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Global existence of solutions to the semilinear heat equation on Riemannian manifolds with negative sectional curvature

Abstract. We address local existence, blow-up and global existence of mild solutions to the semilinear heat equation on Riemannian manifolds with negative sectional curvature. We deal with a power nonlinearity multiplied by a time-dependent positive function $h(t)$, and initial conditions $u_0 \in L^p(M)$. We show that depending on the behavior at infinity of h, either every solution blows up in finite time, or a global solution exists, if the initial datum is small enough. In particular, for any power nonlinearity, if $h \equiv 1$ we have global existence for small initial data, whereas if $h(t) = e^{\alpha t}$ a Fujita type phenomenon prevails varying the parameter $\alpha > 0$.

Keywords. Local existence, finite time blow-up; global existence; mild solutions; Laplace-Beltrami operator; heat kernel.

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1 - Introduction

Local existence, finite time blow-up and global existence of solutions to the following Cauchy problem for the semilinear heat equation:

$$
\begin{cases} \partial_t u = \Delta u + u^\nu & \text{in } \mathbb{R}^N \times (0,T) \\ u = u_0 & \text{in } \mathbb{R}^N \times \{0\}, \end{cases} \tag{1}
$$

where $\nu > 1, u_0 \geq 0, u_0 \in L^\infty(\mathbb{R}^N)$, have been largely investigated. Indeed (see [**5**], [**6**] and [**11**]), problem (1) does not admit global bounded solutions for $1 < \nu \leq 1 + \frac{2}{N}$. Instead, for $\nu > 1 + \frac{2}{N}$ global bounded solutions exist, provided that u_0 is sufficiently small. This dichotomy is usually said Fujita's phenomenon.

In [**16**]-[**18**], similar results have been stated for mild solutions from the space $C\big([0,T); L^p(\mathbb{R}^N)\big)$, supposing $u_0 \in L^p(\mathbb{R}^N)$.

Moreover, the blow-up result given in [**5**] has been extended on Riemannian manifolds M, endowed with a Riemannian metric g (see [**19**]), provided there exist $C > 0$ and a suitable $\alpha > 0$ such that:

(a) $\mu(B(x,r)) \leq Cr^\alpha$, when r is large and for all $x \in M$;

(b) $\frac{\partial \log \sqrt{g}}{\partial r} \leq \frac{C}{r}$, when $r = d(x_0, x)$, for some $x_0 \in M$, is smooth. Here μ is the Riemannian volume on M, $\sqrt{g}$ is the volume density of $M, B(x,r)$ is the geodesics ball with center $x \in M$ and radius $r > 0$.

Observe that if the Ricci curvature of M is nonnegative, then $(a) - (b)$ are satisfied. On the other hand (see Theorem 5.2.10 in [4], or [8], Section 10.1), hypotheses $(a) - (b)$ imply that $\lambda_1(M) = 0$, where $\lambda_1(M)$ is the infimum of the L^2- spectrum of the operator $-\Delta$ on M.

In [**1**], the semilinear Cauchy problem

$$
\begin{cases} \partial_t u = \Delta u + h(t) u^\nu & \text{in } \mathbb{H}^N \times (0,T) \\ u = u_0 & \text{in } \mathbb{H}^N \times \{0\} \end{cases} \tag{2}
$$

has been studied, where $\mathbb{H}^N$ is the $N-$dimensional hyperbolic space, u_0 is nonnegative and bounded on M, h is a positive continuous function defined in $[0, \infty)$. Note that $\lambda_1(\mathbb{H}^N) = \frac{(N-1)^2}{4}$.

To be specific, it has been shown that if $h(t) \equiv 1$ $(t \geq 0)$, or for some $\alpha_1 > 0, \alpha_2 > 0, t_0 > 0$ and $q > -1$

$$\alpha_1 t^q \leq h(t) \leq \alpha_2 t^q \quad \text{for any } \ t > t_0, \tag{3}$$

then there exist global bounded solutions for sufficiently small initial data u_0. Moreover, when $h(t) = e^{\alpha t}$ $\quad (t \geq 0)$ for some $\alpha > 0$, we have the following results:

- (i) if $1 < v < 1 + \frac{\alpha}{\lambda_1(\mathbb{H}^N)}$, then every nontrivial bounded solution of problem (2) blows up in finite time;
- (ii) if $v > 1 + \frac{\alpha}{\lambda_1(\mathbb{H}^N)}$, then problem (2) posses global bounded solutions for small initial data ;
- (iii) if $v = 1 + \frac{\alpha}{\lambda_1(\mathbb{H}^N)}$ and $\alpha > \frac{2}{3}\lambda_1(\mathbb{H}^N)$, then there exist global bounded solutions of problem (2) for small initial data.

Analogous results to those established in [**1**] have been obtained in [**14**], for problem

$$\begin{cases} \partial_t u = \Delta u + h(t)u^v & \text{in } M \times (0, T) \\ u = u_0 & \text{in } M \times \{0\}, \end{cases} \tag{4}$$

where M is a Cartan-Hadamard Riemannian manifold with sectional curvature bounded above by a negative constant, while Δ denotes the Laplace-Beltrami operator on M. For this type of Riemannian manifolds we have $\lambda_1(M) > 0$. Hence the hypotheses (a)-(b) cannot be satisfied.

Observe that, in [**14**], local and global existence have been shown, supposing that

(j) there exists a supersolution to equation

$$\Delta\phi = \lambda\phi \quad \text{in } \ M,$$

for some $\lambda \geq 0$, such that $\phi(x) \to +\infty$ as $x \to \infty$.

Moreover, to prove global existence it is also assumed that

(jj) there exists a positive bounded solution to equation

$$\Delta\phi + \lambda_1(M)\phi = 0 \quad \text{in } \ M. \tag{5}$$

Note that (j) implies that comparison principle holds for bounded weak solutions to problem (4). Moreover, regarding (jj), it is known that a classical positive solution to equation (5) exists, but it seems not to be known under which hypotheses on M such a solution is bounded. In particular, (j) is satisfied when Ricci curvature of M is bounded form below, while (jj) is satisfied for $M = \mathbb{H}^N$ (see [**1**], [**14**]).

Let us underline that in both [**1**] and [**14**] the initial datum u_0 is bounded, thus only bounded solutions are considered. Moreover, at first, by comparison principles and compactness arguments, it is proved that a local weak solution to problem (4) exists, after having provided suitable sub- and supersolutions to problem (4). Then, under appropriate assumptions on h, global existence follows by comparison principles, exhibiting a bounded supersolution to problem (4), constructed by means of a bounded positive solution to equation (5). Finally, by an argument based on comparison principle it is proved that such weak solutions are mild solutions, too, in the sense of Definition 3.1. Instead, if h satisfies specific conditions, the blow-up is shown directly using mild solutions.

In this paper we shall extend results given in [**14**], for mild solutions $u \in C\big([0,T); L^p(M)\big) \cap C\big((0,T); L^{p\nu}(M)\big)$ with $u_0 \in L^p(M)$. Let us underline that we remove assumptions $(j) - (jj)$ made in [**14**]; furthermore, we make only use of mild solutions to problem (4), and we do not make use of comparison principles.

To be specific, local existence is proved using a little variation of abstract results given in [**16**]; to apply such general results, preliminarily we discuss some properties of heat semigroup on M. Blow-up in finite time is shown by similar arguments to those used in [**14**] (see also [**1**]), based on heat kernel estimates on M. Instead, global existence is obtained differently from [**1**] and [**14**]; indeed, we do not use $(j) - (jj)$, but adapt to the present situation some ideas of [**17**], where problem (1) was addressed.

