and the algebra $A = \mathrm{Mat}(n, n'; B)$ or $A = \mathrm{Mat}(n, B)$ according as B is trivially graded or not. Since A is isomorphic to $B \otimes \mathrm{Mat}(n, n'; \mathbb{R})$ or to $B \otimes \mathrm{Mat}(n, \mathbb{R})$, Theorem 5.2 implies that A admits positive involutions: it suffices to extend the positive involution on B to a positive involution on A by means of the transposition T on $\mathrm{Mat}(n, n'; \mathbb{R})$ or $\mathrm{Mat}(n, \mathbb{R})$. Since $\mathrm{cl}(T) = 1$, the positive involutions on A have the same class as the positive involution on B. The conclusion follows. □

The next theorem is an obvious consequence of the previous ones; it gives the multiplication in the group BW($\mathbb{R}$) when BW($\mathbb{R}$) is treated as the set of the eight algebras listed in Theorem 5.1.

Theorem 5.5. *By associating with every graded central simple algebra the class of its positive involutions we obtain a canonical isomorphism* $\mathrm{BW}(\mathbb{R}) \to R_8$.

As an application of Theorem 5.5, let us prove that

$$\mathbb{H}'^g \,\hat{\otimes}\, (\mathbb{H}^2)^g \;\cong\; \mathrm{Mat}(4, \mathbb{C}^g)\,; \tag{5.1}$$

the left hand member of (5.1) is an algebra of dimension $4 \times 8 = 32$ in which the class of the positive involutions is $\exp(2i\pi/4) \times \exp(5i\pi/4) = \exp(7i\pi/4)$; therefore, it is isomorphic to $\mathrm{Mat}(n, \mathbb{C}^g)$ if $n = \sqrt{32/2} = 4$.

Theorem 5.5 also proves that the couples (A, τ) (where τ is an involution on a real graded central simple algebra A) are classified by the group $R_8 \times_2 R_8$ (subgroup of all $(w, \gamma) \in R_8 \times R_8$ such that $w^4 = \gamma^4$). This is a group of order 32, which in particular classifies the real Clifford algebras.

Theorem 5.6. *Let E be a real vector space provided with a quadratic form q of type (m, n). In the Clifford algebra $\mathrm{C}\ell(E, q)$, the class of the positive involutions and the class of the reversion are respectively*

$$\exp\left(\frac{i(m-n)\pi}{4}\right) \quad \textit{and} \quad \exp\left(\frac{i(m+n)\pi}{4}\right).$$

Proof. As in the proof of Theorem 4.5, it suffices to verify these equalities when $\dim(E) = 1$. This is quite easy since $\mathrm{C}\ell(E, q) \cong (\mathbb{R}^2)^g$ for the type $(1, 0)$, and $\mathrm{C}\ell(E, q) \cong \mathbb{C}^g$ for the type $(0, 1)$. □

Thus the examination of quadratic spaces of dimension 1 suffices to reveal the 32 classes of real Clifford algebras. The class of the positive involution gives the wanted information by means of Theorems 5.5, 5.1 and 2.6, while the information given by the class of the reversion shall be explained later in Sections 8 and 12.

6 - Comparison with Wall's group GB$(K, 1)$

The starting problem of all my research in this domain was the problem that I recalled in Section 3: a calculation of dimensions of eigenspaces. To solve it, I associated with every graded involutive endomorphism a class in $\mathbb{C}$, which proved to belong to R_8 in all cases under consideration. At that time I did not know the existence of [**7**] which was seldom referred to. When later I discovered it, I was anxious to know whether my classification of involutions was already present in it. The answer to this question is rather no than yes.

In [**7**], after recalling the group GB(K) (which here I denote by BW(K)), Wall presented the group GB(K,j) associated with a field K of characteristic $\neq 2$ and an involution j of K; it classifies the couples (A, J) where A is a graded central simple algebra over K, and J an involution of A that induces j on K. Since here I accept only K-linear involutions, I am concerned only with GB$(K, 1)$, where 1 means the identity mapping of K. Let me recall Wall's construction of GB$(K, 1)$. When P was a graded vector space, and τ' an involution of End(P), Wall decided that the class of $(\text{End}(P), \tau')$ was trivial if there was a nondegenerate even symmetric bilinear form $\beta : P \times P \to K$ such that $\beta(f(x), y) = \beta(x, \tau'(f)(y))$ for all $f \in \text{End}(P)$ and all $x, y \in P$. Then a couple (A, τ) was said to be trivial if it was isomorphic to a trivial couple $(\text{End}(P), \tau')$. This means that (A, τ) is trivial if (and only if) A is isomorphic to some algebra Mat$(n, n'; K)$, and if τ corresponds by this isomorphism to an involution $z \longmapsto cT(z)c^{-1}$, where T is the transposition of matrices, and c is an invertible even symmetric matrix. The argument that settles the first part of the proof of Theorem 4.4, shows that this condition is equivalent to $\text{cl}(\tau) = 1$. Since a couple (A, τ) is trivial according to Wall's definition if and only if its class in BW$(K) \times_2 R_8$ is trivial, there is a canonical isomorphism GB$(K, 1) \to$ BW$(K) \times_2 R_8$.

From Wall's definition of GB$(K, 1)$, it followed that every field extension $K \to L$ determined a group morphism GB$(K, 1) \to$ GB$(L, 1)$. And Wall mentioned two particular groups GB$(K, 1)$ in [**7**]:

$$\text{(6.1)} \qquad \text{GB}(\mathbb{C}, 1) \cong \mathbb{Z}/8\mathbb{Z}, \qquad \text{GB}(\mathbb{R}, 1) \cong (\mathbb{Z}/8\mathbb{Z}) \oplus (\mathbb{Z}/4\mathbb{Z}).$$

Wall never classified involutions, he only classified couples (A, τ). He could have classified τ by means of the algebraic closure K' of K, by saying that the class of τ might be the image of the class of (A, τ) by the morphism $\mathrm{GB}(K,1) \to \mathrm{GB}(K',1)$; thus the class of τ would belong to $\mathrm{GB}(K',1)$ which is canonically isomorphic to $\mathrm{GB}(\mathbb{C},1)$. Nevertheless, he never suggested such an idea, and after mentioning $\mathrm{GB}(\mathbb{R},1)$, he did not mention the canonical isomorphism $\mathrm{GB}(\mathbb{R},1) \to \mathrm{BW}(\mathbb{R}) \times_2 \mathrm{GB}(\mathbb{C},1)$.

Wall defined positive involutions over $\mathbb{R}$, and despite the scarcity of his explanation, it is certain that he knew the statements which here are Theorems 5.2 and 5.4. He defined a group $\mathrm{GB}^c(\mathbb{R},1)$ classifying the couples (A,τ) with a positive involution, and he discovered a canonical isomorphism $\mathrm{GB}^c(\mathbb{R},1) \to \mathrm{GB}(\mathbb{C},1)$. This isomorphism does not give the same information as Theorem 5.5; in particular, it does not help us in the calculation (5.1) here above.

After the advances achieved in [**6**] and [**7**], the concept of "class of involution" (also called "complex divided trace") brings new simplifications of arguments and new reductions of calculations. Besides, the definition of $\mathrm{cl}(\tau)$ does not require A to be an algebra, and the formulas (3.8), ..., (3.11) only require that $\mathrm{cl}(\tau) \in R_8$; this feature is especially noticeable in the main theorems 8.3, 8.4, 12.3, 12.4 which are devoted to the class of some θ which is just an involutive transformation of a graded vector space.

The references [**8**] and [**4**] are also important for involutions of algebras, but their concern is still farther than [**7**] from the subject treated here.

7 - Graded centralizers in bimodules

Every module M over an algebra A is a left module unless it is said to be a right module. If A and M are graded, M is a graded module if the mapping $A \times M \to M$ is even. When P is a left module over A and a right module over B, it is called a bimodule over A and B (or just over A in case $A = B$) if the equality $(az)b = a(zb)$ holds for all $a \in A$, $b \in B$ and $z \in P$. The definition of a graded bimodule follows. A graded bimodule over A and B is also a module over $A \hat{\otimes} B^{to}$ if we set $(a \otimes b^{to})\, z = (-1)^{\partial b\, \partial z} azb$. Conversely, a module over a tensor product $A \hat{\otimes} B$ is often treated as a bimodule over A and B^{to}.

If P is a bimodule over A, the centralizer $\mathrm{Z}(A,P)$ is the subspace of all $z \in P$ such that $az = za$ for all $a \in A$. When P is a graded bimodule, $\mathrm{Z}(A,P)$ is a graded subspace of P, and often we prefer the graded centralizer $\mathrm{Z}^g(A,P)$ such that $\mathrm{Z}^g_0(A,P) = \mathrm{Z}_0(A,P)$, and $\mathrm{Z}^g_1(A,P)$ is the subspace of all $z \in P_1$ such that $az = z\,\sigma(a)$ for all $a \in A$.

When M is a module over A, the algebra morphism $A \to \mathrm{End}(M)$ makes $\mathrm{End}(M)$ become a bimodule over A, and $\mathrm{Z}(A, \mathrm{End}(M))$ is a subalgebra denoted by $\mathrm{End}_A(M)$. When M is a graded module over a graded algebra A, we obtain the subalgebra $\mathrm{Z}^g(A, \mathrm{End}(M)) = \mathrm{End}^g_A(M)$ which is characterized by the property of *twisted A-*

linearity: $f(ax) = (-1)^{\partial a\, \partial f} a f(x)$ (for all homogeneous $a \in A$, $f \in \mathrm{End}^g_A(M)$ and $x \in M$). Its homogeneous components are denoted by $\mathrm{End}^g_{A,p}(M)$ with $p = 0, 1$.

When M is a module over an algebra A provided with an involution τ, the space $\mathcal{B}(M)$ of all bilinear forms $\beta : M \times M \to K$ is a bimodule over A: the bilinear form $a\beta b$ (with $a,\ b \in A$) is defined by $(a\beta b)(x, y) = \beta(bx,\ \tau(a)y)$. The resulting centralizer is denoted by $\mathrm{Z}(A, \mathcal{B}(M); \tau)$. The space $\mathcal{B}(M)$ is a right module over $\mathrm{End}(M)$ when βf is defined by $(\beta f)(x, y) = \beta(f(x), y)$, and it is easy to verify that this definition lets $\mathrm{Z}(A, \mathcal{B}(M); \tau)$ become a right module over $\mathrm{End}_A(M)$. When parity gradations are taken into account, we must write $(a\beta b)(x, y) = (-1)^{\partial a\, \partial\beta} \beta(bx,\ \tau(a)y)$. Thus we obtain $\mathrm{Z}^g(A, \mathcal{B}(M); \tau)$ which is a graded right module over $\mathrm{End}^g_A(M)$. The following equalities (where the symbol ∂ silently requires the following element to be homogeneous) are immediate consequences of the definitions:

$$\forall \beta \in \mathrm{Z}^g(A, \mathcal{B}(M); \tau),\ \forall a \in A, \quad \beta(ax, y) \;=\; (-1)^{\partial a\, \partial \beta} \beta(x,\ \tau(a)y)\,; \tag{7.1}$$

$$\mathrm{Z}^g_p(A, \mathcal{B}(M); \tau) = \mathrm{Z}_p(A, \mathcal{B}(M);\ \sigma^p \tau) \quad \text{for } p = 0, 1\,. \tag{7.2}$$

When $\mathrm{Z}^g(A, \mathcal{B}(M); \tau)$ contains a nondegenerate homogeneous bilinear form β_n that is symmetric or skew symmetric, this bilinear form induces an involution τ_E on $\mathrm{End}^g_A(M)$ that is characterized by this equality:

$$\forall f \in \mathrm{End}^g_A(M), \quad \beta_n(f(x), y) \;=\; (-1)^{\partial f\, \partial \beta_n} \beta_n(x,\ \tau_E(f)(y))\,. \tag{7.3}$$

It follows from (7.1) that there is a graded involutive endomorphism θ of $\mathrm{Z}^g(A, \mathcal{B}(M); \tau)$ that is defined by the equality $\theta(\beta)(x, y) = \beta(y, x)$. When A is graded central simple, $\mathrm{cl}(\tau_E)$ and $\mathrm{cl}(\theta)$ shall prove to belong to R_8 like $\mathrm{cl}(\tau)$, and the equality $\mathrm{cl}(\theta) = \mathrm{cl}(\tau)^{-1}$ is the main purpose of Section 8.

