Riv. Mat. Univ. Parma, Vol. 4 (2013), 81-111

Luisa Consiglieri

Dynamic bilateral boundary conditions on interfaces

Abstract. Two boundary value problems for an elliptic equation in divergence form with bounded discontinuous coefficient are studied in a bidomain. On the interface, generalized dynamic boundary conditions such as of the Wentzell-type and Signorini-type transmission are considered in a subdifferential form. Several non-constant coefficients and nonlinearities are the main objective of the present work. Generalized solutions are built via time discretization.

Keywords. Wentzell transmission, Signorini transmission, subdifferential, Rothe method.

Mathematics Subject Classification (2010): 35J87, 49M25, 78A70.

Contents

Received: May 23, 2012; accepted in revised form: September 28, 2012.

1 - Introduction

In the description of real life phenomena, challenges in science and technology such as diffusion problems with transmission conditions are being addressed (cf. for instance [**7**] and the references therein). We refer to [**14, 15**] a general framework which allows to prove, in a unified and systematic way, the analyticity of semigroups generated by operators with generalized Wentzell boundary conditions on function spaces with bounded trace operators. The thin obstacle problem (also called the Signorini problem) models threshold phenomena like contact problem, thermostatic device or semipermeable membranes [**4**]. In [**1**] the study relies on the presence of differential operators. We point out that their method is based on a fixed point argument. Under continuous or even constant coefficients, the regularity was shown for the Laplace-Wentzell problem [**13**] or the thin obstacle problem [**5**]. The question of dynamic boundary conditions can be found in frictional contact problems (see [**21**] and the references therein). Their theoretical and numerical achievements are based on the time discretization method being closely related to ours.

With the aim of forcing to make realistic assumptions and then deal with the mathematical consequences, we prove the well-posedness of boundary value problems subject to dynamic nonlinear and friction-type boundary conditions. The present work extends the known results of Laplacian operator to a general elliptic operator in divergence form with bounded measurable coefficient in the context of diffusion processes. The motivation comes essentially from the models for the electrical conduction in biological tissues [**1, 6, 10, 11**]. The construction of generalized solutions is shown via time discretization, following the Rothe method [**17, 19, 20**].

Let Ω_1 and Ω_2 be two disjoint bounded domains of $\mathbb{R}^n$ $(n \geq 2)$ such that $\bar{\Omega} = \bar{\Omega}_1 \cup \bar{\Omega}_2$ is connected with Lipschitz boundary. Let $\Gamma = \partial\Omega_1 \cap \Omega \subset \partial\Omega_2$ denote a nonempty interface. We are interested on that one of the following physical descriptions can occur.

1. If $\partial\Omega_1 \subset \Omega$ then Γ is a closed curve ($n = 2$) or surface ($n \geq 3$). Currently, Ω_1 and Ω_2 are called the inner and the outer domains of Ω, respectively.
2. If $\Gamma_1 := \partial\Omega_1 \setminus \overline{\Gamma} = \operatorname{int}(\partial\Omega_1 \cap \partial\Omega) \neq \emptyset$ then
   - if $n = 2$, Γ is relatively open (see Figure 1 (a));
   - if $n = 3$, Ω_1 stands for a cylindrical-type domain such that Γ_1 represents its top and/or bottom (see Figure 1 (b)). Other situations such as three versions of the illustration shown in Figure 1 (a) can also be of interest.
3. The case of $\partial\Omega_1 \cap \partial\Omega \neq \emptyset$ with $\operatorname{meas}(\partial\Omega_1 \cap \partial\Omega) = 0$ can be clearly included whenever $\partial\Omega_2$ is Lipschitz continuous (see Figure 1 (c)).

In conclusion, we assume that Γ is a $(n-1)$-dimensional interface, and $\partial\Omega_k$ $(k = 1, 2)$ are Lipschitz continuous. The domains have neither cuts (cracks) nor cusps, and situations as in Figure 1 (d) are excluded. Define a relatively open $(n-1)$-dimensional set $\Gamma_2 \subset \partial\Omega_2 \setminus \Gamma$, with $\operatorname{meas}(\Gamma_2) > 0$, and $\Gamma_D = \Gamma_1 \cup \Gamma_2$ where we will impose Dirichlet boundary conditions.

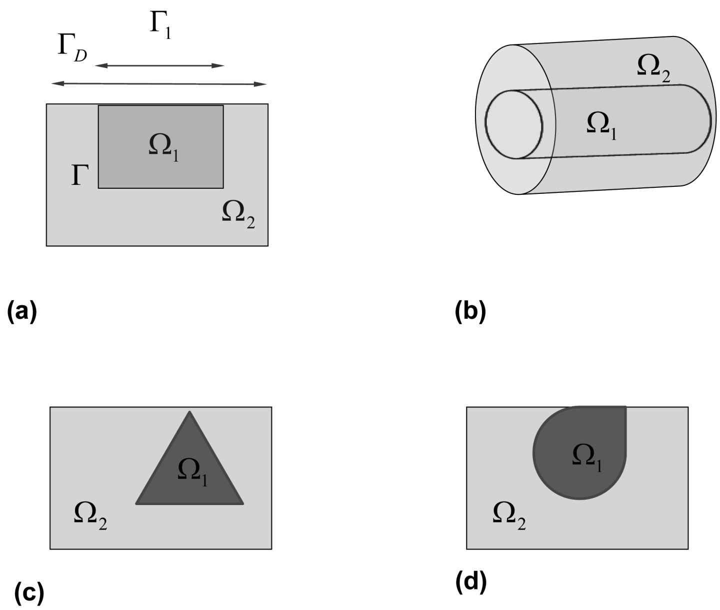

Fig. 1. The geometry and interface conditions: 2D (a) and 3D (b) models when $\Gamma_1 \neq 0$; (c) other possible situation; (d) 2D counterexample.

Let us introduce the problems under study. For $T > 0$, find $u_k : \Omega_k \times]0, T[\to \mathbb{R}$ satisfying

$$-\nabla \cdot (\sigma_k \nabla u_k) = f_k \text{ in } \Omega_k \quad (k = 1, 2). \tag{1}$$

The first mathematical interest of this problem is due to the discontinuous coefficient which reflects the spatial dependence of the conductivity on the electrical conduction in different materials.

On the exterior boundary $\partial\Omega = (\partial\Omega_2 \setminus \Gamma) \cup \Gamma_1$, we have homogeneous mixed boundary condition

$$\nabla u_2 \cdot \mathbf{n} = 0 \text{ on } \partial\Omega \setminus \Gamma_D \quad \text{ and } u_k = 0 \text{ on } \Gamma_D. \tag{2}$$

On the interface Γ, we study two different types of dynamic bilateral conditions.

Wentzell-type transmission

The generalized Wentzell transmission boundary condition is given by

$$u_1 = u_2 \quad \text{and} \tag{3}$$

$$[\sigma\nabla u \cdot \mathbf{n}] + \beta\Delta u_1 - \alpha\partial_t u_1 \in \partial j(u_1) \qquad \text{on } \Sigma := \Gamma \times]0, T[, \tag{4}$$

under the initial condition

$$u_1(\cdot, 0) = S \text{ on } \Gamma \tag{5}$$

where α and S are known functions and β is a non-negative constant. Indeed, the coefficients can be obtained as limits of certain integrals [**22**]. If $\beta = 0$, the transmission boundary condition (3)-(5) accounts for the transmission in a thin (or lower dimensional) porous layer. Here $\mathbf{n}$ is the normal unit vector to Γ pointing into Ω_2, ∂ is the subdifferential with respect to the argument of the function j, and $[\,\cdot\,]$ denotes the jump of a quantity across the interface in direction of $\mathbf{n}$, e.g. $[\sigma\nabla u \cdot \mathbf{n}] := \sigma_2\nabla u_2 \cdot \mathbf{n} - \sigma_1\nabla u_1 \cdot \mathbf{n}$.

Signorini-type transmission

The transmission that characterizes the thin obstacle problems such as the semipermeable membrane is constituted by the jump condition

$$[\sigma\nabla u \cdot \mathbf{n}] = g \text{ on } \Gamma, \tag{6}$$

and the Signorini-type boundary condition

$$\sigma_2\nabla u_2 \cdot \mathbf{n} - \alpha\partial_t[u] \in \partial j([u]) \text{ on } \Sigma = \Gamma \times]0, T[, \tag{7}$$

accomplished with the initial condition

$$[u](\cdot, 0) = S \text{ on } \Gamma \tag{8}$$

where g, α, j and S are known functions [**1**].

The most common application appears when ∂j represents the indicatrice Heaviside. These boundary value problems also model some of the slip phenomena observed in contact problems [**12, 21**]. Other related problems are the unilateral problems [**3**].

The paper is organized as follows. In next Section we set the functional space framework, the assumptions on the data, and the main results. Sections 3 and 6 are devoted to the proofs of existence and uniqueness of weak solutions of each problem, namely provided by the Wentzell-type and Signorini-type transmission, respectively. These two sections have similar structures based on the time discretization technique and they are split into several subsections in order to clarify the exposition. In Section 5, we show how the unique solution to the boundary value problem provided by a thin porous layer can be obtained as the limit of perturbed problems. Finally, some additional regularity is shown in Sections 4 and 7 corresponding to the generalized solutions of Sections 3 and 6, respectively.

2 - Functional space framework and main results

The data are given under the following regularity assumptions. Here we assume that

$$\sigma_k \in L^\infty(\Omega_k): \ \exists \sigma_\#, \sigma^\# > 0, \quad \sigma_\# \leq \sigma_k(x) \leq \sigma^\#, \quad \text{for a.a. } x \in \Omega_k; \tag{9}$$

for $k = 1, 2$,

$$\alpha \in L^\infty(\Gamma): \ \exists \alpha_\#, \alpha^\# > 0, \quad \alpha_\# \leq \alpha(x) \leq \alpha^\#, \quad \text{for a.a. } x \in \Gamma; \tag{10}$$

and

(11) $j : \mathbb{R} \to \mathbb{R}$ is a convex and lower semicontinuous function, $j \geq 0$ and $j(0) = 0$.

Let us define

$$\begin{aligned} H^1_{\Gamma_D}(\Omega) &= \{v \in H^1(\Omega): \ v|_{\Gamma_D} = 0\}; \\ H^1_{\Gamma_k}(\Omega_k) &= \{v \in H^1(\Omega_k): \ v|_{\Gamma_k} = 0\}, \qquad (k = 1, 2). \end{aligned}$$

For a Lipschitz domain Ω_1, the trace operator $H^1_{\Gamma_1}(\Omega_1) \to H^{1/2}_{00}(\Gamma)$ has bounded linear right inverse, that is, for every element S of the trace space

$$H^{1/2}_{00}(\Gamma) = \{v \in L^2(\Gamma): \ \text{its zero extension belongs to } H^{1/2}(\partial\Omega_1)\}$$

there exists $u_1^0 \in H^1_{\Gamma_1}(\Omega_1)$ such that $u_1^0 = S$ on Γ [**16**]. However, the trace mapping considered as a mapping from $H^1_{\Gamma_2}(\Omega_2)$ in $L^2(\partial\Omega_2)$ is surjective on $H^{1/2}_{00}(\partial\Omega_2 \setminus \overline{\Gamma}_2)$.

