Rinaldo M. Colombo and Francesca Marcellini

Smooth and discontinuous junctions in the p-system and in the 3×3 Euler system

Abstract. Consider the p-system describing the subsonic flow of a fluid in a pipe with section $a = a(x)$. We analyze the mathematical problem related to a junction, i.e., a sharp discontinuity in the pipe's geometry, we consider the case of a picewise constant pipe's section and then, the smooth case. In particular, through a limit procedure, we prove the well posedness of the smooth case from the discontinuous one and also the opposite case for the full 3×3 Euler system. Then, all the basic analytical properties of the equations governing a fluid flowing in a duct with varying section are extended to the Euler system. In both cases of the p-system and the Euler system, a key assumption is the boundedness of the total variation of the pipe's section. We provide explicit examples to show that this bound is necessary.

Keywords. Conservation laws at junctions, nozzle flow, coupling conditions at junctions.

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1 - Introduction

Consider a gas pipe with smoothly varying section. In the isentropic or isothermal approximation, the dynamics of the fluid in the pipe are described by the p-*system*:

$$
(1) \quad
\begin{cases}
\partial_t(a\rho) + \partial_x(aq) = 0 \\
\partial_t(aq) + \partial_x\left[a\left(\dfrac{q^2}{\rho} + p(\rho)\right)\right] = p(\rho)\,\partial_x a,
\end{cases}
$$

where, as usual, ρ is the fluid density, q is the linear momentum density, $p = p(\rho)$ is the pressure and $a = a(x)$ is *cross-sectional area* of the tube.

Here, the source term takes into account the inhomogeneities in the tube geometry, see for instance [13]. In this case, the regularity of the pipe automatically selects the appropriate definition of weak solution.

The problem related to a junction has been widely considered in the recent literature, see [3, 6, 9] and the references therein. Analytically, it consists of a *sharp discontinuity* in the pipe's geometry, say sited at $x = 0$. More precisely, it corresponds to the section $a(x) = a^-$ for $x < 0$ and $a(x) = a^+$ for $x > 0$. Thus, in each of the two pipes, the model reads

$$(2) \qquad \begin{cases} \partial_t \rho + \partial_x q = 0 \\[2mm] \partial_t q + \partial_x\left(\dfrac{q^2}{\rho} + p(\rho)\right) = 0 \end{cases}$$

where the *coupling condition*

$$(3) \qquad \Psi(a^-, (\rho, q)(t, 0-); a^+, (\rho, q)(t, 0+)) = 0$$

imposes suitable physical requirements, such as the conservation of mass and the equality of the hydrostatic pressure, see [3], or the partial conservation of momentum, see [6].

In Section 2 we study the proprerties of the *coupling condition* and we introduce a choice of Ψ motivated as limit of the smooth case, see Figure 1.
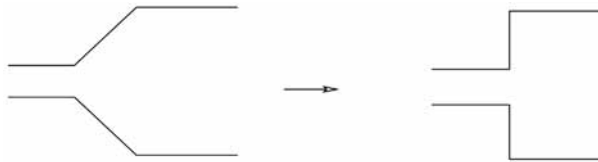

Fig. 1. The unique concept of solution in the case of a smooth section induces a definition of solution in the case of the junction.

With this particular definition of the *coupling condition*, we prove the well posedness of the resulting model first for a single junction, than also in the case of a piecewise constant pipe's section. The bounds on the total variation obtained in this construction allow to pass, through a suitable limit, to the case of a pipe's section a of class $W^{1,1}$, see Figure 2. In particular, by means of this latter limit, we also prove the well posedness of the smooth case.
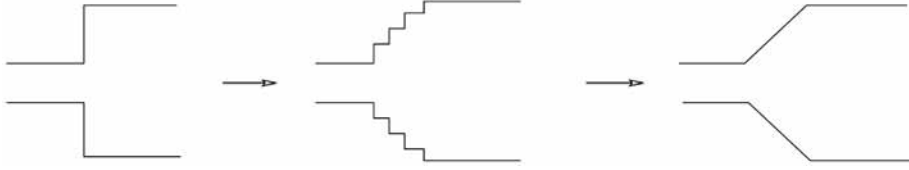
Fig. 2. The construction for a single jump is first extended to general piecewise constant sections and then, through an approximating procedure, to a section function a of class $W^{1,1}$.

Above, as usual in the theory of conservation laws, by *well posedness* we mean that we construct an L^1 Lipschitz semigroup whose orbits are solutions to the Cauchy problem. Moreover, the formal convergence of the problem with piecewise constant section to that one with $W^{1,1}$ section is completed by the rigorous proof of the convergence of the corresponding semigroups.