In particular, we will prove that local solutions to problem (4) exist for any $u_0 \in L^p(M)$ with $p \geq \frac{N}{2}(\nu - 1)$. Furthermore, if $h(t) = e^{\alpha t}$ $(t \geq 0)$ with $\alpha > \lambda_1(M)(\nu - 1)$, then we have finite time blow-up in $L^{p\nu}(M)$. Instead, problem

$$\begin{cases} \partial_t u = \Delta u + u^\nu & \text{in } M \times (0,T) \\ u = u_0 & \text{in } M \times \{0\}, \end{cases} \tag{6}$$

admits a global mild solution, for every $\nu > 1$. Moreover, global existence prevails also when condition (3) is satisfied, or $h(t) = e^{\alpha t}$ $(t \geq 0)$ for appropriate $\alpha \in \mathbb{R}$.

The paper is organized as follows. In Section 2 we recall preliminaries concerning heat semigroup on M. In Section 3 we state our results about local existence, finite time blow-up and global existence, that will be shown in Sections 4, 5 and 6, respectively.

2 - Preliminaries

Consider a complete noncompact Riemannian manifold M. Let $\{e^{-\Delta t}\}_{t\geq 0}$ be the analytical contraction semigroup generated by $-\Delta$ on $L^2(M)$ (see [8]-[**10**]).

The semigroup $\{e^{-\Delta t}\}_{t\geq 0}$ admits a *heat kernel*, more precisely there exists a function $\mathcal{P} \in C^\infty(M \times M \times (0,\infty))$, $\mathcal{P} > 0$ in $M \times M \times (0,\infty)$ such that

$$e^{-\Delta t}f(x) = \int_M \mathcal{P}(x,y,t)f(y)d\mu_y \quad \big(x \in M, t > 0\big) \tag{7}$$

for any $f \in L^2(M)$. Moreover, there hold:

$$\mathcal{P}(x,y,t) = \mathcal{P}(y,x,t) \quad \text{for all } \ x,y \in M, t > 0;$$

$$\int_M \mathcal{P}(x,y,t)d\mu_y \leq 1 \quad \text{for all } \ x \in M, t > 0; \tag{8}$$

$$\mathcal{P}(x,y,t+s) = \int_M \mathcal{P}(x,z,t)\mathcal{P}(z,y,s)d\mu_z \quad \text{for all } \ x,y \in M, t > 0. \tag{9}$$

We have the following propositions (see [**4**], [**12**] and Section **7** (Exercises included) in [**10**]).

Proposition 2.1. *The semigroup* $\{e^{-\Delta t}\}_{t\geq 0}$ *on* $L^2(M)$

(*i*) *is positive preserving*;
(*ii*) *can be extended to a positive contraction semigroup on* $L^p(M)$ *for every* $p \in [1,\infty]$;
(*iii*) *can be extended to a holomorphic contraction semigroup in* $L^p(M)$ *for every* $p \in (1,\infty)$.

Proposition 2.2. (*i*) *There holds*:

$$e^{-\Delta t}f(x) = \int_M \mathcal{P}(x,y,t)f(y)d\mu_y \quad \big(x \in M, t > 0\big) \tag{10}$$

for any $f \in L^p(M)$, $p \in [1,\infty]$. (*ii*) *Suppose that for every* $t > 0$

$$\sup_{x\in M} \mathcal{P}(x,x,t) < \infty.$$

Then

$$\|e^{-\Delta t}f\|_q \leq \left[\sup_{x\in M}\mathcal{P}(x,x,t)\right]^{\left(\frac{1}{p}-\frac{1}{q}\right)}\|f\|_p \tag{11}$$

for any $f\in L^p(M), 1\leq p<q\leq\infty$.

Let $spec(-\Delta)$ be the spectrum in $L^2(M)$ of the operator $-\Delta$. Note that (see [**10**], Chapter 4)

$$spec(-\Delta)\subseteq[0,\infty).$$

Denote by $\lambda_1(M)$ the bottom of $spec(-\Delta)$, that is

$$\lambda_1(M) := \inf spec(-\Delta).$$

For every $x\in M$ and for every plane $\pi\subseteq T_xM$ denote by $K_\pi(x)$ the *sectional curvature* of the plane π (see [**7**]).

Let M be a Cartan-Hadamard manifold (*i.e.* is a geodesically complete, simply connected Riemannian manifold with nonpositive sectional curvature), suppose that $K_\pi(x)\leq -k^2$ for some constant $k>0$ and for any $x\in M$ and any plane $\pi\subseteq T_pM$. Then (see [**13**]; see also [**8**])

$$\lambda_1(M)\geq\frac{(N-1)^2}{4}k^2. \tag{12}$$

Moreover, (see [**10**], Corollary 15.17 and Remark 14.6)

$$\mathcal{P}(x,y,t)\leq\frac{C}{\left(\min\{t,T\}\right)^{N/2}}\left(1+\frac{d^2}{t}\right)^{N/2}\exp\left\{-\frac{d^2}{4t}-\lambda_1(M)(t-T)_+\right\} \tag{13}$$

for all $x,y\in M, t>0, T>0$ and for some positive constant C; here we have set $d\equiv dist(x,y)$.

From (11) and (13) it immediately follows that, for some constant $C>0$,

$$\|e^{-\Delta t}f\|_q\leq C[\mathcal{F}_T(t)]^{\left(\frac{1}{p}-\frac{1}{q}\right)}\|f\|_p \tag{14}$$

for all $T>0, f\in L^p(M), 1\leq p<q\leq\infty$, where

$$\mathcal{F}_T(t) := \left(\min\{t,T\}\right)^{-N/2}\exp\{-\lambda_1(M)(t-T)_+\}\quad(t>0). \tag{15}$$

Finally, let us recall that if M is a noncompact Riemannian manifold, then (see [**3**], Corollary 1)

$$\lim_{t\to\infty}\frac{\log\mathcal{P}(x,y,t)}{t} = -\lambda_1(M)\quad\text{locally uniformly in } M\times M. \tag{16}$$

3 - Results

In what follows we always make the following assumptions:

$$(A_0) \qquad h \in C([0,\infty)), \ h > 0 \text{ in } [0,\infty);$$

$$(A_1) \qquad \begin{cases} M \text{ is a complete simply connected Riemannian manifold;} \\ \text{moreover, there exists } k>0 \text{ such that for any } x \in M \\ \text{and for any plane } \pi \subseteq T_xM \text{ there holds } K_\pi(x) \le -k^2. \end{cases}$$

Observe that assumption (A_1) implies (12) and that M is a *Cartan-Hadamard* manifold.

To study existence and uniqueness of solutions to problem (4) we shall think of it as an abstract Cauchy problem, namely

$$\begin{cases} u' - \Delta u = h(t)u^\nu, & 0<t<T \\ u(0) = u_0. \end{cases}$$

As a consequence, we make the following

Definition 3.1. *Let* $u_0 \in L^p(M), p \ge 1, u_0 \ge 0, \nu > 1$. *By a mild solution to problem* (4) *we mean any nonnegative function* $u \in C\big([0,T]; L^p(M)\big) \cap C\big((0,T]; L^{p\nu}(M)\big) \cap L^1((0,T); L^{p\nu}(M))$ *such that*

$$u(t) = e^{-\Delta t}u_0 + \int_0^t e^{-\Delta(t-s)}[h(s)u^\nu(s)]ds \quad (t \in [0,T)). \tag{17}$$

From now on we only say solution, instead of mild solution.

A solution to problem (4) is *global*, if it exists for all $t > 0$. Clearly, a solution to problem (4) also satisfies

$$u(t) = \int_M \mathcal{P}(x,y,t)u_0(y)d\mu_y + \int_0^t \int_M \mathcal{P}(x,y,t-s)h(s)u^\nu(y,s)d\mu_y ds \tag{18}$$

$(t \in [0,T))$.