On one side, if P is a graded bimodule over A, the space $\mathrm{L}(P, A)$ of all linear mappings $F : P \to A$ becomes a graded bimodule over $A \hat{\otimes} A$ if we set (for all $a,\ a',\ b,\ b' \in A$ and for all $z \in P$)

$$\big((a \otimes a')\, F\, (b \otimes b')\big)\,(z) \;=\; (-1)^{\partial a'(\partial F + \partial b + \partial z)}\, (-1)^{\partial b' \partial z}\, a\, F(bza')\, b'\,;$$

the twisting exponent $\partial a'(\partial F + \partial b + \partial z)$ comes from the letter a' which jumps above the letters F, b, z, while $\partial b' \partial z$ comes from b' which jumps over z. On the other side, if M is a graded module over A, then $M \otimes M$ is a graded bimodule over A if we set (for all $a,\ b \in A$ and all $x,\ y \in M$)

$$a\,(x \otimes y)\, b \;=\; ax \otimes \tau(b) y\,. \tag{7.4}$$

Therefore, the space $\mathrm{L}^2(M, A)$ of all linear mappings $F : M \otimes M \to A$ is a graded bimodule over $A \hat{\otimes} A$ if we set

$$\big((a \otimes a') F (b \otimes b')\big)(x \otimes y) = (-1)^{\partial a'(\partial F + \partial b + \partial x + \partial y)} (-1)^{\partial b'(\partial x + \partial y)} a F(bx \otimes \tau(a') y) b'.$$

This allows to define the graded centralizer $\mathrm{Z}^g(A\hat{\otimes}A,\ \mathrm{L}^2(M,A);\tau)$; its homogeneous elements F are characterized by this property, which means twisted linearity with respect to a, and τ-semi-linearity with respect to b:

$$\forall a,\, b\in A,\quad \forall x,\, y\in M,\quad F(ax\otimes by)\ =\ (-1)^{\partial a\,\partial F}\, a\,F(x\otimes y)\,\tau(b)\,. \tag{7.5}$$

Theorem 7.1. *Let us suppose that there is a linear form* Scal : $A\to K$ *that vanishes on* A_1 *and that gives a* nondegenerate symmetric *bilinear form* $(a,b)\longmapsto \mathrm{Scal}(ab)$. *If we map every* $F\in \mathrm{L}^2(A,M)$ *to the bilinear form* β *defined by* $\beta(x,y)=\mathrm{Scal}(F(x\otimes y))$, *we obtain by restriction a graded bijection from* $\mathrm{Z}^g(A\hat{\otimes}A,\ \mathrm{L}^2(M,A);\tau)$ *onto* $\mathrm{Z}^g(A,\mathcal{B}(M);\tau)$.

Proof. The equality $\partial\beta=\partial F$ follows from $\mathrm{Scal}(A_1)=0$. And from the equality $\mathrm{Scal}(ab)=\mathrm{Scal}(ba)$, it follows that β satisfies (7.1) whenever F satisfies (7.5). Besides, for every homogeneous $F\in \mathrm{Z}^g(A\hat{\otimes}A,\ \mathrm{L}^2(M,A);\tau)$ we have:

$$\forall a\in A,\quad \forall x,\, y\in M,\quad \mathrm{Scal}(a\,F(x\otimes y))\ =\ (-1)^{\partial a\,\partial\beta}\,\beta(ax,\,y)\,. \tag{7.6}$$

Since the bilinear form $(a,b)\longmapsto \mathrm{Scal}(ab)$ is nondegenerate, this equality (7.6) allows us to derive a unique element $F\in \mathrm{L}^2(M,A)$ from every $\beta\in\mathcal{B}(M)$, and obviously $\beta(x,y)=\mathrm{Scal}(F(x\otimes y))$. If β satisfies (7.1), it follows form (7.6) that, for all $a,\, b,\, c\in A$, and for all $x,\, y\in M$,

$$\mathrm{Scal}(c\,F(ax\otimes\tau(b)\,y))\ =\ (-1)^{\partial a\partial F}\,\mathrm{Scal}(ca\,F(x\otimes y)\,b)\,;$$

consequently F satisfies (7.5). □

Theorem 7.2. *If the hypothesis of Theorem* 7.1 *is true, and if* $\mathrm{Z}^g(A,\mathcal{B}(M);\tau)$ *contains a homogeneous, nondegenerate, symmetric or skew symmetric element* β_n, *then the image* F_n *of* β_n *in* $\mathrm{Z}^g(A\hat{\otimes}A,\ \mathrm{L}^2(M,A);\tau)$ *satisfies the following property, where* $\pm$ *means* $+$ *or* $-$ *according as* β_n *is symmetric or skew symmetric, and* $\sigma^{\partial\beta_n}\tau$ *means* τ *or* $\sigma\tau$ *according to the parity of* β_n:

$$\forall x,\, y\in M,\quad F_n(y\otimes x)\ =\ \pm\,\sigma^{\partial\beta_n}\tau(F_n(x\otimes y))\,. \tag{7.7}$$

If τ_E *is the involution defined by* (7.3), *then, for every* $f\in \mathrm{End}^g_A(M)$,

$$F_n(f(x)\otimes y)\ =\ (-1)^{\partial f\,\partial\beta_n}\ \sigma^{\partial f}\big(F_n\big(x\otimes\tau_E(f)(y)\big)\big)\,. \tag{7.8}$$

Proof. Let G be the element of $\mathrm{L}^2(M,A)$ such that $G(x\otimes y)=\sigma^{\partial\beta_n}\tau(F_n(y\otimes x))$; it belongs to $\mathrm{Z}^g(A\hat{\otimes}A,\ \mathrm{L}^2(M,A);\tau)$ because

$$\begin{aligned} G(ax\otimes by)\ &=\ (-1)^{\partial b\partial F_n}\,\sigma^{\partial\beta_n}\tau\big(b\,F_n(y\otimes x)\,\tau(a)\big)\\ &=\ (-1)^{\partial b\partial F_n+(\partial a+\partial b)\partial\beta_n}\,a\ \sigma^{\partial\beta_n}\tau\big(F_n(y\otimes x)\big)\ \tau(b)\\ &=\ (-1)^{\partial a\partial G}\,a\,G(x\otimes y)\,\tau(b)\,. \end{aligned}$$

Because of Theorem 7.1, the comparison of F_n and G is reduced to the comparison of $\mathrm{Scal}(F_n(x\otimes y)) = \beta_n(x\otimes y)$ and $\mathrm{Scal}(G(x\otimes y)) = \beta_n(y\otimes x)$.

Now let G_1 and G_2 be the elements of $\mathrm{L}^2(M,A)$ defined by $G_1(x\otimes y) = F_n(f(x)\otimes y)$ and $G_2(x\otimes y) = \sigma^{\partial f}(F_n(x\otimes \tau_E(f)(y)))$. Thus $\partial G_1 = \partial G_2 = \partial\beta_n + \partial f$. A similar calculation shows that G_1 and G_2 belong to $\mathrm{Z}^g(A\hat{\otimes} A,\ \mathrm{L}^2(M,A);\tau)$; to compare them, it suffices to compare $\mathrm{Scal}(G_1(x\otimes y)) = \beta_n(f(x), y)$ and $\mathrm{Scal}(G_2(x\otimes y)) = \beta_n(x,\ \tau_E(f)(y))$. □

If all hypotheses of Theorem 7.2 are true, and if there is also a linear form $\mathrm{Scal}_E : \mathrm{End}^g_A(M)\to K$ with similar properties, then β_n determines a linear mapping $F'_n : M\otimes M \to \mathrm{End}^g_A(M)$, because (7.3) shows that β_n belongs to $\mathrm{Z}^g(\mathrm{End}^g_A(M), \mathcal{B}(M);\tau_E)$. Theorem 7.2 gives the similar properties of F'_n.

We can even treat M as a graded module over $A\hat{\otimes}\,\mathrm{End}^g_A(M)$ provided with the involution $\tau\tilde{\otimes}\tau_E$; thus we get a linear mapping $F''_n : M\otimes M \to A\hat{\otimes}\,\mathrm{End}^g_A(M)$. When A is graded central simple, all hypotheses ensuring the existence of F''_n will be true, and F''_n will prove to be bijective.

Besides, when A is graded central simple, we can also use the next Theorem 7.3, which is interpreted in this way in [**1**]: the functors $P\longmapsto \mathrm{Z}^g(A,P)$ and $V\longmapsto A\otimes V$ determine an equivalence between the category of graded bimodules P over A and the category of graded vector spaces V over K. If V is a graded vector space, then $A\otimes V$ is a graded bimodule over A if we set $a(x\otimes v)b = (-1)^{\partial b\,\partial v}(axb)\otimes v$ (for all $a,\ b,\ x\in A$ and all $v\in V$). It is easy to verify that $\mathrm{Z}^g(A,\ A\otimes V) = \mathrm{Z}^g(A)\otimes V$, whence $\mathrm{Z}^g(A,\ A\otimes V)\cong V$ if A is graded central simple. Conversely, we find this theorem in [**1**].

Theorem 7.3. *If P is a graded bimodule over a graded central simple algebra A, the mapping $A\otimes \mathrm{Z}^g(A,P)\to P$ defined by $a\otimes z\longmapsto az$ is bijective.*

Corollary 7.4. *When a graded central simple algebra A is a graded subalgebra of a graded algebra C, then $\mathrm{Z}^g(A,C)$ is a graded subalgebra of C, and the mapping $A\hat{\otimes}\,\mathrm{Z}^g(A,C)\to C$ (that is $a\otimes z\longmapsto az$) is an algebra isomorphism. If A and C are graded central simple, the same holds for $\mathrm{Z}^g(A,C)$.*

8 - Graded modules

This section is devoted to a graded module M over a graded central simple algebra A provided with an involution τ.

Every graded module over A is semi-simple; therefore it is worth looking at the graded irreducible modules. A graded irreducible module is not always an irreducible module. For instance, if B is a graded central division algebra such that $B_1 \neq 0$, every graded module N over B is free, because every B_0-basis of N_0 is a B-basis of N; therefore every graded irreducible module over B has the same dimension (over K) as B; but if B is not a division algebra when the gradation is forgotten, there are irreducible modules of dimension $\dim(B)/2$; when $K = \mathbb{R}$, this happens when B is $(\mathbb{R}^2)^g$ or $\mathbb{H}'^g$ or $\mathbb{H} \otimes \mathbb{C}^g$ or $(\mathbb{H}^2)^g$.

When B is a graded central division algebra such that $B_1 \neq 0$, all graded irreducible modules over $A = \mathrm{Mat}(n, B)$ are isomorphic to B^n. But when $B_1 = 0$, the graded irreducible modules over $\mathrm{Mat}(n, n'; B)$ constitute two isomorphy classes; there is a graded irreducible module M where $M_0 \cong B^n$ and $M_1 \cong B^{n'}$, but M is not isomorphic to the module M^s with shifted gradation (where $\partial x^s = 1 - \partial x$, see Section 1). Therefore, every graded module over $\mathrm{Mat}(n, n'; B)$ admits a decomposition $M' \oplus M''$ into a direct sum of isotypical components: all graded irreducible submodules of M' are isomorphic to one another, and the same for M''; of course, M' or M'' is reduced to 0 if M is already isotypical. When $nn' = 0$, the decomposition $M' \oplus M''$ coincides with the decomposition $M_0 \oplus M_1$. When $nn' \neq 0$, the center of $\mathrm{Mat}_0(n, n'; B)$ contains an Arf element ω such that $\omega^2 = 1$, and the four subspaces M'_0, M'_1, M''_0 and M''_1 are characterized by their parity and this property: $M'_0 \oplus M''_1$ (resp. $M''_0 \oplus M'_1$) is the subspace of all $x \in M$ such that $\omega x = x$ (resp. $\omega x = -x$).

When M is a graded module over an arbitrary graded central simple algebra A, there is an isotypical decomposition $M = M' \oplus M''$ if and only if the class of A in $\mathrm{BW}(K)$ belongs to the subgroup $\mathrm{B}(K)$.

