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Some applications of the *SBV* Regularity Theorem for entropy solutions of 1D scalar conservation laws to Convection Theory and sticky particles

Abstract. We show how it is possible to apply the *SBV* Regularity Theorem for entropy solutions of one-dimensional scalar conservation laws, proved by Ambrosio and De Lellis, to Convection Theory and sticky particles. In the multi-dimensional case we present a counterexample which prevent us from using the same approach.

Keywords. *SBV* regularity, pressureless gasses, sticky particles, Convection Theory, Generalized Hydrostatic Boussinesq equations, Hamilton-Jacobi equations, scalar conservation laws.

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1 - Introduction

We present some simple applications of the *SBV* Regularity Theorem for entropy solutions of one-dimensional scalar conservation laws presented in [**1**]. There, Ambrosio and De Lellis studied the regularity of entropy solutions of the scalar conservation law

$$\partial_t u + D_x(H(u)) = 0 \qquad \text{in } \Omega := \mathbb{R}^+ \times (a, b), \tag{1}$$

and proved the following

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Theorem 1.1 (One-dimensional *SBV* Regularity Theorem, Ambrosio and De Lellis). *Let $u \in L^\infty(\Omega)$ be an entropy solution of* (1) *with $H \in C^2(\mathbb{R})$ locally uniformly convex. Then u belongs to $SBV_{loc}(\Omega)$.*

For a detailed description of the spaces BV and SBV we refer to [**2**], Chapters 3, 4. For completeness, we briefly recall that, given $u \in BV(\mathbb{R}^d)$, it is possible to decompose its distributional derivative into three mutually singular measures: $Du = D^a u + D^c u + D^j u$. $D^a u$ is the absolutely continuous part with respect to the Lebesgue measure. $D^j u$ is the part of the measure which is concentrated on the rectifiable $m-1$ dimensional set J, where the function u has jump discontinuities, and for this reason is called jump part. $D^c u$, the Cantor part, is the singular part which satisfies $D^c u(E) = 0$ for every Borel set E with $\mathcal{H}^{m-1}(E) < \infty$. If this part vanishes, i.e. $D^c u = 0$, we say that $u \in SBV(\mathbb{R}^d)$.

Theorem 1.1 can be easily extended to one-dimensional Hamilton-Jacobi equations. Indeed, the potential, given by

$$\begin{cases} \partial_t U \ = -H(u) \\ D_x U = u, \end{cases}$$

is a viscosity solution (in the sense of Crandall-Lions) of the Hamilton-Jacobi equation

$$\partial_t U + H(D_x U) = 0 \tag{2}$$

if and only if u is an entropy solution to (1). Therefore, the *SBV* Regularity Theorem applies also to the distributional derivative of a viscosity solution of Hamilton-Jacobi equation (2) when H is $C^2(\Omega)$ and locally uniformly convex.

A recent generalization to the multi-dimensional case has been proved by Bianchini, De Lellis and Robyr in [**4**].

Theorem 1.2 (Multi-dimensional *SBV* Regularity Theorem, Bianchini, De Lellis, Robyr). *Let U be a viscosity solution of* (2) *in $\Omega \subset [0,T] \times \mathbb{R}^d$, assume H belongs to $C^2(\mathbb{R}^d)$ and*

$$c_H^{-1} Id_n \le D^2 H \le c_H Id_n$$

for some $c_H > 0$. Then $D_x U, \partial_t U$ belong to $SBV_{loc}(\Omega)$.

In Sections 2 and 3 we describe Generalized Hydrostatic Boussinesq (GHB) equations and the model of sticky particles, then, in Section 4, we show how the *SBV* Regularity Theorem applies to them in the one-dimensional case. In the last section we present a counterexample which prevent us from using the same approach for the multi-dimensional case. A similar counterexample was shown by Vasseur in [**12**], but never published.