3.1 - *Local existence*

For every $1 \le p \le q$ define

$$a(p,q) := \frac{N}{2}\left(\frac{1}{p} - \frac{1}{q}\right)$$

$$b(p,q) := \frac{p}{q-p}[1 - a(p,q)].$$

Let

$$\frac{N}{2}\,\frac{\nu-1}{\nu} < p < \frac{N}{2}(\nu-1). \tag{19}$$

Define

$$J : L^{p\nu}(M) \to L^{p}(M), \quad J(u) := u^{\nu} \quad (u \in L^{p\nu}(M)). \tag{20}$$

For any $\phi, \psi \in L^{p\nu}(\Omega)$ such that $\|\phi\|_{p\nu} \le r$ and $\|\psi\|_{p\nu} \le r$ $(r > 0)$ we have:

$$\|J(\phi) - J(\psi)\|_{p} \le l(r)\|\phi - \psi\|_{p\nu} \tag{21}$$

with $l : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$l(r) = O(r^{\nu-1}) = O(r^{\frac{1-a(p,p\nu)}{b(p,p\nu)}}) \quad \text{as} \quad r \to \infty; \tag{22}$$

notice that

$$0 < a(p, p\nu) < 1. \tag{23}$$

Furthermore, under the present hypotheses on p and ν there hold:

$$0 < b(p, p\nu) < a(p, p\nu) < 1. \tag{24}$$

We can choose $\tilde{C} > 0$ such that

$$l(r) \le \tilde{C} r^{\frac{1-a(p,p\nu)}{b(p,p\nu)}} \quad \text{for all} \quad r \ge 1.$$

Take positive constants K and τ that satisfy

$$\tilde{C}(2K)^{\frac{1-a(p,p\nu)}{b(p,p\nu)}} \int_0^{\tau} s^{a(p,p\nu)-1-b(p,p\nu)} \max_{[0,\tau]} h < 1$$

and

$$\tilde{C}C \int_0^1 (1-s)^{-a(p,p\nu)} s^{a(p,p\nu)-1-b(p,p\nu)} ds\, K^{\frac{1-a(p,p\nu)}{b(p,p\nu)}} \max_{[0,\tau]} h \le 1,$$

where C is the same constant as in (14).

Then we have the following existence and uniqueness result.

Theorem 3.1. *Let assumptions* $(A_0) - (A_1)$ *be satisfied. Let* $\nu > 1$, $p \ge 1$, $u_0 \in L^p(M)$, $u_0 \ge 0$; *suppose that condition* (19) *is satisfied. Moreover, assume that*

$$\limsup_{t \to 0^+} t^{b(p,p\nu)} \|e^{-\Delta t} u_0\|_{p\nu} < K. \tag{25}$$

Then, for some $T > 0$*, there exists a solution* u *to problem* (4) *such that*

$$\|t^{b(p,p\nu)}u(t)\|_{p\nu} \le \tilde{K} \quad \text{for any} \ \ t \in (0, T), \tag{26}$$

for some constant $\tilde{K} > 0$.

Moreover, if v *is a solution to problem* (4) *satisfying condition* (26) *with* u *replaced by* v*, then* $v(t) \equiv u(t)$ *for any* $t \in [0, T]$.

By Theorem 3.1 we will deduce next

Corollary 3.1. *Let assumptions* $(A_0) - (A_1)$ *be satisfied. Let* $\nu > 1, p \ge 1$; *suppose that condition* (19) *is satisfied. Let* $u_0 \in L^{\frac{N}{2}(\nu-1)}(M) \cap L^p(M), u_0 \ge 0$. *Then there exists a solution* $u \in C\big([0, T]; L^{\frac{N}{2}(\nu-1)}(M)\big)$ *to problem* (4), *for some* $T > 0$, *satisfying condition*

$$\lim_{t\to 0^+} \|t^{b(p,p\nu)}u(t)\|_{p\nu} \ = \ 0. \tag{27}$$

Now, instead of (19), suppose that

$$p > \frac{N}{2}(\nu - 1); \tag{28}$$

then we have the following existence and uniqueness result.

Theorem 3.2. *Let assumptions* $(A_0) - (A_1)$ *be satisfied. Let* $\nu > 1, p \ge 1$; *suppose that condition* (28) *is satisfied. Let* $u_0 \in L^p(M), u_0 \ge 0$. *Then, for some* $T > 0$, *there exists a solution* $u \in L^\infty\big((\varepsilon, T); L^\infty(M)\big)$ $(0<\varepsilon<T)$ *to problem* (4) *such that*

$$\lim_{t\to 0^+} \|t^{a(p,p\nu)}u(t)\|_{p\nu} \ = \ 0. \tag{29}$$

Moreover, if $p \ge \nu$, *then the solution is unique in* $C([0, T]; L^p(M))$.

Remark 3.1. (*i*) Similar results to Theorem 3.1, Corollary 3.1 and Theorem 3.2 have been stated in [**17**], for problem (1) (see also [**16**]).

(*ii*) In Theorem 3.1 and Corollary 3.1, if the maximal existence time $T<\infty$, then

$$\lim_{t\to T^-} \|u(t)\|_p = \infty, \ \ \lim_{t\to T^-} \|u(t)\|_{p\nu} = \infty. \tag{30}$$

This follows from Remark 4.1. The same holds for Theorem 3 in [**16**].

(*iii*) In Theorem 3.2, if the maximal existence time $T<\infty$, then

$$\lim_{t\to T^-} \|u(t)\|_{p\nu} = \infty. \tag{31}$$

This follows from Remark 4.1. The same holds for Theorem 3 in [**16**] and Theorem 3 in [**17**].

(*iv*) If $p > 1$, for the solution u provided by Theorem 3.1, Corollary 3.1 and Theorem 3.2 we have that $u \in C^1\big((0,T);L^p(M)\big)$. This immediately follows by their proofs, in view of Theorem 4.1 and Proposition 2.1-(*iii*).

In view of Remark 3.1, we say that a solution u to problem (4) *blows-up in finite time*, if, for some $T > 0$, (30) and (28) hold true, or (31) and (19) hold true.

Remark 3.2. As it can be easily seen by their proofs, results stated in this Section remain true if instead of (A_1), we assume that for any $T > 0$ there exists $C > 0$ such that

$$\mathcal{P}(x,x,t) \le Ct^{-\frac{N}{2}} \quad \text{for all } \; x \in M, t \in (0,T).$$

Sufficient geometric conditions for the previous inequality can be found in [8]-[**10**].

3.2 - *Finite time blow-up*

Set

$$H(t) := \int_0^t h(s)ds \quad \text{for any } \; t \ge 0.$$

We shall prove the following finite time blow-up result.

Theorem 3.3. *Let assumptions* $(A_0) - (A_1)$ *be satisfied; suppose that* $u_0 \in L^p(M)\; (p \ge 1), u_0 \ge 0,$ $\operatorname{essinf}_\Omega u_0 > 0$ *for some open subset* $\Omega \subset M$ *with* $\mu(\Omega) < \infty$. *Moreover, assume that*

$$\lim_{t\to\infty} \frac{[H(t)]^{\frac{1}{\nu-1}}}{e^{[\lambda_1(M)+\varepsilon]t}} = \infty \tag{32}$$

for some $\varepsilon > 0$. *Then any solution to problem* (4) *blows-up in finite time.*

Remark 3.3. Let

$$\frac{\alpha}{\nu - 1} > \lambda_1(M).$$

If

$$h(t) = e^{\alpha t} \quad \text{for any } \; t \ge 0,$$

then (32) is satisfied for appropriate $\varepsilon > 0$.

3.3 - *Global existence*

We shall prove next theorems, concerning global existence of solutions given by Corollary 3.1 and Theorem 3.2.