Considering that the Poincaré inequality occurs whenever $\Gamma_D \cap \partial\Omega_k \neq \emptyset$, for $k = 1, 2$, then the above Hilbert spaces are endowed with the norms

$$\|v\|_{H^1_{\Gamma_k}(\Omega_k)} = \|\nabla v\|_{2,\Omega_k}.$$

When $\Gamma_1 = \emptyset$ we endow $H^1_{\Gamma_1}(\Omega_1)$ with any of the equivalent norms

$$\|v\|_{2,\Omega_1} + \|\nabla v\|_{2,\Omega_1} \sim \|v\|_{2,\Gamma} + \|\nabla v\|_{2,\Omega_1}.$$

Indeed, we recognize that $H^1_{\Gamma_1}(\Omega_1) \equiv H^1(\Omega_1)$ and $H^{1/2}_{00}(\Gamma) \equiv H^{1/2}(\partial\Omega_1)$.

2.1 - *Wentzell-type transmission*

We can interpret the solutions $u_k : \Omega_k \times]0, T[\to \mathbb{R}$ $(k = 1, 2)$ as the uniquely (almost everywhere) determined function $u : \Omega \times]0, T[\to \mathbb{R}$ such that $u|_{\Omega_1} = u_1$, $u|_{\Omega_2} = u_2$ and $u_1 = u_2$ on Γ.

Let us define H_β as the Hilbert space

$$\{v \in H^1_{\Gamma_D}(\Omega) : \ v_1 = v|_{\Omega_1};\ v_2 = v|_{\Omega_2};\ v_1 = v_2 \text{ on } \Gamma\} \quad \text{if} \quad \beta = 0;$$
$$\{v \in H^1_{\Gamma_D}(\Omega) : \ v_1 = v|_{\Omega_1};\ v_2 = v|_{\Omega_2};\ v_1 = v_2 \text{ on } \Gamma;\ \nabla v \in L^2(\Gamma)\} \quad \text{if} \quad \beta > 0,$$

endowed with the inner product

$$(u, v)_\beta = \int_\Omega \nabla u \cdot \nabla v dx + \beta \int_\Gamma \nabla u \cdot \nabla v ds.$$

The identity on the interface Γ should be understood as an identity of the corresponding trace functions, that is, $T_1 v = T_2 v$ with T_1 and T_2 denoting the trace operators from $H^1_{\Gamma_D}(\Omega_1)$ and $H^1_{\Gamma_D}(\Omega_2)$, respectively, into $L^2(\Gamma)$.

Definition 2.1. We say that a function $u \in L^2(0, T; H_\beta)$ is a *weak solution to the problem* (1)-(5) if $\partial_t u \in L^2(\Sigma)$ and it satisfies (5) and the variational formulation

$$\begin{aligned} &\int_0^T \int_\Omega \sigma \nabla u \cdot \nabla(v - u) dx dt + \beta \int_0^T \int_\Gamma \nabla u \cdot \nabla(v - u) ds dt \\ &+ \int_0^T \int_\Gamma \alpha \partial_t u (v - u) ds dt + \int_0^T \int_\Gamma \{j(v) - j(u)\} ds dt \geq \int_0^T \langle f, v - u\rangle_\Omega dt, \end{aligned} \tag{12}$$

for all $v \in L^2(0, T; H_\beta)$, with $\sigma = \sigma_1 \chi_{\Omega_1} + \sigma_2 \chi_{\Omega_2}$, and $f \in C([0, T]; (H_\beta)')$.

The symbol $\langle \cdot, \cdot \rangle_\Omega$ denotes the duality pairing $\langle \cdot, \cdot \rangle_{(H_\beta)' \times H_\beta}$.

For $u : \Omega \times]0,T[\to \mathbb{R}$ such that the homogeneous Neumann boundary condition in (2) is satisfied, the Green formula yields

$$-\langle \nabla \cdot (\sigma \nabla u), v \rangle_\Omega = \int_\Omega \sigma \nabla u \cdot \nabla v dx + \langle [\sigma \nabla u \cdot \mathbf{n}], v \rangle_\Gamma, \qquad \forall v \in H_\beta.$$

Thus, using (1) and (4) it follows (12).

Remark 2.1. In the statement of Definition 2.1 the framework is rather general on f. Indeed, we consider $f = f_1 \chi_{\Omega_1} + f_2 \chi_{\Omega_2}$, whenever $f_k \in C([0,T]; L^q(\Omega_k))$ $(k = 1,2)$, where $q \geq (2^*)'$, with $(2^*)'$ being the conjugate exponent of $2^* = 2n/(n-2)$ if $n > 2$, and any real value if $n = 2$. This definition of f itself as a combination of f_1, f_2 implies that, for each $t \in [0,T]$, $f(t) \in L^q(\Omega) \hookrightarrow (H_\beta)'$.

Theorem 2.1. *Under the assumptions* (9)-(11),

$$\exists u^0 \in H_\beta : \quad u^0 = S \text{ on } \Gamma; \tag{13}$$

$$\int_\Gamma j(S) ds \leq C(\|S\|_{2,\Gamma}^2 + 1), \tag{14}$$

where C stands for a positive constant, and $f \in C^{0,1}(0,T;(H_\beta)')$ with the Lipschitz constant d, that is,

$$\|f(\tau) - f(t)\|_{(H_\beta)'} \leq d|\tau - t|, \qquad \forall \tau, t \in]0,T[, \tag{15}$$

there exists $u \in L^\infty(0,T;H_\beta)$ a unique weak solution in accordance to Definition 2.1.

Remark 2.2. The assumption (14) yields if for instance j verifies $j(d) \leq C(d^2 + 1)$ for all $d \in \mathbb{R}$. Notice that (13) guarantees that $S \in L^2(\Gamma)$ is such that $\beta \nabla S \in L^2(\Gamma)$.

Theorem 2.2. *Let the assumptions of Theorem* 2.1 *be fulfilled. Moreover, if the compatibility condition*

$$\int_\Omega \sigma \nabla u^0 \cdot \nabla(v - u^0) dx + \beta \int_\Gamma \nabla u^0 \cdot \nabla(v - u^0) ds + \int_\Gamma \{j(v) - j(S)\} ds \geq \langle f(0), v - u^0 \rangle_\Omega \tag{16}$$

holds for all $v \in H_\beta$, then $\partial_t u \in L^2(0,T;H_\beta) \cap L^\infty(0,T;L^2(\Gamma))$. In particular, $u \in C([0,T];H_\beta)$.

The transmission problem in a thin porous layer, (1)-(5) with $\beta = 0$, can be obtained as the asymptotic limit, when a small parameter ε goes to zero, of the following perturbed problem, whenever the domain Ω verifies the ε_0-property: There exists $\varepsilon_0 > 0$ such that, for every $0<\varepsilon \leq \varepsilon_0$,

$$S_\varepsilon := \{\xi + \tau\mathbf{n}(\xi) : \ \xi \in \Gamma, \ 0<\tau<\varepsilon\gamma(\xi)\} \subset \Omega_2,$$

with $\gamma \in C^{0,1}(\Gamma)$ such that $0<\gamma_\# \leq \gamma(\xi) \leq \gamma^\#$ for all $\xi \in \Gamma$, and $\Omega_{2,\varepsilon} := \Omega_2 \setminus \overline{S_\varepsilon}$ is a Lipschitz domain.

For instance, all domains consisting of inner and outer subdomains, i.e. the interface $\Gamma = \partial\Omega_1 \subset \Omega$, $\Gamma_1 = \emptyset$ and $\Gamma_D = \Gamma_2$, satisfy the ε_0-property with $\gamma \equiv 1$ and $0<\varepsilon_0<\mathrm{dist}(\Gamma, \partial\Omega)$. Consequently, $\overline{S_\varepsilon} \subset \Omega$ for every $0<\varepsilon \leq \varepsilon_0$.

($\mathbf{P}_\varepsilon$) Find $u_\varepsilon : \Omega = \Omega_1 \cup \overline{S_\varepsilon} \cup \Omega_{2,\varepsilon} \to \mathbb{R}$ satisfying

$$\begin{aligned}
-\nabla \cdot (\sigma_1 \nabla u_\varepsilon) &= f_1 && \text{in} \quad \Omega_1;\\
-\nabla \cdot (\sigma_2 \nabla u_\varepsilon) &= f_2 && \text{in} \quad \Omega_{2,\varepsilon};\\
\varepsilon\gamma\Delta u_\varepsilon - \alpha\partial_t u_\varepsilon &\in \partial j(u_\varepsilon) && \text{in} \quad S_\varepsilon \times]0,T[;\\
u_\varepsilon(\cdot,0) &= u^0 && \text{in} \quad S_\varepsilon;\\
[u_\varepsilon] = [\sigma\nabla u_\varepsilon \cdot \mathbf{n}] &= 0 && \text{on} \quad \Gamma;\\
[u_\varepsilon] = [\sigma\nabla u_\varepsilon \cdot \mathbf{n}] &= 0 && \text{on} \quad \Gamma_\varepsilon := \partial S_\varepsilon \setminus \Gamma;\\
\nabla u_2 \cdot \mathbf{n} &= 0 && \text{on} \quad \partial\Omega \setminus \Gamma_2;\\
u_2 &= 0 && \text{on} \quad \Gamma_2.
\end{aligned} \tag{17}$$

Let us define the Hilbert space

$$X_\varepsilon = \{v \in H^1_{\Gamma_2}(\Omega_\varepsilon) : \ v_1 = v|_{\Omega_1}, \ v_{S_\varepsilon} = v|_{S_\varepsilon}, \ v_{2,\varepsilon} = v|_{\Omega_{2,\varepsilon}};$$
$$v_1 = v_{S_\varepsilon} \text{ on } \Gamma, \ v_{S_\varepsilon} = v_{2,\varepsilon} \text{ on } \Gamma_\varepsilon\},$$

where $\Omega_\varepsilon = \Omega_1 \cup S_\varepsilon \cup \Omega_{2,\varepsilon}$. Set $f_\varepsilon = f_1\chi_{\Omega_1} + f_2\chi_{(S_\varepsilon \cup \Omega_{2,\varepsilon})}$ (compare with Remark 2.1). We emphasize that neither Γ nor Γ_ε belong to Ω_ε. This means that the identities of a admissible function on these interfaces should be understood as the corresponding identities between trace functions as above.

Proposition 2.1. *Let the assumptions* (9)-(11), (13), *and* $\beta = 0$ *be fulfilled, and* (14) *be replaced by* $j(d) \leq C(d^2 + 1)$ *for all* $d \in \mathbb{R}$. *Let* u *be the unique solution of the problem* (1)-(5) *in accordance to Theorem* 2.1, *under the admissible test function space* $\mathcal{X} := L^2(0,T;H_0) \cap H^1(0,T;H^1(\Omega \setminus \overline{\Omega_1}))$, *and* $f_1 \in C([0,T];L^2(\Omega_1))$ *and* $f_2 \in C([0,T];L^2(\Omega_2))$ *such that* (15) *is replaced by*

$$\|f_1(\tau) - f_1(t)\|_{2,\Omega_1} + \|f_2(\tau) - f_2(t)\|_{2,\Omega_2} \leq d|\tau - t|, \qquad \forall \tau, t \in]0,T[,$$

for some $d > 0$*. Then,* u *is the limit of the sequence of the unique solutions* u_ε *to the variational formulation of the perturbed problem* $(\mathbf{P}_\varepsilon)$

$$\begin{aligned} &\int_0^T\int_{\Omega_\varepsilon} \sigma_\varepsilon \nabla u_\varepsilon \cdot \nabla(v-u_\varepsilon)dxdt + \int_0^T\int_{S_\varepsilon} \frac{\alpha}{\varepsilon\gamma}\partial_t u_\varepsilon (v-u_\varepsilon)dxdt \\ &+\int_0^T\int_{S_\varepsilon}\frac{1}{\varepsilon\gamma}\{j(v)-j(u_\varepsilon)\}dxdt \geq \int_0^T \langle f_\varepsilon, v-u_\varepsilon\rangle_{\Omega_\varepsilon}dt, \quad \forall v\in L^2(0,T;X_\varepsilon), \end{aligned} \tag{18}$$

with (17), and $\sigma_\varepsilon = \sigma_1\chi_{\Omega_1} + \chi_{S_\varepsilon} + \sigma_2\chi_{\Omega_{2,\varepsilon}}$.

