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Stability of scalar balance laws and scalar non-local conservation laws

Abstract. We present here a stability result for the solutions of scalar balance laws. The estimates we obtained are then used to study the continuity equation with a non-local flow, which appears for example in a new model of pedestrian traffic and in a model of supply-chain.

Keywords. Scalar balance law, $\mathbf{L}^1$ stability estimate, pedestrian traffic, non-local flow, continuity equation.

Mathematics Subject Classification (2000): 35L65, 49K20, 93C20.

1 - Introduction

We study here the Cauchy problem for scalar balance laws:

$$(1) \qquad \begin{cases} \partial_t u + \operatorname{Div} f(t,x,u) = F(t,x,u) & (t,x) \in \mathbb{R}_+^* \times \mathbb{R}^N \\ u(0,x) = u_0(x) & x \in \mathbb{R}^N , \end{cases}$$

where $u_0 \in (\mathbf{L}^1 \cap \mathbf{L}^\infty)(\mathbb{R}^N;\mathbb{R})$ is the initial condition, $f \in \mathscr{C}^2([0,T] \times \mathbb{R}^N \times \mathbb{R};\mathbb{R}^N)$ is the flow and $F \in \mathscr{C}^1([0,T] \times \mathbb{R}^N \times \mathbb{R};\mathbb{R})$ is the source. This kind of equation often appears in physics and their properties have already been intensively investigated, see for example [**17, 22, 27**]. In particular, Kružkov's Theorem [**22**] states that this kind of equation admits a unique weak entropy solution and describes the dependence on the initial condition of the solution.

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In the first part, we want to describe the dependence of the solutions with respect to flow and source in the case the flow f and the source F depend and the three variables t, x and u. Some cases were already studied: for example Lucier [**23**] or Bouchut & Perthame [**5**] have considered the case in which the flow depends only on u and in which there is no source. We treat here the general case, which includes the preceding results. In this very general setting, under natural assumptions (see **(K), (H1*)**), we obtain an estimate on the total variation of the solution

$$\begin{aligned}\mathrm{TV}(u(t)) \le{}& \mathrm{TV}(u_0)e^{\kappa_0 t}\\ &+ NW_N \int_0^t e^{\kappa_0(t-\tau)} \int_{\mathbb{R}^N} \|\nabla_x(F-\mathrm{div}f)(\tau,x,\cdot)\|_{\mathbf{L}^\infty(\mathrm{d}u)}\mathrm{d}x\,\mathrm{d}\tau\,.\end{aligned}$$

Assuming furthermore that $(f-g, F-G)$ satisfies **(H2*)**, we obtain a stability estimate of the solution with respect to flow, source and initial condition:

$$\begin{aligned}\|(u-v)(t)\|_{\mathbf{L}^1} \le{}& e^{\kappa t}\|u_0-v_0\|_{\mathbf{L}^1} + \frac{e^{\kappa_0 t}-e^{\kappa t}}{\kappa_0-\kappa}\mathrm{TV}(u_0)\|\,\partial_u(f-g)\|_{\mathbf{L}^\infty}\\ &+\int_0^t \frac{e^{\kappa_0(t-\tau)}-e^{\kappa(t-\tau)}}{\kappa_0-\kappa}\int_{\mathbb{R}^N}\|\nabla_x(F-\mathrm{div}f)(\tau,x,\cdot)\|_{\mathbf{L}^\infty(\mathrm{d}u)}\mathrm{d}x\mathrm{d}\tau\\ &\quad\times NW_N\|\,\partial_u(f-g)\|_{\mathbf{L}^\infty}\\ &+\int_0^t e^{\kappa(t-\tau)}\int_{\mathbb{R}^N}\|((F-G)-\mathrm{div}\,(f-g))(\tau,x,\cdot)\|_{\mathbf{L}^\infty(\mathrm{d}u)}\mathrm{d}x\mathrm{d}\tau\,.\end{aligned}$$

These estimates are very satisfactory since we obtain the same results as the ones we already knew, when we look at some particular case, for example the homogeneous case without source. The results we present here come from a collaboration with R. Colombo and M. Rosini and are more precisely described in [**13**].

The second part is devoted to the study of the continuity equation:

$$\partial_t u + \mathrm{Div}(u\,V(x,u(t))) = 0\,, \qquad u(0,\cdot) = u_0 \in (\mathbf{L}^1\cap\mathbf{L}^\infty\cap\mathbf{BV})\,, \tag{2}$$

where $V : \mathbb{R}^N \times \mathbf{L}^1(\mathbb{R}^N;\mathbb{R}) \to \mathscr{C}^2(\mathbb{R}^N;\mathbb{R})$ is a non-local averaging functional. Our driving examples are, if $\varphi:\mathbb{R}\to\mathbb{R}$ is a regular function:

- $V(u) = \varphi\Big(\int_0^1 u\,\mathrm{d}x\Big)$ for a supply-chain model [**2, 3, 15**]. Here, u is the density of products at stage x of the production, at a time t. We scale $x\in[0,1]$ so that $x=0$ represents the beginning of the production line and $x=1$ its end. The functional V describes the velocity for elaborating the products and it is assumed to depend on the total load on the production line. We assume furthermore that φ is a positive, decreasing function.

- $V(x, u) = \varphi(\eta *_x u)w(x)$ for a pedestrian traffic model. It follows several other macroscopic models [**4, 7, 14, 18, 24, 26**]. In this model, u stands for the density of pedestrian at the place $x \in \mathbb{R}^2$ and at time t. The functional V describes the velocity of the pedestrian after two rules: the first one is that the pedestrians have all the same behavior. This behavior is described by the vector field w: for example, the pedestrians all want to exit a room and follow some paths directed to the door. The second rule is that the velocity at point x depends on the average density of pedestrians around this point, which means that the pedestrians react to their environment and to what they see, in average, around them.