A careful estimate on the total variation of the solution shows that, at lower fluid speeds, higher total variations of the pipe's section are acceptable for the solution to exist. On the contrary, an explicit example computed in the case of the isothermal pressure law shows that, if the fluid speed is sufficiently close to the sound speed, a shock entering a pipe may have its strength arbitrarily magnified due to its inter-action with the pipe's walls. In other words, near to the sonic speed and with a having large total variation, the total variation of the solution may grow arbitrarily in finite time.

In Section 3, with reference to the results in [**14**], we study the limit illustrated in Figure 1. To this aim, we consider the Cauchy problem for an $n \times n$ strictly hyperbolic system of balance laws

$$(4) \qquad \begin{cases} \partial_t u + \partial_x f(u) = g(x, u) \\ u(0, x) = u_o(x) \end{cases}$$

with $u \in \mathbb{R}^n, u_o \in L^1 \cap BV(\mathbb{R}; \mathbb{R}^n)$ and with each characteristic field being genuinely nonlinear or linearly degenerate. Under the *nonresonance* assumption

$$|\lambda_i(u)| \geq c > 0 \text{ for all } i \in \{1, \ldots, n\} \text{ and for all } u,$$

and the boundedness condition

$$\|g(x, \cdot)\|_{C^2} \leq M(x) \text{ with } M \in L^1(\mathbb{R}; \mathbb{R}),$$

we prove the global existence, uniqueness and regularity of entropy solutions with bounded total variation provided, as usual, that the L^1 norm of $\|g(x, \cdot)\|_{C^1}$ and $TV(u_o)$ are small enough. In [**1**] an analogous result was proved, but under the stronger condition $M \in \left(L^\infty(\mathbb{R}; \mathbb{R}) \cap L^1(\mathbb{R}; \mathbb{R})\right)$.

This general result allows to compute the limit illustrated in Figure 1, also in the case of the full 3×3 Euler system. Indeed, in Section 3, we derive existence and uniqueness of solutions in the case of a discontinuous pipe's section as limit of solutions corresponding to smooth pipe's section.

In Section 4, the basic analytical properties of the equations governing a fluid flowing in a pipe with varying section, proved in Section 2 for the case of the p-system, are extended to the full 3×3 Euler system.

In particular, we consider Euler equations for the evolution of a fluid flowing in a pipe with varying section $a = a(x)$, see [**16**, Section 8.1] or [**13**, **15**]:

$$
(5) \qquad \begin{cases} \partial_t(a\rho) + \partial_x(aq) = 0 \\ \partial_t(aq) + \partial_x[aP(\rho,q,E)] = p(\rho,e)\,\partial_x a \\ \partial_t(aE) + \partial_x[aF(\rho,q,E)] = 0 \end{cases}
$$

where, as usual, ρ is the fluid density, q is the linear momentum density and E is the total energy density. Moreover

$$
(6) \qquad E(\rho,q,E) = \frac{1}{2}\frac{q^2}{\rho} + \rho e, \quad P(\rho,q,E) = \frac{q^2}{\rho} + p, \quad F(\rho,q,E) = \frac{q}{\rho}(E + p),
$$

with e being the internal energy density, P the flow of the linear momentum density and F the flow of the energy density.

The case of a sharp discontinuous change in the pipe's section due to a junction sited at, say, $x = 0$, corresponds to $a(x) = a^-$ for $x < 0$ and $a(x) = a^+$ for $x > 0$. Then, the motion of the fluid can be described by

$$
(7) \qquad \begin{cases} \partial_t\rho + \partial_x q = 0 \\ \partial_t q + \partial_x P(\rho,q,E) = 0 \\ \partial_t E + \partial_x F(\rho,q,E) = 0, \end{cases}
$$

for $x \neq 0$, together with a *coupling condition* at the junction of the form:

$$
(8) \qquad \Psi(a^-, (\rho,q,E)(t,0-); a^+, (\rho,q,E)(t,0+)) = 0.
$$

This framework comprises various choices of the coupling condition found in the literature, such as for instance in [**2, 3, 6, 12**], once they are extended to the 3×3 case and [**12**] for the full 3×3 system. We also extend the condition inherited from the smooth case introduced for the p-system in Section 2.

Within this setting, we prove the well posedness of the Cauchy problem for (7)-(8) and, then, the extension to pipes with several junctions and to pipes with a $W^{1,1}$ section.

As in the 2×2 case of the p-system, here a key assumption is the boundedness of the total variation of the pipe section. We provide explicit examples to show that this bound is necessary for each of the different coupling conditions considered.