Theorem 8.1. *When M is a graded module over a graded central simple algebra A, then* $\mathrm{End}^g_A(M)$ (*defined in Section* 7) *is a graded central simple algebra, with Brauer-Wall class and dimension given by these equalities:*

$$\mathrm{cl}(A)\,\mathrm{cl}(\mathrm{End}^g_A(M)) = 1\,, \qquad \dim(A)\,\dim(\mathrm{End}^g_A(M)) \;=\; \big(\dim(M)\big)^2\,.$$

When the class of A in $\mathrm{BW}(K)$ *does not belong to* $\mathrm{B}(K)$, *the gradation of* $\mathrm{End}^g_A(M)$ *is always balanced. When it belongs to* $\mathrm{B}(K)$, *the dimensions of the homogeneous components of* $\mathrm{End}^g_A(M)$ *are given by the following formulas where d and d' are the dimensions of the isotypical components of M:*

$$\dim(\mathrm{End}^g_{A,0}(M)) = \frac{d^2 + d'^2}{\dim(A)}\ , \qquad \dim(\mathrm{End}^g_{A,1}(M)) = \frac{2dd'}{\dim(A)}\ .$$

Proof. Corollary 7.4 implies that $A \hat{\otimes} \mathrm{End}^g_A(M)$ is isomorphic to $\mathrm{End}(M)$, and this fact gives all pieces of information except the dimensions of the homogeneous components of $\mathrm{End}^g_A(M)$. When the class of A does not belong to $\mathrm{B}(K)$, neither does the class of $\mathrm{End}^g_A(M)$ which is its inverse; therefore the gradation of $\mathrm{End}^g_A(M)$ is balanced. Now we assume that the class of A belongs to $\mathrm{B}(K)$; thus A is isomorphic to an algebra $B \otimes \mathrm{End}(V)$ where B is a trivially graded central division algebra, and V a graded vector space over K; let us set $\delta = \dim(B)$, $n = \dim(V_0)$ and $n' = \dim(V_1)$. Every graded module M over $A = B \otimes \mathrm{End}(V)$ is isomorphic to some module $B \otimes V \otimes U$ (where U also is a graded vector space) if the operation of $c \otimes g$ (for all $c \in B$ and $g \in \mathrm{End}(V)$) is defined by $x \otimes v \otimes u \longmapsto cx \otimes g(v) \otimes u$ (for all $x \in B$, $v \in V$ and $u \in U$). If we assume that M is precisely this module, its isotypical components are $M' = B \otimes V \otimes U_0$ and $M'' = B \otimes V \otimes U_1$. If the dimensions of U_0 and U_1 are denoted by m and m', then

$$(8.1)\qquad \begin{aligned} \dim(M'_0) &= \delta n m\,, & \dim(M''_0) &= \delta n' m'\,,\\ \dim(M'_1) &= \delta n' m\,, & \dim(M''_1) &= \delta n m'\,. \end{aligned}$$

Let the algebra $B^o \otimes \mathrm{End}(U)$ act on M in this way: every $b^o \otimes f$ maps every $x \otimes v \otimes u$ to $(-1)^{\partial f \partial v}\, xb \otimes v \otimes f(u)$. The operation of $b^o \otimes f$ commutes or anticommutes with the operation of $c \otimes g$ according to the parity of $\partial f\, \partial g$; therefore $B^o \otimes \mathrm{End}(U) \subset \mathrm{End}^g_A(M)$, and this inclusion is an equality because both algebras have the same dimension. The dimensions of the homogeneous components of $B^o \otimes \mathrm{End}(U)$ are $\delta(m^2 + m'^2)$ and $2\delta m m'$; they agree with the announced values because $\dim(A) = \delta(n+n')^2$, $d = \delta(n+n')m$ and $d' = \delta(n+n')m'$. □

The graded space $\mathrm{Z}^g(A, \mathcal{B}(M); \tau)$ (defined in Section 7) is a graded right module over $\mathrm{End}^g_A(M)$; the next theorem enables us to calculate the dimensions of its homogeneous components.

Theorem 8.2. *The space* $\mathrm{Z}^g(A, \mathcal{B}(M); \tau)$ *has the same dimension as the algebra* $\mathrm{End}^g_A(M)$. *It contains a nondegenerate homogeneous bilinear form* β_n *that is symmetric or skew symmetric. The mapping* $f \longmapsto \beta_n f$ *is a homogeneous linear bijection from* $\mathrm{End}^g_A(M)$ *onto* $\mathrm{Z}^g(A, \mathcal{B}(M); \tau)$*; its parity is the parity of* β_n. *When the class of* A *in* $\mathrm{BW}(K)$ *belongs to* $\mathrm{B}(K)$, *and when the isotypical components of* M *have different dimensions, then* β_n *must be even or odd according as the operation of* τ *in* $\mathrm{Z}(A_0)$ *is trivial or not.*

Proof. Since $\dim(\mathcal{B}(M)) = \dim(\mathrm{End}(A))$, the bijection $A \otimes \mathrm{Z}^g(A, \mathcal{B}(M); \tau) \to \mathcal{B}(M)$ given by Theorem 7.3 proves that $\mathrm{Z}^g(A, \mathcal{B}(M); \tau)$ has the same dimension as $\mathrm{End}^g_A(M)$. To prove the existence of bilinear forms like β_n it suffices to consider a graded irreducible module M, because every other graded module is a direct sum of graded submodules isomorphic to M or (in some cases) to M^s. When M is graded irreducible, every nonzero homogeneous element of $Z^g(A, \mathcal{B}(M); \tau)$ is nondegenerate, because its kernel (on the left or right side) is a graded submodule; therefore β_n may be any nonzero eigenvector of θ in either homogeneous component $Z^g_p(A, \mathcal{B}(M); \tau)$. The existence of β_n is now sure, and the properties of the mapping $f \longmapsto \beta_n f$ are evident. It remains to study the parity of β_n when $\mathrm{cl}(A) \in \mathrm{B}(K)$. Let $M = M' \oplus M''$ be the isotypical decomposition; when $A_1 \neq 0$, there is an Arf element $\omega \in \mathrm{Z}(A_0)$ such that $\omega x = x$ (resp. $\omega x = -x$) for all x in $M'_0 \oplus M''_1$ (resp. $M''_0 \oplus M'_1$); when $A_1 = 0$, then $M''_0 \oplus M'_1 = 0$, and these equalities still hold with $\omega = 1$. Either $\tau(\omega) = \omega$, or $\tau(\omega) = -\omega$, and the second case occurs only when the gradation of A is balanced. In the first case the equality $\beta_n(\omega x, y) = \beta_n(x, \omega y)$ implies that $M'_0 \oplus M''_1$ is orthogonal for β_n to $M''_0 \oplus M'_1$; if β_n is odd, this implies that β_n induces a duality between M'_0 and M''_1, and between M''_0 and M'_1; the formulas (8.1) show that this is impossible if $m \neq m'$; in this case β_n is even. When $\tau(\omega) = -\omega$, the equality $\beta_n(\omega x, y) = -\beta_n(x, \omega y)$ shows that $M'_0 \oplus M''_1$ and $M''_0 \oplus M'_1$ are totally isotropic for β_n; if β_n is even, this implies that β_n induces a duality between M'_0 and M''_0, and between M''_1 and M'_1; again this is impossible if $m \neq m'$; in this case β_n is odd. □

Now we consider the involution θ of $Z^g(A, \mathcal{B}(M); \tau)$ defined by $\theta(\beta)(x, y) = \beta(y, x)$, and the involution τ_E of $\mathrm{End}^g_A(M)$ determined by a choice of β_n according to the definition (7.3).

Theorem 8.3. *When β_n is even, then $\mathrm{cl}(\theta) = \pm\,\mathrm{cl}(\tau_E)$, and when β_n is odd, then $\mathrm{cl}(\theta) = \pm\, i\,\mathrm{cl}(\tau_E)$; in both cases, the sign $\pm$ means $+$ if β_n is symmetric, $-$ if β_n is skew symmetric.*

Proof. It follows from (7.3) that $(\beta_n f)(x, y) = \pm(\beta_n \tau_E(f))(y, x)$ for every homogeneous $f \in \mathrm{End}^g_A(M)$ (and for all x, $y \in M$); the sign $\pm$ depends on the symmetry property of β_n and on $\partial f\, \partial\beta_n$. Therefore the mapping $f \longmapsto \beta_n f$ maps homogeneous eigenvectors of τ_E to homogeneous eigenvectors of θ. According to the symmetry property of β_n and its parity, it is easy to determine onto which eigenspace of θ_0 or θ_1 each of the eigenspaces of $\tau_{E,0}$ and $\tau_{E,1}$ is mapped. The announced conclusions soon follow. □

Theorem 8.4. *The equality $\mathrm{cl}(\tau)\,\mathrm{cl}(\theta) = 1$ holds in R_8.*

Proof. If we treat M as a graded module over the trivially graded algebra $A^\dagger = K$ provided with the involution $\tau^\dagger = \mathrm{id}$, then we meet the graded algebra $\mathrm{End}^g_{A^\dagger}(M) = \mathrm{End}(M)$, the graded space $Z^g(A^\dagger, \mathcal{B}(M); \tau^\dagger) = \mathcal{B}(M)$, and the involutive endomorphism $\theta^\dagger$ of $\mathcal{B}(M)$; moreover, β_n determines also an involution $\tau^\dagger_E$ of $\mathrm{End}(M)$.

The formulas (7.1) and (7.3) show that $\tau \mathbin{\tilde{\otimes}} \tau_E$ corresponds to $\tau_E^{\dagger}$ by the isomorphism $A \mathbin{\hat{\otimes}} \mathrm{End}_A^g(M) \to \mathrm{End}(M)$. This gives the first equality in (8.2), and the second one comes from Corollary 4.2:

$$\text{(8.2)} \qquad \mathrm{cl}(\tau_E^{\dagger}) = \mathrm{cl}(\tau)\,\mathrm{cl}(\tau_E)\ , \qquad \mathrm{cl}(\theta^{\dagger}) = 1\ .$$

According to Theorem 8.3, there is $r \in R_4$ such that

$$\text{(8.3)} \qquad \mathrm{cl}(\theta) = r\,\mathrm{cl}(\tau_E)\,, \qquad \mathrm{cl}(\theta^{\dagger}) = r\,\mathrm{cl}(\tau_E^{\dagger})\ .$$

Theorem 8.4 is a consequence of the four equalities (8.2) and (8.3). □

Remark. Theorem 4.4 is not involved in the proof of Theorem 8.4; on the contrary, Theorem 8.4 may help to prove Theorem 4.4. Indeed, if the Brauer-Wall class of A is trivial, A admits a graded irreducible module M of dimension $\sqrt{\dim(A)}$, whence $\dim(\mathrm{Z}^g(A, \mathcal{B}(M); \tau)) = 1$; this enforces $\mathrm{cl}(\theta)$ to be a fourth root of 1, and the same for $\mathrm{cl}(\tau)$ because of Theorem 8.4. When A is an arbitrary graded central simple algebra, it follows from Theorem 2.2 that the class of the involution $\tau \mathbin{\tilde{\otimes}} \tau$ on $A \mathbin{\hat{\otimes}} A^{to}$ is a fourth root of 1, and consequently $\mathrm{cl}(\tau)$ is an eighth root of 1. The final assertions in Theorem 4.4 follow from the study of the bijection $\mathrm{Z}^g(A, \mathcal{B}(M); \tau) \to \mathrm{Z}^g(A, \mathcal{B}(M); \sigma\tau)$ defined by $\beta \longmapsto \beta\omega$, where ω is an Arf element.

Now let us consider the space $\mathrm{L}^2(M, A)$ of all linear mappings $M \otimes M \to A$. *Every nonzero homogeneous element F of $\mathrm{Z}^g(A \mathbin{\hat{\otimes}} A,\ \mathrm{L}^2(M, A); \tau)$ is surjective onto A*; indeed, the formula (7.5) shows that its image is a graded ideal of A, which must be equal to A. By Theorem 7.1, *every choice of a suitable linear form* $\mathrm{Scal} : A \to K$ *determines a graded bijection from* $\mathrm{Z}^g(A \mathbin{\hat{\otimes}} A,\ \mathrm{L}^2(M, A); \tau)$ *onto* $\mathrm{Z}^g(A, \mathcal{B}(M); \tau)$. Thus every homogeneous, nondegenerate, symmetric or skew symmetric element β_n of $\mathrm{Z}^g(A, \mathcal{B}(M); \tau)$ determines an element F_n of $\mathrm{Z}^g(A \mathbin{\hat{\otimes}} A,\ \mathrm{L}^2(M, A); \tau)$. Since β_n induces an involution τ_E on the graded central simple algebra $\mathrm{End}_A^g(M)$, it still determines similar linear mappings $F'_n : M \otimes M \to \mathrm{End}_A^g(M)$, and $F''_n : M \otimes M \to A \mathbin{\hat{\otimes}} \mathrm{End}_A^g(M)$. *The mapping F''_n is bijective*, because it is a surjective mapping between two spaces of equal dimensions. When $M \otimes M$ is treated as a bimodule over A according to the formula (7.4), F''_n allows us to compare the action of A on $M \otimes M$ to its action on the bimodule $A \mathbin{\hat{\otimes}} \mathrm{End}_A^g(M)$.