We describe more deeply these two models at the beginning of Section 3.

In the study of equation (2) our goals are: first, prove existence and uniqueness of a weak entropy solution, second find the extrema of a cost functional depending on the initial condition.

Using the estimates we obtained in the first part, we show not only that this model admits a unique weak entropy solution, but also that the linearized equation admits a weak entropy solution. Furthermore, the non-linear local semi-group obtained by solving the initial value problem is Gâteaux-differentiable with respect to the initial condition and the Gâteaux-derivative is the solution of the linearized equation. This fact allows us to characterize the minima or maxima of a given cost functional depending on the initial condition. This is of interest in pedestrian traffic if for example we want to minimize the time of exit out of a room, avoiding high density in the crowd. These results come from a collaboration with R. Colombo and M. Herty and are presented in [**11**].

2 - $\mathbf{L}^1$ Stability for scalar balance laws

In this part, we are concerned by the Cauchy problem (1) when the flow f and the source F depend on the three variables t, x and u. We give here an estimate on the total variation of the solution and a stability estimate with respect to flow and source in a general setting. These results are described in the note [**12**] and in the articles [**13, 25**].

2.1 - *Previous Results*

Let us first recall the Kružkov Theorem:

Theorem 2.1 [Kružkov]. *We denote* $\Omega_A = [0,T] \times \mathbb{R}^N \times [-A,A]$ *for all* $A \geq 0$. *Under the conditions* $f \in \mathscr{C}^0(\Omega_\infty;\mathbb{R}^N)$, $F \in \mathscr{C}^0(\Omega_\infty;\mathbb{R})$ *and*

$$(\mathbf{K}) \quad \begin{cases} f\,,\ F \ \text{have continuous derivatives}: \ \partial_u f\,,\ \partial_u \nabla f\,,\ \nabla^2 f\,,\ \partial_u F\,,\ \nabla F\,, \\ \forall A > 0\,, \quad \partial_u f \in \mathbf{L}^\infty(\Omega_A)\,, \quad F - \mathrm{div} f \in \mathbf{L}^\infty(\Omega_A), \quad \partial_u(F - \mathrm{div} f) \in \mathbf{L}^\infty(\Omega_A) \end{cases}$$

there exists a unique weak entropy solution $u \in \mathbf{L}^\infty([0,T];\mathbf{L}^1(\mathbb{R}^N;\mathbb{R}))$ *of* (1) *that is right-continuous in time.*

Let $v_0 \in (\mathbf{L}^1 \cap \mathbf{L}^\infty)(\mathbb{R}^N;\mathbb{R})$. *Let* u *be the solution associated to the initial condition* u_0 *and* v *be the solution associated to the initial condition* v_0. *Let* M *be such that* $M \geq \sup\left(\|u\|_{\mathbf{L}^\infty([0,T]\times\mathbb{R}^N;\mathbb{R})}, \|v\|_{\mathbf{L}^\infty([0,T]\times\mathbb{R}^N;\mathbb{R})}\right)$. *Then, for all* $t \in [0,T]$, *with* $\gamma = \|\partial_u F\|_{\mathbf{L}^\infty(\Omega_M)}$, *we have*

$$\|(u-v)(t)\|_{\mathbf{L}^1} \leq e^{\gamma t}\|u_0 - v_0\|_{\mathbf{L}^1}\,. \tag{3}$$

Some other results concerning the dependence of the solution with respect to flow and source were already known. The following was first proved by Lucier [**23**], and later improved by Bouchut & Perthame [**5**]. Their results are about the homogeneous conservation laws: the flow depends only on u and there is no source. More precisely, if $f,g:\mathbb{R}\to\mathbb{R}^N$ are globally lipschitz, then for all $u_0, v_0 \in (\mathbf{L}^1 \cap \mathbf{L}^\infty)(\mathbb{R}^N;\mathbb{R})$ initial conditions for

$$\partial_t u + \mathrm{Div} f(u) = 0\,, \qquad \partial_t v + \mathrm{Div} g(v) = 0\,,$$

with furthermore $v_0 \in \mathbf{BV}(\mathbb{R}^N;\mathbb{R})$ (see Definition 2.1 below), we have for all $t \geq 0$,

$$\|(u-v)(t)\|_{\mathbf{L}^1} \leq \|u_0 - v_0\|_{\mathbf{L}^1} + C\,t\,\mathrm{TV}(v_0)\,\mathbf{Lip}\,(f-g)\,.$$

A flow depending also on x was considered by Chen & Karlsen [**9**], in the special case $f(x,u) = \lambda(x) l(u)$. There, under appropriate hypotheses, with $f(t,x,u) = \lambda(x)\,l(u)$, $g(t,x,v) = \mu(x)\,m(v)$, and without source ($F = G = 0$), they obtained the estimate:

$$\|(u-v)(t)\|_{\mathbf{L}^1} \leq \|u_0 - v_0\|_{\mathbf{L}^1} + C_1\,t\big(\|\lambda-\mu\|_{\mathbf{L}^\infty} + \|\lambda-\mu\|_{\mathbf{W}^{1,1}} + \|l-m\|_{\mathbf{L}^\infty} + \|l-m\|_{\mathbf{W}^{1,\infty}}\big)$$

where $C_1 = C\sup_{[0,T]}(\mathrm{TV}(u(t)), \mathrm{TV}(v(t)))$. However, this general settings contains the following Cauchy problem:

$$\partial_t u + \partial_x(\cos x) = 0\,, \qquad u_0 = 0\,.$$

The solution of this problem is $u(t,x) = t\sin x$ for which $\mathrm{TV}(u_0) = 0$ and $\mathrm{TV}(u(t)) = +\infty$ for any $t > 0$. Hence, the coefficient C_1 is also $+\infty$. This fact motivated us for searching first an estimate on the total variation in the case the flow and source depend on the three variables t, x and u.