2 - The p-System

This section is devoted to the study of (1). We refer to [11, Section 2] for the notations. In particular the pressure law p in (1) is assumed to satisfy the following requirement:

(**P**) $p \in C^2(\mathbb{R}^+; \mathbb{R}^+)$ is such that for all $\rho > 0$, $p'(\rho) > 0$ and $p''(\rho) \geq 0$,

and, throughout this section, we will refer to the *subsonic* region, given by

$$(9) \qquad A_0 = \left\{ u \in \overset{\circ}{\mathbb{R}}{}^+ \times \mathbb{R} : \lambda_1(u) < 0 < \lambda_2(u) \right\}.$$

As a tool in the study of (1) we use the system (2) recently proposed for the case of a sharp discontinuous change in the pipe's section between the values a^- and a^+, equipped with the coupling condition (3), whose role is essentially that of selecting stationary solutions. The introduction of this condition is necessary as soon as the section of the pipe is not smooth.

We specify the choice of (3) writing

$$(10) \qquad \Psi(a^-, u^-; a^+, u^+) = \begin{bmatrix} a^+ q^+ - a^- q^- \\ a^+ P(u^+) - a^- P(u^-) \end{bmatrix} - \Sigma(a^-, a^+; u^-),$$

where $\Sigma = \Sigma(a^-, a^+; u^-)$ describes the effects of the junction when the section changes from a^- to a^+ and the state to the left of the junction is u^-. We fix the section $\bar{a} > \Delta$, with $\Delta > 0$ and the state $\bar{u} \in A_0$ and we pose the following assumptions on Σ:

(**Σ0**) $\Sigma \in C^1\left([\bar{a} - \Delta, \bar{a} + \Delta] \times B(\bar{u}; \delta); \mathbb{R}^2 \right)$.

(**Σ1**) $\Sigma(a, a; u^-) = 0$ for all $a \in [\bar{a} - \Delta, \bar{a} + \Delta]$ and all $u^- \in B(\bar{u}; \delta)$.

(**Σ2**) $\Sigma(a^-, a^0; u^-) + \Sigma(a^0, a^+; T(a^-, a^0; u^-)) = \Sigma(a^-, a^+; u^-)$.

For the definition of T see [11, Lemma 2.1]. Condition (**Σ0**) is a natural regularity condition. Condition (**Σ1**) is aimed to comprehend the standard *"no junction"* situation: if $a^- = a^+$, then the junction has no effects and Σ vanishes. Finally condition (**Σ2**) says that if the two Riemann problems with initial states $(a^-, u^-), (a^0, u^0)$ and $(a^0, u^0), (a^+, u^+)$ both yield the stationary solution, then also the Riemann problem with initial state (a^-, u^-) and (a^+, u^+) is solved by the stationary solution.

Conditions (**Σ0**)–(**Σ1**) ensure the existence of stationary solutions to problem (2)-(3).

With this definition of coupling condition, we prove the well posedness for system (2)-(3), see [11, Theorem 2.3], also in the case of picewise constant pipe's section, see [11, Theorem 2.4]. In this case, at each junction x_j, we require con-

dition (3), namely

(11) $\Psi(a_{j-1}, u_j^-; a_j, u_j^+) = 0$ for all $j = 1, \ldots, n$, where $u_j^\pm = \lim_{x \to x_j\pm} u_j(x)$.

A key role in the result above is played by wave front tracking solutions to conservation laws and by the *operator splitting method*. In the present case, the standard wave front tracking procedure [4, Chapter 7] needs to be adapted to the presence of the junction.

Next we consider the case of a section a smooth, in particular we assume that $a \in W^{1,1}$. In general, if a is smooth, the product in the right hand side of the second equation in (1) is well defined and system (1) is equivalent to the 2×2 system of conservation laws

(12)
$$\begin{cases} \partial_t \rho + \partial_x q = -\dfrac{q}{a} \, \partial_x a \\ \partial_t q + \partial_x \left(\dfrac{q^2}{\rho} + p(\rho) \right) = -\dfrac{q^2}{a\rho} \, \partial_x a. \end{cases}$$

See [11, Lemma 2.6] for the equivalence between (1) and (12). In this case the section of the pipe is sufficiently regular to select stationary solutions as solutions of the system:

(13)
$$\begin{cases} \partial_x(a(x) \, \hat{q}) = 0 \\ \partial_x \left(a(x) \left(\dfrac{q^2}{\rho} + p(\rho) \right) \right) = p(\hat{\rho}) \, \partial_x a \end{cases} \quad \text{or} \quad \begin{cases} \partial_x q = -\dfrac{q}{a} \, \partial_x a \\ \partial_x \left(\dfrac{q^2}{\rho} + p(\rho) \right) = -\dfrac{q^2}{a\rho} \, \partial_x a \, . \end{cases}$$

Hence, the smoothness of a also single out a specific choice of Σ. Thus we introduce:

(14)
$$\Sigma(a^-, a^+, u^-) = \begin{bmatrix} 0 \\ \displaystyle\int_{-X}^{X} p(R^a(x; u^-)) a'(x) \, dx \, , \end{bmatrix}$$

where we call $\rho = R^a(x; u^-)$ the ρ-component of the corresponding solution to the Cauchy problem (13). The function (14) satisfies **(Σ0)-(Σ2)**, see [11, Proposition 2.7].