Put

$$\gamma := \frac{1}{p} - \frac{1}{p\nu}, \quad \beta := \frac{2}{N(\nu-1)} - \frac{1}{p\nu}.$$

Theorem 3.4. *Let assumptions of Corollary* 3.1 *be satisfied. Take* $p \geq 1$ *such that condition* (19) *is satisfied. Suppose that* $\|u_0\|_{\frac{N}{2}(\nu-1)}$ *is small enough. Furthermore, assume that*

$$\sup_{t\in(1,\infty)} \left\{ e^{\lambda_1(M)(\beta-\gamma)t} \int_1^t e^{\lambda_1(M)(\gamma-\beta\nu)s} h(s)ds + e^{-\lambda_1(M)\beta(\nu-1)t} \max_{[1,t]} h \right\} < \infty. \tag{33}$$

Then the solution u to problem (4) *is global. Moreover, there exists a constant* $\check{C} > 0$ *such that*

$$[\mathcal{F}_1(t)]^{-\beta} \|u(t)\|_{p\nu} \leq \check{C} \quad \textit{for all } t > 0 \tag{34}$$

(*with* $\mathcal{F}_1$ *defined as in* (15)) *and*

$$\|u(t)\|_p \leq \check{C} \quad \textit{for all } t > 0. \tag{35}$$

Clearly, from standard interpolation inequalities it follows that $t \mapsto \|u(t)\|_q$ is bounded in $(0,\infty)$, for any $q \in (p, p\nu)$, hence, in particular, for $q = \frac{N}{2}(\nu-1)$.

Under the same hypotheses on p and ν as in Theorem 3.4, in [**17**], it has been proved that problem (1) admits a global solution such that

$$t^{\frac{N}{2}\beta} \|u(t)\|_{p\nu} \leq \bar{C} \quad \text{for all } t > 0, \tag{36}$$

for some $\bar{C} > 0$, provided that $\|u_0\|_{\frac{N}{2}(\nu-1)}$ is sufficiently small.

In (36) the term $t^{\frac{N}{2}\beta}$ appears, since in $\mathbb{R}^N$,

$$\|e^{-\Delta t} f\|_q \leq C t^{-\frac{N}{2}\left(\frac{1}{p}-\frac{1}{q}\right)} \|f\|_p$$

for $f \in L^p(\mathbb{R}^N), 1 \leq p < q \leq \infty$. Hence, in view of (14), it is natural that in (34) there is the term $[\mathcal{F}_1(t)]^{-\beta}$, instead of $t^{\frac{N}{2}\beta}$.

Theorem 3.5. *Let assumptions of Theorem* 3.2 *be satisfied. Suppose that* $\|u_0\|_p$ *is small enough. Furthermore, assume that*

$$\sup_{t\in(1,\infty)} \left\{ \int_1^t e^{-\lambda_1(M)\gamma(\nu-1)s} h(s)ds + e^{-\lambda_1(M)\gamma(\nu-1)t} \max_{[1,t]} h \right\} < \infty. \tag{37}$$

Then the solution u to problem (4) *is global. Moreover, there exists a constant $\check{C} > 0$ such that*

$$[\mathcal{F}_1(t)]^{-\gamma}\|u(t)\|_{p\nu} \leq \check{C} \quad \textit{for all} \;\; t > 0, \tag{38}$$

(*with $\mathcal{F}_1$ defined as in* (15)) *and*

$$\|u(t)\|_{p} \leq \check{C} \quad \textit{for all} \;\; t > 0. \tag{39}$$

Clearly, from standard interpolation inequalities it follows that $t \mapsto \|u(t)\|_q$ is bounded in $(0, \infty)$, for any $q \in (p, p\nu)$.

In Theorems 3.4 and 3.5 the request that $\|u_0\|_{\frac{N}{2}(\nu-1)}$ and $\|u_0\|_p$ are small enough is meant in the sense that conditions (62) and (71) are satisfied.

Remark 3.4. (i) Let $h \equiv 1$. Then assumptions (33) and (37) are satisfied.
(ii) If condition (3) is satisfied, then assumptions (33) and (37) are satisfied.
(iii) Let

$$h(t) = e^{\alpha t} \quad \text{for any} \;\; t \geq 0.$$

If

$$\alpha < \beta(\nu - 1)\lambda_1(M),$$

then assumption (33) is satisfied.

If

$$\alpha < \gamma(\nu - 1)\lambda_1(M),$$

then assumption (37) is satisfied.

Let $h(t) = e^{\alpha t} \quad (t \geq 0)$. Comparing Remark 3.4-($iii$) with Remark 3.3, we see that it is an open problem to understand whether global existence holds for

$$\lambda_1(M)\gamma(\nu - 1) \leq \alpha \leq \lambda_1(M)(\nu - 1),$$

and for

$$\lambda_1(M)\beta(\nu - 1) \leq \alpha \leq \lambda_1(M)(\nu - 1).$$

Note that in [**14**], under hypotheses $(j) - (jj)$, global existence of bounded solutions is deduced when

$$\alpha < \lambda_1(M)(\nu - 1),$$

and $\|u_0\|_\infty$ is small enough.

4 - Local existence: proofs

Consider the initial value problem

$$(40)\qquad \begin{cases} u'(t) - A[u(t)] = h(t)J[u(t)] & 0<t<T \\ u(0) = \phi, \end{cases}$$

where $h \in C([0,\infty)), h > 0$ in $[0,\infty)$, u is a curve in a Banach space $E, u : [0.T] \to E, \phi \in E$, A is the infinitesimal generator of a C_0 semigroup $\{e^{-At}\}_{t\geq 0}$ on E, and J is a nonlinear function from a Banach space E_J into E.

We shall deal with *mild solutions* to problem (40), that is solutions to the integral equation

$$u(t) = e^{-At}\phi + \int_0^t e^{-A(t-s)}[h(s)J(u(s))]ds \quad \big(t \in [0,T]\big).$$

We suppose the following:

- $J(0) = 0$;
- $J : E_J \to E$ is locally Lipschitz, thus
$$\|J(\phi) - J(\psi)\|_E \leq l(r)\|\phi - \psi\|_{E_J}$$
whenever $\|\phi\|_{E_J} \leq r, \|\psi\|_{E_J} \leq r\ (r > 0)$, for some nondecreasing function $l : (0,\infty) \to (0,\infty)$;
- $E_J \cap E$ is dense in E; if $\phi_n \to \phi_1$ in E and $\phi_n \to \phi_2$ in E_J (as $n \to \infty$), then $\phi_1 = \phi_2$;
- $\|e^{-At}\phi\|_E \leq Me^{\gamma t}\|\phi\|_E$ for all $\phi \in E, t > 0$, for some $M > 0, \gamma \geq 0$;
- there exist $\bar{C} = \bar{C}(T) > 0$ and $0<a<1$ such that
$$\|e^{-At}\phi\|_{E_J} \leq \bar{C}t^{-a}\|\phi\|_E \quad \text{for all } \phi \in E, t \in (0,T];$$
- for each $\phi \in E$, the map $t \mapsto e^{-At}\phi$ is continuous into E_J for any $t > 0$.

Now, we can state the following local existence and uniqueness result.

Theorem 4.1. *Let E, J, E_J and $\{e^{-At}\}_{t\geq 0}$ be as described above.*
(*a*) *Suppose that*

$$\int_\tau^\infty r^{-\frac{1}{a}}l(r)dr < \infty, \quad \textit{for some } \tau > 0.$$

Then, for some $T > 0$, there exists a solution $u(t)$ to problem (40) *such that, for* *some* $k > 0$,

$$\|t^a u(t)\|_{E_J} \leq k \quad \textit{for any} \quad t \in (0, T).$$

If $\{e^{-At}\}_{t\geq 0}$ *is an analytic semigroup on both* E *and* E_j*, then* $u \in C^1\big((0,T); E\big)$. *Moreover, if* $v(t)$ *is another solution to problem* (40) *with* $\|t^a v(t)\|_{E_J}$ *bounded in* $(0, T)$*, then* $u \equiv v$ *in* $[0, T]$.