2.2 - *Estimate on the total variation*

Let us recall here the definition of total variation.

Definition 2.1. *For* $u \in \mathbf{L}^1_{loc}(\mathbb{R}^N;\mathbb{R})$ *we denote the* total variation of *u:*

$$\mathrm{TV}(u) = \sup\left\{ \int_{\mathbb{R}^N} u \,\mathrm{div}\, \Psi\,; \quad \Psi \in \mathscr{C}^1_c(\mathbb{R}^N;\mathbb{R}^N)\,, \quad \|\Psi\|_{\mathbf{L}^\infty} \leq 1 \right\}.$$

The space of function with bounded variation *is then defined as*

$$\mathbf{BV}(\mathbb{R}^N;\mathbb{R}) = \left\{ u \in \mathbf{L}^1_{loc}; \mathrm{TV}(u) < \infty \right\}.$$

When f and F depend only on u we already know that $u_0 \in (\mathbf{L}^\infty \cap \mathbf{BV})(\mathbb{R}^N;\mathbb{R})$ implies that for all $t \geq 0$, $u(t) \in (\mathbf{L}^\infty \cap \mathbf{BV})(\mathbb{R}^N;\mathbb{R})$. Besides, we have $\mathrm{TV}(u(t)) \leq \mathrm{TV}(u_0)e^{\gamma t}$, where $\gamma = \|\partial_u F\|_{\mathbf{L}^\infty(\Omega_U)}$ and $U = \|u\|_{\mathbf{L}^\infty([0,T]\times\mathbb{R}^N)}$.

Now we give a more general estimate on the total variation. Let $W_N = \int_0^{\pi/2} (\cos\theta)^N \,\mathrm{d}\theta$, $\Omega = \Omega_\infty$ and

$$(\mathbf{H1}): \begin{cases} f \in \mathscr{C}^2(\Omega;\mathbb{R}^N)\,, & F \in \mathscr{C}^1(\Omega;\mathbb{R})\,, \\ \nabla\partial_u f \in \mathbf{L}^\infty(\Omega;\mathbb{R}^{N\times N})\,, & \partial_t \,\mathrm{div} f \in \mathbf{L}^\infty(\Omega;\mathbb{R})\,, \\ \partial_t\partial_u f \in \mathbf{L}^\infty(\Omega;\mathbb{R}^N)\,, & \partial_t F \in \mathbf{L}^\infty(\Omega;\mathbb{R})\,, \\ \displaystyle\int_{\mathbb{R}_+^*}\int_{\mathbb{R}^N} \|\nabla(F - \mathrm{div} f)(t,x,\cdot)\|_{\mathbf{L}^\infty(\mathbb{R};\mathbb{R}^N)} \,\mathrm{d}x\,\mathrm{d}t < +\infty\,. \end{cases}$$

Theorem 2.2 (see Theorem 2.5 in [13]). *Assume* (f,F) *satisfies* (**K**) *and* (**H1**). *Let* $u_0 \in (\mathbf{L}^\infty \cap \mathbf{BV})(\mathbb{R}^N;\mathbb{R})$, *then for all* $t \in [0,T]$, $u(t) \in (\mathbf{L}^\infty \cap \mathbf{BV})(\mathbb{R}^N;\mathbb{R})$. *Furthermore, denoting* $U = \|u\|_{\mathbf{L}^\infty([0,T]\times\mathbb{R}^N)}$ *and* $\kappa_0 = NW_N((2N+1)\|\nabla_x\partial_u f\|_{\mathbf{L}^\infty(\Omega_U)} + \|\partial_u F\|_{\mathbf{L}^\infty(\Omega_U)})$, *we have*

$$\mathrm{TV}(u(t)) \leq \mathrm{TV}(u_0)e^{\kappa_0 t} + NW_N \int_0^t e^{\kappa_0(t-\tau)} \int_{\mathbb{R}^N} \|\nabla_x(F - \mathrm{div} f)(\tau,x,\cdot)\|_{\mathbf{L}^\infty([-U,U])} \mathrm{d}x\,\mathrm{d}\tau\,.$$

Remark 2.1. *In some cases, we recover known estimates.*

When f *depends only on u and* $F = 0$, *we have a result similar to the one that was already known :* $\mathrm{TV}(u(t)) \leq \mathrm{TV}(u_0)$.

When f *and* F *do not depend on u, the equation reduces in fact to the ODE*

$\partial_t u = (F - \mathrm{div} f)(t,x)$, whose solution writes

$$u(t,x) = u_0(x) + \int_0^t (F - \mathrm{div} f)(\tau, x) \mathrm{d}\tau \,.$$

Meanwhile, the bound above reduces to

$$\mathrm{TV}(u(t)) \leq \mathrm{TV}(u_0) + NW_N \int_0^t \int_{\mathbb{R}^N} |(F - \mathrm{div} f)(\tau, x)| \mathrm{d}\tau$$

which is essentially what we expected.

Remark 2.2. *The set of hypotheses* (**H1**) *is in fact very strong. We expect it can be relaxed to*

$$(\mathbf{H1}^*) \ : \ \begin{cases} f \in \mathscr{C}^0(\Omega; \mathbb{R}^N) \,, \qquad F \in \mathscr{C}^0(\Omega; \mathbb{R}) \,, \\ f\,,\ F \ \text{ have continuous derivatives}: \ \partial_u \nabla f \,,\ \nabla^2 f \,,\ \partial_u F \,,\ \nabla F \,, \\ \forall A > 0 : \qquad \nabla \partial_u f \in \mathbf{L}^\infty(\Omega_A; \mathbb{R}^{N\times N}) \,, \qquad \partial_u F \in \mathbf{L}^\infty(\Omega_A; \mathbb{R}) \,, \\ \displaystyle\int_0^T \int_{\mathbb{R}^N} \| \nabla (F - \mathrm{div} f)(t,x,\cdot)\|_{\mathbf{L}^\infty([-A,A];\mathbb{R}^N)} < \infty \,, \end{cases}$$

which is useful for example in [**11**]. *Furthermore, we can replace κ_0 by the better coefficient*

$$\kappa_0^* = (2N+1) \| \nabla_x \partial_u f \|_{\mathbf{L}^\infty(\Omega_U)} + \| \partial_u F \|_{\mathbf{L}^\infty(\Omega_U)} \,.$$

This result is described in the paper [**25**].

