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Weakly well posed characteristic hyperbolic problems

Abstract. We present recent results about the mixed initial-boundary value problem for a linear hyperbolic system with characteristic boundary of constant multiplicity. We assume the problem to be "weakly" well posed, namely that a unique L^2-solution exists, for sufficiently smooth data, and obeys an a priori energy estimate with a finite loss of conormal regularity. Under the assumption of the loss of one conormal derivative, we obtain the regularity of solutions in the natural framework of the anisotropic Sobolev spaces, provided the data are sufficiently smooth.

Keywords. Symmetric and symmetrizable hyperbolic systems, initial-boundary value problem, weak well posedness, characteristic boundary, anisotropic Sobolev spaces, tangential regularity.

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1 - Introduction and main results

For $n \geq 2$, let $\mathbb{R}^n_+$ denote the $n-$dimensional positive half-space

$$\mathbb{R}^n_+ := \{x = (x_1, x'),\ x_1 > 0,\ x' := (x_2, \ldots, x_n) \in \mathbb{R}^{n-1}\}.$$

The boundary of $\mathbb{R}^n_+$ will be identified with $\mathbb{R}^{n-1}_{x'}$. For $T > 0$, we set $Q_T := \mathbb{R}^n_+ \times]0, T[$ and $\Sigma_T := \mathbb{R}^{n-1} \times]0, T[$; also $\Omega_T := \mathbb{R}^n_+ \times]-\infty, T[$ and $\omega_T := \mathbb{R}^{n-1} \times]-\infty, T[$. If time t spans the whole real line $\mathbb{R}$, we set $Q := \mathbb{R}^n_+ \times \mathbb{R}_t$ and $\Sigma := \mathbb{R}^{n-1} \times \mathbb{R}_t$. We are

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interested in the following initial-boundary value problem (shortly written IBVP)

$$Lu = F \quad \text{in } Q_T, \tag{1}$$

$$Mu = G \quad \text{on } \Sigma_T, \tag{2}$$

$$u_{|t=0} = f \quad \text{in } \mathbb{R}^n_+, \tag{3}$$

where L is a first-order linear partial differential operator

$$L = \partial_t + \sum_{i=1}^{n} A_i(x,t)\partial_i + B(x,t), \tag{4}$$

$\partial_t := \dfrac{\partial}{\partial t}$ and $\partial_i := \dfrac{\partial}{\partial x_i}$, $i = 1, \dots, n$.

The coefficients A_i, B, for $i = 1, \dots, n$, are real $N \times N$ matrix-valued functions, defined on Q. The unknown $u = u(x,t)$, and the data $F = F(x,t)$, $G = G(x,t)$, $f = f(x)$ are vector-valued functions. M is a given real matrix of size $d \times N$ and maximal rank $d < N$.

We study the problem (1)-(3) under the following assumptions. The function spaces involved in (D), (E) and in the statement of Theorem 1.1 below, as well as the norms appearing in (9), (10), (12) will be described in the next Section 2. The square brackets [ ] of a real number denote its integer part.

(A) L is *Friedrichs symmetrizable*, namely there exists a matrix S_0, definite positive on $\overline{Q}$ (there exists $\rho > 0$ such that $S_0(x,t) \geq \rho$ for every $(x,t) \in Q$), symmetric and such that $S_0 A_i$, for $i = 1, \dots, n$, are also symmetric.

(B) The IBVP is *characteristic of constant multiplicity* $1 \leq r < N$, namely the coefficient A_1 of the normal derivative in L displays the structure

$$A_1(x,t) = \begin{pmatrix} A_1^{I,I} & A_1^{I,II} \\ A_1^{II,I} & A_1^{II,II} \end{pmatrix}, \tag{5}$$

where $A_1^{I,I}$, $A_1^{I,II}$, $A_1^{II,I}$, $A_1^{II,II}$ are respectively $r \times r$, $r \times (N-r)$, $(N-r) \times r$, $(N-r) \times (N-r)$ sub-matrices, such that

$$A^{I,II}_{1\,|\,x_1=0} = 0\,, \quad A^{II,I}_{1\,|\,x_1=0} = 0\,, \quad A^{II,II}_{1\,|\,x_1=0} = 0\,, \tag{6}$$

and $A_1^{I,I}$ is uniformly invertible on the boundary Σ, namely there exists $\mu > 0$ such that $|\det A_1^{I,I}(x,t)| \geq \mu$, for any $(x,t) \in \Sigma$. Accordingly we split the unknown u as $u = (u^I, u^{II})$; $u^I \in \mathbb{R}^r$ and $u^{II} \in \mathbb{R}^{N-r}$ are said respectively the *noncharacteristic* and the *characteristic* components of u.

(C) $M = (I_d \ 0)$, where I_d is the identity matrix of size d, 0 is the zero matrix of size $d \times (N-d)$ and $d \leq r$ is the (constant) number of positive eigenvalues of $A^{I,I}_{1\,|\,\{x_1=0\}}$ (namely the *incoming characteristics* of problem (1)-(3)).

(D) *Existence of the L^2 weak solution.* Assume that $S_0, A_i \in W^{2,\infty}(Q)$, for $i = 1, \dots, n$. For all $T > 0$ and $B \in W^{1,\infty}(\Omega_T)$, there exist constants $\gamma_0 \geq 1$ and $C_0 > 0$ (depending on T, ρ, μ, $\|S_0\|_{W^{2,\infty}(\Omega_T)}$, $\|A_i\|_{W^{2,\infty}(\Omega_T)}$, $\|B\|_{W^{1,\infty}(\Omega_T)}$) such that for all $\gamma \geq \gamma_0$ and $F \in H^1_{tan,\gamma}(\Omega_T)$, $G \in H^1_\gamma(\omega_T)$, vanishing for $t<0$, the boundary value problem (shortly written BVP)

$$Lu = F \quad \text{in } \Omega_T, \tag{7}$$

$$Mu = G \quad \text{on } \omega_T, \tag{8}$$

with B in L, admits a unique solution $u \in L^2(\Omega_T)$, vanishing for $t<0$, such that $u^I_{|\,\omega_T} \in L^2(\omega_T)$. Furthermore $u \in C([0,T]; L^2(\mathbb{R}^n_+))$, and it satisfies an a priori estimate of the form

$$\begin{aligned} &\gamma\|u_\gamma\|^2_{L^2(\Omega_t)} + \|u_\gamma(t)\|^2_{L^2(\mathbb{R}^n_+)} + \|u^I_{\gamma|\,\omega_T}\|^2_{L^2(\omega_t)} \\ &\leq C_0\left(\frac{1}{\gamma^3}\|F_\gamma\|^2_{H^1_{tan,\gamma}(\Omega_t)} + \frac{1}{\gamma^2}\|G_\gamma\|^2_{H^1_\gamma(\omega_t)}\right) \end{aligned} \tag{9}$$