2 - Generalized Hydrostatic Boussinesq equations

Generalized Hydrostatic Boussinesq (GHB) equations can be seen as the most degenerate version of Generalized Navier-Stokes Boussinesq (GNSB) equations, where both the inertia terms and the dissipative operator are neglected. These equations rule the dynamic of a fluid under fast convection. In terms of the temperature of the fluid they take the form

$$y = x + \nabla p, \qquad \nabla \cdot v = 0, \tag{3}$$

$$\partial_t y + (v \cdot \nabla) y = G(x), \tag{4}$$

here, being $D \subset \mathbb{R}^d$ a smooth bounded domain where the fluid is placed, the function $y(t,x) : \mathbb{R}^+ \times D \to \mathbb{R}^d$ is the generalized temperature field of the fluid, $v(t,x) : \mathbb{R}^+ \times D \to \mathbb{R}^d$ its velocity, $p(t,x) : \mathbb{R}^+ \times D \to \mathbb{R}$ the pressure, $G(x) : \mathbb{R}^d \to \mathbb{R}^d$ the generalized heat source term, an $[L^\infty(\mathbb{R}^d)]^d$ function. Equation (4) can be seen as a generalization of the hydrostatic balance in Convection Theory.

The fact that G depends only on the position of the fluid allows us to apply the previous results. However, this is a very particular assumption since the heat source can depend also on time and temperature $G = G(t,x,y)$.

The complete description of this system can be found in [**7**], Chapter 3. Passing to Lagrangian coordinates, Brenier proved here that a generalized solution can be found.

Since we need some concepts of Optimal Transport Theory let us recall some preliminary definitions and results.

First, we introduce rearrangements and measure preserving maps.

Definition 2.1. *Given two* $[L^2(D)]^d$ *maps* Y *and* Z*, we say that they are* rearrangement of each other *if they define the same image measure, i.e. for all continuous* f *on* $\mathbb{R}^d$*, such that* $|f(y)| \le 1 + |y|^2$*,*

$$\int_D f(Y(a))da = \int_D f(Z(a))da.$$

Definition 2.2. *We say that* Y *in* $[L^2(D)]^d$ *is a* measure preserving map, *when it is a rearrangement of the identity map, i.e. for all continuous* f *on* $\mathbb{R}^d$*, such that* $|f(y)| \le 1 + |y|^2$*,*

$$\int_D f(a)da = \int_D f(Y(a))da.$$

Next we define the class of maps with convex potential.

Definition 2.3. *We say that an* $[L^2(D)]^d$ *map* Y belongs to the class C of maps with a convex potential, *if there is a lower semi-continuous convex function* $p : \mathbb{R}^d \to (-\infty, +\infty]$ *such that, for* $\mathcal{L}^d$*-a.e. point* x *in* D*, the gradient* $\nabla p(x)$ *coincides with* Y.

Then, looking for a rearrangement with convex potential, we have the following Brenier's Theorem which can be found in [8]:

Theorem 2.1 (Polar Factorization Theorem, Brenier). *Let* Y *be a non degenerate* $[L^2(D)]^d$ *map. Then there is a unique polar factorization*

$$Y = Y^R \circ X,$$

where Y^R *belongs to* C *and* X *is a Lebesgue measure preserving map of* D.

In this decomposition, Y^R *is the unique rearrangement of* Y *in* C *and* X *is the unique measure preserving map of* D *that minimizes*

$$\int_D |X(a) - Y(a)|^2 da.$$

In addition, X *can be written:*

$$X(a) = (\nabla \phi)(Y(a)), \quad a.e.\ a \in D,$$

where ϕ *is a convex Lipschitz function defined on* $\mathbb{R}^d$.

Coming back to our problem and passing to Lagrangian coordinates the system (3,4) looks like

$$Y(t, a) = X(t, a) + \nabla p(t, X(t, a)), \tag{5}$$

$$\partial_t Y(t, a) = G(X(t, a)), \tag{6}$$

where, for all $t, D \ni a \longmapsto X(t, a)$ is a measure preserving map as a consequence of the fact that v is a smooth divergence-free vector field. $X(t, a)$ denotes the position of a fluid parcel a at a time t, therefore its velocity and its temperature are

$$\partial_t X(t, a) = v(t, X(t, a)), \qquad Y(t, a) = y(t, X(t, a)). \tag{7}$$

Note that, due to the above equations, particles that collide, get stuck together. If particles reach the same position they will have the same velocity, the same temperature and will no more separate. In fact GHB equations are very closed to systems of sticky particles as we can see in the next section.