(b) *Suppose that*

$$l(r) = O(r^{\frac{1-a}{b}}) \quad \textit{as} \quad r \to \infty$$

for some $b \in (0, a)$. *Take* $C > 0$ *such that* $l(r) \leq Cr^{\frac{1-a}{b}}$ *for all* $r \geq 1$, *and* $K > 0$ *and* $\tau > 0$ *such that*

$$Me^{\gamma t}C(2K)^{\frac{1-a}{b}} \int_0^\tau s^{a(p,pv)-1-b(p,pv)} \max_{[0,\tau]} h < 1$$

and

$$\bar{C}C \int_0^1 (1-s)^{-a} s^{a-1-b} ds\, K^{\frac{1-a}{b}} \max_{[0,\tau]} h \leq 1.$$

Suppose that

$$\limsup_{t\to 0^+} \|t^b e^{-At}\phi\|_{E_J} < K,$$

then, for some $T > 0$*, there exists a solution to problem* (40). *If* $\{e^{-At}\}_{t\geq 0}$ *is an analytic semigroup on both* E *and* E_j*, then* $u \in C^1\big((0,T); E\big)$. *Moreover, if* v *is another solution to problem* (40) *with* $\|t^b v(t)\|_{E_J} \leq 2K$*, then* $v \equiv u$ *in* $[0, T]$.

In both cases (a) *and* (b)*, if* $\{e^{-At}\}_{t\geq 0}$ *is positivity preserving,* J *takes nonnegative functions into nonnegative functions, and* $\phi \geq 0$*, then* $u \geq 0$.

Remark 4.1. In Theorem 4.1-(a), if the maximal existence time $T < \infty$, then

$$\lim_{t\to T^-} \|u(t)\|_E = \infty, \quad \lim_{t\to T^-} \|u(t)\|_{E_J} = \infty.$$

Furthermore, in Theorem 4.1-(b), if the maximal existence time $T < \infty$, then

$$\lim_{t\to T^-} \|u(t)\|_{E_J} = \infty.$$

Since $h \in C([0, \infty))$ and $h > 0$ in $[0, \infty)$, Theorem 4.1 and Remark 4.1 can be proved by minor changes in the arguments used to show Theorem 2, Corollaries 2.1 and 3.1 in [**16**], where $h \equiv 1$; we omit the details.

Proof of Theorem 3.1. Let J and l be as in (20) and (21), respectively. Since $p < \frac{N}{2}(\nu - 1) < p\nu$, from (14) we obtain for any $T > 0$

$$\|e^{-\Delta t} u_0\|_{p\nu} \leq C t^{-a(p,p\nu)} \|u_0\|_p \quad (t \in (0, T)). \tag{41}$$

Now, in view of hypothesis (25), conditions (22)-(24), Propositions 2.1 and 2.2 we can apply Theorem 4.1-(a). Hence the conclusion follows. □

Proof of Corollary 3.1. Keep the notation as in the proof of Theorem 3.1. We claim that

$$\limsup_{t \to 0^+} t^{b(p,p\nu)} \|e^{-\Delta t} u_0\|_{p\nu} = 0. \tag{42}$$

In fact, observe that

$$a\left(\frac{N}{2}(p-1), p\nu\right) = b(p, p\nu). \tag{43}$$

Take a sequence $\{u_{0,m}\} \subseteq L^{p\nu}(M) \cap L^{\frac{N}{2}(\nu-1)}(M)$ such that

$$u_{0,m} \to u_0 \quad \text{in } L^{\frac{N}{2}(\nu-1)}(M) \text{ as } m \to \infty.$$

Since $\{e^{-\Delta t}\}_{t \geq 0}$ is a continuous semigroup of contractions in $L^{p\nu}(M)$, using (14) with $p = \frac{N}{2}(\nu - 1)$ and $q = p\nu$, we obtain

$$\begin{aligned}
&\limsup_{t \to 0^+} t^{b(p,p\nu)} \|e^{-\Delta t} u_0\|_{p\nu} \\
&\leq \limsup_{t \to 0^+} t^{b(p,p\nu)} \left[\|e^{-\Delta t}(u_0 - u_{0,m})\|_{p\nu} + \|e^{-\Delta t} u_{0,m}\|_{p\nu}\right] \\
&\leq C \limsup_{t \to 0^+} t^{b(p,p\nu)} \left[t^{-a(\frac{N}{2}(\nu-1),p\nu)} \|u_0 - u_{0,m}\|_{\frac{N}{2}(\nu-1)} + \|u_{0,m}\|_{p\nu}\right] = C\|u_0 - u_{0,m}\|_{\frac{N}{2}(\nu-1)};
\end{aligned}$$

here use of (24), (43) has been made. Sending $m \to \infty$, we get (42).

In view of (22), (42) we can apply Theorem 4.1-(b) to get the existence and uniqueness of a mild solution to problem (4).

It remains to show that $u \in C\big([0, T]; L^{\frac{N}{2}(\nu-1)}(M)\big)$. In fact, $u \in C\big([0, T); L^p(M)\big) \cap C\big((0, T); L^{p\nu}(M)\big)$; hence, by usual interpolation inequalities and (19), $u \in C\big((0, T); L^{\frac{N}{2}(\nu-1)}(M)\big)$. Thus we must only show that $u(t)$ is continuous into $L^{\frac{N}{2}(\nu-1)}(M)$ at $t = 0$. Hence we have to prove that

$$\lim_{t \to 0} \int_0^t \|e^{-(t-s)\Delta} u^\nu(s)\|_{\frac{N}{2}(\nu-1)} ds = 0. \tag{44}$$

In order to prove (44), note that, as it can be easily seen using (42),

$$\sup_{(0,T]} \|t^{b(p,pv)} u(t)\|_{pv} \le 2 \sup_{(0,T]} \|t^{b(p,pv)} e^{-t\Delta} u_0\|_{pv}. \tag{45}$$

Furthermore, by Proposition 2.2,

$$\begin{aligned} &\int_0^t \|e^{-(t-s)\Delta} u^v(s)\|_{\frac{N}{2}(v-1)} ds \\ &\le C \int_0^t (t-s)^{-\frac{N}{2}r} l(\|u(s)\|_{pv}) \|u(s)\|_{pv} ds \\ &\le C \int_0^t (t-s)^{-\frac{N}{2}r} s^{-b(p,pv)(v-1)} s^{-b(p,pv)} ds \sup_{(0,T]} \|s^{b(p,pv)} u(s)\|_{pv} \\ &= C t^{1-\frac{N}{2}r - b(p,pv)v} \sup_{(0,T]} \|s^{b(p,pv)} u(s)\|_{pv}, \end{aligned} \tag{46}$$

where $r := \dfrac{1}{p} - \dfrac{2}{N(v-1)}$ and $C > 0$ is a positive constant. Now, (44) follows from (45), (46) and (42). $\square$

Proof of Theorem 3.2. Under the present hypotheses

$$0 < a(p,pv) < \frac{1}{v} < b(p,pv).$$

Hence

$$\int_1^\infty r^{-\frac{1}{a(p,pv)} + v - 1} \, dr < \infty.$$

Then, due to Propositions 2.1 and 2.2, the existence of a mild solution follows by Theorem 4.1$-(a)$.

We claim that

$$\limsup_{t \to 0^+} t^{a(p,pv)} \|e^{-\Delta t} u_0\|_{pv} = 0. \tag{47}$$

In fact, choose a sequence $\{u_{0,m}\} \subseteq L^{pv}(M) \cap L^p(M)$ such that

$$u_{0,m} \to u_0 \quad \text{in} \;\; L^p(M) \;\; \text{as} \;\; m \to \infty.$$

Since $\{e^{-\Delta t}\}_{t \ge 0}$ is a continuous semigroup of contractions in $L^{pv}(M)$, using (14) with

$q = pv$, we obtain

$$\limsup_{t\to 0^+} t^{a(p,pv)} \|e^{-\Delta t} u_0\|_{pv}$$

$$\leq \limsup_{t\to 0^+} t^{a(p,pv)} \left[\|e^{-\Delta t}(u_0 - u_{0,m})\|_{pv} + \|e^{-\Delta t} u_{0,m}\|_{pv}\right]$$

$$\leq C \limsup_{t\to 0^+} t^{a(p,pv)} \left[t^{-a(p,pv)} \|u_0 - u_{0,m}\|_p + \|u_{0,m}\|_{pv}\right] = C\|u_0 - u_{0,m}\|_p;$$

here use of (24), (43) has been made. Sending $m \to \infty$, we get (47).