With the particular coupling condition in (14) and thanks to [11, Theorem 2.4] we prove the well posedness for the smooth case, when the section $a \in W^{1,1}$.

Theorem 2.1.   *Let p satisfy* **(P)**. *For any $\bar{a} > 0$ and any $\bar{u} \in A_0$ there exist positive M, Δ, δ, L such that for any profile a satisfying:*

(A1)
$$\begin{cases} a \in W^{1,1}(\mathbb{R}; \,]\bar{a} - \Delta, \bar{a} + \Delta[) \text{ for suitable } \Delta > 0, \ \bar{a} > \Delta \\ TV(a) < M \\ a'(x) = 0 \text{ for a.e. } x \in \mathbb{R} \setminus [-X, X] \text{ for a suitable } X > 0, \end{cases}$$

there exists a stationary solution $\hat{u}$ to (1) satisfying

$$\hat{u} \in A_0 \text{ with } \|\hat{u}(x) - \bar{u}\| < \delta \text{ for all } x \in \mathbb{R}$$

and a semigroup $S^a \colon \mathbb{R}^+ \times \mathcal{D}^a \to \mathcal{D}^a$ such that

1. *$\mathcal{D}^a \supseteq \left\{ u \in \hat{u} + L^1(\mathbb{R}; A_0) \colon TV(u - \hat{u}) < \delta \right\}$;*

2. *S_0^a is the identity and for all $t, s \geq 0$, $S_t^a S_s^a = S_{s+t}^a$;*

3. *for all $u, u' \in \mathcal{D}^a$ and for all $t, t' \geq 0$,*

$$\|S_t^a u - S_{t'}^a u'\|_{L^1} \leq L \cdot (\|u - u'\|_{L^1} + |t - t'|);$$

4. *for all $u \in \mathcal{D}^a$, the orbit $t \to S_t^a u$ is a weak entropy solution to (2); (see [**11**, Definition 2.5]).*

5. *Let $\hat{\lambda}$ be an upper bound for the moduli of the characteristic speeds in $\bar{B}(\hat{u}(\mathbb{R}), \delta)$. For all $u \in \mathcal{D}$, the orbit $u(t) = S_t u$ satisfies the integral conditions*

   - *For all $\tau < 0$ and $\xi \in \mathbb{R}$,*

$$(15) \qquad \lim_{h \to 0} \frac{1}{h} \int_{\xi - h\hat{\lambda}}^{\xi + h\hat{\lambda}} \|u(\tau + h, x) - U_{(u;\tau,\xi)}^{\sharp}(\tau + h, x)\|\, dx = 0 \,.$$

   - *There exists a $C > 0$ such that, for all $\tau > 0$, $a, b \in \mathbb{R}$ and $\xi \in \,]a, b[$,*

$$(16) \qquad \frac{1}{h} \int_{a + h\hat{\lambda}}^{b - h\hat{\lambda}} \|u(\tau + h, x) - U_{(u;\tau,\xi)\|(\tau + h, x)}^{\flat}\, dx$$

$$\leq C [TV\{u(\tau); \,]a, b[\} + TV\{a; \,]a, b[\}]^2 \,.$$

6. *If a Lipschitz map $w \colon \mathbb{R} \to \mathcal{D}$ solves (1), then it coincides with the semigroup orbit: $w(t) = S_t(w(0))$.*

Above $U_{(u;\tau,\xi)}^{\sharp}$ and $U_{(u;\tau,\xi)}^{\flat}$ are defined following [**4**, Theorem 9.2]; see also [**14**, Definition 1, Definition 2]. The proof is obtained approximating a with a piecewise constant function a_n. The corresponding problems (2)-(11) generate semigroups defined on domains characterized by uniform bounds on the total variation and with a uniformly bounded Lipschitz constants for their time dependence. Then, we pass to the limit, see [**11**, Theorem 2.8] for the proof, and we get the well posedness for the smooth case.