Here, the duality product should be understood as $\langle f, v\rangle_{\Omega_\varepsilon} = \int_{\Omega_1} f_1 v_1 dx + \int_{S_\varepsilon} f_2 v_{S_\varepsilon} dx + \int_{\Omega_{2,\varepsilon}} f_2 v_{2,\varepsilon} dx$.

2.2 - *Signorini-type transmission*

Here, we keep the notation of jump $[v] = v_2 - v_1$ for any vector $\mathbf{v} = (v_1, v_2)$. However, in order to differentiate this case from the above, let us set every vector by boldface. In general $v_1 \neq v_2$ on Γ. Thus, their weak derivatives do not exist. Let us define the Hilbert space

$$\mathbf{V} = \{\mathbf{v} = (v_1, v_2):\ v_1 \in H^1_{\Gamma_1}(\Omega_1);\ v_2 \in H^1_{\Gamma_2}(\Omega_2)\} \hookrightarrow L^2(\Omega_1)\times L^2(\Omega_2)$$

endowed with the norm (cf. Lemma 6.1)

$$\|\mathbf{v}\|_{\mathbf{V}} = \|\nabla v_1\|_{2,\Omega_1} + \|\nabla v_2\|_{2,\Omega_2} + \|[v]\|_{2,\Gamma}.$$

For $\mathbf{v}\in\mathbf{V}$, $\mathbf{v}|_\Gamma \in H^{1/2}_{00}(\Gamma)\times H^{1/2}_{00}(\partial\Omega_2\setminus\overline{\Gamma}_2)$.

Definition 2.2. We say that a function $\mathbf{u} = (u_1, u_2) \in L^2(0,T;\mathbf{V})$ is a *weak solution to the problem* (1)-(2) *with* (6)-(8) if $\partial_t[u]\in L^2(\Sigma)$ and it satisfies (8) and the variational formulation

$$\begin{aligned} &\int_0^T\int_{\Omega_1}\sigma_1\nabla u_1\cdot\nabla(v_1-u_1)dxdt + \int_0^T\int_{\Omega_2}\sigma_2\nabla u_2\cdot\nabla(v_2-u_2)dxdt \\ &+\int_0^T\langle g, v_1-u_1\rangle_\Gamma dt + \int_0^T\int_\Gamma \alpha\partial_t[u]([v]-[u])dsdt + \int_0^T\int_\Gamma\{j([v])-j([u])\}dsdt \\ &\geq \int_0^T\langle \mathbf{f}, \mathbf{v}-\mathbf{u}\rangle_\Omega dt, \qquad \forall \mathbf{v} = (v_1, v_2)\in L^2(0,T;\mathbf{V}), \end{aligned} \tag{19}$$

with $\mathbf{f} = (f_1, f_2)$.

Here, we use the same notation $\langle\cdot,\cdot\rangle_\Omega$ to denote the duality pairing $\langle\cdot,\cdot\rangle_{\mathbf{V}'\times\mathbf{V}}$, with $\mathbf{V}' = (H^1_{\Gamma_1}(\Omega_1))' \times (H^1_{\Gamma_2}(\Omega_2))'$ being the dual space of $\mathbf{V}$. The symbol $\langle\cdot,\cdot\rangle_\Gamma$ stands for the duality pairing $\langle\cdot,\cdot\rangle_{Y'\times Y}$, using the notation $Y = H^{1/2}_{00}(\Gamma)$.

For $\mathbf{u} = (u_1, u_2)$ such that the homogeneous Neumann boundary condition in (2) is satisfied, the Green formula yields

$$-\langle\nabla\cdot(\sigma\nabla\mathbf{u}),\mathbf{v}\rangle_\Omega = \sum_{k=1}^{2}\int_{\Omega_k}\sigma_k\nabla u_k\cdot\nabla v_k dx + \langle[\sigma\nabla u\cdot\mathbf{n}], v_1\rangle_\Gamma + \langle\sigma_2\nabla u_2\cdot\mathbf{n}, [v]\rangle_\Gamma,$$

for all $\mathbf{v}\in\mathbf{V}$. Thus, using (1) and (6)-(7) it follows (19).

Theorem 2.3. *Assuming* (9)-(11), (14),

$$\exists\mathbf{u}^0\in\mathbf{V}: \quad [u^0] = S \text{ on } \Gamma, \tag{20}$$

and $\mathbf{f}$ *and* g *are Lipschitz functions in the following sense: there exist two positive constants* d_1 *and* d_2 *such that*

$$\|\mathbf{f}(\tau)-\mathbf{f}(t)\|_{\mathbf{V}'} \le d_1|\tau - t| \tag{21}$$

$$\|g(\tau)-g(t)\|_{Y'} \le d_2|\tau - t|, \qquad \forall\tau, t\in\,]0,T[, \tag{22}$$

there exists $\mathbf{u}\in L^\infty(0,T;\mathbf{V})$ *a unique weak solution in accordance to Definition* 2.2.

Remark 2.3. The assumption (20) implies that

$$\|S\|_{2,\Gamma} \le \|[u^0]\|_{2,\Gamma} \le \|\mathbf{u}^0\|_{\mathbf{V}}.$$

Theorem 2.4. *Let the assumptions of Theorem* 2.3 *be fulfilled. Moreover, if the compatibility condition*

$$\begin{aligned}\sum_{k=1}^{2}\int_{\Omega_k}\sigma_k\nabla u_k^0\cdot\nabla(v_k - u_k^0)dx + \langle g(0), v_1 - u_1^0\rangle_\Gamma\\ + \int_\Gamma\{j([v]) - j(S)\}ds \ge \langle\mathbf{f}(0),\mathbf{v}-\mathbf{u}^0\rangle_\Omega,\end{aligned} \tag{23}$$

holds for all $\mathbf{v}\in\mathbf{V}$, *then* $\partial_t\mathbf{u}\in L^2(0,T;\mathbf{V})\cap L^\infty(0,T;L^2(\Gamma))$. *In particular,* $\mathbf{u}\in C([0,T];\mathbf{V})$.

3 - Proof of Theorem 2.1

3.1 - *Discretization in time*

In the following we use similar arguments from the methods described in [**19**]. We decompose the time interval $I = [0,T]$ into m subintervals $I_{i,m} = [t_{i,m}, t_{i+1,m}]$ of size

$h = T/m,\ i \in \{0, 1, \cdots, m-1\},\ m \in \mathbb{N}$. We define, for all $i \in \{0, 1, \cdots, m-1\}$, $u^{i+1} = u(t_{i+1,m})$ as solutions given at the following proposition.

Proposition 3.1. *Let $i \in \{0, 1, \cdots, m-1\}$ be fixed, $u^i \in L^2(\Gamma)$, and*

$$f^{i+1} = f(t_{i+1,m}) \in (H_\beta)'.$$

Then there exists $u^{i+1} \in H_\beta$ a solution to the problem

$$\begin{aligned} &\int_\Omega \sigma \nabla u^{i+1} \cdot \nabla(v - u^{i+1}) dx + \beta \int_\Gamma \nabla u^{i+1} \cdot \nabla(v - u^{i+1}) ds \\ &\quad + \int_\Gamma \frac{\alpha}{h} u^{i+1} (v - u^{i+1}) ds + \int_\Gamma \{j(v) - j(u^{i+1})\} ds \\ &\geq \langle f^{i+1}, v - u^{i+1} \rangle_\Omega + \int_\Gamma \frac{\alpha}{h} u^i (v - u^{i+1}) ds, \quad \forall v \in H_\beta. \end{aligned} \tag{24}$$

Proof. The existence of a solution to (24) is deduced from the general theory on maximal monotone mappings applied to elliptic variational inequalities [**23**, pp. 874-875, 892-893]. Indeed, the mapping $A : H_\beta \to (H_\beta)'$ defined by

$$\langle Au, v \rangle = \int_\Omega \sigma \nabla u \cdot \nabla v dx + \beta \int_\Gamma \nabla u \cdot \nabla v ds + \int_\Gamma \frac{\alpha}{h} uv ds$$

is single-valued, linear and hemicontinuous; the mapping $\varphi : H_\beta \to [0, +\infty]$ defined by

$$\varphi(v) = \begin{cases} \int_\Gamma j(v) ds, & \text{if } j(v) \in L^1(\Gamma) \\ +\infty, & \text{otherwise} \end{cases}$$

is convex, lower semicontinuous and $\varphi \not\equiv +\infty$; and the coercivity condition

$$\langle Au, u \rangle + \varphi(u) = \int_\Omega \sigma |\nabla u|^2 dx + \beta \int_\Gamma |\nabla u|^2 ds \geq \min\{\sigma_\#, 1\} \|u\|_{H_\beta}^2$$

is valid under the assumptions (9)-(11). Then, for $b \in (H_\beta)'$ such that

$$\langle b, v \rangle = -\langle f^{i+1}, v \rangle_\Omega - \int_\Gamma \frac{\alpha}{h} u^i v ds,$$

the variational inequality (24) has a unique weak solution $u = u^{i+1} \in H_\beta$. □

Remark 3.1. Since $u^0 = S$ on Γ means that $u^0 \in L^2(\Gamma)$, then Proposition 3.1 guarantees the existence of $u^1 \in H_\beta$ and consequently $u^1 \in L^2(\Gamma)$. Therefore, Proposition 3.1 successively guarantees the existence of $u^{i+1} \in H_\beta$ for every $i = 1, \cdots, m-1$.