for $\gamma \geq \gamma_0$ and $0<t \leq T$, where $u_\gamma := e^{-\gamma t}u, F_\gamma := e^{-\gamma t}F, G_\gamma := e^{-\gamma t}G$. Furthermore, if $T = +\infty$, for all $B_1 \in W^{1,\infty}(Q)$ and all *conormal pseudo-differential operators* B_2 with symbol in Γ^0, there exist constants $\gamma_0' \geq 1$ and $C_0' > 0$ (depending on ρ, μ, $\|S_0\|_{W^{2,\infty}(Q)}$, $\|A_i\|_{W^{2,\infty}(Q)}$, $\|B_1\|_{W^{1,\infty}(Q)}$, and on a finite number of seminorms of the symbol of B_2) such that for all $F \in e^{\gamma t}H^1_{tan,\gamma}(Q)$, $G \in e^{\gamma t}H^1_\gamma(\Sigma)$, the BVP (7), (8) on Q, with $B = B_1 + B_2$ in L, admits a unique solution $u \in e^{\gamma t}L^2(Q)$ such that $u^I_{|\,\Sigma} \in e^{\gamma t}L^2(\Sigma)$. Furthermore, for $\gamma \geq \gamma_0'$, u satisfies the a priori estimate

$$\begin{aligned} &\gamma\|u_\gamma\|^2_{L^2(Q)} + \|u^I_{\gamma|\,\Sigma}\|^2_{L^2(\Sigma)} \\ &\leq C_0'\left(\frac{1}{\gamma^3}\|F_\gamma\|^2_{H^1_{tan,\gamma}(Q)} + \frac{1}{\gamma^2}\|G_\gamma\|^2_{H^1_\gamma(\Sigma)}\right). \end{aligned} \tag{10}$$

(E) Given $T > 0$, let $(S_0, A_i) \in \mathcal{C}_T(H^\sigma_{*,\gamma}) \times \mathcal{C}_T(H^\sigma_{*,\gamma})$, where $\sigma \geq [(n+1)/2] + 4$, be matrices enjoying properties (A) - (D) on Q. Assume there exist matrix-valued functions $(S_0^{(k)}, A_i^{(k)})$ in C^∞ converging, as $k \to \infty$, to (S_0, A_i) in $\mathcal{C}_T(H^\sigma_{*,\gamma}) \times \mathcal{C}_T(H^\sigma_{*,\gamma})$ on $[0,T]$, and in $W^{2,\infty}(Q) \times W^{2,\infty}(Q)$ on $\mathbb{R}_t$. Assume also that $(S_0^{(k)}, A_i^{(k)})$ satisfy (A), (B) on Q. Then, for k large enough, property (D) holds for the approximating problems with coefficients $(S_0^{(k)}, A_i^{(k)})$.

When an IBVP admits the solution u enjoying an a priori estimate of type (9) or (10), with $F = Lu$, $G = Mu$, the IBVP is said to be *weakly L^2-well posed*. This is the case of problems that do not satisfy the *uniform Kreiss-Lopatinskiĭ condition*. More specifically, an energy inequality of type (9), (10) occurs when the *Lopatinskiĭ determinant* has one simple root in the *hyperbolic region* of the frequency domain, see e.g. [**2, 3**] for the definitions. In [8], Coulombel and Guès show that the loss of regularity displayed in (9), (10) in such a case is optimal. They also prove that the well posedness result with loss of regularity is independent of Lipschitzean zero order terms B but is *not* independent of bounded zero order terms. This is a major difference with the *strongly* L^2*–well posed* problems studied in [**14**], where there is no loss of derivatives and one can treat lower order terms as source terms in energy estimates. Thus the stability of the problem under lower order perturbations is no longer a trivial consequence of the well posedness itself, and it is assumed as an additional hypothesis about the IBVP, see (D). Under an a priori estimate of type (9), (10), Coulombel [**7**] has proven the well posedness of the problem, namely the existence of the L^2 solution for all H^1 data. As for (E), hyperbolic IBVPs that do not satisfy the uniform Kreiss-Lopatinskiĭ condition in the hyperbolic region belong to the WR class defined by Benzoni-Gavage, Rousset, Serre and Zumbrun [**2**]. This class of problems is *stable* under small perturbations of the coefficients A_i, B, in agreement with (E). Examples of problems where the uniform Kreiss-Lopatinskiĭ condition breaks down are given by elastodynamics (with the Rayleigh waves [**19, 24**]), shock waves or contact discontinuities in compressible fluids, see e.g. [**13, 10**]. An a priori estimate similar to (9), (10) holds for linearized compressible vortex sheets, see Coulombel and Secchi [**9, 10, 11**], provided that $S_0, A_i \in W^{2,\infty}(Q)$ and $B \in W^{1,\infty}(Q)$.

Under the assumptions (A)-(D) it is not hard to get the L^2 solvability of the IBVP (1)-(3) on $[0, T]$, with initial data $f \neq 0$, cf. [**16,** Theorem 1.1].

In this paper, we are mainly interested in the regularity of solutions to the IBVP (1)-(3), for which some *compatibility conditions* on the data F, G, f are needed. These conditions are defined in the usual way, see [**18**]. Given the equation (1), we recursively define $f^{(h)}$ by formally taking $h-1$ t-derivatives of $Lu = F$, solving for $\partial_t^h u$ and evaluating it at $t = 0$. For $h = 0$ we set $f^{(0)} := f$. The *compatibility condition* of order $m \geq 0$ for the IBVP (1)-(3) reads as

$$Mf^{(h)} = \partial_t^h G_{|\,t=0}\,, \quad \text{on } \mathbb{R}^{n-1}\,,\ h = 0, \dots, m\,. \tag{11}$$

In [**16**] we have proved the following theorem. This note is devoted to the presentation of this result. For a detailed proof of Theorem 1.1, the reader is addressed to [**16**].