There are two conspicuous involutive endomorphisms on $M \otimes M$, the grade automorphism and the mapping $x \otimes y \longmapsto y \otimes x$. They commute with each other and their common eigenspaces constitute a decomposition of $M \otimes M$ into a direct sum of four subspaces. By F''_n these four subspaces of $M \otimes M$ correspond to the four subspaces of $A \mathbin{\hat{\otimes}} \mathrm{End}_A^g(M)$ that are the common eigenspaces of $\sigma \otimes \sigma_E$ and $\tau \mathbin{\tilde{\otimes}} \tau_E$; the formula (7.7) explains how F''_n behaves with respect to the symmetry properties.

9 - An example (first part)

Let $A = \mathrm{C}\ell(E, q)$ be the Clifford algebra generated by a real vector space E provided with a quadratic form q of type $(1, 3)$ (according to the definition at the beginning of Section 5), and M a graded module over A of dimension 8 over $\mathbb{R}$. The class of the positive involutions of A is $\exp(-2i\pi/4) = -i$ (see Theorem 5.6), and $\dim(A) = 16$; therefore $A \cong \mathrm{Mat}(2, \mathbb{H}^g)$, and M is an irreducible module. The involution τ that plays the essential role in Sections 7 and 8 is now the reversion ρ or the conjugation $\rho\sigma$. Theorem 4.4 gives us this piece of information: $\mathrm{cl}(\rho) = \mathrm{cl}(\rho\sigma) = -1$.

A line contained in E is called a time line (resp. a light line) if it is spanned by a vector v such that $q(v) > 0$ (resp. $q(v) = 0$). A subspace of dimension 3 is called a euclidian subspace if q is negative definite on this subspace. As it happens in all spaces provided with a quadratic form of non constant sign, there are two orientations in E: if one time line is oriented, all time lines (and light lines) are oriented by continuity; and if one euclidian subspace is oriented, all euclidian subspaces are oriented by continuity. This double orientation explains why the orthogonal group $\mathcal{L}$ of q (called the Lorentz group) has 4 connected components; let $\mathcal{L}_{nc}$ be the neutral component. The orientation of time is physically relevant, whereas the euclidian orientation is just a mathematical convention. A basis (e_0, e_1, e_2, e_3) of E is said to be a *normal basis* if it is an orthogonal basis, if $q(e_0) = -q(e_1) = -q(e_2) = -q(e_3) = 1$, if e_0 respects the orientation of time (in other words, if it is oriented toward future), and if (e_1, e_2, e_3) respects the euclidian orientation. The group $\mathcal{L}_{nc}$ acts in a simply transitive way on the set of normal bases. If we set $\omega = e_0e_1e_2e_3$, then $\omega e_r = -e_r\omega$ for $r = 0, 1, 2, 3$, and consequently, ω is an Arf element; moreover, $\omega^2 = -1$. All normal bases give the same Arf element ω, and ω determines the global orientation of E.

The spin group $\mathcal{S}$ is the subgroup all homogeneous elements $a \in A$ such that $a\rho(a) = \pm 1$; since $\dim(E) = 4$, this implies $av\rho(a) \in E$ for all $v \in E$, whence the orthogonal transformation g_a defined by $g_a(v) = av\,\sigma(a)^{-1}$. Thus we obtain a surjective morphism $\mathcal{S} \to \mathcal{L}$; its kernel is $\{1, -1\}$. The neutral connected component $\mathcal{S}_{nc}$ (the subgroup of all $a \in A_0$ such that $a\rho(a) = 1$) is a 2-sheet covering group over $\mathcal{L}_{nc}$.

Since the Brauer-Wall class of $\mathrm{End}^g_A(M)$ is $\mathrm{cl}(A)^{-1}$, and since its dimension is 4, we know that $\mathrm{End}^g_A(M) \cong \mathbb{H}'^g$. But to follow the common use, instead of treating M as a module over $A \hat{\otimes} \mathrm{End}^g_A(M)$, we will rather treat it as a bimodule over A and $(\mathrm{End}^g_A(M))^{to} \cong \mathbb{H}^g$. The algebra $\mathbb{H}'^g$ is generated by an even i' and an odd j' such that $-i'^2 = j'^2 = 1$ and $i'j' = -j'i'$, while $\mathbb{H}^g$ (here identified with $(\mathbb{H}'^g)^{to}$) is generated by an even $i = i'^{to}$ and an odd $j = j'^{to}$ such that $i^2 = j^2 = -1$ and $ij = -ji$. Every graded left module N over $\mathbb{H}'^g$ is a graded right module over $\mathbb{H}^g$:

$$(9.1) \qquad \forall z \in N, \qquad zi = i'z \quad \text{and} \quad zj = j'\sigma_N(z).$$

Here it is forbidden to identify i and i'; indeed, when $N = \mathbb{H}'^g$, then (9.1) implies $j'i = i'j'$ although $j'i' = -i'j'$. The even subalgebras of $\mathbb{H}'^g$ and $\mathbb{H}^g$ are denoted by $\mathbb{C}'$ and $\mathbb{C}$; here $\mathbb{C}$ acts on the right side.

These explanations determine the operation of i in M up to a sign $\pm$, but Section 10 shall prove that it may be determined precisely by this property:

$$\text{(9.2)} \qquad \text{for } p = 0, 1, \quad \forall x \in M_p\,, \qquad xi = (-1)^p\,\omega x\,.$$

When physicists are concerned with a spinor space M over this Clifford algebra, usually they do not mention any parity gradation, but they emphasize its complex structure: there is an operator $x \longmapsto xi$ such that $(xi)i = -x$ and $(vx)i = v(xi)$ for all $v \in E$ and $x \in M$. Later, they use (9.2) to define the subspace M_0 of right hand Weyl spinors, and the subspace M_1 of left hand Weyl spinors; since every operator $x \longmapsto vx$ permutes M_0 and M_1 (because $v\omega = -\omega v$ for all $v \in E$), they have turned M into a graded module. When ω (or equivalently, the global orientation of E) has been chosen, the equality (9.2) establishes a bijective correspondence between the complex structures on the module M and the parity gradations of the module M.

The existence of a physically meaningful gradation on M has an important consequence: the quadratic form $-q$ would give the same theory as q. Indeed, the Clifford algebra of $-q$ can be identified with the twisted algebra $C\ell(E,q)^t$ (defined in Section 1), and with every graded module M over $C\ell(E,q)$, a twisted module M^t over $C\ell(E,q)^t$ is associated according to the definition $a^t x^t = (-1)^{\partial a\, \partial x}(ax)^t$. If $-q$ were used instead of q, then $\mathbb{H}^g$ would act on the left side, and $\mathbb{H}'^g$ on the right side. Without its gradation, $C\ell(E,q)^t$ is isomorphic to $\mathrm{Mat}(4,\mathbb{R})$ and admits irreducible modules of dimension 4; this may explain why q is preferred to-day, although $-q$ was preferred formerly.

A gauge group Γ acts on M: it is the group of all $\exp(i\alpha) \in \mathbb{C}$ with $\alpha \in \mathbb{R}$; the observable quantities are not modified when all spinors are multiplied by $\exp(i\alpha)$. Several observable quantities are derived from a spinor $x \in M$, and they depend on x by means of quadratic mappings $M \to P$ that are constant on the orbits of Γ; the target P is a vector space on which $\mathcal{S}$ acts. Consequently, we are interested in symmetric bilinear mappings $\psi : M \times M \to P$ such that $\psi(xi, y) = -\psi(x, yi)$ for all $x, y \in M$, and we look for a bilinear form β_n in $Z^g(A, \mathcal{B}(M); \tau)$ such that $\beta_n(xi, y) = -\beta_n(x, yi)$; this means that $\tau_E(i') = -i'$, whence $\mathrm{cl}(\tau_E) = \pm i$. Since $\mathrm{cl}(\rho) = \mathrm{cl}(\rho\sigma) = -1$, we deduce from Theorem 8.3 that β_n *must be odd.*

Because of the complex structure of M, there is a tensor product $M \otimes_{\mathbb{C}} M$ (resp. $M \bar{\otimes}_{\mathbb{C}} M$) which is the quotient of $M \otimes M = M \otimes_{\mathbb{R}} M$ by the subspace spanned by all $xi \otimes y - x \otimes yi$ (resp. $xi \otimes y + x \otimes yi$). We can identify $M \otimes_{\mathbb{C}} M$ and $M \bar{\otimes}_{\mathbb{C}} M$ with supplementary subspaces of $M \otimes M$ by setting

$$\text{(9.3)} \qquad x \otimes_{\mathbb{C}} y = \frac{1}{2}(x \otimes y - xi \otimes yi)\,, \qquad x \bar{\otimes}_{\mathbb{C}} y = \frac{1}{2}(x \otimes y + xi \otimes yi)\,.$$

Because of the gauge group Γ, only $M \bar{\otimes}_{\mathbb{C}} M$ shall be studied, and $M \otimes_{\mathbb{C}} M$ will be forgotten as long as there is no evidence of its physical usefulness.

After these unavoidable preliminaries, let us consider the involutive operator θ on $\mathrm{Z}^g(A, \mathcal{B}(M); \rho)$ defined by $\theta(\beta)(x,y) = \beta(y,x)$, and the similar operator θ' on $\mathrm{Z}^g(A, \mathcal{B}(M); \rho\sigma)$. Theorem 8.4 reveals that $\mathrm{cl}(\theta) = \mathrm{cl}(\theta') = -1$. Consequently, all elements of $\mathrm{Z}^g_0(A, \mathcal{B}(M); \rho)$ and $\mathrm{Z}^g_0(A, \mathcal{B}(M); \rho\sigma)$ are skew symmetric, while the kernels of $\theta_1 - \mathrm{id}$, $\theta_1 + \mathrm{id}$, $\theta'_1 - \mathrm{id}$, $\theta'_1 + \mathrm{id}$, all have dimension 1. Since we are looking for quadratic mappings $M \to P$, it is preferable to choose a symmetric β_n; shall it be chosen in $\ker(\theta_1 - \mathrm{id})$ or in $\ker(\theta'_1 - \mathrm{id})$? The choice in $\ker(\theta'_1 - \mathrm{id})$ shall be justified later; henceforth, β_n is a symmetric element in $\mathrm{Z}^g_1(A, \mathcal{B}(M); \rho\sigma) = \mathrm{Z}_1(A, \mathcal{B}(M); \rho)$. From Theorem 8.3 we deduce $\mathrm{cl}(\tau_E) = i$, whence $\tau_E(i') = -i'$ and $\tau_E(j') = j'$.

From this β_n we derive a linear mapping $F_n : M \otimes M \to A$ which satisfies the following properties, consequences of (7.5), (7.7) and (7.8):

$$\forall a,\, b \in A, \quad F_n(ax \otimes by) = \sigma(a)\, F_n(x \otimes y)\, \rho\sigma(b)\,, \tag{9.4}$$

$$F_n(y \otimes x) = \rho(F_n(x \otimes y))\,, \tag{9.5}$$

$$\forall h \in \mathbb{H}^g, \quad F_n(xh \otimes y) = F_n(x \otimes y\bar{h})\,. \tag{9.6}$$

Only (9.6) needs an explanation. The positive involutions of $\mathbb{C}$ and $\mathbb{H}^g$ are denoted by $\tau_{\mathbb{C}}$ and $\tau_{\mathbb{H}}$; the classical notation $\bar{h} = \tau_{\mathbb{H}}(h)$ has been used in (9.6). On one side, from (7.8) we know that $F_n(h'x \otimes y)$ is equal to $\pm F_n(x \otimes \tau_E(h')y)$ for every $h' \in \mathbb{H}'^g$, and that the sign $\pm$ depends on the parity of $\partial h'(1 + \partial x + \partial y)$. On the other side, if $h = h'^{to}$, then $xh = (-1)^{\partial h' \partial x}\, h'x$, $yh = (-1)^{\partial h' \partial y}\, h'y$, and the comparison of τ_E and $\tau_{\mathbb{H}}$ shows that $\tau_E(h')^{to} = (-1)^{\partial h'}\, \tau_{\mathbb{H}}(h)$; now (9.6) is clear.