3.2 - *Existence of a limit u*

Proposition 3.2. *For all $i \in \{0, 1, \cdots, m-1\}$, the following estimate holds:*

$$\alpha_\# \|u^{i+1}\|^2_{2,\Gamma} \le \max\left\{\frac{1}{\sigma_\#}, 1\right\} \|f\|^2_{L^2(0,T;(H_\beta)')} + \alpha^\# \|S\|^2_{2,\Gamma}. \tag{25}$$

Moreover, if $\{\tilde{u}_m\}_{m\in\mathbb{N}}$ is the sequence defined by the step functions $\tilde{u}_m : I \to H_\beta$

$$\tilde{u}_m(t) = \begin{cases} u^1 & \text{for } t = 0 \\ u^{i+1} & \text{in }]t_{i,m}, t_{i+1,m}] \end{cases}$$

then there exists u such that

$$\tilde{u}_m \rightharpoonup u \text{ in } L^2(0,T; H_\beta).$$

Proof. Choosing $v = 0$ as a test function in (24), we get

$$\int_\Omega \sigma |\nabla u^{i+1}|^2 dx + \beta \int_\Gamma |\nabla u^{i+1}|^2 ds + \int_\Gamma \frac{\alpha}{h} (u^{i+1})^2 ds \le \langle f^{i+1}, u^{i+1} \rangle_\Omega + \int_\Gamma \frac{\alpha}{h} u^i u^{i+1} ds,$$

for all $i \in \{0, 1, \cdots, m-1\}$. Observing that $\alpha/h > 0$, and

$$|\langle f^{i+1}, u^{i+1} \rangle_\Omega| \le \frac{1}{2\min\{\sigma_\#, 1\}} \|f^{i+1}\|^2_{(H_\beta)'} + \frac{\min\{\sigma_\#, 1\}}{2} \|u^{i+1}\|^2_{H_\beta};$$

$$\left| \int_\Gamma \frac{\alpha}{h} u^i u^{i+1} ds \right| \le \frac{1}{2} \int_\Gamma \frac{\alpha}{h} (u^i)^2 ds + \frac{1}{2} \int_\Gamma \frac{\alpha}{h} (u^{i+1})^2 ds,$$

then, after multiplying by the factor 2, it follows

$$\min\{\sigma_\#, 1\} \|u^{i+1}\|^2_{H_\beta} + \int_\Gamma \frac{\alpha}{h} (u^{i+1})^2 ds \le \max\left\{\frac{1}{\sigma_\#}, 1\right\} \|f^{i+1}\|^2_{(H_\beta)'} + \int_\Gamma \frac{\alpha}{h} (u^i)^2 ds.$$

Summing on $j = 0, ..., i$, it follows

$$\min\{\sigma_\#, 1\} h \sum_{j=0}^{i} \|u^{j+1}\|^2_{H_\beta} + \alpha_\# \|u^{i+1}\|^2_{2,\Gamma} \le \max\left\{\frac{1}{\sigma_\#}, 1\right\} h \sum_{j=1}^{i+1} \|f^j\|^2_{(H_\beta)'} + \alpha^\# \|S\|^2_{2,\Gamma}.$$

Consequently, we get (25) and, for $i = m-1$

$$\min\{\sigma_\#, 1\}\|\tilde{u}_m\|^2_{L^2(0,T;H_\beta)} \le \max\left\{\frac{1}{\sigma_\#}, 1\right\}\|f\|^2_{L^2(0,T;(H_\beta)')} + \alpha^\#\|S\|^2_{2,\Gamma}. \tag{26}$$

Thus we can extract a subsequence, still denoted by $\tilde{u}_m$, weakly convergent to $u \in L^2(0,T;H_\beta)$. □

Next, let us study the discrete derivative with respect to t at the time $t = t_{i+1}$:

$$Z^{i+1} := \frac{u^{i+1} - u^i}{h}.$$

Proposition 3.3. *Let $Z_m : [0,T[\to L^2(\Omega)$ be defined by*

$$Z_m(t) = \begin{cases} Z^1 & \text{for } t = 0 \\ Z^{i+1} & \text{in }]t_{i,m}, t_{i+1,m}] \end{cases} \quad \text{in } \Omega.$$

If the assumptions (9)-(11) *and* (13)-(15) *are fulfilled, then the following estimate holds:*

$$\|\tilde{u}_m\|^2_{L^\infty(0,T;H_\beta)} + \|Z_m\|^2_{2,\Sigma} \le C(\|f\|^2_{L^2(0,T;(H_\beta)')} + \|u^0\|^2_{H_\beta}). \tag{27}$$

Hence, we can extract a subsequence, still denoted by Z_m, weakly convergent to $Z \in L^2(\Sigma)$.

Proof. For a fixed t, there exists $i \in \{0, \cdots, m-1\}$ such that $t \in]t_{i,m}; t_{i+1,m}]$. Choosing $v = u^i$ as a test function in (24), we have

$$\int_\Omega \sigma\nabla u^{i+1}\cdot\nabla(u^{i+1}-u^i)dx + \beta\int_\Gamma \nabla u^{i+1}\cdot\nabla(u^{i+1}-u^i)ds$$
$$+\int_\Gamma \frac{\alpha}{h}(u^{i+1}-u^i)^2 ds + \int_\Gamma j(u^{i+1})ds \le \int_\Gamma j(u^i)ds + \langle f^{i+1}, u^{i+1}-u^i\rangle_\Omega.$$

In order to sum the above expression on $j = 0, ..., i$, consider the relation $2(a-b)a = a^2 + (a-b)^2 - b^2$ to obtain

$$\sum_{j=0}^{i}\int_\Omega \sigma\nabla u^{j+1}\cdot\nabla(u^{j+1}-u^j)dx = \frac{1}{2}\int_\Omega \sigma|\nabla u^{i+1}|^2dx - \frac{1}{2}\int_\Omega \sigma|\nabla u^0|^2dx + \frac{1}{2}\sum_{j=0}^{i}\int_\Omega \sigma|\nabla(u^{j+1}-u^j)|^2dx;$$

$$\sum_{j=0}^{i}\int_\Gamma \nabla u^{j+1}\cdot\nabla(u^{j+1}-u^j)ds = \frac{1}{2}\int_\Gamma |\nabla u^{i+1}|^2ds - \frac{1}{2}\int_\Gamma |\nabla u^0|^2ds + \frac{1}{2}\sum_{j=0}^{i}\int_\Gamma |\nabla(u^{j+1}-u^j)|^2ds.$$

Now, using the assumptions (9)-(11) we find

$$\frac{\min\{\sigma_\#,1\}}{2}\|u^{i+1}\|^2_{H_\beta}+\alpha_\#\sum_{j=0}^{i}h\int_\Gamma\left(\frac{u^{j+1}-u^j}{h}\right)^2ds$$
$$\le\frac{\sigma^\#}{2}\|\nabla u^0\|^2_{2,\Omega}+\frac{\beta}{2}\|\nabla u^0\|^2_{2,\Gamma}+\int_\Gamma j(S)ds \tag{28}$$
$$-\langle f^1,u^0\rangle_\Omega-\sum_{j=1}^{i}\langle f^{j+1}-f^j,u^j\rangle_\Omega+\langle f^{i+1},u^{i+1}\rangle_\Omega.$$

By (15) it follows

$$\sum_{j=1}^{i}\langle f^{j+1}-f^j,u^j\rangle_\Omega\le dh\sum_{j=1}^{i}\|u^j\|_{H_\beta}.$$

Therefore, inserting the above inequality in (28) and applying (26), we conclude (27). □

From the Rothe function defined by

$$u_1(x,t)=u^0(x)+t\,\frac{u^1(x)-u^0(x)}{h}\ \text{ in } I_{0,1}=I,$$

consider the following definition.

Definition 3.1. We say that $\{u_m\}_{m\in\mathbb{N}}$ is the *Rothe sequence* if

$$u_m(x,t)=u^i(x)+(t-t_{i,m})\frac{u^{i+1}(x)-u^i(x)}{h}\ \text{ in } I_{i,m},$$

for all $i\in\{0,1,\cdots,m-1\}$.

Proposition 3.4. *If Z satisfies Proposition 3.3, then*

$$\partial_t u=Z\ \textit{in}\ L^2(\Gamma),\ \textit{for almost all}\ t\in I.$$

Proof. For a fixed t, there exists $i\in\{0,\cdots,m-1\}$ such that $t\in\,]t_{i,m};t_{i+1,m}]$. Thus we obtain

$$\int_0^t Z_m(\tau)d\tau=\sum_{j=0}^{i-1}\int_{jh}^{(j+1)h}\frac{u^{j+1}-u^j}{h}d\tau+\int_{ih}^{t}\frac{u^{i+1}-u^i}{h}d\tau\ \text{ in } \Omega.$$

From Definition 3.1 we have $\int_0^t Z_m(\tau)d\tau = u_m(t) - S$. By the Riesz theorem we get

$$(u_m(t) - S, v) = \int_0^t (Z_m(\tau), v)d\tau, \quad \forall v \in L^2(\mathit{\Gamma}).$$

Indeed, the right hand side of the above equation is a bounded linear functional in $L^2(\mathit{\Gamma})$, representable thus (uniquely) by the element $u_m(t) - S$ from $L^2(\mathit{\Gamma})$.

Because there exists $w \in C([0,T]; L^2(\mathit{\Gamma}))$ such that

$$(w(t), v) = \int_0^t (Z(\tau), v)d\tau, \qquad \forall v \in L^2(\mathit{\Gamma}),$$

then it follows

$$\lim_{m\to+\infty} (u_m(t) - S - w(t), v) = \lim_{m\to+\infty} \int_0^t (Z_m(\tau) - Z(\tau), v)d\tau = 0. \tag{29}$$

Let us prove that the norms of the functions u_m are uniformly bounded with respect to $t \in I$ and m. From the estimates (25) independent on i and m, and considering

$$\|u_m(t)\|_{2,\mathit{\Gamma}} = \left\| u^i \left(1 + \frac{t - t_{i,m}}{h}\right) + u^{i+1} \frac{t - t_{i,m}}{h} \right\|_{2,\mathit{\Gamma}}$$

then, we get

$$\|u_m\|^2_{L^\infty(0,T;L^2(\mathit{\Gamma}))} \leq C(\|f\|^2_{L^2(0,T;(H_\beta)')} + \|S\|^2_{2,\mathit{\Gamma}}).$$

Hence, the Lebesgue Dominated Convergence Theorem can be applied in (29) giving

$$\lim_{m\to+\infty} \int_0^T (u_m(t) - S - w(t), v)dt = 0, \qquad \forall v \in L^2(\mathit{\Gamma}).$$

In the same manner this result can be derived for the case when $v(t)$ is a piecewise constant function of $t \in I$. Since these functions are dense in $L^2(\mathit{\Sigma})$, it remains valid for every function $v \in L^2(\mathit{\Sigma})$. From the uniqueness of the weak limit, we conclude

$$u(t) - S = \int_0^t Z(\tau)d\tau,$$

which corresponds to the claim. □

3.3 - *Passage to the limit on $m \to +\infty$*

Set $Q = \Omega \times]0,T[$. Denoting $f_m(t) = f^{i+1}$ for $t \in]t_{i,m}, t_{i+1,m}]$ and $i \in \{0, \cdots, m-1\}$, we have

$$\int_Q \sigma \nabla \tilde{u}_m \cdot \nabla v dx dt + \beta \int_\Sigma \nabla \tilde{u}_m \cdot \nabla v ds dt + \int_\Sigma \alpha Z_m v ds dt$$
$$+ \int_\Sigma j(v) ds dt \geq \int_Q \sigma |\nabla \tilde{u}_m|^2 dx dt + \beta \int_\Sigma |\nabla \tilde{u}_m|^2 ds dt$$
$$+ \int_\Sigma \alpha Z_m \tilde{u}_m ds dt + \int_\Sigma j(\tilde{u}_m) ds dt + \int_0^T \langle f_m, v - \tilde{u}_m \rangle_\Omega dt.$$

Using Propositions 3.2 and 3.3, and recalling the weak lower semicontinuity property for the first and second terms on the right hand side of the above inequality, to pass to the limit the above inequality it remains to prove that

$$\tilde{u}_m \to u \text{ in } L^2(\Sigma).$$

Taking $\tilde{u}_m - u = \tilde{u}_m - u_m + u_m - u$ first let us prove that

$$\tilde{u}_m - u_m \to 0 \text{ in } L^2(\Sigma).$$

Since we have $0 < t - t_{i,m} \leq h$ in $]t_{i,m}; t_{i+1,m}]$ we obtain

$$\|\tilde{u}_m(t) - u_m(t)\|_{2,\Gamma} = \|Z_m\|_{2,\Gamma}(h - (t - t_{i,m})) < h \|Z_m\|_{2,\Gamma}$$

and from (27) then it follows

$$\|\tilde{u}_m - u_m\|_{2,\Sigma} \leq \frac{CT}{m}(\|f\|^2_{L^2(0,T;(H_\beta)')} + \|u^0\|^2_{H_\beta})^{1/2} \to 0.$$

Secondly the Rothe sequence $\{u_m\}$ is bounded in $L^2(0,T;H_\beta)$, and, from Proposition 3.4, the functions $\partial_t u_m$ are bounded in $L^2(\Sigma)$ then, for a subsequence still denoted by u_m, the strong convergence holds

$$u_m \to u \text{ in } L^2(\Sigma).$$

Then it results

$$\int_0^T \int_\Gamma Z_m \tilde{u}_m ds dt \to \int_0^T \int_\Gamma Z u ds dt = \int_0^T \int_\Gamma \partial_t u u ds dt.$$

Therefore we are in the conditions to pass to the limit concluding the weak formulation (12).