We have to show that $u \in L^\infty\big((\varepsilon, T); L^\infty(M)\big)$ $(0<\varepsilon<T)$. This easily follows by standard arguments (see, *e.g.* [**15**]), in view of Proposition 2.2. In fact, choose $\varepsilon > 0$ small, let $\kappa_1 := vp$. By Definition 3.1,

$$u \in L^\infty\big((\varepsilon, T); L^{\kappa_1}(M)\big).$$

Clearly,

$$u(t+\varepsilon) = e^{-t\Delta} u(\varepsilon) + \int_0^t e^{-(t-s)\Delta} u^v(s+\varepsilon) ds. \tag{48}$$

Take $\kappa_2 > \kappa_1$ such that $\beta_1 := \frac{N}{2}\left(\frac{v}{\kappa_1} - \frac{1}{\kappa_2}\right) < 1$; this is possible since $p > \frac{N}{2}(v-1)$. Set $\beta_2 := \frac{N}{2}\left(\frac{1}{\kappa_1} - \frac{1}{\kappa_2}\right)$. From (48) and Proposition 2.2 we get

$$\|u(t+\varepsilon)\|_{\kappa_2} \leq t^{-\beta_2} \|u(\varepsilon)\|_{\kappa_1} + \max_{[0,T]} h \int_0^t (t-s)^{-\beta_1} \|u(s+\varepsilon)\|_{\kappa_1}^v \leq C(\varepsilon) \quad (\varepsilon < t < T - \varepsilon).$$

Hence $u \in L^\infty\big((2\varepsilon, T); L^{\kappa_2}(M)\big)$. Now a standard bootstrap argument shows that $u \in L^\infty\big((\varepsilon, T); L^\infty(M)\big)$.

To prove the uniqueness we follow an argument used in the proof of Proposition 1-(*e*) in [**2**]. In fact, take $A > 0$ so that

$$\|u(t)\|_p \leq A, \quad \|v(t)\|_p \leq A \quad \text{for all } \ t \in [0, T].$$

We have:

$$u(t) - v(t) = \int_0^t e^{(t-s)\Delta}[u^v(s) - v^v(s)] ds.$$

Since $p \geq v$ we get

$$\|u^v - v^v\|_p \leq v\|u^{v-1} + v^{v-1}\|_{\frac{p}{v}-1} \|u - v\|_p \leq 2vA^{v-1}\|u - v\|_p.$$

From (14) it follows

$$\|u(t)-v(t)\|_p \le 2C_2A^{\nu-1}\,\frac{t^{1-\frac{N(\nu-1)}{2p}}}{1-\dfrac{N(\nu-1)}{2p}}\,\max_{[0,t]}\|u-v\|_p,$$

which is impossible for $t>0$ small enough, if $u\not\equiv v$; hence $u\equiv v$. □

5 - Blow-up: proofs

Theorem 3.3 will be proved by the same arguments as in [**14**].

Lemma 5.1. *Let assumptions* $(A_0)-(A_1)$ *be satisfied; suppose that* $u_0\in L^p(M)$ $(p\ge 1), u_0\ge 0,$ $\operatorname{essinf}_\Omega u_0>0$ *for some open subset* $\Omega\subset M$ *with* $\mu(\Omega)<\infty$. *Let* $\varepsilon>0$. *Then there exist* $t_0>0$ *and* $C_1>0$ *such that*

$$(e^{-t\Delta}u_0)(x)\ge \frac{C_1}{e^{[\lambda_1(M)+\varepsilon]t}} \quad \textit{for any } x\in\Omega, t>t_0. \tag{49}$$

Proof. Let $\varepsilon>0$. By (16) there exists $t_0>0$ such that for any $x,y\in\Omega$ and $t>t_0$ there holds

$$\mathcal{P}(x,y,t)\ge \frac{1}{e^{[\lambda_1(M)+\varepsilon]t}};$$

hence

$$(e^{-t\Delta}u_0)(x)\ge \int_\Omega \mathcal{P}(x,y,t)u_0(y)d\mu_y \ge \frac{\mu(\Omega)\operatorname{essinf}_\Omega u_0}{e^{[\lambda_1(M)+\varepsilon]t}}.$$

This completes the proof. □

Lemma 5.2. *Let assumptions* $(A_0)-(A_1)$ *be satisfied; suppose that* $u_0\in L^p(M)$ $(p\ge 1), u_0\ge 0,$ $\operatorname{essinf}_\Omega u_0>0$ *for some open subset* $\Omega\subset M$ *with* $\mu(\Omega)<\infty$. *Let* $\varepsilon>0$. *Let there exists a solution to problem* (4). *Then*

$$\left(e^{\tau\Delta}u_0\right)^{\nu-1}\le \frac{1}{(\nu-1)H(\tau)} \quad \textit{for any } x\in M, \tau\in(0,T). \tag{50}$$

Proof. Let $\tau\in(0,T)$. Let u be a solution to problem (4). We multiply by $\mathcal{P}(x,z,\tau-t)$ equality (18) with x replaced by z, then integrate over M and use (9). So, we get

$$\int_M \mathcal{P}(x,z,\tau-t)u(z,t)d\mu_z = \int_M \mathcal{P}(x,y,\tau)u_0(y)d\mu_y$$

$$+\int_0^t \int_M \mathcal{P}(x,y,\tau-s)h(s)u^\nu(y,s)d\mu_y ds \qquad (t\in(0,\tau)),$$

that is

$$\phi_\tau(x,t) = \phi_\tau(x,0) + \int_0^t \int_M \mathcal{P}(x,y,\tau-s)h(s)u^\nu(y,s)d\mu_y ds, \tag{51}$$

where we have set

$$\phi_\tau(x,t) := \int_M \mathcal{P}(x,z,\tau-t)u(z,t)d\mu_z \quad (x\in M, t\in(0,\tau)).$$

By Jensen's inequality (applied, for each $s\in(0,\tau)$ and $x\in M$, with respect to the measure $d\nu_y := \mathcal{P}(x,y,\tau-s)d\mu_y$),

$$[\phi_\tau(x,s)]^\nu \le \int_M \mathcal{P}(x,y,\tau-s)u^\nu(y,s)d\mu_y \quad \big(x\in M, s\in(0,\tau)\big).$$

This combined with (51) yields

$$\int_0^t h(s)[\phi_\tau(x,s)]^\nu ds \le \phi_\tau(x,t) - \phi_\tau(x,0) \quad \big(x\in M, t\in(0,\tau)\big).$$

Then by a Gronwall type argument,

$$(\nu-1)H(t) \le \frac{1}{[\phi_\tau(x,0)]^{\nu-1}} - \frac{1}{[\phi_\tau(x,t)]^{\nu-1}} \quad \big(x\in M, t\in(0,\tau)\big),$$

hence the conclusion immediately follows. □

Now we can show Theorem 3.3.

Proof of Theorem 3.3. By contradiction, suppose that there exists a global solution u to problem (4).

Now, take $\Omega\subseteq M$ and $\varepsilon>0$ as in Lemma 5.1. Hence

$$\phi_\tau(x,0) \ge \frac{C_1}{e^{[\lambda_1(M)+\varepsilon]\tau}} \quad \text{for any } x\in\Omega; \tau>t_0 \tag{52}$$

for some $C_1>0$ and $t_0>0$.