Theorem 1.1. *Let* $m \geq 1$ *be an arbitrary integer and* $s = \max\{m+1, [(n+1)/2] + 7\}$. *Given* $T > 0$, *assume that* $S_0, A_i \in \mathcal{C}_T(H^s_{*,\gamma})$, *for* $i = 1, \dots, n$, *and* $B \in \mathcal{C}_T(H^{s-1}_{*,\gamma})$ *(or* $B \in \mathcal{C}_T(H^s_{*,\gamma})$ *if* $m+1 = s$*). Assume also that* (A)-(E) *are satisfied. Then there exist constants* $C_m > 0$ *and* $\gamma_m \geq 1$, *depending only on* A_i, B, *such that for all* $\gamma \geq \gamma_m$, $F \in H^{m+1}_{*,\gamma}(Q_T)$, $G \in H^{m+1}_\gamma(\Sigma_T)$, $f \in H^{m+1}_\gamma(\mathbb{R}^n_+)$, *satisfying the compatibility condition* (11) *of order* m, *the unique solution* u *to* (1)-(3) *belongs to*

$\mathcal{C}_T(H^m_{*,\gamma})$ *and* $u^I_{|\Sigma_T} \in H^m_\gamma(\Sigma_T)$. *Moreover* u *satisfies the a priori estimate*

$$\begin{aligned} &\gamma\|u_\gamma\|^2_{H^m_{*,\gamma}(Q_T)} + \max_{t\in[0,T]} |||u_\gamma(t)|||^2_{m,*,\gamma} + \|u^I_{\gamma|\,\Sigma_T}\|^2_{H^m_\gamma(\Sigma_T)} \\ &\le C_m\left(\frac{1}{\gamma^2}|||f|||^2_{m+1,*,\gamma} + \frac{1}{\gamma^3}\|F_\gamma\|^2_{H^{m+1}_{*,\gamma}(Q_T)} + \frac{1}{\gamma^2}\|G_\gamma\|^2_{H^{m+1}_\gamma(\Sigma_T)}\right). \end{aligned} \tag{12}$$

In [**14**], the regularity of weak solutions to the characteristic IBVP (1)-(3) is studied, under the assumption that the problem is *strongly L^2-well posed*, namely a unique L^2-solution exists for arbitrary L^2-data, and the solution obeys an a priori energy inequality *without loss of regularity with respect to the data*; this means that the L^2-norms of the interior and boundary values of the solution are measured by the L^2-norms of the corresponding data F, G, f. The statement of Theorem 1.1 extends the result of [**14**], to the case where only a *weak well posedness* property is satisfied by the IBVP (1)-(3). Here, the L^2-solvability of (1)-(3) requires an additional regularity of the data F, G, f, cf. (D). Correspondingly, the regularity of the solution of order m is achieved provided the data have a regularity of order $m+1$. To prove the result of [**14**], the solution u to (1)-(3) is regularized by a family of *tangential mollifiers* J_ε, $0<\varepsilon<1$, defined by Nishitani and Takayama in [**17**] as a suitable combination of the operator $\sharp$ (see the next Section 3) and the standard Friedrichs'mollifiers. The essential point of the analysis performed in [**14**] is to notice that the mollified solution $J_\varepsilon u$ solves the same problem (1)-(3), as the original solution u. The data of the problem for $J_\varepsilon u$ come from the regularization, by J_ε, of the data involved in the original problem for u; furthermore, an additional term $[J_\varepsilon, L]u$, where $[J_\varepsilon, L]$ is the commutator between the differential operator L and J_ε, appears into the equation satisfied by $J_\varepsilon u$. Because the strong L^2-well posedness is preserved under lower order perturbations, actually this term can be incorporated into the source term of the equation satisfied by $J_\varepsilon u$. In the case of Theorem 1.1, where the L^2 a priori estimate exhibits a finite loss of regularity with respect to the data, this technique seems to be unsuccesful, since $[J_\varepsilon, L]u$ cannot be absorbed into the right-hand side without losing derivatives on the solution u; on the other hand it seems that the same term cannot be merely reduced to a lower order term involving the smoothed solution $J_\varepsilon u$. These observations lead to develop another technique, where the mollifier J_ε is replaced by the operator $\lambda^{m-1,\gamma}_\delta(Z)$ in (30), arising from the characterization of regularity given by Proposition 3.1 (see also (31)). Instead of studying the problem satisfied by $J_\varepsilon u$, here we consider the problem satisfied by $\lambda^{m-1,\gamma}_\delta(Z)u$. As before, a new term $[\lambda^{m-1,\gamma}_\delta(Z), L]u$ appears which takes account of the commutator between L and $\lambda^{m-1,\gamma}_\delta(Z)$. Since we assume the weak well posedness of the IBVP (1)-(3) to be preserved under lower order terms, the approach is to treat the commutator $[\lambda^{m-1,\gamma}_\delta(Z), L]u$ as a lower order term within the interior equation for $\lambda^{m-1,\gamma}_\delta(Z)u$ (see (39)); this is made possible by taking advantage from the invertibility of the operator $\lambda^{m-1,\gamma}_\delta(Z)$.

The paper is organized as follows. In Section 2 we introduce the function spaces and some notations. In Section 3 we recall some technical results useful to prove the tangential regularity of solutions. The last Section 4 contains a very short description of the main steps of the proof of Theorem 1.1.

2 - Function spaces

In this Section, the function spaces to be used in the following are introduced.

2.1 - *Weighted Sobolev spaces*

For $\gamma \geq 1$ and $s \in \mathbb{R}$, we set $\lambda^{s,\gamma}(\xi) := (\gamma^2 + |\xi|^2)^{s/2}$ and, in particular, $\lambda^s := \lambda^{s,1}$.

For real $\gamma \geq 1$, $H^s_\gamma(\mathbb{R}^n)$ will denote the Sobolev space of order s, equipped with the $\gamma-$depending norm $\|\cdot\|_{s,\gamma}$ defined by

$$\|u\|^2_{s,\gamma} := (2\pi)^{-n} \int_{\mathbb{R}^n} \lambda^{2s,\gamma}(\xi)|\widehat{u}(\xi)|^2 d\xi \,, \tag{13}$$

$\widehat{u}$ being the Fourier transform of u. The norms defined by (13), with different values of γ, are equivalent each other. For $\gamma = 1$ we set for brevity $\|\cdot\|_s := \|\cdot\|_{s,1}$ and, accordingly, $H^s(\mathbb{R}^n) := H^s_1(\mathbb{R}^n)$. For $s \in \mathbb{N}$, (13) is equivalent, *uniformly with respect to* γ, to the norm

$$\|u\|^2_{H^s_\gamma(\mathbb{R}^n)} := \sum_{|\alpha|\leq s} \gamma^{2(s-|\alpha|)} \|\partial^\alpha u\|^2_{L^2(\mathbb{R}^n)} \,. \tag{14}$$

2.2 - *Conormal and anisotropic Sobolev spaces*

Let us introduce some classes of function spaces of Sobolev type, defined over the half-space $\mathbb{R}^n_+$. For $j = 1, 2, \ldots, n$, we set

$$Z_1 := x_1\partial_1 \,, \quad Z_j := \partial_j \,, \quad \text{for } j \geq 2 \,.$$

For every $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$, the *conormal* (or *tangential*) derivative Z^α is defined by $Z^\alpha := Z_1^{\alpha_1} \ldots Z_n^{\alpha_n}$; we also write $\partial^\alpha = \partial_1^{\alpha_1} \ldots \partial_n^{\alpha_n}$ for the partial derivative corresponding to α.