From the standard technique to prove uniqueness of solution (see, for instance, [18]), the solution u to (12) with (8) is unique. Then the whole sequence $\{\tilde{u}_m\}$ converges *-weakly to $u \in L^\infty(0,T;H_\beta)$.

4 - Regularity in time

Proof of Theorem 2.2. The proof follows the time discretization argument as in Theorem 2.1, considering the existence of the integral inequality (24). Choosing $v = (u^{i+1} + u^i)/2$ as a test function in (24) for the solutions u^{i+1} and u^i, summing the consecutive integral inequalities, and dividing by h, we deduce

$$\int_\Omega h\sigma|\nabla Z^{i+1}|^2 dx + h\beta \int_\Gamma |\nabla Z^{i+1}|^2 ds + \int_\Gamma \alpha(Z^{i+1} - Z^i)Z^{i+1} ds \leq \langle f^{i+1} - f^i, Z^{i+1}\rangle_\Omega$$

taking the convexity of j into account. Applying the assumptions (9) and (15), it results

$$\min\{\sigma_\#, 1\} h\|Z^{i+1}\|^2_{H_\beta} + \int_\Gamma \alpha(Z^{i+1} - Z^i)Z^{i+1} ds \leq dh\|Z^{i+1}\|_{H_\beta}.$$

Considering the relation $2(a-b)a = a^2 + (a-b)^2 - b^2$, with $a = Z^{i+1}$ and $b = Z^i$, and summing on $j = 1, \cdots, i$ ($i \in \{1, \cdots, m-1\}$) we obtain

$$\min\{\sigma_\#, 1\} \sum_{j=1}^{i} h\|Z^{j+1}\|^2_{H_\beta} + \alpha_\# \|Z^{i+1}\|^2_{2,\Gamma} \leq 2\int_\Gamma \alpha \left(\frac{u^1 - S}{h}\right)^2 ds + d^2 \max\left\{\frac{1}{\sigma_\#}, 1\right\} \sum_{j=0}^{i} h.$$

Notice that $mh = T$.

Let us determine the estimate for the first term on the right hand side of the above inequality. Rewrite the integral inequality (24) for $i = 0$ in the form

$$\int_\Omega \sigma\nabla(u^1 - u^0)\cdot\nabla(v - u^1) dx + \int_\Omega \sigma\nabla u^0 \cdot \nabla(v - u^1) dx + \beta\int_\Gamma \nabla(u^1 - u^0)\cdot\nabla(v - u^1) ds + \beta\int_\Gamma \nabla u^0 \cdot \nabla(v - u^1) ds + \int_\Gamma \alpha\frac{u^1 - S}{h}(v - u^1) ds + \int_\Gamma \{j(v) - j(u^1)\} ds \geq \langle f^1 - f(0), v - u^1\rangle_\Omega + \langle f(0), v - u^1\rangle_\Omega,$$

for all $v \in V$, and in particular $v = u^0$. Thus, we apply the assumption (16) with $v = u^1$ and divide by h we deduce

$$\frac{\sigma_\#}{2h}\int_\Omega |\nabla(u^1 - u^0)|^2 dx + \frac{\beta}{2h}\int_\Gamma |\nabla(u^1 - u^0)|^2 ds + \int_\Gamma \alpha\left(\frac{u^1 - S}{h}\right)^2 ds \leq \frac{C}{2h}\|f^1 - f(0)\|^2_{(H_\beta)'}.$$

Then, using (15), we have

$$\int_\Gamma \alpha\left|\frac{u^1 - S}{h}\right|^2 ds \leq Chd^2 < C.$$

Since the above regularity estimates are independent on m the proof of the passage to the limit is similar to the one of Section 3. Moreover, the uniqueness of the weak solution implies that the weak solution is the strong solution in the sense $u \in C([0, T]; H_\beta)$ by appealing to the Aubin-Lions Theorem.

5 - Proof of Proposition 2.1

5.1 - *Existence of* u_ε

The time discretization described in Section 3.1 reads, for the perturbed problem, as

$$\int_{\Omega_\varepsilon} \sigma_\varepsilon \nabla u^{i+1} \cdot \nabla(v - u^{i+1}) dx + \int_{S_\varepsilon} \frac{\alpha}{\varepsilon h \gamma}(u^{i+1} - u^i)(v - u^{i+1}) dx + \int_{S_\varepsilon} \frac{1}{\varepsilon\gamma}\{j(v) - j(u^{i+1})\} dx \geq \langle f^{i+1}, v - u^{i+1}\rangle_{\Omega_\varepsilon}, \quad \forall v \in X_\varepsilon. \tag{30}$$

The existence and uniqueness of a solution $u_\varepsilon^{i+1} \equiv u^{i+1} \in X_\varepsilon$ is due to standard results for elliptic variational inequalities as in the proof of Proposition 3.1 (cf. [**18**]). Indeed, the bilinear symmetric form

$$a(u, v) = \int_{\Omega_\varepsilon} \sigma_\varepsilon \nabla u \cdot \nabla v dx + \int_{S_\varepsilon} \frac{\alpha}{\varepsilon h \gamma} uv dx$$

is coercive in the following sense

$$a(u, u) \geq \sigma_\# \|\nabla u\|^2_{2,\Omega_\varepsilon} + \frac{\alpha_\#}{\varepsilon h \gamma^\#}\|u\|^2_{2,S_\varepsilon}.$$

Taking $v = 0$ as a test function in (30), analogously to the proof of Proposition 3.2, we get the estimates

$$
\begin{aligned}
&\frac{\alpha_\#}{\varepsilon\gamma^\#}\|u^{i+1}\|^2_{2,S_\varepsilon} \le \frac{\alpha^\#}{\varepsilon\gamma_\#}\|u^0\|^2_{2,S_\varepsilon} + \frac{1}{\sigma_\#}\|f_\varepsilon\|^2_{L^2(0,T;(X_\varepsilon)')};\\
(31)\qquad &\sigma_\#\int_0^T \|\tilde{u}_m\|^2_{X_\varepsilon} dt \le \frac{\alpha^\#}{\varepsilon\gamma_\#}\|u^0\|^2_{2,S_\varepsilon} + \frac{1}{\sigma_\#}\|f_\varepsilon\|^2_{L^2(0,T;(X_\varepsilon)')}.
\end{aligned}
$$

Next taking $v = u^i$ in (30) and arguing as the proof of Proposition 3.3, we obtain

$$
\begin{aligned}
&\sigma_\#\|\nabla u^{i+1}\|^2_{2,\Omega_\varepsilon} + \frac{\alpha_\# h}{\varepsilon\gamma^\#}\sum_{j=0}^{i}\|Z^{j+1}\|^2_{2,S_\varepsilon}\\
&\le \int_{S_\varepsilon}\frac{1}{\varepsilon\gamma_\#} j(u^0)dx + C(\|\nabla u^0\|^2_{2,\Omega_\varepsilon} + \|f_\varepsilon\|^2_{L^2(0,T;(X_\varepsilon)')} + \frac{1}{\varepsilon}\|u^0\|^2_{2,S_\varepsilon}).
\end{aligned}
$$

Applying (14), it results that $\tilde{u}_m$ and Z_m are uniformly bounded in $L^\infty(0,T;X_\varepsilon)$ and $L^2(S_\varepsilon \times]0,T[)$, respectively. Therefore the existence of a solution $u_\varepsilon \in L^2(0,T;X_\varepsilon)$ to (18) can be proven by similar arguments of passage to the limit as in the proof of Theorem 2.1 (cf. Section 3.3).

5.2 - *Passage to the limit on ε*

In order to let $\varepsilon \to 0$, we utilize the following equivalent variational inequalities to (18) and (12) with $\beta = 0$, respectively,

$$
\begin{aligned}
&\int_0^T\int_{\Omega_\varepsilon}\sigma_\varepsilon\nabla u_\varepsilon\cdot\nabla(v-u_\varepsilon)dxdt + \int_0^T\int_{S_\varepsilon}\frac{\alpha}{\varepsilon\gamma}\partial_t v(v-u_\varepsilon)dxdt\\
(32)\qquad &+\int_{S_\varepsilon}\frac{\alpha}{2\varepsilon\gamma}|v(0)-u^0|^2dx + \int_0^T\int_{S_\varepsilon}\frac{1}{\varepsilon\gamma}\{j(v)-j(u_\varepsilon)\}dxdt \ge \int_0^T\langle f_\varepsilon, v-u_\varepsilon\rangle_{\Omega_\varepsilon}dt,\\
&\forall v\in\mathcal{X}_\varepsilon := L^2(0,T;X_\varepsilon)\cap H^1(0,T;H^1(S_\varepsilon));
\end{aligned}
$$

and

$$
\begin{aligned}
&\int_0^T\int_{\Omega}\sigma\nabla u\cdot\nabla(v-u)dxdt + \int_0^T\int_{\Gamma}\alpha\partial_t v(v-u)dsdt + \int_{\Gamma}\frac{\alpha}{2}|v(0)-u^0|^2ds\\
&+\int_0^T\int_{\Gamma}\{j(v)-j(u)\}dsdt \ge \int_0^T\langle f, v-u\rangle_{\Omega}dt, \quad \forall v\in\mathcal{X}.
\end{aligned}
$$

Let u_ε be the solution of (18), or equivalently (32), satisfying (17). By appealing to Section 5.1 we have

$$\|u_\varepsilon\|_{L^\infty(0,T;L^2(S_\varepsilon))} \leq C(\|u^0\|_{2,\Omega} + \|f\|_{2,Q}).$$

Using the result (cf. [8])

$$\frac{1}{\varepsilon}\|u^0\|^2_{2,S_\varepsilon} \leq C(\|u^0\|^2_{2,\Gamma} + \varepsilon\|\nabla u^0\|^2_{2,S_\varepsilon})$$

in the estimate (31) it follows

$$\|u_\varepsilon\|_{L^2(0,T;H^1(\Omega_\varepsilon))} \leq C(\|u^0\|_{H_0} + \|f\|_{2,Q}).$$

Thus there exists a subsequence $\varepsilon \to 0$ and a function $u \in L^\infty(0,T;L^2(S_\varepsilon)) \cap L^2(0,T;H^1(\Omega_\varepsilon))$ such that

$$u_\varepsilon \rightharpoonup u \quad \text{*-weakly in } L^\infty(0,T;L^2(S_\varepsilon)); \tag{33}$$

$$u_\varepsilon \rightharpoonup u \quad \text{weakly in } L^2(0,T;H^1(\Omega_\varepsilon)). \tag{34}$$

Next we recall the following lemma which is an extension the one proved in [**8, 9**].