The next step is the calculation of $F''_n : M \otimes M \to A \hat{\otimes} \mathbb{H}'^g$:

$$\begin{aligned} F''_n(x \otimes y) &= F_n(x \otimes y) \otimes 1 - F_n(i'x \otimes y) \otimes i' \\ &\quad - \sigma F_n(j'x \otimes y) \otimes j' - \sigma F_n(i'j'x \otimes y) \otimes i'j'\,. \end{aligned} \tag{9.7}$$

Of course $\sigma F_n(j'x \otimes y)$ is a short writing for $\sigma_A(F_n(j'x \otimes y))$. A direct calculation using (9.3), (9.7) and (7.8) confirms this predictable property: $F''_n(x \bar{\otimes}_{\mathbb{C}} y)$ is the sum of the first two terms in the right hand member of (9.7). Thus F''_n induces a bijection $M \bar{\otimes}_{\mathbb{C}} M \to A \otimes \mathbb{C}'$. To follow the common use, we will rather use the bijection $F_{\mathbb{C}} : M \bar{\otimes}_{\mathbb{C}} M \to A \otimes \mathbb{C}$ defined in this way:

$$F_{\mathbb{C}}(x \bar{\otimes}_{\mathbb{C}} y) = F_n(x \otimes y) \otimes 1 - F_n(x \otimes yi) \otimes i\,. \tag{9.8}$$

For all $a,\, b \in A$ and all $\lambda,\, \mu \in \mathbb{C}$, we have

$$F_{\mathbb{C}}(ax\lambda \,\bar{\otimes}_{\mathbb{C}}\, by\mu) = (\sigma(a) \otimes \bar{\lambda})\, F_{\mathbb{C}}(x \bar{\otimes}_{\mathbb{C}} y)\, (\rho\sigma(b) \otimes \mu)\,. \tag{9.9}$$

Thus $F_{\mathbb{C}}$ has the property of sesqui-linearity over $\mathbb{C}$ that is usually preferred in Hilbert spaces: semi-linearity with respect to λ, and linearity with respect to μ. The bijection $F_{\mathbb{C}}$ is odd: it maps even elements of $M\,\bar{\otimes}_{\mathbb{C}}\,M$ into $A_1\otimes\mathbb{C}$, and odd elements into $A_0\otimes\mathbb{C}$.

Let us search for the image of the subspace $\mathrm{Sym}(M\,\bar{\otimes}_{\mathbb{C}}\,M)$ of all symmetric elements; it has the same dimension 16 as the subspace $\mathrm{SSym}(M\,\bar{\otimes}_{\mathbb{C}}\,M)$ of skew symmetric elements, because the mapping $x\,\bar{\otimes}_{\mathbb{C}}\,y\longmapsto x\,\bar{\otimes}_{\mathbb{C}}\,yi$ permutes these two subspaces. Firstly, we need the property of symmetry of $F_{\mathbb{C}}$:

$$F_{\mathbb{C}}(y\,\bar{\otimes}_{\mathbb{C}}\,x) \;=\; (\rho\otimes\tau_{\mathbb{C}})\big(F_{\mathbb{C}}(x\,\bar{\otimes}_{\mathbb{C}}\,y)\big)\,. \tag{9.10}$$

Secondly, because of (9.10), we need the eigenspaces of ρ. They are given by the canonical bijection $\bigwedge(E)\to\mathrm{C}\ell(E,q)$ that maps every exterior product of vectors to their Clifford product when they are pairwise orthogonal; remember that orthogonal vectors anticommute also in $\mathrm{C}\ell(E,q)$. By this bijection the $\mathbb{Z}$-gradation of $\bigwedge(E)$ gives a $\mathbb{Z}$-gradation $A=A^0\oplus A^1\oplus A^2\oplus A^3\oplus A^4$ of the vector space $A=\mathrm{C}\ell(E,q)$; it is clear that $A^0=\mathbb{R}$, $A^4=\mathbb{R}\omega$ and $A^1=E$, and it is well known that

$$\ker(\rho-\mathrm{id})=A^0\oplus A^4\oplus A^1 \quad\text{and}\quad \ker(\rho+\mathrm{id})=A^2\oplus A^3\,. \tag{9.11}$$

Therefore, $F_{\mathbb{C}}$ maps $\mathrm{Sym}(M\,\bar{\otimes}_{\mathbb{C}}\,M)$ bijectively onto the direct sum of $(A^0\oplus A^4\oplus A^1)\otimes 1$ and $(A^2\oplus A^3)\otimes i$. There is a bijective correspondence between all symmetric bilinear mappings $\psi: M\times M\to P$ (with the property $\psi(xi,y)=-\psi(x,yi)$) and all linear mappings from this subspace of $A\otimes\mathbb{C}$ into P.

Let us remember that the group $\mathcal{S}$ acts on the target P of ψ, and that ψ must behave correctly with respect to this action of $\mathcal{S}$; it must be equivariant at least under the action of $\mathcal{S}_{nc}$. The orthogonal transformation g_a induced by each $a\in\mathcal{S}$ extends to an automorphism of A, which maps every homogeneous $c\in A$ to $(-1)^{\partial a\,\partial c}\,aca^{-1}$; by the bijection $\bigwedge(E)\to A$, its corresponds exactly to the automorphism $\bigwedge(g_a)$ of $\bigwedge(E)$, and consequently, the subspaces A^k are the irreducible invariant subspaces under the action of $\mathcal{S}$ on A. From (9.4) we know how F_n behaves under the operation of $a\in\mathcal{S}$: $F_n(ax\otimes ay)=(a\rho(a))\,a\,F_n(x\otimes y)\,a^{-1}$; since both factors $a\rho(a)$ and $(-1)^{\partial a\,\partial c}$ are equal to 1 when a is in $\mathcal{S}_{nc}$, fortunately the bijection $F_{\mathbb{C}}: M\bar{\otimes}_{\mathbb{C}}M\to A\otimes\mathbb{C}$ is equivariant under the action of $\mathcal{S}_{nc}$.

Physicists are interested in the quadratic mappings $Q_k: M\to A^k$ such that

$$F_{\mathbb{C}}(x\,\bar{\otimes}_{\mathbb{C}}\,x) \;=\; (Q_0(x)+Q_4(x)-Q_1(x))\otimes 1 \;-\; (Q_2(x)-Q_3(x))\otimes i\,.$$

Since the subspaces A^k are orthogonal to one another for the bilinear form $(a,b)\longmapsto\mathrm{Scal}(ab)$, the mappings Q_k are characterized by these properties:

$$\text{when } k=0,\,4,\,1,\quad \forall a\in A^k,\quad \mathrm{Scal}(a\,Q_k(x))=\beta_n(ax,\,x)\,, \tag{9.12}$$

$$\text{when } k=2,\,3,\quad \forall a\in A^k,\quad \mathrm{Scal}(a\,Q_k(x))=\beta_n(ax,\,xi)\,. \tag{9.13}$$

In particular $Q_0(x)=\beta_n(x,x)$ and $Q_4(x)=-\beta_n(\omega x,\,x)\,\omega$.

In Section 10, the vector $Q_1(x)$ proves to be a time vector (or a light vector) which is oriented either toward future (for all x), or toward past (for all x). Since Q_1 depends linearly on β_n, the orientation of the line $\ker(\theta'_1 - \mathrm{id})$ is related to the orientation of time; this explains why β_n has been chosen in $\ker(\theta'_1 - \mathrm{id})$. The bilinear form $(x,y) \longmapsto \beta_n(\omega x, y)$ belongs to $\ker(\theta_1 - \mathrm{id})$; consequently, the orientation of the line $\ker(\theta_1 - \mathrm{id})$ is related to the euclidian orientation, which is not physically meaningful.

10 - An example (second part)

The previous argument can be translated into precise calculations using the normal bases of E defined in Section 9. A precise isomorphism $C\ell(E,q) \to \mathrm{Mat}(2,\mathbb{H}^g)$ can be constructed by means of a normal basis (e_0, e_1, e_2, e_3) and a family of four *odd* matrices $(\gamma_0, \gamma_1, \gamma_2, \gamma_3)$ satisfying these two conditions in $\mathrm{Mat}(2,\mathbb{H}^g)$: firstly, $\gamma_0^2 = \mathbf{1}$ (where $\mathbf{1}$ is the unit matrix) and $\gamma_r^2 = -\mathbf{1}$ for $r = 1,2,3$; secondly, $\gamma_r\gamma_s = -\gamma_s\gamma_r$ when $r \neq s$. Here is such a family:

$$\gamma_0 = \begin{pmatrix} 0 & -j \\ j & 0 \end{pmatrix}, \quad \gamma_1 = \begin{pmatrix} j & 0 \\ 0 & -j \end{pmatrix}, \quad \gamma_2 = \begin{pmatrix} ij & 0 \\ 0 & ij \end{pmatrix}, \quad \gamma_3 = \begin{pmatrix} 0 & j \\ j & 0 \end{pmatrix}.$$

There is a positive involution $T_{\mathbb{H}}$ on $\mathrm{Mat}(2,\mathbb{H}^g)$ that corresponds to $T \otimes \tau_{\mathbb{H}}$ by the isomorphism $\mathrm{Mat}(2,\mathbb{H}^g) \to \mathrm{Mat}(2,\mathbb{R}) \otimes \mathbb{H}^g$; thus $T_{\mathbb{H}}(\gamma_0) = \gamma_0$ and $T_{\mathbb{H}}(\gamma_r) = -\gamma_r$ for $r = 1,2,3$. The involution of $\mathrm{Mat}(2,\mathbb{H}^g)$ that leaves invariant the four matrices γ_r is denoted by ρ because it corresponds to the reversion ρ by the above isomorphism $C\ell(E,q) \to \mathrm{Mat}(2,\mathbb{H}^g)$. There is a matrix c such that $\rho(a) = c\,T_{\mathbb{H}}(a)\,c^{-1}$ for all $a \in \mathrm{Mat}(2,\mathbb{H}^g)$; since $c\gamma_0 c^{-1} = \gamma_0$ whereas $c\gamma_r c^{-1} = -\gamma_r$ for $r = 1,2,3$, we can choose $c = \gamma_0$:

$$\forall a \in \mathrm{Mat}(2,\mathbb{H}^g), \ \rho(a) = \gamma_0\, T_{\mathbb{H}}(a)\, \gamma_0 \ , \tag{10.1}$$

$$\forall a_1, a_2, a_3, a_4 \in \mathbb{H}^g, \quad \rho\begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} = j \begin{pmatrix} \bar{a}_4 & -\bar{a}_2 \\ -\bar{a}_3 & \bar{a}_1 \end{pmatrix} j^{-1} . \tag{10.2}$$

We must know the image of the connected spin group $\mathcal{S}_{nc}$ in $\mathrm{Mat}(2,\mathbb{H}^g)$; it is the subgroup of all even matrices a such that $a\rho(a) = 1$; because of (10.2), this means that the entries of a are complex numbers such that $a_1a_4 - a_2a_3 = 1$. Therefore the image of $\mathcal{S}_{nc}$ in $\mathrm{Mat}(2,\mathbb{H}^g)$ is the group $\mathrm{SL}(2,\mathbb{C})$ of all $a \in \mathrm{Mat}(2,\mathbb{C})$ such that $\det_{\mathbb{C}}(a) = 1$. This corroborates the well known isomorphism $\mathcal{S}_{nc} \cong \mathrm{SL}(2,\mathbb{C})$.

The notation Scal will be used for the algebras $A = C\ell(E,q)$, $\mathrm{Mat}(2,\mathbb{H}^g)$ and $\mathbb{H}^g$. If $(a_1, a_2; a_3, a_4)$ are the entries of the matrix a as in (10.2), then $\mathrm{Scal}(a)$ is the scalar part of the quaternion $(a_1 + a_4)/2$.

Since $\gamma_0\gamma_1\gamma_2\gamma_3$ is the image of the Arf element ω (an element of $Z(A_0)$ such that $\omega^2 = -1$), it is a diagonal matrix $\pm i\mathbf{1}$; the four matrices γ_r have been chosen so that $\gamma_0\gamma_1\gamma_2\gamma_3 = i\mathbf{1}$. The space $M_{\mathbb{H}}$ of all column matrices with two entries in $\mathbb{H}^g$ is a graded bimodule over $\mathrm{Mat}(2, \mathbb{H}^g)$ and $\mathbb{H}^g$; its even (resp. odd) component is the subspace $M_{\mathbb{H},0}$ (resp. $M_{\mathbb{H},1}$) of all column matrices with two entries in $\mathbb{C}$ (resp. $\mathbb{C}j$). Thus $\gamma_0\gamma_1\gamma_2\gamma_3\, x = (-1)^p xi$ for all $x \in M_{\mathbb{H},p}$ (if $p = 0, 1$), in agreement with (9.2). For each normal basis (e_0, e_1, e_2, e_3) of E there are $\mathbb{H}^g$-bases of M such that the operation on M of each e_r (with $r = 0, 1, 2, 3$) is given by the matrix γ_r. Let (w_1, w_2) be such a basis; it is unique up to an invertible factor in $Z(\mathrm{Mat}(2, \mathbb{H}^g)) \cong \mathbb{R}$, and it is also a $\mathbb{C}$-basis of M_0 because the natural $\mathbb{H}^g$-basis of $M_{\mathbb{H}}$ is also a $\mathbb{C}$-basis of $M_{\mathbb{H},0}$.