Given an integer $m \geq 1$ the *conormal* (*or tangential*) *Sobolev space* $H^m_{tan}(\mathbb{R}^n_+)$ is defined as the set of functions $u \in L^2(\mathbb{R}^n_+)$ such that $Z^\alpha u \in L^2(\mathbb{R}^n_+)$, for all α with $|\alpha| \leq m$. For $\gamma \geq 1$, $H^m_{tan,\gamma}(\mathbb{R}^n_+)$ will denote the conormal space of order m equipped with the $\gamma-$depending norm

(15) $$\|u\|^2_{H^m_{tan,\gamma}(\mathbb{R}^n_+)} := \sum_{|\alpha|\leq m} \gamma^{2(m-|\alpha|)} \|Z^\alpha u\|^2_{L^2(\mathbb{R}^n_+)}$$

and $H^m_{tan}(\mathbb{R}^n_+) := H^m_{tan,1}(\mathbb{R}^n_+)$.

With the same notations as above, for an integer $m \geq 1$ the *anisotropic Sobolev space* $H^m_*(\mathbb{R}^n_+)$ is defined as

$$H^m_*(\mathbb{R}^n_+) := \{w \in L^2(\mathbb{R}^n_+) : \quad Z^\alpha \partial_1^k w \in L^2(\mathbb{R}^n_+) \,, \ |\alpha| + 2k \leq m\} \,.$$

For $\gamma \geq 1$, $H^m_{*,\gamma}(\mathbb{R}^n_+)$ is the same space equipped with the γ-depending norm

(16) $$\|w\|^2_{H^m_{*,\gamma}(\mathbb{R}^n_+)} := \sum_{|\alpha|+2k\leq m} \gamma^{2(m-|\alpha|-2k)} \|Z^\alpha \partial_1^k w\|^2_{L^2(\mathbb{R}^n_+)} \,.$$

We have $H^m_*(\mathbb{R}^n_+) = H^m_{*,1}(\mathbb{R}^n_+)$. For an extensive study of the anisotropic spaces, we refer the reader to [**14, 16, 23**] and references therein.

$H^m_{tan,\gamma}(\mathbb{R}^n_+)$, $H^m_{*,\gamma}(\mathbb{R}^n_+)$, endowed with norms (15), (16) respectively, are Hilbert spaces. In a similar way we define the spaces $H^m_{tan,\gamma}(Q_T)$, $H^m_{*,\gamma}(Q_T)$, equipped with their natural norms.

For a Banach space X, let $C^j([0,T];X)$ denote the space of all X-valued j-times continuously differentiable functions of $t \in [0,T]$.

We define the space $\mathcal{C}_T(H^m_{*,\gamma}) := \bigcap_{j=0}^m C^j([0,T]; H^{m-j}_{*,\gamma}(\mathbb{R}^n_+))$ provided with the norm $\|u\|^2_{\mathcal{C}_T(H^m_{*,\gamma})} := \sum_{j=0}^m \sup_{t\in[0,T]} \|\partial_t^j u(t)\|^2_{H^{m-j}_{*,\gamma}(\mathbb{R}^n_+)}$; the space $\mathcal{C}_T(H^m_{tan,\gamma})$ is defined in a completely similar way, with the natural norm $\|\cdot\|_{\mathcal{C}_T(H^m_{tan,\gamma})}$.

For the initial data we set

$$|||f|||^2_{m,*,\gamma} := \sum_{j=0}^m \|f^{(j)}\|^2_{H^{m-j}_{*,\gamma}(\mathbb{R}^n_+)} \,, \quad |||f|||^2_{m,tan,\gamma} := \sum_{j=0}^m \|f^{(j)}\|^2_{H^{m-j}_{tan,\gamma}(\mathbb{R}^n_+)} \,.$$

3 - Preliminaries and technical tools

In this Section, we collect several technical tools that will be used in the analysis of the next Section 4. We start by recalling the definition of the operators $\sharp$ and $\natural$, introduced by Nishitani and Takayama in [**17**]. The mappings $\sharp : L^2(\mathbb{R}^n_+) \to L^2(\mathbb{R}^n)$ and $\natural : L^\infty(\mathbb{R}^n_+) \to L^\infty(\mathbb{R}^n)$ are respectively defined by

$$w^\sharp(x) := w(e^{x_1}, x')e^{x_1/2}, \quad a^\natural(x) = a(e^{x_1}, x') \,, \quad \forall\, x = (x_1, x') \in \mathbb{R}^n \,.$$

They are both norm preserving bijections and it is easy to see (cf. [**17**]) that

$$(\psi u)^\sharp = \psi^\natural u^\sharp\,, \tag{17}$$

$$\partial_j(u^\natural) = (Z_j u)^\natural, \quad j = 1, \ldots, n\,, \tag{18}$$

$$\partial_1(u^\sharp) = (Z_1 u)^\sharp + \frac{1}{2}u^\sharp\,, \quad \partial_j(u^\sharp) = (Z_j u)^\sharp\,, \quad j = 2, \ldots, n\,. \tag{19}$$

From (19) and the L^2-boundedness of $\sharp$, it can be also proved that

$$\sharp : H^m_{tan,\gamma}(\mathbb{R}^n_+) \to H^m_\gamma(\mathbb{R}^n) \tag{20}$$

is a topological isomorphism, for each integer $m \geq 1$ and real $\gamma \geq 1$.

Let us denote by $C^\infty_{(0)}(\mathbb{R}^n_+)$ the set of restrictions to $\mathbb{R}^n_+$ of functions of $C^\infty_0(\mathbb{R}^n)$. Then we observe that the operator $\sharp$ continuously maps the space $C^\infty_{(0)}(\mathbb{R}^n_+)$ into the Schwartz space $\mathcal{S}(\mathbb{R}^n)$ of rapidly decreasing functions in $\mathbb{R}^n$.

3.1 - *Parameter depending norms on Sobolev spaces*

In order to show the regularity result stated in Theorem 1.1, it is useful providing the conormal Sobolev space $H^{m-1}_{tan,\gamma}(\mathbb{R}^n_+)$, $m \in \mathbb{N}$, $\gamma \geq 1$, with a family of parameter-depending norms satisfying the same properties of similar norms defined by Hörmander [**12**] in the framework of usual Sobolev spaces in $\mathbb{R}^n$. Following [**17**], for $\gamma \geq 1$, $\delta \in\,]0,1]$ and all $u \in H^{m-1}_{tan}(\mathbb{R}^n_+)$ we set

$$\|u\|^2_{\mathbb{R}^n_+,m-1,tan,\gamma,\delta} := (2\pi)^{-n} \int\limits_{\mathbb{R}^n} \lambda^{2m,\gamma}(\xi)\lambda^{-2,\gamma}(\delta\xi)|\widehat{u^\sharp}(\xi)|^2 d\xi\,. \tag{21}$$

For each fixed $\delta \in\,]0,1]$, $\|\cdot\|_{\mathbb{R}^n_+,m-1,tan,\gamma,\delta}$ defines a norm in $H^{m-1}_{tan}(\mathbb{R}^n_+)$ and it is equivalent, *uniformly with respect to* γ, to the norm $\|\cdot\|_{H^{m-1}_{tan,\gamma}(\mathbb{R}^n_+)}$ defined in (15) (with $m-1$ instead of m). Starting from a result by Hörmander [**12**], and exploiting the boundedness of (20), we can prove (cf. [**14, 17**]).