Hence by Lemma 5.2,

$$\phi_\tau(x,0) \le \left(\frac{1}{\nu-1}\right)^{\frac{1}{\nu-1}} [H(\tau)]^{-\frac{1}{\nu-1}} \quad \text{for any } x \in M, \tau > 0. \tag{53}$$

From (52)-(53) it follows that for any $\tau > t_0$ we have

$$\frac{[H(\tau)]^{\frac{1}{\nu-1}}}{e^{[\lambda_1(M)+\varepsilon]\tau}} \le \frac{1}{C_1}\left(\frac{1}{\nu-1}\right)^{\frac{1}{\nu-1}}.$$

If we send $\tau \to \infty$ in the previous inequality, we get a contradiction with (32); hence the proof is complete. □

6 - Global existence: proofs

Proof of Theorem 3.4. Let $\tau \in (0,T)$. By (17), using (14), we get:

$$\begin{aligned}
&[\mathcal{F}_\tau(t)]^{-\beta}\|u(t)\|_{p\nu} \le [\mathcal{F}_\tau(t)]^{-\beta}\|e^{-t\Delta}u_0\|_{p\nu} \\
&+ [\mathcal{F}_\tau(t)]^{-\beta}\int_0^t \|e^{-(t-s)\Delta}[h(s)u^\nu(s)]\|_{p\nu} ds \le C\|u_0\|_{\frac{N}{2}(\nu-1)} \\
&+ C[\mathcal{F}_\tau(t)]^{-\beta}\int_0^t [\mathcal{F}_\tau(t-s)]^\gamma h(s)\|u^\nu(s)\|_p ds \le C\|u_0\|_{\frac{N}{2}(\nu-1)} \\
&+ C[\mathcal{F}_\tau(t)]^{-\beta}\int_0^t [\mathcal{F}_\tau(t-s)]^\gamma [\mathcal{F}_\tau(s)]^{\beta\nu} h(s) ds \sup_{s\in(0,t)} \|[\mathcal{F}_\tau(s)]^{-\beta}u(s)\|_{p\nu}^\nu ds
\end{aligned} \tag{54}$$

for all $t \in (0,T)$. Observe that

$$\frac{N}{2}\gamma < 1 \tag{55}$$

and

$$\frac{N}{2}\beta\nu < 1. \tag{56}$$

Define

$$\psi(t) := [\mathcal{F}_\tau(t)]^{-\beta}\int_0^t [\mathcal{F}_\tau(t-s)]^\gamma [\mathcal{F}_\tau(s)]^{\beta\nu} h(s) ds \quad (t \ge 0).$$

If $t \le \tau$, then

$$\psi(t) = t^{\frac{N}{2}\beta} \int_0^t (t-s)^{-\frac{N}{2}\gamma} s^{-\frac{N}{2}\beta\nu} h(s)\, ds. \tag{57}$$

Note that, for any $t \in (0, \tau]$,

$$\begin{aligned} 0 &\le t^{\frac{N}{2}\beta} \int_0^t (t-s)^{-\frac{N}{2}\gamma} s^{-\frac{N}{2}\beta\nu} h(s) \\ &\le t^{\frac{N}{2}\beta} \Big[\int_0^{\frac{t}{2}} (t-s)^{-\frac{N}{2}\gamma} s^{-\frac{N}{2}\beta\nu} + \int_{\frac{t}{2}}^t (t-s)^{-\frac{N}{2}\gamma} s^{-\frac{N}{2}\beta\nu} \Big] \max_{[0,\tau]} h \\ &\le t^{\frac{N}{2}\beta} \Big[\Big(\frac{t}{2}\Big)^{-\frac{N}{2}\gamma} \int_0^{\frac{t}{2}} s^{-\frac{N}{2}\beta\nu} ds + \Big(\frac{t}{2}\Big)^{-\frac{N}{2}\beta\nu} \int_{\frac{t}{2}}^t (t-s)^{-\frac{N}{2}\gamma} ds \Big] \max_{[0,\tau]} h \\ &\le t^{\frac{N}{2}\beta} \left(\frac{1}{1-\frac{N}{2}\beta\nu} + \frac{1}{1-\frac{N}{2}\gamma} \right) \Big(\frac{t}{2}\Big)^{-\frac{N}{2}\gamma + 1 - \frac{N}{2}\beta\nu} \max_{[0,\tau]} h \\ &= \left(\frac{1}{1-\frac{N}{2}\beta\nu} + \frac{1}{1-\frac{N}{2}\gamma} \right) \frac{\max_{[0,\tau]} h}{2^{-\frac{N}{2}\gamma+1-\frac{N}{2}\beta\nu}}, \end{aligned} \tag{58}$$

since $\dfrac{N}{2}(\beta - \gamma - \beta\nu) + 1 = 0$ and (55)-(56) hold true.

If $t > 2\tau$, then

$$\begin{aligned} \psi(t) = \tau^{\frac{N}{2}\beta} e^{\lambda_1(M)(t-\tau)\beta} \Big\{ &\tau^{-\frac{N}{2}\gamma} e^{-\lambda_1(M)(t-\tau)\gamma} \int_0^\tau e^{\lambda_1(M)\gamma s} s^{-\frac{N}{2}\beta\nu} h(s)\, ds \\ &+ \tau^{-\frac{N}{2}(\beta\nu+\gamma)} e^{-\lambda_1(M)(\gamma t - \gamma\tau - \beta\nu\tau)} \int_\tau^{t-\tau} e^{\lambda_1(M)(\gamma-\beta\nu)s} h(s) ds \\ &+ \tau^{-\frac{N}{2}\beta\tau} e^{\lambda_1(M)\beta\nu\tau} \int_{t-\tau}^t (t-s)^{-\frac{N}{2}\gamma} e^{-\lambda_1(M)\beta\nu s} h(s) ds \Big\}. \end{aligned} \tag{59}$$

Since $\beta < \gamma$, and $\psi \in C((0,\infty))$, from (33), (56), (57), (58) and (59) we deduce that

$$\kappa := C \sup_{t \in [0,\infty)} \psi(t) < \infty.$$

Define

$$\varphi(t) := \sup_{s\in(0,t)} \|[\mathcal{F}_\tau(s)]^{-\beta} u(s)\|_{p\nu}^{\nu} \quad \big(t \in [0,\infty)\big).$$

Note that $\varphi \in C\big([0,\infty)\big), \varphi$ is increasing in $(0,\infty)$; moreover, by (27) and Jensen's inequality,

$$\varphi(0) = \lim_{t\to 0^+} \|s^{\frac{N}{2}\beta} u(s)\|_{p\nu}^{\nu} = 0.$$

From (54) it follows

$$\varphi(t) \leq C\|u_0\|_{\frac{N}{2}(\nu-1)} + \kappa[\varphi(t)]^{\nu} \quad \text{for all} \quad t \geq 0. \tag{60}$$

Take $\varepsilon > 0$ such that

$$\varepsilon < \big(2^{\nu+1}\kappa\big)^{-\frac{1}{\nu-1}}. \tag{61}$$

We can assume that

$$\|u_0\|_{\frac{N}{2}(\nu-1)} \leq \frac{\varepsilon}{C}. \tag{62}$$

We claim that

$$\varphi(t) \leq 2\varepsilon \quad \text{for all} \quad t > 0.$$

In fact, suppose by contradiction that there is $t_0 \in (0,\infty)$ such that $\varphi(t_0) = 2\varepsilon$. By (60)-(62) we obtain

$$2\varepsilon \leq \varepsilon + \frac{(2\varepsilon)^{\nu}}{2^{\nu+1}\varepsilon^{\nu-1}},$$

which is absurd. Hence the claim and (34) follow.

By (17), Proposition 2.1 and (34) we get for any $t > 0$

$$\|u(t)\|_p \leq \|u_0\|_p + \int_0^t \|u^{\nu}(t)\|_p \, h(s)ds$$

$$\leq \|u_0\|_p + \int_0^t \|u(t)\|_{p\nu}^{\nu} h(s)ds \leq \|u_0\|_p + C\int_0^t [\mathcal{F}_1(s)]^{\beta\nu} h(s)ds$$

$$\leq \|u_0\|_p + C\int_0^1 s^{-\frac{N}{2}\beta\nu} ds \max_{[0,1]} h + C\int_1^t e^{-\lambda_1(M)\beta\nu s} h(s)ds.$$

This combined with (33) and (56) yields (35); thus the proof is complete. □

Theorem 3.4 will be proved by minor changes in the previous proof; we give all details for reader's convenience.