Moreover we show that, in the case of Σ as in (14) and with the isothermal pressure law $p(\rho) = c^2\rho$, the total variation of the solution to (2)-(11) may grow un-

boundedly if $TV(a)$ is large. For computations releted to this part, in particular related on the explicit estimate on the bound on the total variation of the section, see [**11**, Section 2.2].
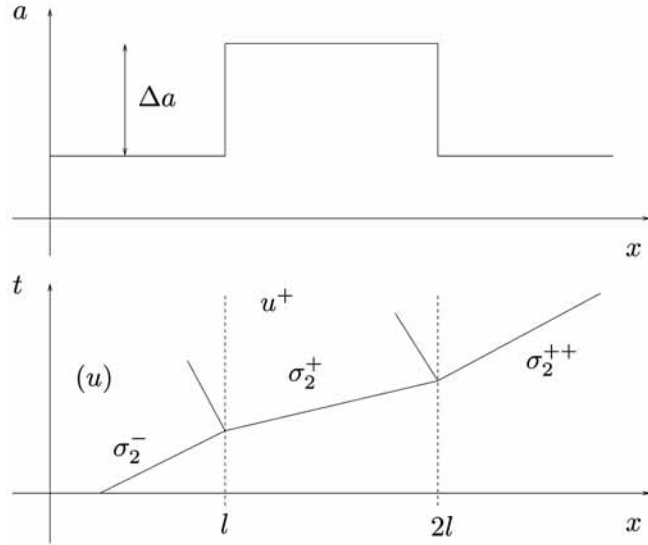

Fig. 3. A wave σ_2^- hits a junction, giving rise to σ_2^+ which hits a second junction.

Consider the case in Figure 3. A wave σ_2^- hits a junction where the pipe's section increases by $\varDelta a > 0$. From this interaction, the wave σ_2^+ of the second family arises, which hits the second junction where the section diminishes by $\varDelta a$. At the leading term in $\varDelta a$, we have the estimate

$$(17) \qquad |\sigma_2^{++}| \leq \left(1 + \mathcal{K}(\bar{v}/c)\left(\frac{\varDelta a}{a}\right)^2\right)|\sigma_2^-| \qquad \text{where}$$

$$(18) \qquad \mathcal{K}(\xi) = \frac{-1 + 8\,\xi^2 - 7\,\xi^4 + 2\,\xi^6}{2(1-\xi)^3\,(1+\xi)^3}\,.$$

Observe that $\mathcal{K}(0) = -1/2$ whereas $\lim\limits_{\xi \to 1-} \mathcal{K}(\xi) = +\infty$. Therefore, for any fixed $\varDelta a$, if $\bar{v}$ is sufficiently near to c, that is the fluid speed is sufficiently near to the sound speed, repeating the interactions in Figure 3 a sufficient number of times makes the 2 shock waves arbitrarily large (see [**11**, Section 3.2] for the proof). This example shows the necessity of a bound on the total variation of the pipe section in any well posedness theorem for (1).

3 - The $n \times n$ **Balance Law**

Throughout this section, we study the limit illustrated in Figure 1. At first, we consider the Cauchy problem related to the system of balance laws in (4) and concerning the source term g, we assume that it satisfies the following Caratheodory−type conditions:

(P_1) $g : \mathbb{R} \times \Omega \to \mathbb{R}^n$ is measurable with respect to (w.r.t.) x, for any $u \in \Omega$, and is C^2 w.r.t. u, for any $x \in \mathbb{R}$;

(P_2) there exists a L^1 function $\tilde{M}(x)$ such that $\|g(x, \cdot)\|_{C^2} \leq \tilde{M}(x)$;

(P_3) there exists a function $\omega \in L^1(\mathbb{R})$ such that $\|g(x, \cdot)\|_{C^1} \leq \omega(x)$.

Moreover, we assume that a *nonresonance* condition holds, that is the characteristic speeds of the system (4) are bounded away from zero:

$$(19) \qquad \hat{\lambda} \geq |\lambda_i(u)| \geq c > 0, \qquad \forall u \in \Omega, \ i \in \{1, \dots, n\},$$

for some $\hat{\lambda} > c > 0$.