Idea of the proof of Theorem 2.2. We state first a very useful proposition, characterizing functions with bounded total variation:

Proposition 2.1. *Let $\mu \in \mathscr{C}_c^\infty(\mathbb{R}_+; \mathbb{R}_+)$ be such that $\|\mu\|_{\mathbf{L}^1} = 1$ and $\mu' < 0$ on $\mathbb{R}_+^*$. We denote $\mu_\lambda(x) = \dfrac{1}{\lambda^N} \mu\left(\dfrac{\|x\|}{\lambda}\right)$. If there exists $C_0 > 0$ such that for all $\lambda > 0$,*

$$\frac{1}{\lambda} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} |u(x+y) - u(x)| \mu_\lambda(y) \mathrm{d}x \, \mathrm{d}y \leq C_0,$$

then $u \in \mathbf{BV}(\mathbb{R}^N;\mathbb{R})$ *and*

$$\mathrm{TV}(u)\int_{\mathbb{R}^N}|y_1|\mu(\|y\|)\mathrm{d}y=\lim_{\lambda\to 0}\frac{1}{\lambda}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}|u(x+y)-u(x)|\mu_\lambda(y)\mathrm{d}x\,\mathrm{d}y\le C_0\,.$$

Let us now give the idea of the proof of Theorem 2.2.

Proof. Let us introduce

$$\mathcal{F}(T,\lambda)=\int_0^T\int_{\mathbb{R}^N}\int_{B(x_0,R+M(T_0-t))}|u(x+y)-u(x)|\mu_\lambda(y)\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t.$$

The doubling variables method introduced by Kružkov [22] gives the estimate:

$$\partial_T\mathcal{F}(T,\lambda)\le\ \partial_T\mathcal{F}(0,\lambda)+C\lambda\partial_\lambda\mathcal{F}(T,\lambda)+C'\mathcal{F}(T,\lambda)+\lambda\int_0^T A(t)\mathrm{d}t\,,\tag{4}$$

with $A(t)=\int_{\mathbb{R}^N}\|\nabla(F-\mathrm{div}f)(t,x,\cdot)\|_{\mathbf{L}^\infty([-U,U])}\mathrm{d}x$ and $U=\|u\|_{\mathbf{L}^\infty([0,T]\times\mathbb{R}^N)}$. Then, we integrate in time and divide by $CT\lambda$ to obtain:

$$0\le\frac{1}{C\lambda}\mathcal{F}(0,\lambda)+\partial_\lambda\mathcal{F}(T,\lambda)+\frac{\alpha(T)}{\lambda}\mathcal{F}(T,\lambda)+\frac{1}{C}\int_0^T A(t)\mathrm{d}t\,,$$

where $\alpha(T)=N+C'/C-1/T$ satisfies $\alpha(T)\to-\infty$ when $T\to 0$. We choose T small enough so that $\alpha(T)<-1$ and we integrate on $[\lambda,+\infty[$.

We obtain

$$\mathcal{F}(T,\lambda)\le\frac{\lambda}{-\alpha-1}K\mathrm{TV}(u_0)+\frac{\lambda}{C(-\alpha-1)}\int_0^T A(t)\mathrm{d}t\,.$$

Next, we can re-introduce this estimate in the line (4), divide by λ and make λ go to 0. This gives us a first estimate ensuring that $u(t)\in\mathbf{BV}(\mathbb{R}^N;\mathbb{R})$ thanks to Proposition 2.1. This estimate can be then improved using that $u(t)\in\mathbf{BV}(\mathbb{R}^N;\mathbb{R})$, which allows us to use tools on $\mathbf{BV}(\mathbb{R}^N;\mathbb{R})$ functions. □

2.3 - $\mathbf{L}^1$ *Stability of the solution*

Now, we can study the dependence of the solution with respect to flow and source. Let us introduce the set of hypotheses:

$$
(\mathbf{H2}) \ : \ \begin{cases} f \in \mathscr{C}^1(\Omega;\mathbb{R}^N)\,, \qquad F \in \mathscr{C}^0(\Omega;\mathbb{R})\,,\\ \partial_u F \in \mathbf{L}^\infty(\Omega;\mathbb{R})\,, \quad \partial_u f \in \mathbf{L}^\infty(\Omega;\mathbb{R}^N)\,,\\ \displaystyle\int_{\mathbb{R}_+^*}\int_{\mathbb{R}^N} \|(F-\mathrm{div} f)(t,x,\cdot)\|_{\mathbf{L}^\infty(\mathbb{R};\mathbb{R})}\,\mathrm{d}x\,\mathrm{d}t < +\infty\,. \end{cases}
$$