Lemma 5.1. a) *For any function* $w \in W^{1,1}(\Omega \setminus \overline{\Omega_1})$ *we have*

$$\int_{S_\varepsilon} \frac{w}{\varepsilon\gamma}dx \to \int_\Gamma w ds \quad \textit{as } \varepsilon \to 0.$$

b) *For any sequence of functions* $w_\varepsilon \in L^1((\Omega \setminus \overline{\Omega_1}) \times]0,T[)$ *and any* $w \in L^1(\Gamma \times]0,T[)$ *such that*

$$\|\nabla w_\varepsilon\|_{q,S_\varepsilon} \leq C \quad \textit{and} \quad \int_0^T\int_\Gamma (w_\varepsilon - w)dsdt \to 0,$$

for some constant $C > 0$ *and some exponent* $q > 1$, *we have*

$$\int_0^T\int_{S_\varepsilon} \frac{w_\varepsilon}{\varepsilon\gamma}dxdt \to \int_0^T\int_\Gamma w dsdt \qquad \textit{as } \varepsilon \to 0.$$

For an arbitrary $v \in \mathcal{X}_\Gamma \hookrightarrow \mathcal{X}_\varepsilon \cap C([0,T];H^1(\Omega \setminus \overline{\Omega_1}))$, by Lemma 5.1 a) we have

$$\int_{S_\varepsilon} \frac{1}{2\varepsilon\gamma}|v(0) - u^0|^2 dx \to \int_\Gamma \frac{1}{2}|v(0) - u^0|^2 ds.$$

In order to apply Lemma 5.1 b), we define $w_\varepsilon = (v - u_\varepsilon)\partial_t v$ and $w = (v - u)\partial_t v$. By (33) we obtain

$$\int_0^T \int_\Gamma (w_\varepsilon - w) ds dt \to 0.$$

Since $\partial_t \nabla v \in L^2(\Omega \times]0, T[)$ we have

$$\|\nabla w_\varepsilon\|_{q,S_\varepsilon} \leq \|\nabla(v - u_\varepsilon)\|_{2,S_\varepsilon} \|\partial_t v\|_{\frac{2q}{2-q},S_\varepsilon} + \|v - u_\varepsilon\|_{\frac{2q}{2-q},S_\varepsilon} \|\partial_t \nabla v\|_{2,S_\varepsilon}$$

for $q > 1$ satisfying $2q/(2 - q) \leq 2n/(n - 2)$ that means $q \leq n/(n - 1)$.

Thus we can pass to the limit on $\varepsilon \to 0$ in (32) to obtain the desired solution.

6 - Proof of Theorem 2.3

The generalized version of the Poincaré inequality applied to functions admitting jumps [**2**] can be once more extended to the following version.

Lemma 6.1. *Let* $\mathbf{v} \in \mathbf{V}$. *Then*

$$\int_{\Omega_1} v_1^2 dx \leq C\left\{ \int_\Omega |\nabla \mathbf{v}|^2 dx + \int_\Gamma [v]^2 ds \right\}. \tag{35}$$

Proof. If $\Gamma_1 \neq \emptyset$, the classical Poincaré inequality is valid and then (35) clearly holds. If $\Gamma_1 = \emptyset$, we will prove (35) by contradiction. Assuming that (35) is not true, there exists a sequence $\{\mathbf{v}_m\} \subset \mathbf{V}$ such that for all $m \in \mathbb{N}$

$$\|v_{1m}\|_{2,\Omega_1} = 1 \quad \text{and} \quad \|\nabla \mathbf{v}_m\|_{2,\Omega}^2 + \|[v_m]\|_{2,\Gamma}^2 \leq 1/m.$$

Hence $\nabla \mathbf{v}_m \to \mathbf{0}$ in $\mathbf{L}^2(\Omega)$ and $[v_m] \to 0$ in $L^2(\Gamma)$. Since $\mathbf{V}$ is a reflexive Banach space, we can extract a subsequence of $\mathbf{v}_m$, still denoted by $\mathbf{v}_m$, such that $\mathbf{v}_m \rightharpoonup \mathbf{v}$ in $\mathbf{V}$. Thus $\nabla \mathbf{v} = \mathbf{0}$ in Ω and $v_1 = v_2$ on Γ. Consequently $v_1 \in H^1_{\Gamma_1}(\Omega_1)$ and $v_2 \in H^1_{\Gamma_2}(\Omega_2)$ satisfy $v_1 \equiv v_2 \equiv 0$. From the compact embedding $\mathbf{V} \hookrightarrow \hookrightarrow L^2(\Omega_1) \times L^2(\Omega_2)$ it follows that

$$\mathbf{v}_m \to \mathbf{0} \quad \text{in } L^2(\Omega_1) \times L^2(\Omega_2).$$

Then we conclude that

$$\|v_{1m}\|_{2,\Omega_1} = 1 \to \|0\|_{2,\Omega_1} = 1,$$

which is a contradiction. □

6.1 - *Discretization in time*

As in Section 3.1, we will construct weak solutions $\mathbf{u}^{i+1} = \mathbf{u}(t_{i+1,m})$, $i \in \{0, 1, \cdots, m-1\}$, of an approximate time discrete problem.

Proposition 6.1. *Let the assumptions* (9)-(11) *be valid,* $m \geq \sigma_\# T/\alpha_\#$ *and* $i \in \{0, 1, \cdots, m-1\}$ *be fixed,* $[u^i] \in L^2(\Gamma)$,

$$\mathbf{f}^{i+1} = \mathbf{f}(t_{i+1,m}) \in \mathbf{V}' \quad \textit{and} \quad g^{i+1} = g(t_{i+1,m}) \in Y'.$$

Then there exists a time discrete solution $\mathbf{u}^{i+1} \in \mathbf{V}$ *to the problem*

$$\begin{aligned} &\sum_{k=1}^{2} \int_{\Omega_k} \sigma_k \nabla u_k^{i+1} \cdot \nabla (v_k - u_k^{i+1}) dx + \langle g^{i+1}, v_1 - u_1^{i+1} \rangle_\Gamma \\ &+ \int_\Gamma \frac{\alpha}{h} [u^{i+1}] \big([v] - [u^{i+1}]\big) ds + \int_\Gamma \{ j([v]) - j([u^{i+1}]) \} ds \\ &\geq \langle \mathbf{f}^{i+1}, \mathbf{v} - \mathbf{u}^{i+1} \rangle_\Omega + \int_\Gamma \frac{\alpha}{h} [u^i] \big([v] - [u^{i+1}]\big) ds, \quad \forall \mathbf{v} \in \mathbf{V}. \end{aligned} \tag{36}$$

Proof. We show the existence of a solution to (36) with the aid of the general theory on maximal monotone mappings applied to elliptic variational inequalities [23, pp. 874-875, 892-893]. To this end, we define the mapping $A : \mathbf{V} \to \mathbf{V}'$ by

$$\langle A\mathbf{u}, \mathbf{v} \rangle = \sum_{k=1}^{2} \int_{\Omega_k} \sigma_k \nabla u_k \cdot \nabla v_k dx + \int_\Gamma \frac{\alpha}{h} [u][v] ds$$

which is single-valued, linear and hemicontinuous; and the mapping $\varphi : \mathbf{V} \to [0, +\infty]$ by

$$\varphi(\mathbf{v}) = \begin{cases} \int_\Gamma j([v]) ds, & \text{if } j([v]) \in L^1(\Gamma) \\ +\infty, & \text{otherwise} \end{cases}$$

which is convex, lower semicontinuous and $\varphi \not\equiv +\infty$. Because of (9)-(11) the coercivity condition

$$\langle A\mathbf{u}, \mathbf{u} \rangle + \varphi(\mathbf{u}) = \sum_{k=1}^{2} \int_{\Omega_k} \sigma_k |\nabla u_k|^2 dx + \int_\Gamma \frac{\alpha}{h} [u]^2 ds + \int_\Gamma j([u]) ds \geq \sigma_\# \|\mathbf{u}\|_{\mathbf{V}}^2,$$

is valid for any $h \leq \alpha_\#/\sigma_\#$. Then, for $\mathbf{b} \in \mathbf{V}'$ such that

$$\langle \mathbf{b}, \mathbf{v}\rangle = -\langle \mathbf{f}^{i+1}, \mathbf{v}\rangle_\Omega + \langle g^{i+1}, v_1\rangle_\Gamma - \int_\Gamma \frac{\alpha}{h}[u^i][v]ds,$$

the variational inequality (36) has a unique weak solution $\mathbf{u} = \mathbf{u}^{i+1} \in \mathbf{V}$. □

Remark 6.1. Since $[u^0] = S$ on Γ means that $[u^0] \in L^2(\Gamma)$, then Proposition 6.1 guarantees the existence of $\mathbf{u}^1 \in \mathbf{V}$ and consequently $[u^1] \in L^2(\Gamma)$. Therefore, Proposition 6.1 successively guarantees the existence of $\mathbf{u}^{i+1} \in \mathbf{V}$ for every $i = 1, \cdots, m-1$.

6.2 - *Existence of a limit* $\mathbf{u}$

In the sequel, let us suppose (9)-(11), $\mathbf{f} \in L^2(0,T;\mathbf{V}')$, $g \in L^2(0,T;Y')$, and $S \in L^2(\Gamma)$.

Proposition 6.2. *Let $m \geq \sigma_\# T/\alpha_\#$. For all $i \in \{0, 1, \cdots, m-1\}$, the following estimate holds:*

$$\alpha_\# \|[u^{i+1}]\|^2_{2,\Gamma} \leq C(\|\mathbf{f}\|^2_{L^2(0,T;\mathbf{V}')} + \|g\|^2_{L^2(0,T;Y')} + \|S\|^2_{2,\Gamma}). \tag{37}$$

Moreover, if $\{\widetilde{\mathbf{u}}_m\}_{m\in\mathbb{N}}$ is the sequence defined by the step functions $\widetilde{\mathbf{u}}_m : I \to \mathbf{V}$

$$\widetilde{\mathbf{u}}_m(t) = \begin{cases} \mathbf{u}^1 & \text{for } t = 0 \\ \mathbf{u}^{i+1} & \text{in }]t_{i,m}, t_{i+1,m}] \end{cases}$$

then there exists $\mathbf{u}$ such that

$$\widetilde{\mathbf{u}}_m \rightharpoonup \mathbf{u} \text{ in } L^2(0,T;\mathbf{V}).$$

Proof. Testing in (36) with $\mathbf{v} = \mathbf{0}$ and using (9) and (11), we get

$$\sigma_\# \|\nabla \mathbf{u}^{i+1}\|^2_{2,\Omega} + \int_\Gamma \frac{\alpha}{h}[u^{i+1}]^2 ds \leq \langle \mathbf{f}^{i+1}, \mathbf{u}^{i+1}\rangle_\Omega - \langle g^{i+1}, u_1^{i+1}\rangle_\Gamma + \int_\Gamma \frac{\alpha}{h}[u^i][u^{i+1}]ds,$$

for all $i \in \{0, 1, \cdots, m-1\}$. Hence, applying Lemma 6.1 it follows

$$\frac{\sigma_\#}{2}\|\nabla \mathbf{u}^{i+1}\|^2_{2,\Omega} + \int_\Gamma \frac{\alpha}{2h}[u^{i+1}]^2 ds \leq \frac{1}{2\sigma_\#}\left(\|\mathbf{f}^{i+1}\|_{\mathbf{V}'} + C_Y\|g^{i+1}\|_{Y'}\right)^2$$
$$+\frac{\sigma_\#}{2}\|[u^{i+1}]\|^2_{2,\Gamma} + \int_\Gamma \frac{\alpha}{2h}[u^i]^2 ds,$$

with C_Y standing for the continuity constant of $H^1_{\Gamma_1}(\Omega_1) \hookrightarrow Y$. Summing on $j = 0, \ldots, i$, multiplying by $2h$ and applying (10), we find

$$\sigma_\# h \sum_{j=0}^{i} \|\mathbf{u}^{j+1}\|^2_{\mathbf{V}} + \alpha_\# \|[u^{i+1}]\|^2_{2,\Gamma} \leq \frac{2}{\sigma_\#} h \sum_{j=1}^{i+1} (\|\mathbf{f}^j\|^2_{\mathbf{V}'} + C_Y^2 \|g^j\|^2_{Y'})$$
$$+ \sigma_\# h \sum_{j=0}^{i} \|[u^{j+1}]\|^2_{2,\Gamma} + \alpha^\# \|S\|^2_{2,\Gamma}.$$