Assuming a priori that the map $x \mapsto x + \nabla p(t, x)$ has convex potential, we deduce from (5) that $x \mapsto x + \nabla p(t, x)$ is the unique convex rearrangement $Y^R(t, \cdot)$ of $Y(t, \cdot)$, due to the fact that $Y(t,a) = Y^R(t,\cdot) \circ X(t,a)$. Moreover (6) implies that for all C^1 function f compactly supported on $\mathbb{R}^d$, $Y^R(t,\cdot)$ satisfies

$$\frac{d}{dt}\int_D f(Y^R(t,a))da = \int_D (\nabla f)(Y^R(t,a)) \cdot G(a)da. \tag{8}$$

With these considerations Brenier naturally introduced a more general concept of solution to GHB system.

Definition 2.4. *We say that* Y^{CR} *in* $C^0([0,T],[L^2(D)]^d)$ *is the* convex rearrangement (*CR*) solution *to the GHB equations* (5, 6)*, if:*

- $Y^{CR}(t,\cdot)$ *belongs to the set C of all maps with convex potential, for all* $t \in [0,T]$,
- *for all compactly supported* C^1 *function* f *on* $\mathbb{R}^d$, $Y^{CR}(t,\cdot)$ *satisfies* (8).

In [7], he proved the following existence theorem.

Theorem 2.2 (Brenier). *For each initial condition* Y^0 *in* $[L^2(D)]^d$*, there is at least one CR-solution* $Y^{CR}(t,a)$ *such that* $Y^{CR}(0,\cdot) = (Y^0)^R(\,\cdot\,)$.

This solution can be obtained as the limit in $C^0([0,T],[L^2(D)]^d)$ *as* $h \to 0$*, of a time discrete approximation* $Y^h(t,a)$ *defined, first at discrete times* $t = nh$*, by:*

$$Y^h(nh+h,a) = [Y^h(nh,a) + hG(a)]^R, \quad n = 0,1,2,\ldots$$

(*where, as seen before,* $(\,\cdot\,)^R$ *is the convex rearrangement operator*) *and then linearly interpolated in* t.

The time discrete approximation, given by the theorem above, tells us that starting from an initial temperature data Y^0, the CR-solution evolves linearly as $(Y^0)^R(a) + tG(a)$ as far as this function remains with convex potential. When this is no more the case, it is rearranged in order to preserve the membership to the space of maps with convex potential.

Note that CR-solutions are a.e. equal to functions with convex potential, i.e. for all t there exists a convex function $\psi_t : D \to \mathbb{R}$ such that

$$Y^{CR}(t,a) = D\psi_t(a),$$

for a.e. a in D. Taking now the Legendre transform of this convex function

$$U(t,x) = \sup_{a\in D} x \cdot a - \psi_t(a),$$

we obtain a function $U(t,\cdot)$ which is again convex and its distributional derivative $D_x U(t,\cdot)$ is the generalized inverse of $Y^{CR}(t,\cdot)$. We are interested in the regularity of $D_x U(t,\cdot)$. What we can say so far is that it is a function of bounded variation.

2.1 - *One-dimensional case*

In the one-dimensional case it is possible to look at $D_xU(t,\cdot)$ as a solution of a scalar conservation law.

First we can observe that, taking $D=[0,1]$, the convex rearrangement is the monotone nondecreasing rearrangement defined by

$$Y^R(s)=\inf\{t\in\mathbb{R}\mid \mu_Y(t)>s\},$$

for s in [0,1], where

$$\mu_Y(t)=|\{Y<t\}|,$$

is the distribution function. For a detailed description of monotone nondecreasing rearrangement we refer to [**11**], Chapter 1.

One of the properties of monotone nondecreasing rearrangement is that it is non expansive in $L^2([0,1])$, i.e.

$$\int_D |Y^R(a)-Z^R(a)|^2 da \le \int_D |Y(a)-Z(a)|^2 da.$$

This property guarantees the uniqueness of the solution of (8).