Proposition 3.1. *For $m \in \mathbb{N}$ and $\gamma \geq 1$, $u \in H^m_{tan,\gamma}(\mathbb{R}^n_+)$ if and only if $u \in H^{m-1}_{tan,\gamma}(\mathbb{R}^n_+)$, and the set $\{\|u\|_{\mathbb{R}^n_+,m-1,tan,\gamma,\delta}\}_{0<\delta\leq 1}$ is bounded. In this case*

$$\|u\|_{\mathbb{R}^n_+,m-1,tan,\gamma,\delta} \uparrow \|u\|_{H^m_{tan,\gamma}(\mathbb{R}^n_+)}\,, \qquad as\ \delta \downarrow 0.$$

3.2 - *A class of conormal operators*

The $\sharp$ operator can be also used to allow pseudo-differential operators in $\mathbb{R}^n$ *acting conormally* on functions only defined over the positive half-space $\mathbb{R}^n_+$. Then the standard machinery of pseudo-differential calculus (in the parameter depending version introduced in [**1**], [**5**]) can be re-arranged into a functional calculus properly behaved on conormal Sobolev spaces described in Section 2, see [**15, 16**]. Let us introduce the symbols, with a parameter, to be used later; here we closely follow the terminology and notations of [**6**].

Definition 3.1. *A parameter-depending pseudo-differential symbol of order $m\in\mathbb{R}$ is a real (or complex)-valued measurable function $a(x,\xi,\gamma)$ on $\mathbb{R}^n\times\mathbb{R}^n\times[1,+\infty[$, such that a is C^∞ with respect to x and ξ and for all multi-indices $\alpha,\beta\in\mathbb{N}^n$ there exists a positive constant $C_{\alpha,\beta}$ satisfying:*

$$|\partial_\xi^\alpha\partial_x^\beta a(x,\xi,\gamma)|\le C_{\alpha,\beta}\lambda^{m-|\alpha|,\gamma}(\xi)\,,\quad\forall\,x,\xi\in\mathbb{R}^n,\ \forall\,\gamma\ge1\,. \tag{22}$$

Actually Definition 3.1 straightforwardly extends to matrix-valued symbols. We denote by Γ^m the set of $\gamma-$depending symbols of order $m\in\mathbb{R}$ (the same notation being used either for scalar-valued or matrix-valued functions). For $m\le m'$, the continuous imbedding $\Gamma^m\subset\Gamma^{m'}$ can be easily proven.

For all $m\in\mathbb{R}$, $\lambda^{m,\gamma}$ is a (scalar-valued) symbol in Γ^m. To perform the analysis of Section 4, it is important to consider the behavior of the weight function $\lambda^{m,\gamma}\lambda^{-1,\gamma}(\delta\,\cdot\,)$, involved in the definition of the norms in (21), as a γ-depending symbol. Henceforth the following notations are used

$$\lambda_\delta^{m-1,\gamma}(\xi):=\lambda^{m,\gamma}(\xi)\lambda^{-1,\gamma}(\delta\xi)\quad\widetilde{\lambda}_\delta^{-m+1,\gamma}(\xi):=\left(\lambda_\delta^{m-1,\gamma}(\xi)\right)^{-1}, \tag{23}$$

for all $m\in\mathbb{R}$, $\gamma\ge1$ and $\delta\in\,]0,1]$.

Lemma 3.1. *For every $m\in\mathbb{R}$ and all $\alpha\in\mathbb{N}^n$ there exists $C_{m,\alpha}>0$ such that $\forall\,\xi\in\mathbb{R}^n$, $\forall\,\gamma\ge1$, $\forall\,\delta\in\,]0,1]$:*

$$|\partial_\xi^\alpha\lambda_\delta^{m-1,\gamma}(\xi)|\le C_{m,\alpha}\lambda_\delta^{m-1-|\alpha|,\gamma}(\xi)\,, \tag{24}$$

$$|\partial_\xi^\alpha\widetilde{\lambda}_\delta^{-m+1,\gamma}(\xi)|\le C_{m,\alpha}\widetilde{\lambda}_\delta^{-m+1-|\alpha|,\gamma}(\xi)\,. \tag{25}$$

After (24), $\lambda_\delta^{m-1,\gamma}(\xi)$ can be regarded as a γ-depending symbol, in two different ways. Combining (24) with the trivial inequality $\lambda^{-1,\gamma}(\delta\xi)\le1$ gives at once that $\{\lambda_\delta^{m-1,\gamma}\}_{0<\delta\le1}$ is a bounded subset of Γ^m. Also, the left inequality in

$$\delta\lambda^{1,\gamma}(\xi)\le\lambda^{1,\gamma}(\delta\xi)\le\lambda^{1,\gamma}(\xi)\,,\quad\forall\,\xi\in\mathbb{R}^n\,,\ \forall\,\delta\in\,]0,1]\,, \tag{26}$$

together with (24), gives that $\lambda_\delta^{m-1,\gamma}(\xi)\in\Gamma^{m-1}$ for each fixed δ; however, $\{\lambda_\delta^{m-1,\gamma}\}_{0<\delta\le1}$ is no longer a bounded subset of Γ^{m-1}. Even, the right inequality in (26) and (25) yield that $\{\widetilde{\lambda}_\delta^{-m+1,\gamma}\}_{0<\delta\le1}$ is a bounded subset of Γ^{-m+1}.