We refer to [**14**, Section 1] for the notations and we recall the following result, see [**14**, Theorem 1], which extends the result in [**1**] to unbounded (in L^∞) sources:

Theorem 3.1. *Assume (P_1)-(P_3) and (19). If the norm of ω in $\mathbf{L^1}(\mathbb{R})$ is sufficiently small, there exist a constant $L > 0$, a closed domain $\mathcal{D}$ of integrable functions with small total variation and a unique semigroup $P : [0, +\infty) \times \mathcal{D} \to \mathcal{D}$ satisfying*

1. *$P_0 u = u, \quad P_{t+s} u = P_t \circ P_s u$ for all $u \in \mathcal{D}$ and $t, s \geq 0$;*

2. *$\|P_s u - P_t v\|_{L^1(\mathbb{R})} \leq L(|s - t| + \|u - v\|_{\mathbf{L^1}(\mathbb{R})})$ for all $u, v \in \mathcal{D}$ and $t, s \geq 0$;*

3. *for all $u_o \in \mathcal{D}$ the function $u(t, \cdot) = P_t u_o$ is a weak entropy solution of the Cauchy problem* (4) *with initial data*

$$(20) \qquad u(o, x) = \begin{cases} u_\ell & \text{if } x < x_0 \\ u_r & \text{if } x > x_0 \end{cases}$$

*and for all $\tau > 0$ satisfies the following integral estimates (see [**14**, Definition 1, Definition 2]):*

• *For every ξ, one has*

$$(21) \qquad \lim_{\theta \to 0} \frac{1}{\theta} \int\limits_{\xi - \theta\hat{\lambda}}^{\xi + \theta\hat{\lambda}} \left| u(\tau + \theta, x) - U^\sharp_{(u(\tau); \xi)}(\theta, x) \right| dx = 0.$$

• *There exists a constant C such that, for every $a < \xi < b$ and $0 < \theta < \dfrac{b-a}{2\hat{\lambda}}$, one has*

(22)
$$\frac{1}{\theta} \int_{a+\theta\hat{\lambda}}^{b-\theta\hat{\lambda}} \left| u(\tau + \theta, x) - U^{\flat}_{(u(\tau);\xi)}(\theta, x) \right| dx$$
$$\leq C \left[TV\{u(\tau); (a,b)\} + \int_a^b \omega(x) \, dx \right]^2 .$$

Conversely let $u : [0, T] \to \mathcal{D}$ be Lipschitz continuous as a map with values in $L^1(\mathbb{R}, \mathbb{R}^n)$ and assume that $u(t, x)$ satisfies the integral conditions (21), (22). Then $u(t, \cdot)$ coincides with a trajectory of the semigroup P.

Here, the technique is again based on the wave front tracking algorithm but, differently from the results in Section 2, we do not use the operator splitting procedure. On the contrary, as in [1] here the source is approximated by a sequence of Dirac deltas; careful estimates allow us to use the L^1 norm on the bound on M instead of its L^∞ norm.

Our aim is to apply this result to the fluid flow in a pipe with discontinuous cross sectional area, showing existence and uniqueness of the underlying semigroup. We consider the equations governing the gas flow in a pipe with a smooth varying cross section $a = a(x)$, given by:

(23)
$$\begin{cases} \partial_t \rho + \partial_x q = -\dfrac{q}{a} \partial_x a \\[2mm] \partial_t q + \partial_x(\dfrac{q^2}{\rho} + p) = -\dfrac{q^2}{a\rho} \partial_x a \\[2mm] \partial_t e + \partial_x \left(\dfrac{q}{\rho}(e+p) \right) = -\dfrac{\left(\dfrac{q}{\rho}(e+p) \right)}{a} \partial_x a. \end{cases}$$

One way to obtain coupling conditions at the point of discontinuity of the cross section a is to take the limit of a sequence of Lipschitz continuous cross sections a^ε converging to a in L^1. Unfortunately the results in [1] require L^∞ bounds on the source term and well posedness is proved on a domain depending on this L^∞ bound. Since in the previous equations the source term contains the derivative of the cross sectional area one cannot hope to take the limit $a^\varepsilon \to a$. Indeed when a is discontinuous, the L^∞ norm of $(a^\varepsilon)'$ goes to infinity. Therefore Theorem 3.1 establishes the result in [1] without requiring the L^∞ bound.

In particular Theorem 3.1 provides an existence and uniqueness result for pipes with Lipschitz continuous cross section where the equations governing the gas flow are given by (23), while the case of discontinuous cross sections does not fulfil the requirements of Theorem 3.1. Nevertheless, we can use this result to derive the existence of solutions to the discontinuous problem by a limit procedure. To this end, we approximate the discontinuous function

$$(24) \qquad a(x) = \begin{cases} a^-, & x < 0 \\ a^+, & x > 0 \end{cases}$$

by a sequence $a_l \in C^{0,1}(\mathbb{R}, \mathbb{R}^+)$ with the following properties

$$(25) \qquad a_l(x) = \begin{cases} a^-, & x < -\dfrac{l}{2} \\ \varphi_l(x), & x \in \left[-\dfrac{l}{2}, \dfrac{l}{2}\right] \\ a^+, & x > \dfrac{l}{2} \end{cases}$$

where φ_l is any smooth monotone function which connects the two strictly positive constants a^-, a^+. One possible choice of the approximations a_l as well as the discontinuous pipe with cross section a are shown in Figure 1.