A $\mathbb{C}$-basis (w'_1, w'_2) of M_0 is called a *normal basis* of M if it is the image of the above chosen basis (w_1, w_2) by an element of $\mathcal{S}_{nc}$; therefore, if $\mathrm{Det}_{\mathbb{C}}$ is the alternate $\mathbb{C}$-bilinear form $M_0 \times M_0 \to \mathbb{C}$ such that $\mathrm{Det}_{\mathbb{C}}(w_1, w_2) = 1$, the equality $\mathrm{Det}_{\mathbb{C}}(w'_1, w'_2) = 1$ means that the basis (w'_1, w'_2) is normal. Moreover, a normal basis (e'_0, e'_1, e'_2, e'_3) of E and a normal basis (w'_1, w'_2) of M are said to be *associated* if the four matrices γ_r describe the operations of the vectors e'_r in the basis (w'_1, w'_2). Because of the morphism $\mathcal{S}_{nc} \to \mathcal{L}_{nc}$, every normal basis of M is associated with a normal basis of E; and since this morphism is a 2-sheet covering, conversely every normal basis of E is associated with two normal bases of M. For instance, both bases (w_1, w_2) and $(-w_1, -w_2)$ are associated with the chosen normal basis (e_0, e_1, e_2, e_3), and the relevance of this fact is corroborated by this calculation: if the normal basis (w'_1, w'_2) is related to (w_1, w_2) by $w'_1 = w_1 \exp(i\alpha)$ and $w'_2 = w_2 \exp(-i\alpha)$ (for some $\alpha \in \mathbb{R}$), then the associated basis (e'_0, e'_1, e'_2, e'_3) is made of $e'_0 = e_0$, $e'_1 = e_1 \cos(2\alpha) + e_2 \sin(2\alpha)$, $e'_2 = -e_1 \sin(2\alpha) + e_2 \cos(2\alpha)$ and $e'_3 = e_3$.

Every couple of associated normal bases in E and M determines a graded algebra isomorphism $A \to \mathrm{Mat}(2, \mathbb{H}^g)$ and a graded A-linear bijection $M \to M_{\mathbb{H}}$. Thus the linear mapping $F_n : M \otimes M \to A$ determines a linear mapping $M_{\mathbb{H}} \otimes M_{\mathbb{H}} \to \mathrm{Mat}(2, \mathbb{H}^g)$. Although F_n (like β_n) is determined up to an invertible real scalar, *this mapping* $M_{\mathbb{H}} \otimes M_{\mathbb{H}} \to \mathrm{Mat}(2, \mathbb{H}^g)$ *is the same for all couples of associated normal bases* because every morphism from $\mathcal{S}_{nc}$ into the group of invertible real scalars is trivial. The next theorem shows that it is more convenient to calculate the mapping $\Phi : M_{\mathbb{H}} \otimes M_{\mathbb{H}} \to \mathrm{Mat}(2, \mathbb{H}^g)$ determined by σF_n; in this calculation, the operator $T_{\mathbb{H}}$ maps each column y with entries (y_1, y_2) to the row $T_{\mathbb{H}}(y)$ with entries $(\bar{y}_1, \bar{y}_2)$.

Theorem 10.1. *If the mapping* $\Phi : M_{\mathbb{H}} \otimes M_{\mathbb{H}} \to \mathrm{Mat}(2, \mathbb{H}^g)$ *corresponds to* $\sigma F_n : M \otimes M \to A$, *there is an invertible real scalar* κ *such that*

$$\forall x, y \in M_{\mathbb{H}}, \qquad \Phi(x \otimes y) = \kappa\, x\, T_{\mathbb{H}}(y)\, \gamma_0 . \tag{10.3}$$

If the entries of x and y are (x_1,x_2) and (y_1,y_2), then

$$\Phi(x\otimes y) = \kappa\begin{pmatrix}x_1\\ x_2\end{pmatrix}(\,\bar y_1 \quad \bar y_2\,)\begin{pmatrix}0 & -j\\ j & 0\end{pmatrix} = \kappa\begin{pmatrix}x_1\bar y_2 j & -x_1\bar y_1 j\\ x_2\bar y_2 j & -x_2\bar y_1 j\end{pmatrix}.$$

The corresponding symmetric bilinear form $M_{\mathbb{H}}\times M_{\mathbb{H}}\to\mathbb{R}$ is

$$(x,y)\longmapsto \mathrm{Scal}(\Phi(x\otimes y)) = \frac{\kappa}{2}\,\mathrm{Scal}\,(x_1\bar y_2 j + y_1\bar x_2 j)\,. \tag{10.4}$$

Proof. The mapping σF_n is characterized (up to a real invertible factor) by these properties: firstly, it is odd; secondly, $\sigma F_n(ax\otimes by) = a\,\sigma F_n(x\otimes y)\rho(b)$ for all $a,\,b\in A$ (see (9.4)); thirdly, the bilinear form β_n defined by $\beta_n(x,y)=\mathrm{Scal}(\sigma F_n(x\otimes y))$ is symmetric. If we define Φ by (10.3), then Φ is odd because γ_0 is odd. For all $a,\,b\in\mathrm{Mat}(2,\mathbb{H}^g)$, both $\Phi(ax\otimes by)$ and $a\,\Phi(x\otimes y)\rho(b)$ are equal to $\kappa\,ax\,T_{\mathbb{H}}(y)T_{\mathbb{H}}(b)\,\gamma_0$ (because of (10.1)). Finally, it is easy to deduce (10.4) from (10.3) and to conclude. □

Theorem 10.2. *Let us set $x=x_0+x_1$ with $x_0\in M_0$ and $x_1\in M_1$, and $\mathrm{Det}_{\mathbb{C}}(x_0,\,x_1 j^{-1})=\lambda+\mu i$ (with $\lambda,\,\mu\in\mathbb{R}$); and let us assume $\lambda+\mu i\neq 0$. There is a normal basis (w_1',w_2') of M such that $x_0=w_1'$ and $x_1=w_2'(\lambda+\mu i)j$, and if (e_0',e_1',e_2',e_3') is the associated normal basis of E, then*

$$\begin{aligned}
Q_0(x)+Q_4(x) &= \kappa\lambda+\kappa\mu\omega\,,\\
Q_1(x) &= \frac{\kappa}{2}(\lambda^2+\mu^2+1)\,e_0' + \frac{\kappa}{2}(\lambda^2+\mu^2-1)\,e_3'\ ,\\
Q_2(x) &= -\,\kappa\lambda\,e_1'e_2' + \kappa\mu\,e_0'e_3'\,,\\
Q_3(x) &= -\frac{\kappa}{2}(\lambda^2+\mu^2+1)\,e_0'e_1'e_2' - \frac{\kappa}{2}(\lambda^2+\mu^2-1)\,e_1'e_2'e_3'\ .
\end{aligned}$$

Proof. It suffices to verify these equalities when $\kappa=1$. The matrices representing $\sigma F_n(x\otimes x)$ and $\sigma F_n(x\otimes xi)$ are

$$\begin{pmatrix}1\\ (\lambda+\mu i)j\end{pmatrix}(\,1 \quad -(\lambda+\mu i)j\,)\begin{pmatrix}0 & -j\\ j & 0\end{pmatrix} = \begin{pmatrix}\lambda+\mu i & -j\\ (\lambda^2+\mu^2)j & \lambda+\mu i\end{pmatrix},$$

$$\begin{pmatrix}1\\ (\lambda+\mu i)j\end{pmatrix}(\,-i \quad -(\lambda+\mu i)ji\,)\begin{pmatrix}0 & -j\\ j & 0\end{pmatrix} = \begin{pmatrix}-\lambda i+\mu & ij\\ (\lambda^2+\mu^2)ij & \lambda i-\mu\end{pmatrix}.$$

In the first matrix, the even part $(\lambda+\mu i)\mathbf{1}$ gives $Q_0(x)+Q_4(x)$; the odd part, which gives $Q_1(x)$, is the half of $(\lambda^2+\mu^2+1)\gamma_0+(\lambda^2+\mu^2-1)\gamma_3$. In the second matrix the even and odd parts give respectively $Q_2(x)$ and $Q_3(x)$; the even part is $-\lambda\gamma_1\gamma_2+\mu\gamma_0\gamma_3$, and the odd part is the half of $(\lambda^2+\mu^2+1)(i\gamma_3)+(\lambda^2+\mu^2-1)(i\gamma_0)$; besides, $i\gamma_r=\gamma_0\gamma_1\gamma_2\gamma_3\gamma_r$ for $r=0,1,2,3$. □

In Theorem 10.2, we realize that $Q_1(x)$ is a time vector that is oriented toward future or past according to the sign of κ; if we require the orientation toward future, then *κ must be positive*. It follows from (10.4) that $\kappa\,\mathrm{Det}_{\mathbb{C}}(x,y)=2\beta_n(x,yj)-2i\beta_n(x,yij)$ for all $x,\ y\in M_0$. Since Theorem 10.2 deals with all spinors in an open dense subset, it allows us to prove the Fiersz identities (10.5) and (10.6), and other similar identities:

$$Q_1(x)^2 \;=\; -Q_3(x)^2 \;=\; (Q_0(x)+Q_4(x))(Q_0(x)-Q_4(x))\,, \tag{10.5}$$

$$Q_1(x)Q_3(x) \;=\; Q_3(x)Q_1(x) \;=\; (Q_0(x)-Q_4(x))\,Q_2(x)\,, \tag{10.6}$$

$$Q_2(x)Q_1(x) \;=\; (Q_0(x)+Q_4(x))\,Q_3(x)\,, \tag{10.7}$$

$$Q_2(x)Q_3(x) \;=\; -\,(Q_0(x)+Q_4(x))\,Q_1(x)\,, \tag{10.8}$$

$$Q_2(x)^2 \;=\; -\,(Q_0(x)+Q_4(x))^2\,. \tag{10.9}$$

The spinors x outside the hypotheses of Theorem 10.2 are characterized by $Q_0(x)=Q_4(x)=0$; their study would reveal that $Q_1(x)$ vanishes only when $x=0$. The equality $Q_3(x)=0$ means that $x=w(1+\exp(i\alpha)j)$ for some $w\in M_0$ and some $\alpha\in\mathbb{R}$. The equality $Q_2(x)=0$ means that x is a Weyl spinor (in M_0 or in M_1). For $p=0,1$, the equality $Q_3(x)=(-1)^pQ_1(x)\,\omega$ characterizes the elements x of M_p.

11 - Regular gradations

Sections 9 and 10 show which powerful simplifications occur when parity gradations are taken into account. Nevertheless, many people still study spinor spaces without worrying about gradations. Fortunately, the study of non graded modules can be reduced to the study of graded modules.

The gradation of an algebra A is said to be *regular* if the multiplication mapping $A\times A\to A$ induces a surjective mapping $A_1\otimes A_1\to A_0$; since the image of this mapping is an ideal of A_0, this condition is equivalent to the existence of a sequence $(a_1,b_1,a_2,b_2,\ \dots,a_r,b_r)$ of elements of A_1 such that $\sum_{j=1}^{r}a_jb_j=1$; the length $2r$ of this sequence is arbitrary, and may be reduced to 2 when A_1 contains invertible elements.

Although $\mathrm{Mat}_0(n,n';K)$ contains two non trivial ideals when $nn'\neq 0$, it is easy to prove that the gradation of $\mathrm{Mat}(n,n';K)$ is regular. Because of Theorem 2.6, it follows immediately that *the gradation of a graded central simple algebra A is regular if and only if $A_1\neq 0$.*

The following well known theorem can be used to prove the assertions that begin Section 8 above.