Any symbol $a = a(x, \xi, \gamma) \in \Gamma^m$ defines a *pseudo-differential operator* $\mathrm{Op}^\gamma(a) = a(x, D, \gamma)$ on $\mathcal{S}(\mathbb{R}^n)$, by the standard formula

$$\forall\, x \in \mathbb{R}^n\,, \quad \mathrm{Op}^\gamma(a)u(x) := (2\pi)^{-n} \int\limits_{\mathbb{R}^n} e^{ix\cdot\xi} a(x, \xi, \gamma)\widehat{u}(\xi) d\xi\,, \tag{27}$$

where $u \in \mathcal{S}(\mathbb{R}^n)$ and $x \cdot \xi := \sum_{j=1}^{n} x_j \xi_j$. $\mathrm{Op}^\gamma(a)$ is the pseudo-differential operator with symbol a and m is the order of $\mathrm{Op}^\gamma(a)$. It is well-known that $\mathrm{Op}^\gamma(a) : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$ is a linear bounded operator extending to a linear bounded operator on the space $\mathcal{S}'(\mathbb{R}^n)$ of tempered distributions in $\mathbb{R}^n$. An exhaustive account of the symbolic calculus for pseudo-differential operators with symbols in Γ^m can be found in [**5**] (see also [**15**]). In particular, for arbitrary symbols $a \in \Gamma^m$ and $b \in \Gamma^l$, with $l, m \in \mathbb{R}$, the product $\mathrm{Op}^\gamma(a)\mathrm{Op}^\gamma(b)$ and the commutator $[\mathrm{Op}^\gamma(a), \mathrm{Op}^\gamma(b)]$ are again pseudo-differential operators, whose symbols belong to Γ^{m+l} and are explicitly computable for a and b. If, in addition, one among a or b is scalar-valued, then the symbol of $[\mathrm{Op}^\gamma(a), \mathrm{Op}^\gamma(b)]$ has order $m + l - 1$.

We introduce now the class of *conormal operators* in $\mathbb{R}^n_+$, to be used later.

Definition 3.2. *For a γ–depending symbol $a(x, \xi, \gamma)$ in Γ^m, $m \in \mathbb{R}$, the conormal operator* $\mathrm{Op}^\gamma_\sharp(a)$ (*also denoted by $a(x, Z, \gamma)$*) *is defined by*

$$\forall\, u \in C^\infty_{(0)}(\mathbb{R}^n_+)\,, \quad \left(\mathrm{Op}^\gamma_\sharp(a)u\right)^\sharp = (\mathrm{Op}^\gamma(a))(u^\sharp)\,. \tag{28}$$

Since $u^\sharp \in \mathcal{S}(\mathbb{R}^n)$ for $u \in C^\infty_{(0)}(\mathbb{R}^n_+)$, formula (28) makes sense and gives that $\mathrm{Op}^\gamma_\sharp(a)u$ is a C^∞-function in $\mathbb{R}^n_+$. Also $\mathrm{Op}^\gamma_\sharp(a) : C^\infty_{(0)}(\mathbb{R}^n_+) \to C^\infty(\mathbb{R}^n_+)$ is a linear bounded operator that extends to a linear bounded operator from the space of distributions $u \in \mathcal{D}'(\mathbb{R}^n_+)$ satisfying $u^\sharp \in \mathcal{S}'(\mathbb{R}^n)$ into $\mathcal{D}'(\mathbb{R}^n_+)$ itself. From (28), for given $a \in \Gamma^m$, $b \in \Gamma^l$, with $m, l \in \mathbb{R}$, there holds

$$\forall\, u \in C^\infty_{(0)}(\mathbb{R}^n_+)\,, \quad \mathrm{Op}^\gamma_\sharp(a)\mathrm{Op}^\gamma_\sharp(b)u = \left(\mathrm{Op}^\gamma(a)\mathrm{Op}^\gamma(b)(u^\sharp)\right)^{\sharp^{-1}}\,. \tag{29}$$

Then, a functional calculus of conormal operators can be borrowed from the pseudo-differential calculus in $\mathbb{R}^n$; in particular, products and commutators of conormal operators are operators of the same type, and their symbols are computed according to the standard rules of symbolic calculus.

In Section 4, we will make use of the conormal operators

$$\lambda^{m-1,\gamma}_\delta(Z) := \mathrm{Op}^\gamma_\sharp(\lambda^{m-1,\gamma}_\delta)\,, \quad \widetilde{\lambda}^{-m+1,\gamma}_\delta(Z) := \mathrm{Op}^\gamma_\sharp(\widetilde{\lambda}^{-m+1,\gamma}_\delta)\,. \tag{30}$$

The operators $\lambda^{m-1,\gamma}_\delta(Z)$ are involved in the characterization of conormal regularity provided by Proposition 3.1. Indeed, Plancherel's formula and the fact that the operator $\sharp$ preserves the L^2–norm yield the following identity

$$\|u\|_{\mathbb{R}^n_+, m-1, tan, \gamma, \delta} \equiv \|\lambda^{m-1,\gamma}_\delta(Z)u\|_{L^2(\mathbb{R}^n_+)}. \tag{31}$$

Hence, Proposition 3.1 can be restated in terms of the boundedness, with respect to δ, of the L^2-norms of functions $\lambda_\delta^{m-1,\gamma}(Z)u$. This is the key point that leads to the analysis performed in Section 4. Also notice that from (29) and the rules of symbolic calculus, $\widetilde{\lambda}_\delta^{-m+1,\gamma}(Z)$ is an inverse operator of $\lambda_\delta^{m-1,\gamma}(Z)$.

Eventually, from the Sobolev continuity of usual pseudo-differential operators (see f.i. [**5**]) and the boundedness of (20), we derive that for $m \in \mathbb{Z}$ and every $a \in \Gamma^m$ the following

$$\mathrm{Op}_\sharp^\gamma(a) : H^{s+m}_{tan,\gamma}(\mathbb{R}^n_+) \to H^s_{tan,\gamma}(\mathbb{R}^n_+) \tag{32}$$

is a linear bounded operator, as long as $s \geq 0$ is an integer such that $s + m \geq 0$. Moreover, its operator norm is uniformly bounded with respect to γ.

4 - The scheme of the proof of Theorem 1.1

The proof of Theorem 1.1 is made of several steps.

In order to simplify the forthcoming analysis, we only consider the case when the differential operator L has smooth coefficients. For the general case of coefficients with the finite regularity prescribed in Theorem 1.1, we refer the reader to [**16**]: roughly speaking, this case is treated by a reduction to the smooth coefficients case, based upon the stability assumption (E). Thus, from now on, we assume that S_0, A_i, B are given functions in $C^\infty_{(0)}(Q)$. Just for simplicity, we even assume that the coefficients A_i of L are symmetric matrices (in this case S_0 reduces to the identity matrix of size N).

We make the change of unknown $u_\gamma := e^{-\gamma t}u$ and we set $F_\gamma := e^{-\gamma t}F$, $G_\gamma = e^{-\gamma t}G$. Then the IBVP (1)-(3) becomes equivalent to

$$\begin{aligned} &(\gamma + L)u_\gamma = F_\gamma \quad && \text{in } Q_T\,, \\ &Mu_\gamma = G_\gamma\,, && \text{on } \Sigma_T\,, \\ &u_{\gamma\,|\,t=0} = f\,, && \text{in } \mathbb{R}^n_+\,. \end{aligned} \tag{33}$$

4.1 - *The homogeneous IBVP. Conormal regularity*

We firstly consider the *homogeneous* IBVP

$$\begin{aligned} &(\gamma + L)u_\gamma = F_\gamma \quad && \text{in } Q_T\,, \\ &Mu_\gamma = G_\gamma && \text{on } \Sigma_T\,, \\ &u_{\gamma\,|t=0} = 0 && \text{in } \mathbb{R}^n_+\,, \end{aligned} \tag{34}$$

and we focus on the conormal regularity of its solution, when the data F_γ, G_γ fulfil the compatibility conditions in a stronger form than the one in (11). Namely for a given integer $m \geq 1$, we assume that

$$\partial_t^h F_{\gamma|\,t=0} = 0\,, \quad \partial_t^h G_{\gamma|\,t=0} = 0\,, \quad h = 0, \dots, m\,. \tag{35}$$

Actually (35) implies the compatibility conditions (11) of order m, for $f = 0$.