Moreover, as explained in [**6**], the limit of the time discrete approximation, defined in Theorem 2.2, satisfies the sub-differential inclusion:

$$G(x)\in \partial_t Y+\partial\Psi[Y], \tag{9}$$

where $\Psi[Y]=0$ if Y is a nondecreasing function of x in D, and $\Psi[Y]=+\infty$ otherwise.

The generalized inverse of the solution, in the one-dimensional case, can be found using the Heaviside function. Looking at its behavior, Brenier proved in [**5**], the following theorem. In the proof he used a Transport Collapse method, which involves the same time discrete approximation scheme seen in Theorem 2.2.

Theorem 2.3 (Brenier). *Let $Y^{CR}(t,a)$ be the CR-solution found in Theorem 2.2, then the generalized inverse*

$$u(t,y)=\int_0^1 \mathbb{H}(y-Y^{CR}(t,a))da,$$

where $\mathbb{H}$ is the Heaviside function, is an entropy solution of the scalar conservation law

$$\partial_t u + D_x(H(u))=0,$$

where H is the primitive of G, $D_pH(p)=G(p)$.

We are interested in the regularity of an entropy solution of a scalar conservation law with nondecreasing initial conditions and Lipschitz flux function H. Applying what we have already said, since we are in the one-dimensional case, the entropy solution above can be seen as the derivative of the unique viscosity solution of the following Hamilton-Jacobi equation

$$\partial_t U + H(D_x U) = 0,$$

with a convex initial datum and Lipschitz Hamiltonian.

3 - Sticky particles

At a discrete level pressureless gases with sticky particles can be modeled by a finite collection of particles that get stuck together right after they collide with conservation of mass and momentum. On the other hand at a continuous level the model is governed by the following one-dimensional system of conservation laws in $(0, +\infty) \times \mathbb{R}$

$$\partial_t \rho + D_x(\rho v) = 0, \tag{10}$$

$$\partial_t(\rho v) + D_x(\rho v^2) = 0, \tag{11}$$

where $\rho(t, x)$ is the density field, while $v(t, x)$ is the velocity one. This set of equations can be seen as the limit, when pressure goes to zero, of the usual Euler equations. In [**9**], Brenier and Grenier showed that the continuous model can be fully described, in an alternative way, by scalar conservation laws, with non-decreasing initial conditions, general flux functions and the usual Kruzhkov entropy condition.

In particular they proved that if (ρ, v) is a solution corresponding to sticky particles, then there exist $H \in \mathrm{Lip}(\mathbb{R})$ and u entropy solution of

$$\partial_t u + D_x(H(u)) = 0,$$

where $u(tx) = D_x U(t, x)$ is such that $\rho(t, x) = D_{xx} U(t, x)$ is a cumulative distribution function associated to the probability measure ρ and $D_x H(x) = v(0, x)$.

The proof uses a scheme in which a finite number of particles are described by weight, position and velocity, under the assumption that the speed of a particle is constant as long as it meets no new particles and it changes only when shocks occur. Only a finite number of shocks can occur because particles remain stuck together after a collision. Moreover particles having the same position at a time t move together at the same speed and their total momentum is the sum of their initial momentum. This scheme is strongly reminiscent of Dafermos's polygonal approximation methods for scalar conservation laws, where each particle corresponds to a jump of an entropy solution of a scalar conservation law with a piecewise linear continuous flux function. Thus, it is reasonable to expect, as it is, that the continuous limit of the sticky particles dynamics is properly described by a scalar conservation law.

The fact that the distribution function u is a nondecreasing entropy solution of that scalar conservation law strictly relates sticky particle system to Convection Theory. Indeed if we take the generalized inverse of u, which is precisely the monotone rearrangement of the measure ρ, it turns out that it is exactly the limit of the time discrete approximation seen in Theorem 2.2.

As we did for GHB equations we can relate the nondecreasing entropy solutions to the viscosity solution of an Hamilton-Jacobi equation with convex initial datum.