Theorem 2.3 (see Theorem 2.6 in [13]). *Assume (f,F), (g,G) satisfy* (**K**), *(f,F) satisfies* (**H1**) *and $(f-g,F-G)$ satisfies* (**H2**). *Let $u_0, v_0 \in (\mathbf{L}^1\cap\mathbf{L}^\infty\cap\mathbf{BV})(\mathbb{R}^N;\mathbb{R})$. Let u and v be the solutions associated to (f,F) and (g,G) respectively and with initial conditions u_0 and v_0. We denote $M = \max\left(\|u\|_{\mathbf{L}^\infty([0,T]\times\mathbb{R}^N)}, \|v\|_{\mathbf{L}^\infty([0,T]\times\mathbb{R}^N)}\right)$ and*

$$
\kappa = 2N\|\,\nabla_x\partial_u f\|_{\mathbf{L}^\infty(\Omega_M)} + \|\partial_u F\|_{\mathbf{L}^\infty(\Omega_M)} + \|\partial_u(F-G)\|_{\mathbf{L}^\infty(\Omega_M)}\,.
$$

Then for all $t\in[0,T]$:

$$
\begin{aligned}
\|(u-v)(t)\|_{\mathbf{L}^1} \leq{}& e^{\kappa t}\|u_0-v_0\|_{\mathbf{L}^1} + \frac{e^{\kappa_0 t}-e^{\kappa t}}{\kappa_0-\kappa}\mathrm{TV}(u_0)\|\,\partial_u(f-g)\|_{\mathbf{L}^\infty}\\
&+\int_0^t \frac{e^{\kappa_0(t-\tau)}-e^{\kappa(t-\tau)}}{\kappa_0-\kappa}\int_{\mathbb{R}^N}\|\nabla_x(F-\mathrm{div} f)(\tau,x,\cdot)\|_{\mathbf{L}^\infty([-M,M])}\mathrm{d}x\mathrm{d}\tau\\
&\quad\times NW_N\|\,\partial_u(f-g)\|_{\mathbf{L}^\infty}\\
&+\int_0^t e^{\kappa(t-\tau)}\int_{\mathbb{R}^N}\|((F-G)-\mathrm{div}\,(f-g))(\tau,x,\cdot)\|_{\mathbf{L}^\infty([-M,M])}\mathrm{d}x\mathrm{d}\tau\,.
\end{aligned}
$$

Remark 2.3. *As in Remark* 2.1*, we recover known estimates in some particular cases:*

- *In the standard case of a conservation law, i.e. when $F=G=0$ and f,g are independent of x, we have $\kappa_0=\kappa=0$ and the result of Theorem* 2.3 *becomes*

$$
\begin{aligned}
\|u(T)-v(T)\|_{\mathbf{L}^1(\mathbb{R}^N;\mathbb{R})} \leq{}& \|u_0-v_0\|_{\mathbf{L}^1(\mathbb{R}^n;\mathbb{R})}\\
&+T\ \mathrm{TV}(u_0)\,\|\,\partial_u(f-g)\|_{\mathbf{L}^\infty(\Omega;\mathbb{R}^N)}\,.
\end{aligned}
$$

- *If (f,F) and (g,G) are not dependent only on u, then $\kappa_0=\kappa=0$ and Theorem* 2.3 *now reads*

$$
\begin{aligned}
\|u(T)-v(T)\|_{\mathbf{L}^1(\mathbb{R}^N;\mathbb{R})} \leq{}& \|u_0-v_0\|_{\mathbf{L}^1(\mathbb{R}^N;\mathbb{R})}\\
&+\int_0^T \|[(F-G)-\mathrm{div}(f-g)](t)\|_{\mathbf{L}^1(\mathbb{R}^N;\mathbb{R})}\mathrm{d}t\,.
\end{aligned}
$$

Remark 2.4. *As in Remark* 2.2, *we think the set of hypotheses* (**H2**) *can be weakened* (*see* [**25**]) *into*

$$(\mathbf{H2}^*) \ : \ \forall A > 0\,, \ \int_{\mathbb{R}_+^*} \int_{\mathbb{R}^N} \|(F - \mathrm{div} f)(t,x,\cdot)\|_{\mathbf{L}^\infty([-A,A];\mathbb{R})} \,\mathrm{d}x \,\mathrm{d}t < +\infty\,.$$

Furthermore, κ *can be replaced by*

$$\kappa^* = \|\partial_u F\|_{\mathbf{L}^\infty(\Omega_M;\mathbb{R})}$$

where $M = \sup(\|u\|_{\mathbf{L}^\infty}, \|v\|_{\mathbf{L}^\infty})$, *which is more in agreement with the result* (3) *of Kružkov's Theorem.*

3 - The continuity equation with a non-local flow

This section is a short version of [**11**]. We study here the continuity equation (2) where $V : \mathbb{R}^N \times \mathbf{L}^1(\mathbb{R}^N;\mathbb{R}) \to \mathscr{C}^2(\mathbb{R}^N;\mathbb{R})$ is a non-local averaging functional.

Our goals are: first, prove existence and uniqueness of a weak entropy solution, second find the extrema of a cost functional depending on the initial condition. The second point leads us to differentiate the semi-group in the Gâteaux sense with respect to initial conditions.

Let us remind that our driving examples are a model of supply-chain and a model of pedestrian traffic. We describe below these two models.

Pedestrian traffic. Macroscopic models for pedestrian movements are based on the continuity equation, see [**4, 7, 8, 10, 14, 18, 20**], possibly together with a second equation, as in [**16**]. In these models, pedestrians are assumed to instantaneously adjust their (vector) speed according to the crowd density at their position. The analytical construction here allows to consider the more realistic situation of pedestrians deciding their speed according to the local mean density at their position. We are thus led to consider (2) with

$$V(x,\rho) = v(\rho * \eta)\,\vec{v}(x) \tag{5}$$

where

$$\eta \in \mathscr{C}_c^2\big(\mathbb{R}^2;[0,1]\big) \text{ has support } \mathrm{Supp}\,\eta \subseteq B(0,1) \text{ and } \|\eta\|_{\mathbf{L}^1} = 1\,, \tag{6}$$

so that $(\rho * \eta)(x)$ is an average of the values attained by ρ in $B(x,1)$. Here, $\vec{v} = \vec{v}(x)$ is the given direction of the motion of the pedestrians at $x \in \mathbb{R}^2$. Then, the presence of boundaries, obstacles or other geometric constraint can be described through $\vec{v}$, see [**10**].