Theorem 4.1. *Assume* $A_i, B \in C^\infty_{(0)}(Q)$*, for* $1 \leq i \leq n$ *and that* (34) *satisfies assumptions* (A)-(D)*; then for all* $T > 0$ *and integer* $m \geq 1$ *there exist* $C_m > 0$ *and* $\gamma_m \geq 1$*, with* $\gamma_m \geq \gamma_{m-1}$*, such that for all* $\gamma \geq \gamma_m$*,* $F_\gamma \in H^{m+1}_{tan,\gamma}(Q_T)$ *and* $G_\gamma \in H^{m+1}_\gamma(\Sigma_T)$ *satisfying* (35) *the solution* u_γ *to* (34) *belongs to* $H^m_{tan,\gamma}(Q_T)$*,* $u^I_{\gamma|\Sigma_T} \in H^m_\gamma(\Sigma_T)$ *and the following a priori estimate is satisfied*

$$\gamma\|u_\gamma\|^2_{H^m_{tan,\gamma}(Q_T)} + \|u^I_{\gamma|\,\Sigma_T}\|^2_{H^m_\gamma(\Sigma_T)} \leq C_m\left(\frac{1}{\gamma^3}\|F_\gamma\|^2_{H^{m+1}_{tan,\gamma}(Q_T)} + \frac{1}{\gamma^2}\|G_\gamma\|^2_{H^{m+1}_\gamma(\Sigma_T)}\right). \tag{36}$$

To prove Theorem 4.1, we reduce the problem (34) into a stationary BVP, where the time is allowed to span the whole real line and is treated, consequently, as an additional tangential variable. To make this reduction, we extend the data F_γ, G_γ and the unknown u_γ of (34) to all positive and negative times, following closely the lines of [**14**]. In the sequel, for the sake of simplicity, we remove the subscript γ from the unknown u_γ and the data F_γ, G_γ. Because of (35), we may extend F, G by setting them equal to zero for all negative times and by "reflection" for $t > T$, so that the extended data (that we continue to denote by F and G) vanish also for large $t > T$, and we get $F \in H^{m+1}_{tan,\gamma}(Q)$, $G \in H^{m+1}_\gamma(\Sigma)$. Even the solution u to (34) is extended to negative times, by setting it equal to zero. Then we extend u for $t > T$, by following the arguments of [**14**], based on the assumption (D). The extended solution, again denoted by u, solves the BVP

$$\begin{aligned} &(\gamma + L)u = F\,, && \text{in } Q\,,\\ &Mu = G\,, && \text{on } \Sigma\,. \end{aligned} \tag{37}$$

In (37), the time t is involved with the same role of the tangential space variables, as it spans the whole real line $\mathbb{R}$. Therefore, (37) is now a stationary problem on Q, with data $F \in H^{m+1}_{tan,\gamma}(Q)$, $G \in H^{m+1}_\gamma(\Sigma)$. Furthermore, u enjoys the estimate (10) for γ large enough.

4.2 - *Regularity of the BVP* (37)

The proof of Theorem 4.1 is now derived as a consequence of the conormal regularity of solutions to the BVP (37). Let us argue by induction on m.

We take arbitrary $F \in H^{m+1}_{tan,\gamma}(Q)$, $G \in H^{m+1}_{\gamma}(\Sigma)$. From the inductive hypothesis, we know the L^2-solution u to (37) belongs to $H^{m-1}_{tan,\gamma}(Q)$ and $u^I_{|x_1=0} \in H^{m-1}_{\gamma}(\Sigma)$, provided that γ is large enough; moreover u obeys the estimate

$$\gamma\|u\|^2_{H^{m-1}_{tan,\gamma}(Q)} + \|u^I_{|x_1=0}\|^2_{H^{m-1}_{\gamma}(\Sigma)} \leq C_{m-1}\left(\frac{1}{\gamma^3}\|F\|^2_{H^m_{tan,\gamma}(Q)} + \frac{1}{\gamma^2}\|G\|^2_{H^m_{\gamma}(\Sigma)}\right), \tag{38}$$

where C_{m-1} depends on m, μ, the coefficients A_i $(1 \leq i \leq n)$ and B.

In order to increase the conormal regularity of the solution u to (37) by order one, we apply to u the conormal operator $\lambda^{m-1,\gamma}_{\delta}(Z)$ and consider the problem satisfied by $\lambda^{m-1,\gamma}_{\delta}(Z)u$, following the strategy announced in Section 1. Since $\lambda^{m-1,\gamma}_{\delta} \in \Gamma^{m-1}$ (cf. Lemma 3.1) and $u \in H^{m-1}_{tan,\gamma}(Q)$, from the boundedness of (32) we know that $\lambda^{m-1,\gamma}_{\delta}(Z)u \in L^2(Q)$. Applying $\lambda^{m-1,\gamma}_{\delta}(Z)$ to (37) we find that $\lambda^{m-1,\gamma}_{\delta}(Z)u$ must solve the BVP

$$(\gamma + L)(\lambda^{m-1,\gamma}_{\delta}(Z)u) + [\lambda^{m-1,\gamma}_{\delta}(Z), L]u = \mathcal{F}_{\delta}\,, \quad \text{in } Q\,, \tag{39}$$

$$M(\lambda^{m-1,\gamma}_{\delta}(Z)u) = \mathcal{G}_{\delta}\,, \quad \text{on } \Sigma\,, \tag{40}$$

where the operators L and M are the same as in the original problem for u, and the data $\mathcal{F}_{\delta}$, $\mathcal{G}_{\delta}$ are computed from F and G in such a way that

$$\|\mathcal{F}_{\delta}\|_{H^1_{tan,\gamma}(Q)} \leq C_m\|F\|_{H^{m+1}_{tan,\gamma}(Q)}\,, \quad \|\mathcal{G}_{\delta}\|_{H^1_{\gamma}(\Sigma)} \leq C_m\|G\|_{H^{m+1}_{\gamma}(\Sigma)} \tag{41}$$

hold true with $C_m > 0$ independent of $\delta \in]0,1]$ and $\gamma \geq 1$, cf. [**15**] for details.