4 - Convex solutions of Hamilton-Jacobi equations in the multi-dimensional case

Let us now consider the following Hamilton-Jacobi equation

$$\partial_t U + H(D_x U) = 0,$$

with initial datum $U(0,x) = 1/2|x|^2$, and Lipschitz Hamiltonian H. Thus we are in a particular case of the ones considered above. As proved in [**3**], by Bardi and Evans, the unique viscosity solution to such an equation has the form

$$U(t,x) = \sup_y \inf_z \left\{ \frac{1}{2}|z|^2 + y\cdot(x-z) - tH(y) \right\}.$$

This representation formula is true even in the multi-dimensional case and an analogous one works as well for general initial data but convex Hamiltonians. Moreover it is equivalent to

$$U(t,x) = \sup_y \left\{ x\cdot y - \frac{1}{2}|y|^2 - tH(y) \right\}. \tag{12}$$

Here the sup becomes a maximum under suitable hypotheses on H.

Note that equation (12) is equivalent to saying that U is the Legendre transform of $1/2|y|^2 + tH(y)$. On the other hand since U is, in the GHB equation case, the Legendre transform of $\psi_t(a)$, we have the following geometric representation for the CR-solution, for a.e. a,

$$Y^{CR}(t,a) = \nabla\, convex(\psi_0(a) + tH(a)),$$

where $convex(f) = \max\{g \le f \mid g \text{ convex}\}$.

Define

$$V(t,x) := -\frac{1}{t}\left(U(t,x) - \frac{1}{2}|x|^2\right),$$

then

$$V(t,x) = \min_y \left\{ H(y) + \frac{|x-y|^2}{2t} \right\}$$

is the unique viscosity solution of

$$\partial_t V + \frac{|D_x V|^2}{2} = 0$$

with Lipschitz initial data $V_0(x) = H(x)$.

Since the Hamiltonian $|x|^2/2$ is uniformly convex we can use directly the result obtained by Ambrosio and De Lellis in [**1**], to prove that $D_x V(t,\cdot)$ belongs to SBV for a.e. t and the same is true also for $D_x U(t,\cdot)$.

Remark 4.1. *From what we have seen, in the one-dimensional case, SBV regularity holds for the generalized inverse of a solution of GHB equation with the identity as initial datum and for the cumulative distribution function associated to the density of the pressureless gas.*

5 - Multi-dimensional case

We wonder if Hamilton-Jacobi equations are a good model for GHB systems or sticky particles models even in the multi-dimensional case. Are they able to describe the behavior of our solution? If this was the case we could automatically state SBV regularity applying Theorem 1.2. Unfortunately the answer to our question is negative. In the following subsection we show a counterexample in which a multi-dimensional solution of an Hamilton-Jacobi equation has a behavior which is not allowed for GHB systems or sticky particles models, i.e. Theorem 1.2 does not suit our problem. However, this does not mean that SBV regularity cannot be proved in some other way.

5.1 - *Counterexample*

A first counterexample was found by Vasseur in [**12**] but it was never published. With that counterexample Vasseur showed a discrepancy between the density distribution $\tilde{\rho}(t,x) = \det D_x^2 U(t,x)$, associated to the solution U of the Hamilton-Jacobi equation $\partial_t U + H(D_x U) = 0$ with initial datum $U(0, x) = 1/2$, and the density distribution $\rho(t, x)$, generated from the identity $\rho_0(x) = 1$ in a sticky particles process with speed $v = D_x H(x)$. Indeed, he proved the existence of a time t at which the two density distribution differ.

The following counterexample shows the same discrepancy, underlining in addiction the cause of it. Hamilton-Jacobi equations allow separations of particles after collisions.

Consider the viscosity solution of the Hamilton-Jacobi equation

$$V_t + \frac{1}{2}|D_x V|^2 = 0,$$

in $\mathbb{R}^2$, with initial datum

$$V(0, x) = \begin{cases} -\dfrac{|x|^2}{2} & \text{for } x \in \overline{B(0,1)} \\ f(x) & \text{for } x \in \overline{B(0,2)} \setminus B(0,1) \\ -|x_1| & \text{for } x \in \mathbb{R}^2 \setminus B(0,2), \end{cases}$$

where $f(x)$ joins smoothly $-|x|^2/2$ to $-|x_1|$ and satisfies $f(x) > -|x|^2/2$ in $B(0,2) \setminus \overline{B(0,1)}$, being $B(x, r)$ the open ball with center in x and radius $r > 0$.