Theorem 11.1. *If A is a regularly graded algebra, the category of graded modules over A is equivalent to the category of modules over A_0.*

Proof. If M is a graded module over A, then M_0 is a module over A_0; conversely, if P is a module over A_0, the tensor product over A_0 of A (treated as a right module over A_0) and P is a graded module over A, in which the even component is isomorphic to P. Besides, if M and N are graded modules over A, and if f_0 is an A_0-linear mapping $M_0 \to N_0$, every sequence $(a_1, b_1, a_2, b_2, \ldots, a_r, b_r)$ of elements of A_1 such that $\sum a_j b_j = 1$ allows to extend f_0 to a graded A-linear mapping $f : M \to N$ by setting $f(x) = \sum a_j f_0(b_j x)$ for every $x \in M_1$. □

Regular gradations afford a nice treatment of modules without gradation. If M is just a module over the regularly graded algebra A, we consider the graded space $(M^2)^g$ and the following action of A on $(M^2)^g$:

$$\forall a \in A, \ \ \forall x, y \in M, \qquad a\,(x,y) = (ax, \ \sigma(a)\,y)\,; \tag{11.1}$$

thus $(M^2)^g$ is actually a graded module over A. Besides, let η be the odd linear endomorphism of $(M^2)^g$ defined by

$$\forall x, y \in M, \qquad \eta(x,y) = (-\,y, x)\,; \tag{11.2}$$

since $\eta^2 = -\mathrm{id}$, the space $K \oplus K\eta$ is a graded algebra isomorphic to $((K^2)^g)^{to}$; and since η commutes (resp. anticommutes) with the operation of every even (resp. odd) element of A, the space $(M^2)^g$ is also a graded module over $D = (K \oplus K\eta)\hat{\otimes} A$.

Theorem 11.2. *Let M be a module over the regularly graded algebra A, and let $\mathrm{End}_{A_0}(M)$ be the centralizer of the action of A_0 on M. There is a gradation of $\mathrm{End}_{A_0}(M)$ for which the even subalgebra is $\mathrm{End}_A(M)$, and the odd component is the subspace $\mathrm{End}^t_A(M)$ of all $f \in \mathrm{End}(M)$ such that $f(ax) = \sigma(a)f(x)$ for all $a \in A$ and all $x \in M$. Besides, let us consider the graded algebras $D = (K \oplus K\eta)\hat{\otimes} A$ and $\mathrm{End}^g_D((M^2)^g)$; the mapping $f \longmapsto f \oplus f$ is an isomorphism from $\mathrm{End}_A(M)$ onto the subalgebra $\mathrm{End}^g_{D,0}((M^2)^g)$, and the mapping $f \longmapsto f \oplus (-f)$ is a bijection from $\mathrm{End}^t_A(M)$ onto $\mathrm{End}^g_{D,1}((M^2)^g)$.*

Proof. Let $(a_1, b_1, \ldots, a_r, b_r)$ be a sequence of odd elements of A such that $\sum a_j b_j = 1$. For every $f \in \mathrm{End}_{A_0}(M)$, let $\sigma(f)$ be the endomorphism of M mapping every x to $\sum a_j f(b_j x)$. Easy calculations show that $\sigma^2(f) = f$, and that $\sigma(f)\sigma(f') =$

$\sigma(ff')$ for all $f, f' \in \mathrm{End}_{A_0}(M)$. Therefore, σ is an involutive automorphism of $\mathrm{End}_{A_0}(M)$. Another calculation reveals that $f(ax) = a\,\sigma(f)(x)$ for all $f \in \mathrm{End}_{A_0}(M)$, all $a \in A_1$ and all $x \in M$; therefore the equality $\sigma(f) = f$ (resp. $\sigma(f) = -f$) is equivalent to $f \in \mathrm{End}_A(M)$ (resp. $f \in \mathrm{End}^t_A(M)$). The algebra $\mathrm{End}_{A_0}(M)$ is now graded. Every endomorphism g of $(M^2)^g$ can be represented by a matrix in $\mathrm{Mat}(2, \mathrm{End}(M))$:

$$g = \begin{pmatrix} f_1 & f_2 \\ f_3 & f_4 \end{pmatrix} \quad : \qquad (x, y) \longmapsto (f_1(x) + f_2(y)\,,\; f_3(x) + f_4(y))\ .$$

If this g is a homogeneous element of $\mathrm{End}^g_D((M^2)^g)$, either it commutes with η and the grade automorphism (the swap automorphism) σ of $(M^2)^g$, or it anticommutes with both; therefore, it commutes with $\sigma\eta$ which is the mapping $(x, y) \longmapsto (x, -y)$; therefore, $f_2 = f_3 = 0$, or equivalently, $g = f_1 \oplus f_4$. Then it is easy to verify that g belongs to $\mathrm{End}^g_{D,0}((M^2)^g)$ (resp. $\mathrm{End}^g_{D,1}((M^2)^g)$) if and only if $f_1 = f_4 \in \mathrm{End}_A(M)$ (resp. $f_1 = -f_4 \in \mathrm{End}^t_A(M)$). □

Now let τ be an involution on A, and τ_0 its restriction to A_0.

Theorem 11.3. *The space* $\mathrm{Z}(A_0, \mathcal{B}(M); \tau_0)$ *admits a gradation for which the even component is* $\mathrm{Z}(A, \mathcal{B}(M); \tau)$*, and the odd component is* $\mathrm{Z}(A, \mathcal{B}(M); \sigma\tau)$*. Thus* $\mathrm{Z}(A_0, \mathcal{B}(M); \tau_0)$ *becomes a graded right module over the graded algebra* $\mathrm{End}_{A_0}(M)$*. If D is defined as in Theorem* 11.2*, and if τ_D is the involution on D that induces τ on A and the non trivial involution on $K \oplus K\eta$, then the mapping $\beta \longmapsto \beta \perp \beta$ is a bijection from* $\mathrm{Z}(A, \mathcal{B}(M); \tau)$ *onto* $\mathrm{Z}^g_0(D, \mathcal{B}((M^2)^g); \tau_D)$*, and the mapping $\beta \longmapsto \beta \perp (-\,\beta)$ is a bijection from* $\mathrm{Z}(A, \mathcal{B}(M); \sigma\tau)$ *onto* $\mathrm{Z}^g_1(D, \mathcal{B}((M^2)^g); \tau_D)$*.*

Proof. For every $\beta \in \mathrm{Z}(A_0, \mathcal{B}(M); \tau_0)$, we define a bilinear form $\sigma(\beta)$ by means of the sequence $(a_1, b_1, \ldots, a_r, b_r)$ already used in the previous proof:

$$\sigma(\beta)(x, y) = \sum \beta(b_j x,\ \tau(a_j) y)\,;$$

again it follows that $\sigma^2(\beta) = \beta$, and consequently σ is the grade automorphism of a gradation of $\mathrm{Z}(A_0, \mathcal{B}(M); \tau_0)$. If β is in $\mathrm{Z}(A_0, \mathcal{B}(M); \tau_0)$, another calculation reveals that $\beta(ax,\, y) = \sigma(\beta)(x,\, \tau(a)y)$ for all $a \in A_1$ and all $x,\, y \in M$. Consequently, the even component of $\mathrm{Z}(A_0, \mathcal{B}(M); \tau_0)$ is $\mathrm{Z}(A, \mathcal{B}(M); \tau)$, and its odd one is $\mathrm{Z}(A, \mathcal{B}(M); \sigma\tau)$. Now, if a, β and f are homogeneous elements of A, $\mathrm{Z}(A_0, \mathcal{B}(M); \tau_0)$ and $\mathrm{End}_{A_0}(M)$, then

$$(\beta f)(ax, y) = \beta(f(ax), y) = (-1)^{\partial a \partial f} \beta(af(x), y) =$$

$$(-1)^{\partial a \partial f} (-1)^{\partial a\, \partial\beta} \beta(f(x), \tau(a)y) = (-1)^{\partial a(\partial\beta + \partial f)} (\beta f)(a, \tau(a)y);$$

this calculation proves that $Z(A_0, \mathcal{B}(M); \tau_0)$ is a graded right module over $\mathrm{End}_{A_0}(M)$. Finally, a bilinear form γ on $(M^2)^g$ may be represented by a square matrix of order 2 with entries in $\mathcal{B}(M)$:

$$\gamma = \begin{pmatrix} \beta_1 & \beta_2 \\ \beta_3 & \beta_4 \end{pmatrix}, \qquad \begin{aligned} \gamma((x,y),(x',y')) \ = \ & \beta_1(x,x') + \beta_2(x,y') \\ & + \beta_3(y,x') + \beta_4(y,y') \,. \end{aligned}$$

If γ is a homogeneous element of $Z^g(D, \mathcal{B}((M^2)^g); \tau_D)$, then

$$\gamma(\sigma(x,y), \sigma(x',y')) = \pm\, \gamma((x,y),(x',y'))\,,$$

$$\gamma(\eta(x,y), \eta(x',y')) = \pm\, \gamma((x,y),(x',y')) \quad (\text{because } \tau_D(\eta)\,\eta = 1);$$

since the two signs $\pm$ are equal, and since $\sigma\eta(x,y) = (x, -y)$, this implies $\gamma((x,-y),(x',-y')) = \gamma((x,y),(x',y'))$ (for all $x,\ y,\ x',\ y' \in M$); it follows that $\beta_2 = \beta_3 = 0$ and $\gamma = \beta_1 \perp \beta_4$. Now it is easy to find the homogeneous components of $Z^g(D, \mathcal{B}((M^2)^g); \tau_D)$ and to reach the final conclusions. □

Let us suppose that $Z(A, \mathcal{B}(M); \tau)$ or $Z(A, \mathcal{B}(M); \sigma\tau)$ contains a nondegenerate bilinear form β_n that is symmetric or skew symmetric. Therefore, $\beta_n(ax, y) = (-1)^{\partial a \partial \beta_n}\, \beta_n(x, \tau(a)y)$ for every $a \in A$, and by imitation of this equality we derive from β_n the following involution τ_E of $\mathrm{End}_{A_0}(M)$:

$$(11.3) \qquad \forall f \in \mathrm{End}_{A_0}(M), \quad \beta_n(f(x),\, y) \ = \ (-1)^{\partial\beta_n \partial f}\, \beta_n(x,\, \tau_E(f)(y))\,.$$

By Theorem 11.3, β_n has an image $\beta_n \perp (-1)^{\partial\beta_n}\beta_n$ in $Z^g(D, \mathcal{B}((M^2)^g); \tau_D)$; it is nondegenerate, homogeneous, and symmetric or skew symmetric; therefore, it induces an involution of $\mathrm{End}^g_D((M^2)^g)$. The following easy lemma means that this involution corresponds to τ_E by the isomorphism $\mathrm{End}_{A_0}(M) \to \mathrm{End}^g_D((M^2)^g)$ of Theorem 11.2.

Lemma 11.4. *The involution of* $\mathrm{End}^g_D((M^2)^g)$ *determined by* $\beta_n \perp (-1)^{\partial\beta_n}\beta_n$ *maps every homogeneous element* $f \oplus (-1)^{\partial f} f$ *to* $\tau_E(f) \oplus (-1)^{\partial f}\, \tau_E(f)$.

12 - Modules without gradation

Here M is just a module over a graded central simple algebra A provided with an involution τ. If $A_1 = 0$, then M with the trivial gradation is a graded module over A, and its study follows from Sections 7 and 8. Therefore, it is assumed that $A_1 \neq 0$; thus the gradation of A is regular, and we get a graded algebra $\mathrm{End}_{A_0}(M)$ and a graded centralizer $Z(A_0, \mathcal{B}(M); \tau_0)$ as it is explained in Section 11. Because of Theorem 11.2, the algebra $\mathrm{End}_{A_0}(M)$ is isomorphic to $\mathrm{End}^g_D((M^2)^g)$. Because of Theorem 8.1, its gradation may be unbalanced only when $\mathrm{cl}(D)$ belongs to $\mathrm{B}(K)$; since $\mathrm{cl}(A) = \mathrm{cl}(D)\,\mathrm{cl}((K^2)^g)$, this happens when A is isomorphic to $\mathrm{Mat}(n, (B^2)^g)$ for some central division algebra B. Therefore the assumption $A \cong (A_0^2)^g$ is the critical assumption that may produce unbalance. This is not surprising since A is a simple algebra except when $A \cong (A_0^2)^g$. When $A \cong (A_0^2)^g$, every module M over A is the direct sum of two isotypical components M' and M''; in $\mathrm{Z}(A)$ there is an odd Arf element ω such that $\omega^2 = 1$, and M' and M'' are the eigenspaces of the operation of ω in M.