Consequently, by the Gronwall Lemma we get (37) and, for $i = m-1$,

$$\|\widetilde{\mathbf{u}}_m\|^2_{L^2(0,T;\mathbf{V})} \leq C(\|\mathbf{f}\|^2_{L^2(0,T;\mathbf{V}')} + \|g\|^2_{L^2(0,T;Y')} + \|S\|^2_{2,\Gamma}). \tag{38}$$

Thus we can extract a subsequence, still denoted by $\widetilde{\mathbf{u}}_m$, weakly convergent to $\mathbf{u} \in L^2(0,T;\mathbf{V})$. □

Proposition 6.3. *Let $m \geq \sigma_\# T/\alpha_\#$ and $U_m : [0,T[\to L^2(\Gamma)$ be defined by*

$$U_m(t) = \begin{cases} \dfrac{[u^1] - S}{h} & \text{for } t = 0 \\ \dfrac{[u^{i+1}] - [u^i]}{h} & \text{in }]t_{i,m}, t_{i+1,m}] \end{cases} \quad \text{on } \Gamma.$$

If, in addition, the assumptions (14), (20)-(22) *are fulfilled, then the following estimate holds:*

$$\|\widetilde{\mathbf{u}}_m\|^2_{L^\infty(0,T;\mathbf{V})} + \|U_m\|^2_{2,\Sigma} \leq C(\|\mathbf{f}\|^2_{L^2(0,T;\mathbf{V}')} + \|g\|^2_{L^2(0,T;Y')} + \|\mathbf{u}^0\|^2_{\mathbf{V}}). \tag{39}$$

Hence, we can extract a subsequence, still denoted by U_m, weakly convergent to $U \in L^2(\Sigma)$.

Proof. For a fixed t, there exists $i \in \{0, \cdots, m-1\}$ such that $t \in]t_{i,m}; t_{i+1,m}]$. Choosing $\mathbf{v} = \mathbf{u}^i$ as a test function in (36), we have

$$\sum_{k=1}^{2} \int_{\Omega_k} \sigma_k \nabla u_k^{i+1} \cdot \nabla(u_k^{i+1} - u_k^i) dx + \int_\Gamma \frac{\alpha}{h} ([u^{i+1}] - [u^i])^2 ds + \int_\Gamma j([u^{i+1}]) ds$$
$$\leq \langle g^{i+1}, u_1^i - u_1^{i+1} \rangle_\Gamma + \int_\Gamma j([u^i]) ds + \langle \mathbf{f}^{i+1}, \mathbf{u}^{i+1} - \mathbf{u}^i \rangle_\Omega.$$

Summing on $j = 0, \ldots, i$ and remarking that

$$\sum_{j=0}^{i}\sum_{k=1}^{2}\int\limits_{\Omega_k}\sigma_k\nabla u_k^{j+1}\cdot\nabla(u_k^{j+1}-u_k^j)dx = \frac{1}{2}\sum_{k=1}^{2}\int\limits_{\Omega_k}\sigma_k|\nabla u_k^{i+1}|^2dx$$
$$-\frac{1}{2}\sum_{k=1}^{2}\int\limits_{\Omega_k}\sigma_k|\nabla u_k^0|^2dx + \frac{1}{2}\sum_{j=0}^{i}\sum_{k=1}^{2}\int\limits_{\Omega_k}\sigma_k|\nabla(u_k^{j+1}-u_k^j)|^2dx$$

then we find

$$\frac{\sigma_\#}{2}\|\nabla\mathbf{u}^{i+1}\|_{2,\Omega}^2 + \alpha_\#\sum_{j=0}^{i}h\int\limits_{\Gamma}\left(\frac{[u^{j+1}]-[u^j]}{h}\right)^2ds$$

$$\le\frac{\sigma^\#}{2}\|\nabla\mathbf{u}^0\|_{2,\Omega}^2 + \int\limits_{\Gamma}j(S)ds + \langle g^1,u_1^0\rangle_\Gamma + \sum_{j=1}^{i}\langle g^{j+1}-g^j,u_1^j\rangle_\Gamma - \langle g^{i+1},u_1^{i+1}\rangle_\Gamma$$
$$-\langle\mathbf{f}^1,\mathbf{u}^0\rangle_\Omega - \sum_{j=1}^{i}\langle\mathbf{f}^{j+1}-\mathbf{f}^j,\mathbf{u}^j\rangle_\Omega + \langle\mathbf{f}^{i+1},\mathbf{u}^{i+1}\rangle_\Omega. \tag{40}$$

Using (21)-(22), it follows

$$\sum_{j=1}^{i}\langle\mathbf{f}^{j+1}-\mathbf{f}^j,\mathbf{u}^j\rangle_\Omega \le d_1h\sum_{j=1}^{i}\|\mathbf{u}^j\|_{\mathbf{V}};$$
$$\sum_{j=1}^{i}\langle g^{j+1}-g^j,u_1^j\rangle_\Gamma \le d_2hC_Y\sum_{j=1}^{i}\|\mathbf{u}^j\|_{\mathbf{V}}.$$

Therefore, inserting the above inequalities in (40), applying (38) and gathering (37), we conclude (39). □

We again have to relate the weak limits **u** and U.

Proposition 6.4. *Let* **u** *and* U *be the weak limits obtained in Propositions* 6.2 *and* 6.3, *respectively. Then*

$$\partial_t[u] = U \text{ in } L^2(\Gamma), \text{ for almost all } t\in I.$$

Proof. For a fixed t, there exists $i\in\{0,\cdots,m-1\}$ such that $t\in\,]t_{i,m};t_{i+1,m}]$. By construction

$$\int\limits_0^t U_m(\tau)d\tau = \sum_{j=0}^{i-1}\int\limits_{jh}^{(j+1)h}\frac{[u^{j+1}]-[u^j]}{h}d\tau + \int\limits_{ih}^{t}\frac{[u^{i+1}]-[u^i]}{h}d\tau \quad \text{on } \Gamma.$$

Setting the Rothe sequence $\{\mathbf{u}_m\}_{m\in\mathbb{N}}$ defined by

$$\mathbf{u}_m(x,t)=\mathbf{u}^i(x)+(t-t_{i,m})\frac{\mathbf{u}^{i+1}(x)-\mathbf{u}^i(x)}{h}\ \text{ in } I_{i,m},$$

for all $i\in\{0,1,\cdots,m-1\}$ (compare to Definition 3.1) under $m\geq\sigma_\# T/\alpha_\#$, it results

$$\int_0^t U_m(\tau)d\tau=[u_m](t)-S\ \text{ on } \Gamma.$$

From the Riesz theorem we get

$$([u_m](t)-S,v)=\int_0^t (U_m(\tau),v)d\tau,\quad \forall v\in L^2(\Gamma).$$

Indeed, the right hand side of the above equation is a bounded linear functional in $L^2(\Gamma)$, representable thus (uniquely) by the element $[u_m](t)-S$ from $L^2(\Gamma)$. Also there exists $w\in C([0,T];L^2(\Gamma))$ such that

$$(w(t),v)=\int_0^t (U(\tau),v)d\tau,\qquad \forall v\in L^2(\Gamma).$$

Then we have

$$\lim_{m\to+\infty}([u_m](t)-S-w(t),v)=\lim_{m\to+\infty}\int_0^t (U_m(\tau)-U(\tau),v)d\tau=0.$$

Let us prove that the norms of the functions $[u_m]$ are uniformly bounded with respect to $t\in I$ and m. From the estimates (37) independent on i and m, and considering

$$\|[u_m](t)\|_{2,\Gamma}=\left\|[u^i]\left(1+\frac{t-t_{i,m}}{h}\right)+[u^{i+1}]\frac{t-t_{i,m}}{h}\right\|_{2,\Gamma}$$

then, we get

$$\|[u_m]\|^2_{L^\infty(0,T;L^2(\Gamma))}\leq C(\|\mathbf{f}\|^2_{L^2(0,T;\mathbf{V}')}+\|g\|^2_{L^2(0,T;Y')}+\|S\|^2_{2,\Gamma}).$$

Hence, the Lebesgue Dominated Convergence Theorem yields

$$\lim_{m\to+\infty}\int_0^T ([u_m](t)-S-w(t),v)dt=0,\qquad \forall v\in L^2(\Gamma).$$

Proceeding as in the proof of Proposition 3.4, we end up with

$$[u](t) - S = \int_0^t U(\tau)d\tau.$$

□

6.3 - *Passage to the limit on $m \to +\infty$*

Denoting $\mathbf{f}_m(t) = \mathbf{f}^{i+1}$ and $g_m(t) = g^{i+1}$ for $t \in]t_{i,m}, t_{i+1,m}]$ and $i \in \{0, \cdots, m-1\}$, we have

$$\sum_{k=1}^{2} \int_0^T \int_{\Omega_k} \sigma_k \nabla \widetilde{u}_{mk} \cdot \nabla v_k dx dt + \int_0^T \langle g_m, v_1 - \widetilde{u}_{m1} \rangle_\Gamma dt + \int_\Sigma \alpha U_m[v] ds dt$$
$$+ \int_\Sigma j([v]) ds dt \geq \sum_{k=1}^{2} \int_0^T \int_{\Omega_k} \sigma_k |\nabla \widetilde{u}_{mk}|^2 dx dt + \int_\Sigma \alpha U_m[\widetilde{u}_m] ds dt$$
$$+ \int_\Sigma j([\widetilde{u}_m]) ds dt + \int_0^T \langle \mathbf{f}_m, \mathbf{v} - \widetilde{\mathbf{u}}_m \rangle_\Omega dt.$$

Using Propositions 6.2 and 6.3, and recalling the weak lower s.c. property for the first term on the right hand side of the above inequality, we can pass to the limit the above inequality if we prove that

$$[\widetilde{u}_m] \to [u] \text{ in } L^2(\Sigma).$$

To this end, we take $\widetilde{\mathbf{u}}_m - \mathbf{u} = \widetilde{\mathbf{u}}_m - \mathbf{u}_m + \mathbf{u}_m - \mathbf{u}$ in order to prove that

$$[\widetilde{u}_m] - [u_m] \to 0 \text{ in } L^2(\Sigma).$$

Since we have $0 < t - t_{i,m} \leq h$ in $]t_{i,m}; t_{i+1,m}]$ we obtain

$$\|[\widetilde{u}_m](t) - [u_m](t)\|_{2,\Gamma} = \|U_m\|_{2,\Gamma}(h - (t - t_{i,m})) < h\|U_m\|_{2,\Gamma}.$$

Using (39) we derive

$$\|[\widetilde{u}_m] - [u_m]\|_{2,\Sigma} \leq \frac{CT}{m}(\|\mathbf{f}\|^2_{L^2(0,T;\mathbf{V}')} + \|g\|^2_{L^2(0,T;Y')} + \|\mathbf{u}^0\|^2_{\mathbf{V}})^{1/2} \to 0.$$

Next, the Rothe sequence $\{\mathbf{u}_m\}$ is bounded in $L^2(0, T; \mathbf{V})$, and, from Prop. 6.4, the functions $\partial_t[u_m]$ are bounded in $L^2(\Sigma)$ then, for a subsequence still denoted by $[u_m]$, the strong convergence holds

$$[u_m] \to [u] \text{ in } L^2(\Sigma).$$

Then, it results

$$\int_0^T\int_\Gamma U_m[\widetilde{u}_m]dsdt \to \int_0^T\int_\Gamma U[u]dsdt = \int_0^T\int_\Gamma [\partial_t u][u]dsdt.$$

Therefore, we can pass to the limit to obtain the weak formulation (19). From the standard technique to prove uniqueness of solution (see, for instance, [**18**]), the solution $\mathbf{u}$ to (19) with (8) is unique. Then the whole sequence $\{\widetilde{\mathbf{u}}_m\}$ converges weakly to $\mathbf{u} \in L^2(0,T;\mathbf{V})$.