This model can be especially interesting in the case of a crowd in panic. In some event, indeed, the crowd does not behave rationally: we can think for example at rush phenomena at the end of a football play, or at the pilgrim cramming the Jamarat Bridge in Saudi Arabia on occasion of the pilgrimage to Mecca [**19**]. Other applications of pedestrian modelling arise in transport, political or cultural demonstrations, panic situations such as earthquakes or fire escapes. In such situations, it can happens that the density takes higher value than usual: in standard situations, we want the density to be less than a maximal density, say 5 people per square meter; however for such events, the density can become much higher, say ten people per square meter.

With a usual model (without non-local flow), the maximum principle gives us a maximal density at the beginning, say 1 by renormalization. With our model, the density can possibly increase higher that the initial maximal density. For example look at the following configuration: along a trajectory, let us assume there is a queue in front of a given point x_0, and assume there is nobody at the back of this same point (see Figure 1). Assume furthermore the speed is given as usual by $v(r) = 1 - r$, for $r \in [0, 1]$ and $v \equiv 0$ for $r \geq 1$. Then neither the mean of the density on the ball of center x_0 will be 0, neither the speed in x_0 ! This means precisely that, even with a queue in front of him, a pedestrian in panic could still go on along his trajectory, and consequently we expect the density to become higher than one.

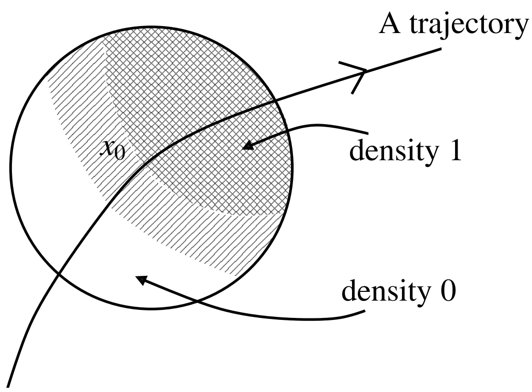

Fig. 1. Case in which the density is expected to become greater than one.

The problem is then naturally to control the increase in the density and to try to find geometry of the trajectories such that the density remains under a given threshold.

Supply-chain. D. Armbruster et al. [**2, 3**], introduced a continuum model to simulate the average behavior of highly re-entrant production systems at an aggregate level appearing, for instance, in large volume semiconductor production line. The factory is described by the density of products $\rho(t,x)$ at stage x of the production at a time t. Typically (see [**1, 3, 15, 21**]) the production velocity V is a given smooth function of the total load $\int_0^1 \rho(t,x)\mathrm{d}x$, for example

$$v(r) = v_{\max}/(1+r) \quad \text{and} \quad V(\rho) = v\left(\int_0^1 \rho(t,s)\,ds\right). \tag{7}$$

3.1 - *Existence and uniqueness of a solution*

Let us introduce the following sets of hypotheses:

(V1) There exists $C \in \mathbf{L}^\infty_{loc}(\mathbb{R}_+;\mathbb{R}_+)$ such that for all $u \in \mathbf{L}^1(\mathbb{R}^N;\mathbb{R})$

$$V(u) \in \mathbf{L}^\infty, \qquad \|\nabla_x V(u)\|_{\mathbf{L}^\infty} \le C(\|u\|_{\mathbf{L}^\infty}),$$
$$\|\nabla_x V(u)\|_{\mathbf{L}^1} \le C(\|u\|_{\mathbf{L}^\infty}), \quad \left\|\nabla_x^2 V(u)\right\|_{\mathbf{L}^1} \le C(\|u\|_{\mathbf{L}^\infty}),$$

and or all $u_1, u_2 \in \mathbf{L}^1(\mathbb{R}^N;\mathbb{R})$

$$\|V(u_1) - V(u_2)\|_{\mathbf{L}^\infty} \le C(\|u_1\|_{\mathbf{L}^\infty})\|u_1 - u_2\|_{\mathbf{L}^1},$$

$$\|\nabla_x(V(u_1) - V(u_2))\|_{\mathbf{L}^1} \le C(\|u_1\|_{\mathbf{L}^\infty})\|u_1 - u_2\|_{\mathbf{L}^1}.$$

(V2) There exists a positive function $C \in \mathbf{L}^\infty_{loc}(\mathbb{R}_+;\mathbb{R}_+)$ such that

$$\left\|\nabla_x^2 V(u)\right\|_{\mathbf{L}^\infty} \le C(\|u\|_{\mathbf{L}^\infty}),$$
$$\left\|\nabla_x^3 V(u)\right\|_{\mathbf{L}^\infty} \le C(\|u\|_{\mathbf{L}^\infty}).$$

Note in particular that through the assumption

$$\|V(u_1) - V(u_2)\|_{\mathbf{L}^\infty} \le C(\|u_1\|_{\mathbf{L}^\infty})\|u_1 - u_2\|_{\mathbf{L}^1},$$

we require V to be non-local.

Theorem 3.1 (see Theorem 2.2 in [**11**]). *Let* $u_0 \in (\mathbf{L}^\infty \cap \mathbf{L}^1 \cap \mathbf{BV})(\mathbb{R}^N;\mathbb{R})$. *If* V *satisfies* **(V1)**, *then there exists a time* $T_{ex} > 0$ *and a unique entropy solution* $u \in \mathscr{C}^0([0,T_{ex}[;\mathbf{L}^1 \cap \mathbf{L}^\infty \cap \mathbf{BV})$ *to* (2) *and we denote* $S_t u_0 = u(t,\cdot)$. *Besides, we have*

$$T_{ex} \ge \sup\left\{\sum_n \frac{\ln(\alpha_{n+1}/\alpha_n)}{C(\alpha_{n+1})}\ ;\ (\alpha_n)_n \textit{ strict. increasing},\ \alpha_0 = \|u_0\|_{\mathbf{L}^\infty}\right\},$$

where the function C *is the one appearing in* **(V1)**.