Theorem 12.1. *The algebra* $\mathrm{End}_{A_0}(M)$ (*defined in Theorem* 11.2) *is graded central simple. Its Brauer-Wall class and its dimension are given by*

$$\mathrm{cl}(A)\,\mathrm{cl}(\mathrm{End}_{A_0}(M)) = \mathrm{cl}((K^2)^g), \quad \dim(A)\,\dim(\mathrm{End}_{A_0}(M)) = 2\,(\dim(M))^2.$$

When A is not isomorphic to $(A_0^2)^g$, the gradation of $\mathrm{End}_{A_0}(M)$ *is balanced. When $A \cong (A_0^2)^g$, the dimension of the homogeneous components of* $\mathrm{End}_{A_0}(M)$ *are given by the following formulas in which d and d′ are the dimensions of the isotypical components of M:*

$$\dim(\mathrm{End}_A(M)) = \frac{2(d^2 + d'^2)}{\dim(A)}, \quad \dim(\mathrm{End}_A^t(M)) = \frac{4dd'}{\dim(A)}.$$

Proof. This follows from Theorems 8.1 and 11.2. In the case $A \cong (A_0^2)^g$, we must find the isotypical components P' and P'' of the D-module $P = (M^2)^g$; we know that $P'_0 \oplus P''_1$ and $P''_0 \oplus P'_1$ are the eigenspaces of the operation of the Arf element $\eta \otimes \omega$ of D. Let us consider $(x + x',\, y + y') \in P$, with $x,\, y \in M'$ and $x',\, y' \in M''$; since ω acts like 1 on M' and like -1 on M'', it soon follows that $\eta \otimes \omega$ maps $(x + x',\, y + y')$ to $(y - y',\, x - x')$. Thus $\eta \otimes \omega$ multiplies by 1 all even (x, x) and all odd $(x', -x')$, but multiplies by -1 all even (x', x') and all odd $(x, -x)$. Therefore, P' is the subspace $M' \oplus M'$ of all (x, y), and P'' is the subspace $M'' \oplus M''$ of all (x', y'). □

Theorem 12.2. *The space* $\mathrm{Z}(A_0, \mathcal{B}(M); \tau_0)$ *has the same dimension as the algebra* $\mathrm{End}_{A_0}(M)$. *It contains a nondegenerate homogeneous bilinear form β_n that is symmetric or skew symmetric. The mapping $f \longmapsto \beta_n f$ is a homogeneous linear bijection from* $\mathrm{End}_{A_0}(M)$ *onto* $\mathrm{Z}(A_0, \mathcal{B}(M); \tau_0)$*; its parity is the parity of β_n. When $A \cong (A_0^2)^g$, and when the isotypical components of M have different dimensions, then β_n must be even or odd according as the operation of τ in* $\mathrm{Z}(A)$ *is trivial or not.*

Proof. This a direct consequence of Theorems 8.2 and 11.3. When $A \cong (A_0^2)^g$, then $\tau_D(\eta \otimes \omega) = \eta \otimes \tau(\omega)$; therefore, τ_D operates trivially in $\mathrm{Z}(D_0)$ if and only if τ operates trivially in $\mathrm{Z}(A)$. □

Now we consider the involution τ_E of $\mathrm{End}_{A_0}(M)$ determined by a choice of β_n according to (11.3), and the involutive endomorphism θ of $\mathrm{Z}(A_0, \mathcal{B}(M); \tau_0)$ defined by $\theta(\beta)(x,y) = \beta(y,x)$.

Theorem 12.3. *When β_n is even, then $\mathrm{cl}(\theta) = \pm\,\mathrm{cl}(\tau_E)$, and when β_n is odd, then $\mathrm{cl}(\theta) = \pm\, i\,\mathrm{cl}(\tau_E)$; in both cases the sign $\pm$ means $+$ or $-$ according as β_n is symmetric or skew symmetric.*

Theorem 12.4. *The equality $\mathrm{cl}(\tau)\,\mathrm{cl}(\theta) = \exp(i\pi/4)$ holds in R_8.*

Proof. Theorems 12.3 and 12.4 follow from Theorems 8.3 and 8.4 since Lemma 11.4 shows that the involution of $\mathrm{End}^g_D((M^2)^g)$ determined by $\beta_n \oplus (-1)^{\partial\beta_n}\beta$ has the same class as τ_E, and since $\mathrm{cl}(\tau_D) = \mathrm{cl}(\tau)\exp(-i\pi/4)$. □

Now let us consider $\mathrm{L}^2(M,A)$ and $\mathrm{L}^2(M, \mathrm{End}_{A_0}(M))$. Since we can permute the roles of τ and $\sigma\tau$, it is sensible to choose β_n in $\mathrm{Z}(A, \mathcal{B}(M); \tau)$, so that *$\beta_n$ is an even element of* $\mathrm{Z}(A_0, \mathcal{B}(M); \tau_0)$. We can apply Theorem 7.1 with trivial gradations, and deduce from it a bijection $\mathrm{Z}(A\otimes A,\ \mathrm{L}^2(M,A); \tau) \to \mathrm{Z}(A, \mathcal{B}(M); \tau)$. Thus β_n determines a linear mapping $F_n : M\otimes M \to A$ such that $\beta_n(x,y) = \mathrm{Scal}(F_n(x\otimes y))$, $F_n(y\otimes x) = \pm\,\tau(F_n(x\otimes y))$ and $F_n(ax\otimes by) = a\,F_n(x\otimes y)\tau(b)$ for all $x,\ y\in M$ and all $a,\ b\in A$.

Let us pay more attention to the mapping $F'_n : M\otimes M \to \mathrm{End}_{A_0}(M)$. Since $\partial\beta_n = 0$, the equality (7.6) here gives $\mathrm{Scal}_E(f\,F'_n(x\otimes y)) = \beta_n(f(x), y)$ (for all $x,\ y\in M$ and all $f\in \mathrm{End}_{A_0}(M)$); it allows us to deduce F'_n from β_n. Moreover, from (7.5), (7.7) and (7.8) we deduce these three properties:

$$\forall f,\, g\in \mathrm{End}_{A_0}(M),\quad F'_n(f(x)\otimes g(y)) = f\,F'_n(x\otimes y)\,\tau_E(g)\,. \tag{12.1}$$

$$F'_n(y\otimes x) = \pm\,\tau_E\big(F'_n(x\otimes y)\big)\,. \tag{12.2}$$

$$\forall a\in A,\quad F'_n(ax\otimes y) = \sigma_E^{\partial a}\big(F'_n(x\otimes \tau(a)y)\big)\,. \tag{12.3}$$

Let us compare these properties of F'_n with the properties that to-day have become most popular among the people concerned with the classification of real Clifford algebras; these properties are often established by means of minimal left ideals of A. It has almost no importance that A is a Clifford algebra $C\ell(E,q)$, and that τ is either the reversion ρ or the conjugation $\rho\sigma$. Now M is an *irreducible* module over A, so that $\mathrm{End}_A(M)$ (the even subalgebra of $\mathrm{End}_{A_0}(M)$) is a division algebra, isomorphic to $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$. When $A \cong (A_0^2)^g$, then $\mathrm{End}^t_A(M) = 0$ (see Theorem 12.1), and $\mathrm{Z}(A, \mathcal{B}(M); \tau) = 0$ if τ does not act trivially on $\mathrm{Z}(A)$ (see Theorem 12.2); in this case, only one of the involutions ρ or $\rho\sigma$ is taken into account; but there are two classes of irreducible modules (the so-called half-spinor spaces). The space M is treated as a bimodule over A and the opposite algebra $\mathrm{End}_A(M)^o$; let us set $xf^o = f(x)$ for every $f \in \mathrm{End}_A(M)$ and every $x \in M$. Those people are interested in $\mathbb{R}$-bilinear mappings $B : M \times M \to \mathrm{End}_A(M)^o$ satisfying these properties for some involution $\tilde{\tau}$ of $\mathrm{End}_A(M)^o$:

$$\forall f,\ g \in \mathrm{End}_A(M),\quad B(xf^o,\ yg^o) = \tilde{\tau}(f^o)\,B(x,y)\,g^o\,, \tag{12.4}$$

$$B(y,x) \;=\; \pm\,\tilde{\tau}\,(B(x,y))\,, \tag{12.5}$$

$$\forall a \in A,\qquad B(ax,\ y) \;=\; B(x,\ \tau(a)y)\,. \tag{12.6}$$

Let β be the bilinear form $M \times M \to \mathbb{R}$ defined by $\beta(x,y) = \mathrm{Scal}_E(B(x,y))$; because of (12.4), B is determined by β. Because of (12.6), β belongs to $\mathrm{Z}(A, \mathcal{B}(M); \tau)$. Because of (12.5), β is symmetric or skew symmetric. Since the module M is irreducible (and since B is assumed to be $\neq 0$), β is nondegenerate; thus we can set $\beta_n = \beta$, and deduce F'_n from β_n as above; obviously only the even component $F'_{n,0} : M \times M \to \mathrm{End}_A(M)$ will be useful here. The properties (12.4), (12.5) and (12.6) are true when $\tilde{\tau}$ is the involution of $\mathrm{End}_A(M)^o$ that corresponds to the even component of τ_E, and when B is the mapping $M \times M \to \mathrm{End}_A(M)^o$ defined by $B(x,y) = (F'_{n,0}(y \otimes x))^o$. Consequently, the precise properties of the bilinear mappings B (according to the type of q) can be deduced very rapidly from the previous theorems.

Nevertheless, one piece of information is not given by these theorems, and must be deduced from other considerations: when $\mathrm{Z}(A, \mathcal{B}(M); \tau)$ contains a *symmetric* element $\beta \neq 0$, what may be its signature? In the general case there exists $a \in E$ such that $\tau(a)\,a = -1$, whence $\beta(ax, ay) = -\beta(x,y)$; this enforces β to be hyperbolic (with a null signature). When such a vector does not exist, then q is either positive definite (if $\tau = \rho$) or negative definite (if $\tau = \rho\sigma$), and β is invariant under the action of a *compact* spin group; since M is an irreducible module, this enforces β to be positive or negative definite.

Example. In the continuation of Sections 9 and 10, let us consider a module M over a real Clifford algebra $\mathrm{C}\ell(E,q)$ of type $(1,3)$ or $(3,1)$, such that $\dim(M) = 8$. The class of the positive involutions of $\mathrm{End}_{A_0}(M)$ is $\exp(3\pi/4)$ for the type (1,3) (see Theorem 12.1); consequently $\mathrm{End}_{A_0}(M) \cong \mathbb{H} \otimes \mathbb{C}^g$ and $\mathrm{End}_A(M)^o \cong \mathbb{H}$. For the type $(3,1)$, it is $\exp(-\,i\pi/4)$, whence $\mathrm{End}_{A_0}(M) \cong \mathrm{Mat}(2, \mathbb{C}^g)$ and $\mathrm{End}_A(M)^o \cong \mathrm{Mat}(2, \mathbb{R})$; the module M is not irreducible for the type (3,1), but Theorems 12.1, ..., 12.4 do not need the hypothesis of irreducibility. Since $\mathrm{cl}(\rho) = \mathrm{cl}(\rho\sigma) = -1$, we have $\mathrm{cl}(\theta) = \exp(5i\pi/4)$ (see Theorem 12.4); the formulas (3.8) and (3.9) show that, both in $\mathrm{Z}(A, \mathcal{B}(M); \rho)$ and in $\mathrm{Z}(A, \mathcal{B}(M); \rho\sigma)$, the symmetric elements constitute a subspace of dimension 1, and the skew symmetric elements a subspace of dimension 3. Let us choose a symmetric β_n in $\mathrm{Z}(A, \mathcal{B}(M); \tau)$ (where τ means ρ or $\rho\sigma$) so that $\partial\beta_n = 0$; this implies $\mathrm{cl}(\tau_E) = \exp(5i\pi/4)$ (see Theorem 12.3), and the formulas (3.8) and (3.9) show that the dimensions of the kernels of $(\tau_{E,0} - \mathrm{id})$ and $(\tau_{E,0} + \mathrm{id})$ are 1 and 3. Therefore, when q has the type (1,3), the involution $\tilde{\tau}$ of $\mathbb{H}$ corresponding to $\tau_{E,0}$ is the positive involution $h \longmapsto \bar{h} = 2\,\mathrm{Scal}(h) - h$; this agrees with (9.6). And when q has the type (3,1), the involution $\tilde{\tau}$ of $\mathrm{Mat}(2,\mathbb{R})$ is $f \longmapsto \mathrm{tr}(f)\mathbf{1} - f$. For the type (1,3) we obtain a mapping $B : M \times M \to \mathbb{H}$, and for the type (3,1) a mapping $B : M \times M \to \mathrm{Mat}(2,\mathbb{R})$, but in both cases B satisfies the equalities (12.4), (12.5) and (12.6), with the sign $+$ in (12.5).

Unfortunately, this little information does not afford any easy solution to the problems tackled in Sections 9 and 10, since it cannot take into account the complex structure of the spinor space M. The complexified algebra $\mathbb{C} \otimes C\ell(E,q)$ would not be more efficient because we are interested in $M \bar{\otimes}_{\mathbb{C}} M$, not in $M \otimes_{\mathbb{C}} M$. It is wiser to notice that a complex structure is equivalent to a parity gradation according to (9.2), and to apply the more powerful graded theory.

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