With the help of Theorem 3.1 we derive the existence of solutions to the discontinuous problem by a limit procedure and we get the following result, see [**14**, Theorem 2]:

Theorem 3.2. *Let $\bar{u}$ a non sonic state. If $|a^+ - a^-|$ is sufficiently small, the semigroups P^l (defined on a domain of functions which take value in a small neighborhood of $\bar{u}$) related with the smooth section a_l converge to a unique semigroup P.*

4 - The Euler System

This section is devoted to the study of (5) and the properties of the equations governing a fluid flowing in a pipe with varying section, proved in Section 2 for the case of the p-system, are extended to the full 3×3 Euler system.

We refer to [**10**, Section 2] for the notations. In particular, concerning the pressure law p in (5), we will often refer to the standard case of the ideal gas, char-

acterized by the relations

(26) $$p = (\gamma - 1)\rho e, \qquad S = \ln e - (\gamma - 1)\ln \rho,$$

for a suitable $\gamma > 1$, and, throughout this section, we will refer to the *subsonic* region, given by

(27) $$A_0 = \left\{ u \in \mathring{\mathbb{R}}^+ \times \mathbb{R}^+ \times \mathring{\mathbb{R}}^+ : \lambda_1(u) < 0 < \lambda_2(u) \right\}.$$

Concerning the case of the sharp discontinuous change in the pipe's section, we refer to system (7) and, extending the 2×2 case of the p-system, we consider some properties of the coupling condition (8), which we rewrite here as

(28) $$\Psi(a^-, u^-; a^+, u^+) = 0.$$

(**Ψ0**) Regularity: $\Psi \in C^1\left((\mathring{\mathbb{R}}^+ \times A_0)^2; \mathbb{R}^3\right)$.

(**Ψ1**) No-junction case: for all $a > 0$ and all $u^-, u^+ \in A_0$, then

$$\Psi(a, u^-; a, u^+) = 0 \Longleftrightarrow u^- = u^+.$$

(**Ψ2**) Consistency: for all positive a^-, a^0, a^+ and all $u^-, u^0, u^+ \in A_0$,

$$\begin{aligned} \Psi(a^-, u^-; a^0, u^0) = 0 \\ \Psi(a^0, u^0; a^+, u^+) = 0 \end{aligned} \Longrightarrow \Psi(a^-, u^-; a^+, u^+) = 0.$$

As in the 2×2 case of the p-system, we prove the well posedness of the Cauchy problem for (7)-(8), by using the techniques in [**7, 8**]. Then, the extension to pipes with several junctions and to pipes with a $W^{1,1}$ section is achieved by the same analytical techniques used in Section 2.

Here, we consider the case of a general coupling condition which comprises all the cases found in the literature.

(**S**)**-Solutions** We consider first the coupling condition inherited from the smooth case. For smooth solutions and pipes' sections, system (7) is equivalent to the 3×3 balance law in (23) and the stationary solutions to (7) are characterized as solutions to

(29) $$\begin{cases} \partial_x(a(x)\, q) = 0 \\ \partial_x(a(x)\, P(\boldsymbol{\rho}, q, E)) = p(\boldsymbol{\rho}, e)\, \partial_x a \quad \text{or} \\ \partial_x(a(x)\, F(\boldsymbol{\rho}, q, E)) = 0 \end{cases} \begin{cases} \partial_x q = -\dfrac{q}{a}\, \partial_x a \\ \partial_x P(\boldsymbol{\rho}, q, E) = -\dfrac{q^2}{a\boldsymbol{\rho}}\, \partial_x a \\ \partial_x F(\boldsymbol{\rho}, q, E) = -\dfrac{F}{a}\, \partial_x a. \end{cases}$$

As in the 2×2 case of the p-system, the smoothness of the sections induces a unique choice for condition (8), which reads

$$(30) \qquad (\mathbf{S}) \quad \boldsymbol{\varPsi} = \begin{bmatrix} a^+ q^+ - a^- q^- \\[6pt] a^+ P(u^+) - a^- P(u^-) + \displaystyle\int_{-X}^{X} p(\mathcal{R}^a(x), \mathcal{E}^a(x)) a'(x)\mathrm{d}x \\[6pt] a^+ F(u^+) - a^- F(u^-) \end{bmatrix}$$

where $a = a(x)$ is a smooth monotone function satisfying $a(-X) = a^-$ and $a(X) = a^+$, for a suitable $X > 0$. $\mathcal{R}^a, \mathcal{E}^a$ are the ρ and e component in the solution to (29) with initial datum u^- assigned at $-X$.