If furthermore, V satisfies **(V2)** *then*

$$u_0 \in \mathbf{W}^{2,1} \cap \mathbf{L}^\infty \Rightarrow \forall t \in [0, T_{ex}[\,, \quad u(t) \in \mathbf{W}^{2,1}\,.$$

Let us give below an idea of the proof.

Proof. We introduce the space $X_\alpha = \mathbf{L}^1(\mathbb{R}^N; [0,\alpha])$ and the application $\mathcal{Q}$ that associates to $w \in \mathcal{X}_\beta = \mathscr{C}^0([0,T[, X_\beta)$ the solution $u \in \mathcal{X}_\beta$ of the Cauchy problem

$$\partial_t u + \operatorname{Div}(uV(w)) = 0\,, \quad u(0,\cdot) = u_0 \in X_\alpha.$$

For $w_1, w_2 \in \mathcal{X}_\beta$, we obtain, thanks to the estimate of Theorem 2.3:

$$\|\mathcal{Q}(w_1) - \mathcal{Q}(w_2)\|_{\mathbf{L}^\infty([0,T[,\mathbf{L}^1)} \leq f(T)\|w_1 - w_2\|_{\mathbf{L}^\infty([0,T[,\mathbf{L}^1)}\,,$$

where f is increasing, $f(0) = 0$ and $f \to \infty$ when $T \to \infty$. Then we apply the Banach Fixed Point Theorem. □

Proposition 3.1. *Let V be defined in* (5) *and η be as in* (6) *in the pedestrian traffic model.*
If $v \in \mathscr{C}^2(\mathbb{R}; \mathbb{R})$ and $\vec{v} \in (\mathscr{C}^2 \cap \mathbf{W}^{2,1})(\mathbb{R}^2; \mathbb{S}^1)$ then V satisfies **(V1)** *and* **(V2)**.

Proposition 3.2. *Let $v \in \mathscr{C}^1([0,1]; \mathbb{R})$. Then, the functional V defined as in* (7) *in the supply-chain model satisfies* **(V1)** *and* **(V2)**.

3.2 - *Gâteaux derivative of the semi-group*

Let us recall the standard (local situation): the semi-group generated by a conservation law is in general lipschitz continuous and *not* differentiable. To cope with this issue, a new differential structure was introduced by Bressan et al. [**6**]. Here, the non-local property implies more regularity and we are able to differentiate the semi-group in the Gâteaux sense. Let us first recall the definition of Gâteaux differentiability.

Definition 3.1. *The application $S : \mathbf{L}^1(\mathbb{R}^N; \mathbb{R}) \to \mathbf{L}^1(\mathbb{R}^N; \mathbb{R})$ is said to be* $\mathbf{L}^1$ Gâteaux-differentiable in $u_0 \in \mathbf{L}^1(\mathbb{R}^N; \mathbb{R})$ in the direction $r_0 \in \mathbf{L}^1(\mathbb{R}^N; \mathbb{R})$ *if there exists a linear continuous application $DS(u_0) : \mathbf{L}^1 \to \mathbf{L}^1$ such that*

$$\left\| \frac{S(u_0 + hr_0) - S(u_0)}{h} - DS(u_0)(r_0) \right\|_{\mathbf{L}^1} \to 0 \qquad \textit{when } h \to 0.$$

Formally, we expect the Gâteaux derivative of the semi-group to be the solution of the linearized problem:

$$\partial_t r + \operatorname{Div}(rV(u) + uDV(u)(r)) = 0\,, \quad r(0,\cdot) = r_0\,.$$

In order to give sense to this equation, we have first to require the differentiability of V. Let us introduce stronger hypotheses:

(V3) $V : \mathbf{L}^1 \to \mathscr{C}^2$ is differentiable and there exists $C \in \mathbf{L}^\infty_{loc}(\mathbb{R}_+;\mathbb{R}_+)$ such that for all $u, r \in \mathbf{L}^1(\mathbb{R}^N;\mathbb{R})$:

$$\|V(u+r) - V(u) - DV(u)(r)\|_{\mathbf{W}^{2,\infty}} \le C\big(\|u\|_{\mathbf{L}^\infty} + \|u+r\|_{\mathbf{L}^\infty}\big)\|r\|^2_{\mathbf{L}^1},$$

$$\|DV(u)(r)\|_{\mathbf{W}^{2,\infty}} \le C(\|u\|_{\mathbf{L}^\infty})\|r\|_{\mathbf{L}^1}.$$

(V4) There exists $C \in \mathbf{L}^\infty_{loc}(\mathbb{R}_+;\mathbb{R}_+)$ such that for all $u, \tilde{u}, r \in \mathbf{L}^1(\mathbb{R}^N;\mathbb{R})$:

$$\|\mathrm{div}(V(\tilde{u}) - V(u) - DV(u)(\tilde{u}-u))\|_{\mathbf{L}^1} \le C(\|\tilde{u}\|_{\mathbf{L}^\infty} + \|u\|_{\mathbf{L}^\infty})\|\tilde{u}-u\|^2_{\mathbf{L}^1}$$

$$\|\mathrm{div}(DV(u)(r))\|_{\mathbf{L}^1} \le C(\|u\|_{\mathbf{L}^\infty})\|r\|_{\mathbf{L}^1}.$$

Proposition 3.3. *Let V be defined in* (5) *and η be as in* (6) *in the pedestrian traffic model.*

1. *If* $v \in \mathscr{C}^3(\mathbb{R};\mathbb{R})$, $\vec{v} \in (\mathscr{C}^3 \cap \mathbf{W}^{2,1})(\mathbb{R}^2;\mathbb{S}^1)$ *and* $\eta \in \mathscr{C}^3(\mathbb{R}^2;\mathbb{R})$, *then V satisfies* **(V3)**.