Proof of Theorem 3.5. Let $\tau \in (0,T)$. By (17) and (14), we get:

$$
\begin{aligned}
[\mathcal{F}_\tau(t)]^{-\gamma}\|u(t)\|_{p\nu} &\le [\mathcal{F}_\tau(t)]^{-\gamma}\|e^{-t\Delta}u_0\|_{p\nu} \\
&+ [\mathcal{F}_\tau(t)]^{-\gamma}\int_0^t \|e^{-(t-s)\Delta}[h(s)u^\nu(s)]\|_{p\nu}ds \le C\|u_0\|_p \\
&+ C[\mathcal{F}_\tau(t)]^{-\gamma}\int_0^t [\mathcal{F}_\tau(t-s)]^{\gamma}h(s)\|u^\nu(s)\|_p ds \le C\|u_0\|_p \\
&+ C[\mathcal{F}_\tau(t)]^{-\gamma}\int_0^t [\mathcal{F}_\tau(t-s)]^{\gamma}[\mathcal{F}_\tau(s)]^{\gamma\nu}h(s)ds \sup_{s\in(0,t)}\|[\mathcal{F}_\tau(s)]^{-\gamma}u(s)\|_{p\nu}^{\nu}ds
\end{aligned}
\tag{63}
$$

for all $t \in (0,T)$. Observe that

$$\frac{N}{2}\gamma < 1 \tag{64}$$

and

$$\frac{N}{2}\gamma\nu < 1. \tag{65}$$

Define

$$\psi(t) := [\mathcal{F}_\tau(t)]^{-\gamma}\int_0^t [\mathcal{F}_\tau(t-s)]^{\gamma}[\mathcal{F}_\tau(s)]^{\gamma\nu}h(s)ds \quad (t \ge 0).$$

If $t \le \tau$, then

$$\psi(t) = t^{\frac{N}{2}\gamma}\int_0^t (t-s)^{-\frac{N}{2}\gamma}s^{-\frac{N}{2}\gamma\nu}h(s)\,ds. \tag{66}$$

Note that, for any $t \in (0,\tau]$,

$$
\begin{aligned}
0 &\le t^{\frac{N}{2}\gamma}\int_0^t (t-s)^{-\frac{N}{2}\gamma} s^{-\frac{N}{2}\gamma\nu} h(s)\\
&\le t^{\frac{N}{2}\beta}\Big[\int_0^{\frac{t}{2}} (t-s)^{-\frac{N}{2}\gamma} s^{\frac{N}{2}\beta\nu} + \int_{\frac{t}{2}}^t (t-s)^{-\frac{N}{2}\gamma} s^{-\frac{N}{2}\beta\nu}\Big]\max_{[0,\tau]} h\\
&\le t^{-\frac{N}{2}\gamma}\Big[\left(\frac{t}{2}\right)^{-\frac{N}{2}\gamma}\int_0^{\frac{t}{2}} s^{-\frac{N}{2}\gamma\nu} ds + \left(\frac{t}{2}\right)^{-\frac{N}{2}\gamma\nu}\int_{\frac{t}{2}}^t (t-s)^{-\frac{N}{2}\gamma} ds\Big]\max_{[0,\tau]} h \qquad (67)\\
&\le t^{\frac{N}{2}\gamma}\left(\frac{1}{1-\frac{N}{2}\gamma\nu} + \frac{1}{1-\frac{N}{2}\gamma}\right)\left(\frac{t}{2}\right)^{-\frac{N}{2}\gamma+1-\frac{N}{2}\gamma\nu}\max_{[0,\tau]} h\\
&= \left(\frac{1}{1-\frac{N}{2}\gamma\nu} + \frac{1}{1-\frac{N}{2}\gamma}\right)\frac{\tau^{1-\frac{N}{2}\gamma\nu}\max_{[0,\tau]} h}{2^{-\frac{N}{2}\gamma+1-\frac{N}{2}\gamma\nu}},
\end{aligned}
$$

since (64)-(65) hold true.

If $t > 2\tau$, then

$$
\begin{aligned}
\psi(t) = \tau^{\frac{N}{2}\gamma} e^{\lambda_1(M)(t-\tau)\gamma}\Big\{&\tau^{-\frac{N}{2}\gamma} e^{-\lambda_1(M)(t-\tau)\gamma}\int_0^\tau e^{\lambda_1(M)\gamma s} s^{-\frac{N}{2}\gamma\nu} h(s)\, ds\\
&+ \tau^{-\frac{N}{2}(\beta\nu+\gamma)} e^{-\lambda_1(M)(\gamma t-\gamma\tau-\gamma\nu\tau)}\int_\tau^{t-\tau} e^{\lambda_1(M)(\gamma-\gamma\nu)s} h(s) ds \qquad (68)\\
&+ \tau^{-\frac{N}{2}\gamma\tau} e^{\lambda_1(M)\gamma\nu\tau}\int_{t-\tau}^t (t-s)^{-\frac{N}{2}\gamma} e^{-\lambda_1(M)\gamma\nu s} h(s) ds\Big\}.
\end{aligned}
$$

Since $\psi \in C((0,\infty))$, from (37), (65), (66), (67) and (68) we deduce that

$$\kappa := C \sup_{t\in[0,\infty)} \psi(t) < \infty.$$

Define

$$\varphi(t) := \sup_{s\in(0,t)} \|[\mathcal{F}_\tau(s)]^{-\gamma} u(s)\|_{p\nu}^{\nu} \quad \big(t \in [0,\infty)\big).$$

Note that $\varphi \in C\big([0,\infty)\big), \varphi$ is increasing in $(0,\infty)$; moreover, by (29) and Jensen's inequality,

$$\varphi(0) = \lim_{t\to 0^+} \|s^{\frac{N}{2}\gamma} u(s)\|_{p\nu}^{\nu} = 0.$$

From (63) it follows

$$\varphi(t) \leq C\|u_0\|_p + \kappa[\varphi(t)]^{\nu} \quad \text{for all} \quad t \geq 0. \tag{69}$$

Take $\varepsilon > 0$ such that

$$\varepsilon < \left(2^{\nu+1}\kappa\right)^{-\frac{1}{\nu-1}}. \tag{70}$$

We can assume that

$$\|u_0\|_{\frac{N}{2}(\nu-1)} \leq \frac{\varepsilon}{C}. \tag{71}$$

We claim that

$$\varphi(t) \leq 2\varepsilon \quad \text{for all} \;\; t > 0.$$

In fact, suppose by contradiction that there is $t_0 \in (0, \infty)$ such that $\varphi(t_0) = 2\varepsilon$. By (69)-(71) we obtain

$$2\varepsilon \leq \varepsilon + \frac{(2\varepsilon)^{\nu}}{2^{\nu+1}\varepsilon^{\nu-1}},$$

which is absurd. Hence the claim and (38) follow.

By (17), Proposition 2.1 and (38) we get for any $t > 0$

$$\|u(t)\|_p \leq \|u_0\|_p + \int_0^t \|u^{\nu}(t)\|_p \, h(s)ds$$

$$\leq \|u_0\|_p + \int_0^t \|u(t)\|_{p\nu}^{\nu} h(s)ds \leq \|u_0\|_p + C\int_0^t [\mathcal{F}_1(s)]^{\gamma\nu} h(s)ds$$

$$\leq \|u_0\|_p + C\int_0^1 s^{-\frac{N}{2}\gamma\nu}ds \max_{[0,1]} h + C\int_1^t e^{-\lambda_1(M)\gamma\nu s}h(s)ds.$$

This combined with (37) and (65) yields (39); thus the proof is complete. □

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