Note that following upside down the passages seen in Section 4 we can recover from V a convex viscosity solution of the equation

$$\partial_t U + H(D_x U) = 0, \qquad U(0, x) = \frac{|x|^2}{2}$$

where

$$U(t, x) = -tV(t, x) + \frac{|x|^2}{2}$$

and $H(x) = V(0, x)$ is a smooth function. We are thus considering a viscosity solution of Hamilton-Jacobi with convex initial datum. If Hamilton-Jacobi were the good model for GHB and sticky particles systems, passing to the Legendre transform of our viscosity solution we should recover the CR-solution limit of the time-discrete approximation scheme.

Using the Hopf-Lax formula for convex Hamiltonians we recover the viscosity solution for any time t

$$V(t, x) = \min_y \left\{ V(0, y) + \frac{(x_1 - y_1)^2 + (x_2 - y_2)^2}{2t} \right\}.$$

Let us compute the value of V for $t = 1$ in the origin:

$$V(1,(0,0)) = \min_y \left\{ V(0,y) + \frac{y_1^2 + y_2^2}{2} \right\}. \tag{13}$$

Observe that for any y in $\overline{B(0,1)}$ we have

$$V(0,y) + \frac{y_1^2 + y_2^2}{2} = 0.$$

For any y in $B(0,2) \setminus \overline{B(0,1)}$

$$V(0,y) + \frac{y_1^2 + y_2^2}{2} > 0.$$

For any y in $\mathbb{R}^2 \setminus B(0,2)$ we have

$$V(0,y) + \frac{y_1^2 + y_2^2}{2} = -|y_1| + \frac{y_1^2 + y_2^2}{2} \geq -|y_1| + \frac{y_1^2}{2},$$

hence we can restrict the minimum in the region $\mathbb{R}^2 \setminus B(0,2)$ to points with $y_2 = 0$, moreover for that points we have

$$-|y_1| + \frac{y_1^2}{2} \geq 0,$$

and the equality occurs only for $(-2,0)$ and $(2,0)$.

Thus the minimum in (13) is obtained if and only if y belongs to the set $B(0,1) \cup \{(-2,0),(2,0)\}$. The origin is therefore a point of non differentiability for $V(1,\cdot)$ with the convex hull of the set of all minima as super-differential. This means that all the points in the set $B(0,1) \cup \{(-2,0),(2,0)\}$, which is of positive $\mathcal{H}^2$-measure, are transported by the flux along straight line trajectories which collide at time $t = 1$ in the position $(0,0)$.

However, for any $\delta > 0$, we have to compute

$$V(1+\delta,(0,0)) = \min_y \left\{ V(0,y) + \frac{y_1^2 + y_2^2}{2(1+\delta)} \right\}. \tag{14}$$

For any y in $\overline{B(0,1)}$ we have

$$V(0,y) + \frac{y_1^2 + y_2^2}{2(1+\delta)} = -\delta \frac{y_1^2 + y_2^2}{2(1+\delta)} \geq -\frac{\delta}{2(1+\delta)}.$$

For any y in $B(0,2) \setminus \overline{B(0,1)}$

$$V(0,y) + \frac{y_1^2 + y_2^2}{2(1+\delta)} > -\delta \frac{y_1^2 + y_2^2}{2(1+\delta)} > -\frac{4\delta}{2(1+\delta)}.$$

Here we note that $-\dfrac{4\delta}{2(1+\delta)} < -\dfrac{\delta}{2(1+\delta)}$.

For any y in $\mathbb{R}^2 \setminus B(0,2)$ we have

$$V(0,y) + \frac{y_1^2 + y_2^2}{2(1+\delta)} = -|y_1| + \frac{y_1^2 + y_2^2}{2(1+\delta)} \geq -|y_1| + \frac{y_1^2}{2(1+\delta)},$$

hence we can restrict the minimum to the points in $\mathbb{R}^2 \setminus B(0,2)$ with $y_2 = 0$. Moreover, for that points we have that the minimum value is reached for $|y_1| = 2$ if $1+\delta < 2$, for $|y_1| = 1+\delta$ otherwise.