7 - Regularity in time

Proof of Theorem 2.4. The proof follows the time discretization argument as in Theorem 2.3, considering the existence of the integral inequality (36). Testing in (36) for the solutions $\mathbf{u}^{i+1}$ and $\mathbf{u}^i$ with $\mathbf{v} = (\mathbf{u}^{i+1} + \mathbf{u}^i)/2$, summing the consecutive integral inequalities, and dividing by h, we deduce

$$\sum_{k=1}^{2}\int_{\Omega_k} h\sigma_k|\nabla Z_k^{i+1}|^2dx + \int_\Gamma \alpha(U^{i+1} - U^i)U^{i+1}ds \leq \langle \mathbf{f}^{i+1} - \mathbf{f}^i, \mathbf{Z}^{i+1}\rangle_\Omega$$
$$+\langle g^i - g^{i+1}, Z_1^{i+1}\rangle_\Gamma$$

with $U^{i+1} = ([u^{i+1}] - [u^i])/h$ on Γ and $\mathbf{Z}^{i+1} = (\mathbf{u}^{i+1} - \mathbf{u}^i)/h \in \mathbf{V}$, and taking into account the convexity of j. Applying the relation $2(a-b)a = a^2 + (a-b)^2 - b^2$ with $a = U^{i+1}$ and $b = U^i$, and the assumptions (9), (21)-(22), it results

$$h\sigma_\#\|\nabla\mathbf{Z}^{i+1}\|_{2,\Omega}^2 + \int_\Gamma \alpha(U^{i+1})^2ds \leq \int_\Gamma \alpha(U^i)^2ds + (d_1 + C_Y d_2)h\|\mathbf{Z}^{i+1}\|_{\mathbf{V}}.$$

Notice that the $\mathbf{V}$-norm may be no equivalent to a seminorm. Thus, summing on $j = 1, \cdots, i$ ($i \in \{1, \cdots, m-1\}$) we obtain

$$\frac{\sigma_\#}{2}\sum_{j=1}^{i} h\|\nabla\mathbf{Z}^{j+1}\|_{2,\Omega}^2 + \alpha_\#\|U^{i+1}\|_{2,\Gamma}^2 \leq \alpha^\#\|U^1\|_{2,\Gamma}^2$$
$$+T\frac{(d_1 + C_Y d_2)^2}{2\sigma_\#} + \frac{\sigma_\#}{2}\sum_{j=1}^{i} h\|U^{j+1}\|_{2,\Gamma}^2, \tag{41}$$

with $mh = T$.

Let us determine the estimate for the first term on the right hand side of the above inequality. Rewrite the integral identity (36) for $i = 0$ in the form

$$\sum_{k=1}^{2} \int_{\Omega_k} \sigma_k \nabla(u_k^1 - u_k^0) \cdot \nabla(v_k - u_k^1) dx + \sum_{k=1}^{2} \int_{\Omega_k} \sigma_k \nabla u_k^0 \cdot \nabla(v_k - u_k^1) dx$$

$$+ \int_{\Gamma} \alpha \frac{[u^1] - S}{h}([v] - [u^1]) ds + \int_{\Gamma} \{ j([v]) - j([u^1]) \} ds \geq \langle \mathbf{f}^1 - \mathbf{f}(0), \mathbf{v} - \mathbf{u}^1 \rangle_{\Omega}$$

$$+ \langle \mathbf{f}(0), \mathbf{v} - \mathbf{u}^1 \rangle_{\Omega} - \langle g^1 - g(0), v_1 - u_1^1 \rangle_{\Gamma} - \langle g(0), v_1 - u_1^1 \rangle_{\Gamma},$$

for all $\mathbf{v} \in \mathbf{V}$, and in particular $\mathbf{v} = \mathbf{u}^0$. Thus, we apply the assumption (23) with $\mathbf{v} = \mathbf{u}^1$, (9), and we divide by h, deducing

$$\sigma_{\#} h \|\nabla \mathbf{Z}^1\|_{2,\Omega}^2 + \int_{\Gamma} \alpha \left(\frac{[u^1] - S}{h} \right)^2 ds \leq \left(\|\mathbf{f}^1 - \mathbf{f}(0)\|_{\mathbf{V}'} + \|g^1 - g(0)\|_{Y'} \right) \|\mathbf{Z}^1\|_{\mathbf{V}}.$$

Then, using (10), (21)-(22) and taking the Young inequality into account for the right hand side, we get

$$\alpha_{\#} \|U^1\|_{2,\Gamma}^2 \leq \frac{(d_1 + C_Y d_2)^2 h}{2\sigma_{\#}} + \frac{\sigma_{\#}}{2} h \|U^1\|_{2,\Gamma}^2.$$

Considering $h < \alpha_{\#} \min\{1/\sigma_{\#}, 1\}$ we insert the resulting estimate for U^1 into (41) concluding

$$\frac{\sigma_{\#}}{2} \sum_{j=1}^{i} h \|\nabla \mathbf{Z}^{j+1}\|_{2,\Omega}^2 + \alpha_{\#} \|U^{i+1}\|_{2,\Gamma}^2 \leq (\alpha^{\#} + T) \frac{(d_1 + C_Y d_2)^2}{\sigma_{\#}} + \frac{\sigma_{\#}}{2} \sum_{j=1}^{i} h \|U^{j+1}\|_{2,\Gamma}^2 .$$

Applying the Gronwall Lemma, U_m is uniformly estimated in $L^\infty(0;T;L^2(\Gamma))$, and successively $\mathbf{Z}_m$ is uniformly estimated in $L^2(0;T;\mathbf{V})$. Therefore, the existence of a solution $\mathbf{u} \in C([0,T];\mathbf{V})$ in accordance to Theorem 2.4 can be proven by similar arguments of passage to the limit (cf. Section 4).

Acknowledgments. The author wishes to express her gratitude to J. F. Rodrigues for suggesting the problem and some stimulating conversations.

References

[1] M. Amar, D. Andreucci, R. Gianni and P. Bisegna, *Evolution and memory effects in the homogenization limit for electrical conduction in biological tissues*, Math. Models Methods Appl. Sci. **14** (2004), no. 9, 1261-1295.

[2] M. Amar, D. Andreucci, P. Bisegna and R. Gianni, *Existence and uniqueness for an elliptic problem with evolution arising in electrodynamics*, Nonlinear Anal. Real World Appl. **6** (2005), 367-380.

[3] S. N. Antontsev, G. Gagneux, R. Luce and G. Vallet, *New unilateral problems in stratigraphy*, M2AN Math. Model. Numer. Anal. **40** (2006), no. 4, 765-784.

[4] F. Ben Belgacem, Y. Renard and L. Slimane, *On mixed methods for Signorini problems*, An. Univ. Craiova Ser. Mat. Inform. **30** (2003), no. 1, 45-52.

[5] W. Chikouche, D. Mercier and S. Nicaise, *Regularity of the solution of some unilateral boundary value problems in polygonal and polyhedral domains*, Comm. Partial Differential Equations **28** (2003), no. 11-12, 1975-2001, or **29** (2004), no. 1-2, 43-70.

[6] Y. S. Choi and R. Lui, *Uniqueness of steady-state solutions for an electrochemistry model with multiple species*, J. Differential Equations **108** (1994), 424-437.

[7] P. Colli Franzone, L. Guerri and M. Pennacchio, *Mathematical models and problems in electrocardiology*, Riv. Mat. Univ. Parma **2*** (1999), 123-142.

[8] P. Colli and J.-F. Rodrigues, *A perturbation problem related to the highly compressible behaviour of a fluid in a thin porous layer*, Appl. Anal. **33** (1989), 191-201.

[9] P. Colli and J.-F. Rodrigues, *Diffusion through thin layers with high specific heat*, Asymptotic Anal. **3** (1990), 249-263.

[10] L. Consiglieri and A. R. Domingos, *An analytical solution for the ionic flux in an axonial membrane model*, in: “Progress in Mathematical Biology Research”, J. T. Kelly, ed., Nova Science Publishers, New York 2008, 321-333.

[11] L. Consiglieri and A. R. Domingos, *On the sodium concentration diffusion with three-dimensional extracellular stimulation*, Int. J. Math. Math. Sci. **2011** (2011), Art. ID 862813, 19 pp.

[12] G. Duvaut and J.-L. Lions, *Les inéquations en mécanique et en physique*, Travaux et Recherches Mathématiques, **21**, Dunod, Paris 1972.

[13] K.-J. Engel, *The Laplacian on $C(\bar{\Omega})$ with generalized Wentzell boundary conditions*, Arch. Math. (Basel) **81** (2003), no. 5, 548-558.

[14] K.-J. Engel and G. Fragnelli, *Analyticity of semigroups generated by operators with generalized Wentzell boundary conditions*, Adv. Differential Equations **10** (2005), no. 11, 1301-1320.

[15] A. Favini, G. R. Goldstein, J. A. Goldstein, E. Obrecht and S. Romanelli, *Elliptic operators with general Wentzell boundary conditions, analytic semigroups and the angle concavity theorem*, Math. Nachr. **283** (2010), no. 4, 504-521.

[16] P. Grisvard, *Elliptic problems in nonsmooth domains*, Monographs and studies in mathematics, **24**, Pitman, Boston, MA 1985.

[**17**] J. Kačur, *Nonlinear parabolic boundary value problems with the time derivatives in the boundary conditions*, Proceedings Equadiff IV, Lectures Notes in Math., **703**, Springer, Berlin 1979, 170-178.

[**18**] J.-L. Lions, *Quelques méthodes de résolution des problèmes aux limites non linéaires*, Dunod et Gauthier-Villars, Paris 1969.

[**19**] K. Rektorys, *The method of discretization in time and partial differential equations*, D. Reidel Publishing Co., Dordrecht-Boston, Mass. 1982.

[**20**] K. Rektorys and M. Ludvíková, *A note on nonhomogeneous initial and boundary conditions in parabolic problems solved by the Rothe method*, Apl. Mat. **25** (1980), 56-72.

[**21**] J.-M. Ricaud and E. Pratt, *Analysis of a time discretization for an implicit variational inequality modelling dynamic contact problems with friction*, Math. Methods Appl. Sci. **24** (2001), 491-511.

[**22**] A. D. Venttsel', *On boundary conditions for multidimensional diffusion processes*, Theor. Probability Appl. **4** (1959), no. 2, 164-177.

[**23**] E. Zeidler, *Nonlinear functional analysis and its applications, II/B*, Springer-Verlag, New York 1990.

Luisa Consiglieri
Independent Research Professor
Portugal
http://sites.google.com/site/luisaconsiglieri
e-mail: lconsiglieri@gmail.com