Now the key point consists of restating the term $[\lambda^{m-1,\gamma}_{\delta}(Z), L]u$, appearing in the interior equation (39), as a lower order operator with respect to $\lambda^{m-1,\gamma}_{\delta}(Z)u$: exploiting the invertibility of the matrix $A^{I,I}_1$ (see the assumption (B)) to express $\partial_1 u^I$ as a function of F and conormal derivatives of u, on the one hand, and acting by the rules of symbolic calculus for conormal operators (using in particular the invertibility of $\lambda^{m-1,\gamma}_{\delta}(Z)$), we manage to represent the commutator $[\lambda^{m-1,\gamma}_{\delta}(Z), L]$ as a conormal operator with symbol in Γ^0 applied to $\lambda^{m-1,\gamma}_{\delta}(Z)u$, see [**15, 16**] for details. Since $\lambda^{m-1,\gamma}_{\delta}(Z)u$ is the unique L^2-solution to (39)-(40), in view of assumption (D) (estimate (10)) we find $\widetilde{\gamma}_m \geq 1$ and $\widetilde{C}_m > 0$ such that for all $\gamma \geq \widetilde{\gamma}_m$ and $\delta \in]0,1]$

$$\begin{aligned} &\gamma\|\lambda^{m-1,\gamma}_{\delta}(Z)u\|^2_{L^2(Q)} + \|(\lambda^{m-1,\gamma}_{\delta}(Z)u^I)_{|x_1=0}\|^2_{L^2(\Sigma)} \\ &\leq \widetilde{C}_m\left(\tfrac{1}{\gamma^3}\|F\|^2_{H^{m+1}_{tan,\gamma}(Q)} + \tfrac{1}{\gamma^2}\|G\|^2_{H^{m+1}_{\gamma}(\Sigma)}\right). \end{aligned} \tag{42}$$

Estimate (42) provides a $\delta-$uniform bound for the L^2-norms of $\lambda^{m-1,\gamma}_{\delta}(Z)u$ in Q and the trace on Σ of its noncharacteristic component. After Proposition 3.1 (and (31)), this gives that $u \in H^m_{tan,\gamma}(Q)$ and $u^I_{|x_1=0} \in H^m_{\gamma}(\Sigma)$; then estimate (38) of order m is recovered letting $\delta \to 0$ into (42) and using again Proposition 3.1. Recalling that the solution to (37) is the extension of the solution u_γ to (34), from the conormal regularity of u we can now derive the conormal regularity of u_γ, namely that $u_\gamma \in H^m_{\tan,\gamma}(Q_T)$ and $u^I_{\gamma\,|\,\Sigma_T} \in H^m_\gamma(\Sigma_T)$. The inequality (36) follows from its "stationary" counterpart (38), where $m-1$ is replaced by m.

4.3 - *The nonhomogeneous IBVP*

We study the general IBVP (1)-(3), where the initial datum f is different from zero. To show the anisotropic regularity of the solution stated by Theorem 1.1 we argue again by induction on the anisotropic order m. Without entering in too many details, we briefly describe the different steps of the proof, for the reader's convenience, addressing to [**16**] for a more extensive discussion.

We firstly prove the statement for $m = 1$; thus let the problem (1)-(3) satisfy the assumptions (A)-(E), where the data $F \in H^2_{*,\gamma}(Q_T)$, such that $\partial_t^i F_{|\,t=0} \in H^{1-i}_\gamma(\mathbb{R}^n_+)$ for $i = 0, 1$, $G \in H^2_\gamma(\Sigma_T)$ and $f \in H^2_\gamma(\mathbb{R}^n_+)$ obey the compatibility conditions $Mf = G_{|\,t=0}$, $Mf^{(1)} = \partial_t G_{|\,t=0}$ on $\mathbb{R}^{n-1}$. Here we refer to [**16**, Theorem 5.1], where actually the result of Theorem 1.1, with $m = 1$, is proved under slightly more general assumptions about the regularity of the data and the coefficients of L. Firstly, we approximate the original data (F, G, f) by regularized functions (F_k, G_k, f_k) satisfying the same compatibility conditions; then we look for the solution u_k of the IBVP with data (F_k, G_k, f_k) in the form $u_k = v_k + w_k$, where $w_k \in H^3_\gamma(Q_T)$ is chosen such that $w_{k\,|t=0} = f_k$, $\partial_t w_{k\,|t=0} = f_k^{(1)}$, $\partial^2_{tt} w_{k\,|t=0} = f_k^{(2)}$, while v_k solves a homogeneous IBVP (with zero initial data). Applying to the latter problem the result of Theorem 4.1 we deduce that $u_k \in H^1_{*,\gamma}(Q_T)$ and $u^I_{k\,|\Sigma_T} \in H^1_\gamma(\Sigma_T)$; also, looking for the IBVPs solved by the first-order conormal derivatives of u_k, we prove that $u_k \in \mathcal{C}_T(H^1_{*,\gamma})$ and it satisfies the estimate (12). Eventually, applying (12) to a difference of two approximating solutions $u_k - u_h$ (and taking γ large enough), we find that $\{u_k\}$ and $\{u^I_{k\,|\Sigma_T}\}$ are respectively Cauchy sequences in $\mathcal{C}_T(H^1_{*,\gamma})$ and $H^1_\gamma(\Sigma_T)$, and passing to the limit as $k \to +\infty$ gives that $u \in \mathcal{C}_T(H^1_{*,\gamma})$ and satisfies (12). Now we assume that Theorem 1.1 holds up to $m-1$. Given the data (F, G, f) as in Theorem 1.1, by the inductive hypothesis there exists a unique solution u of (1)-(3) such that $u \in \mathcal{C}_T(H^{m-1}_{*,\gamma})$ and $u^I_{|\Sigma_T} \in H^{m-1}_\gamma(\Sigma_T)$. In order to show that $u \in \mathcal{C}_T(H^m_{*,\gamma})$, we have to increase the regularity of u by order one, that is by one more conormal derivative and, if m is even, also by one more normal derivative. The idea is the same as in [**20, 21**], revisited as in [**4, 14, 22**]. At every step we estimate some derivatives of u through equations where in the right-hand side we can put other derivatives of u already estimated at previous steps. The big difference is that now we have to deal with the loss of one derivative in the right-hand side. To increase the regularity, we consider the system of equations for purely conormal derivatives, of the type of (1)-(3), where we can use the inductive assumption; as for mixed conormal-normal derivatives, they solve a system where the boundary matrix vanishes identically, so that no boundary condition is needed and we can apply a standard energy method, under the assumption of the symmetrizable system. When we consider the system for purely conormal derivatives, we have the loss of one derivative in the right-hand side. However, the terms in the right-hand side have order $m-1$; after the loss of one derivative they become of order m, and can be absorbed for γ large by similar terms in the left-hand side.

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