(P)-Solutions The particular choice of the coupling condition in [**12**, Section 3] can be recovered in the present setting. Indeed:

$$(31) \qquad (\mathbf{P}) \qquad \varPsi(a^-, u^-, a^+, u^+) = \begin{bmatrix} a^+ q^+ - a^- q^- \\ P(u^+) - P(u^-) \\ a^+ F(u^+) - a^- F(u^-) \end{bmatrix},$$

where a^+ and a^- are the pipe's sections.

(L)-Solutions We can extend the construction in [**2, 3, 5**] to the 3×3 case (7). Indeed, the conservation of the mass and linear momentum in [**5**] with the conservation of the total energy for the third component lead to the choice

$$(32) \qquad (\mathbf{L}) \qquad \varPsi(a^-, u^-, a^+, u^+) = \begin{bmatrix} a^+ q^+ - a^- q^- \\ a^+ P(u^+) - a^- P(u^-) \\ a^+ F(u^+) - a^- F(u^-) \end{bmatrix},$$

where a^+ and a^- are the pipe's sections.

(p)-Solutions Following [**2, 3**], we consider a coupling condition with the conservation of the pressure $p(\rho)$ in the second component of $\varPsi$. Thus

$$(33) \qquad (\mathbf{p}) \qquad \varPsi(a^-, u^-, a^+, u^+) = \begin{bmatrix} a^+ q^+ - a^- q^- \\ p(\rho^+, e^+) - p(\rho^-, e^-) \\ a^+ F(u^+) - a^- F(u^-) \end{bmatrix},$$

where a^+ and a^- are the pipe's sections.

As in Section 2, we show that in each of the cases (30), (31), (32), (33), it is possible to choose an initial datum and a section $a \in BV(\mathbb{R}; [a^-, a^+])$ with $a^+ - a^-$ arbitrarily small, such that the total variation of the corresponding solution to (7)-(8) becomes arbitrarily large.
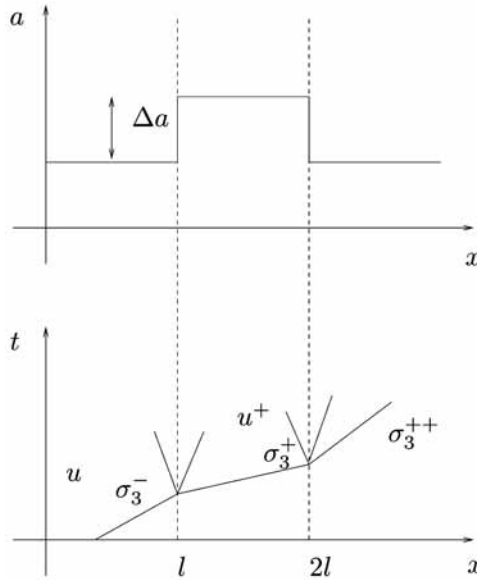
Fig. 4. A wave σ_3^- hits a junction where the pipe's section increases by Δa. From this interaction, the wave σ_3^+ arises, which hits a second junction, where the pipe section decreases by Δa.

Consider the case in Figure 4, where a wave σ_3^- hits a junction where the pipe's section increases by, say, $\Delta a > 0$. The fastest wave arising from this interaction is σ_3^+, which hits the second junction where the section diminishes by Δa.

At the second order in $(\Delta a)/a$ and at the first order in σ_3^-, we get the following estimate:

$$(34) \qquad \sigma_3^{++} = \left(1 + \chi(\theta)\left(\frac{\Delta a}{a}\right)^2\right)\sigma_3^- .$$

It is sufficient to compute the sign of χ. If it is positive, then repeating the interaction in Figure 4 a sufficient number of times leads to an arbitrarily high value of the refracted wave σ_3 and, hence, of the total variation of the solution u.

See [10, Section 5] for the computations of χ in the different cases (30), (31), (32) and (33), where, to reduce the formal complexities of the explicit computations, we consider the standard case of an ideal gas characterized by (26) with $\gamma = 5/3$.

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RINALDO M. COLOMBO
Università degli Studi di Brescia
Dipartimento di Matematica
Via Branze 38
25123 Brescia, Italy
e-mail: Rinaldo.Colombo@UniBs.it

FRANCESCA MARCELLINI
Università di Milano-Bicocca
Dipartimento di Matematica e Applicazioni
Via Cozzi 53
20126 Milano, Italy
e-mail: F.Marcellini@Campus.Unimib.it