In the first case

$$-|y_1| + \frac{y_1^2}{2(1+\delta)} = -\frac{4\delta}{2(1+\delta)}.$$

In the second one

$$-|y_1| + \frac{y_1^2}{2(1+\delta)} = -\frac{(1+\delta)^2}{2(1+\delta)} < -\frac{4\delta}{2(1+\delta)}.$$

Thus (0,0) is a point of non differentiability even for $t = 1+\delta$ for any $\delta > 0$. Moreover its super-differential is the set $[(-2,0),(2,0)]$ for $0 < \delta < 1$, or the set $[(-(1+\delta),0),((1+\delta),0)]$ for $\delta > 1$. In any case it is a set of positive $\mathcal{H}^1$ measure. This set has non-empty intersection with the super-differential of V in the origin at time $t = 1$ but does not contain the whole of it. Recall that, the super-differential of V in the origin contains, at time $t = 1$, the set convex$(B(0,1) \cup \{(-2,0),(2,0)\})$ which is a set of positive $\mathcal{H}^2$-measure.

Points, being in $B(0,1) \cup \{(-2,0),(2,0)\}$ at time $t = 0$, collide at time $t = 1$ and separate at time $t = 1+\delta$ for any $\delta > 0$.

We have thus shown an example of a viscosity solution in which a point of non differentiability of zero codimension evolves in a point of non differentiability of codimension one.

As we have already said, coming back to $U(t,x) = -tV(t,x) + |x|^2/2$ and passing to the Legendre transform of our viscosity solution, we should obtain the CR-solution of the GSB equation. However for this function a flat part of dimension two would evolve in a flat part of dimension one, in contrast with propagation of flat parts. Particles stuck together could have different velocities but this is not the case for GHB and the sticky particles model.

Hence GHB and the sticky particles model cannot be truly described by Hamilton-Jacobi equation in the multidimensional case.

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References

[1] L. Ambrosio and C. De Lellis, *A note on admissible solutions of 1D scalar conservation laws and 2D Hamilton-Jacobi equations*, J. Hyperbolic Differ. Equ. **1** (2004), no. 4, 813-826.

[2] L. Ambrosio, N. Fusco and D. Pallara, *Functions of bounded variation and free discontinuity problems*, Oxford Mathematical Monographs, Oxford University Press, New York 2000.

[3] M. Bardi and L. C. Evans, *On Hopf's formulas for solutions of Hamilton-Jacobi equations*, Nonlinear Anal. **8** (1984), no. 11, 1373-1381.

[4] S. Bianchini, C. De Lellis and R. Robyr, *SBV regularity for Hamilton-Jacobi equations in* $\mathbb{R}^n$, Arch. Ration. Mech. Anal. **200** (2011), 1003-1021.

[5] Y. Brenier, L^2 *formulation of multidimensional scalar conservation laws*, Arch. Ration. Mech. Anal. **193** (2009), no. 1, 1-19.

[6] Y. Brenier, *On the Hydrostatic and Darcy limits of the convective Navier-Stokes equations*, Chin. Ann. Math. Ser. B **30** (2009), no. 6, 683-696.

[7] Y. Brenier, *Optimal transport, convection, magnetic relaxation and generalized Boussinesq equations*, J. Nonlinear Sci. **19** (2009), no. 5, 547-570.

[8] Y. Brenier, *Polar factorization and monotone rearrangement of vector-valued functions*, Comm. Pure Appl. Math. **44** (1991), no. 4, 375-417.

[9] Y. Brenier and E. Grenier, *Sticky particles and scalar conservation laws*, SIAM J. Numer. Anal. **35** (1998), no. 6, 2317-2328.

[10] P. Cannarsa and C. Sinestrari, *Semiconcave functions, Hamilton-Jacobi equations, and optimal control*, Birkhäuser Boston, Boston 2004.

[11] J. Mossino, *Inégalités isopérimétriques et applications en physique*, Hermann, Paris 1984.

[12] A. Vasseur, *Contributions à l'approche cinétique des systèmes de lois de conservation hyperboliques*, Thèse de doctorat de l'Université Paris 6 (1999).

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