2. *If moreover* $v \in \mathscr{C}^4(\mathbb{R};\mathbb{R})$, $\vec{v} \in \mathscr{C}^2(\mathbb{R}^2;\mathbb{R}^2)$ *and* $\eta \in \mathscr{C}^2(\mathbb{R}^2;\mathbb{R})$, *then V satisfies* **(V4)** *and* **(V5)**.

Proposition 3.4. *Let $v \in \mathscr{C}^1([0,1];\mathbb{R})$. Then, the functional V defined as in* (7) *in the supply chain model satisfies* **(V3)**. *Moreover, if $v \in \mathscr{C}^2([0,1];\mathbb{R})$ then the functional V satisfies also* **(V4)** *and* **(V5)**.

With few hypotheses, we obtain the following weak result:

Proposition 3.5 (Weak Gâteaux derivative). *Let us assume that V satisfies* **(V1)** *and* **(V3)**. *Let $u_0 \in (\mathbf{W}^{1,\infty} \cap \mathbf{W}^{1,1})(\mathbb{R}^N;\mathbb{R})$ and $r_0 \in \mathcal{X}_1$. Then there exists $h_* > 0$ and $T_* = T_*(\|u_0\|_{\mathbf{L}^\infty}) > 0$ such that for all $h \in [0, h_*]$ the solutions u and u_h given by Theorem* 3.1, *associated to the initial conditions u_0 and $u_0 + hr_0$ are defined for all $t \in [0, T_*]$.*

Furthermore, if there exists $r \in \mathbf{L}^1([0,T]\times\mathbb{R}^N;\mathbb{R})$ *such that*

$$\frac{u_h - u}{h} \rightharpoonup_{h\to 0} r\,, in\,\mathbf{L}^1$$

then r is a distributional solution of the linearized equation.

We now look for stronger results. We show first that the linearized problem admits a unique entropy solution:

Theorem 3.2 (see Proposition 2.8 in [**11**]). *Assume that V satisfies* **(V1)**, **(V3)**. *Let* $u \in \mathscr{C}^0([0,T_{ex}[;(\mathbf{W}^{1,\infty}\cap\mathbf{W}^{1,1})(\mathbb{R}^N;\mathbb{R}))$, $r_0 \in (\mathbf{L}^1\cap\mathbf{L}^\infty)(\mathbb{R}^N;\mathbb{R})$. *Then the linearized Cauchy problem*

$$\partial_t r + \mathrm{Div}(rV(u) + uDV(u)(r)) = 0\,, \qquad \text{with } r(0,x) = r_0 \tag{8}$$

admits a unique entropy solution $r \in \mathscr{C}^0([0,T_{ex}[;\mathbf{L}^1(\mathbb{R}^N;\mathbb{R}))$ *and we denote* $\Sigma_t^u r_0 = r(t,\cdot)$.

If furthermore V satisfies **(V2)**, *and* $r_0 \in \mathbf{W}^{1,1}$, *then for all* $t\in[0,T_{ex}[$, $r(t)\in\mathbf{W}^{1,1}(\mathbb{R}^N;\mathbb{R})$.

Now, we can prove that the solution of the linearized equation is really the derivative of the semi-group.

Theorem 3.3 (see Theorem 2.10 in [**11**]). *Assume that V satisfies* **(V1)**, **(V2)**, **(V3)**, **(V4)**. *Let* $u_0 \in (\mathbf{W}^{1,\infty}\cap\mathbf{W}^{2,1})(\mathbb{R}^N;\mathbb{R})$, $r_0\in(\mathbf{W}^{1,1}\cap\mathbf{L}^\infty)(\mathbb{R}^N;\mathbb{R})$ *and let T_{ex} be the time of existence for the initial problem given by Theorem* 3.1.

Then, for all $t\in[0,T_{ex}[$ the local semi-group of the pedestrian traffic problem is $\mathbf{L}^1$-Gâteaux differentiable in the direction r_0 and

$$DS_t(u_0)(r_0) = \Sigma_t^{S_t u_0} r_0\,.$$

The following is the idea of the proof.

Proof. Let u, u_h be the solutions of the Cauchy problem

$$\partial_t u + \mathrm{Div}(uV(u)) = 0$$

with initial conditions $u_0, u_0 + hr_0$. Let r be the solution of the linearized equation (8), with $r(0) = r_0$. We define then $z_h = u + hr$ that satisfies $z_h(0) = u_0 + hr_0$ and

$$\partial_t z_h + \mathrm{Div}(z_h(V(u) + hDV(u)(r))) = h^2\,\mathrm{Div}\,(rDV(u)(r))\,.$$

Next, we use Theorem 2.3 to compare u_h and z_h. We obtain

$$\begin{aligned}\|u_h - z_h\|_{\mathbf{L}^\infty([0,T[,\mathbf{L}^1)} \leq{}& F(T)\big[\|u_h-u\|^2_{\mathbf{L}^\infty(\mathbf{L}^1)} + \|u_h - z_h\|_{\mathbf{L}^\infty(\mathbf{L}^1)}\big]\\ &+ h^2 C(\beta) T e^{C(\beta)T}\|r\|_{\mathbf{L}^\infty(\mathbf{W}^{1,1})}\|r\|_{\mathbf{L}^\infty(\mathbf{L}^1)}\,,\end{aligned}$$

where F is increasing and $F(0)=0$. With a good choice of T so that $F(T)\leq 1/2$, we can divide by h, make $h\to 0$ and conclude. □

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