Riv. Mat. Univ. Parma, Vol. **3** (2012), 41-54

CLEOPATRA CHRISTOFOROU and LAURA V. SPINOLO

On the physical and the self-similar viscous approximation of a boundary Riemann problem

Abstract. We deal with the viscous approximation of a system of conservation laws in one space dimension and we focus on initial-boundary value problems. It is known that, in general, different viscous approximations provide different limits because of boundary layer phenomena. We focus on Riemann-type data and we discuss a uniqueness criterion for distributional solutions which applies to both the non characteristic and the boundary characteristic case. As an application, one gets that the limits of the physical viscous approximation

$$\partial_t U^\varepsilon + \partial_x [F(U^\varepsilon)] = \varepsilon \partial_x [B(U^\varepsilon) \partial_x U^\varepsilon]$$

and of the self-similar viscous approximation

$$\partial_t U^\varepsilon + \partial_x [F(U^\varepsilon)] = \varepsilon t \partial_x [B(U^\varepsilon) \partial_x U^\varepsilon]$$

introduced by Dafermos et al. are expected to coincide.

Keywords. Boundary Riemann problem, viscous approximation, self-similar viscous approximation, boundary layer, characteristic boundary.

Mathematics Subject Classification (2010): 35L65.

1 - Systems of conservation laws in one space dimension

We are interested in systems of conservation laws in one space dimension, namely partial differential equations of the form

$$\partial_t U + \partial_x \big[F(U)\big] = 0, \tag{1}$$

Received: October 14, 2010; accepted: January 24, 2011.

where the unknown $U(t, x)$ attains values in $\mathbb{R}^N$ and the flux function $F : \mathbb{R}^N \to \mathbb{R}^N$ is of class $\mathcal{C}^2$. The time variable $t \in [0, +\infty[$ and the space variable x is one-dimensional. This class of equations has several and diverse physical applications, coming from both physics and engineering. In particular, the Euler equations governing the motion of an inviscous fluid take form (1) in the case when the space variable is one-dimensional. For a wide discussion on both the physical motivations and the analysis of (1) we refer to the by-now classical references provided in the books by Dafermos [**9**] and Serre [**26**].

We first consider the Cauchy problem obtained by coupling (1) with the initial datum

$$U(0, x) = U_0(x). \tag{2}$$

It is known that, even if the datum U_0 is smooth, classical solutions of (1), (2) are in general defined only on a finite time interval, and develop discontinuities in finite time. It is thus natural to study distributional solutions. However, one cannot hope for uniqueness: a given Cauchy problem may indeed admit infinitely many distributional solutions. In an attempt at selecting a unique solution, various admissibility criteria have been introduced in the literature. We refer to Dafermos [**9**, Chapters 4 and 8] for an extended discussion on this subject.

Existence and uniqueness results for global in time, admissible, distributional solutions of (1), (2) have been achieved under the hypothesis that the total variation of the initial datum U_0 is sufficiently small and that the system is *strictly hyperbolic*, namely that, for every $U \in \mathbb{R}^N$, the Jacobian matrix $DF(U)$ admits N distinct and real eigenvalues

$$\lambda_1(U) < \lambda_2(U) < \cdots < \lambda_N(U). \tag{3}$$

The proof of these results relies on the construction of suitable approximation procedures, like the Glimm scheme [**13**] or the wave front-tracking algorithm. For an overview, we refer to the books by Dafermos [**9**] and by Bressan [**5**]. If the total variation of the initial datum is finite, but large, then the admissible, distributional solution may experience blow up in finite time: an example is discussed by Jenssen [**14**]. If strict hyperbolicity is violated, then the existence of distributional solutions may fail (see the examples in Dafermos [**9**, Sections 3.1 and 9.6]).

2 - Physical and self-similar viscous approximation of a Riemann problem

Here we are mostly interested in the physical viscous approximation

$$\partial_t U^\varepsilon + \partial_x \big[F(U^\varepsilon)\big] = \varepsilon \partial_x \big[B(U^\varepsilon)\partial_x U^\varepsilon\big], \tag{4}$$

where $U^\varepsilon(t, x) \in \mathbb{R}^N$, the function F is the same as in (1), ε is a small positive parameter and B denotes an $N \times N$ matrix which depends on the physical model under consideration. In particular, if the limit system (1) is the Euler equations, then a natural choice is taking in (4) the Navier-Stokes equations.

Bianchini and Bressan [**3**] established convergence results for (4) under the hypotheses that the total variation of the initial datum is sufficiently small and that the matrix B is constantly equal to the identity. The proof of the convergence in the case of a general matrix B still stands as a major and challenging open problem. We refer again to Dafermos [**9**] for a wider discussion and a complete list of references on this topic.

In the case of the so-called *Riemann problem*, an alternative approach was introduced independently by Dafermos [**8**], Kalasnikov [**17**] and Tupciev [**27**]. The Riemann problem is obtained by coupling the system of conservation laws (1) with an initial datum in the form

$$U(0, x) = \begin{cases} U^- & x < 0 \\ U^+ & x \geq 0, \end{cases} \tag{5}$$

where U^+ and U^- are two given constant values in $\mathbb{R}^N$. The study of the Riemann problem is of key importance for the analysis of systems of conservation laws. This is due to various reasons: first, in general one can find infinitely many solutions of (1), (5) and hence the Riemann problem captures one of the main difficulties in the study of systems of conservation laws. Also, the Riemann problem describes both the local (in space-time) and the long-time behavior of solutions of more general Cauchy problems and, moreover, it constitutes the building block for both the Glimm and the wave front-tracking algorithm. Finally, the Riemann problem has a key role in defining the so-called *Standard Riemann Semigroup* and hence in defining the class of *admissible solutions* for which uniqueness results have been established (see Bressan [**5**, Chapter 8]). As mentioned before, existence and uniqueness results for the Cauchy problem (1), (2) hold under the hypothesis that the total variation of U_0 is sufficiently small. In the present exposition, we always assume that U^+ and U^- are sufficiently close and this reflects the fact that the goal is approximating data of small total variation.

The approach introduced by Dafermos [**8**], Kalasnikov [**17**] and Tupciev [**27**] to study the viscous approximation of a Riemann problem relies on the analysis of the family of parabolic problems

$$\partial_t U^\varepsilon + \partial_x \big[F(U^\varepsilon)\big] = \varepsilon t \partial_x \big[B(U^\varepsilon)\partial_x U^\varepsilon\big]. \tag{6}$$

The advantage in introducing the coefficient t in front of the second order term is that (6) admits self-similar solutions of the form $U^\varepsilon(t, x) = V_\varepsilon(x/t)$. Tzavaras [**29**] obtained compactness results for the family of functions U^ε satisfying (6) with the Riemann datum

$$U^\varepsilon(0,x) = \begin{cases} U^- & x < 0 \\ U^+ & x \geq 0. \end{cases} \tag{7}$$

The analysis in [**29**] relies on the assumptions that $B(U) \equiv I$ and that $|U^+ - U^-|$ is small enough. See also Tzavaras [**28**] and the discussion in Dafermos [**9**, Section 9.8] for other results concerning the self-similar viscous approximation (6).

The results obtained in [**29**] imply, in particular, that one can construct a distributional solution U of the Riemann problem (1), (5) by taking the limit $\varepsilon \to 0^+$ of (6)-(7). This solution is self-similar, namely $U(t,x) = V(x/t)$ for a measurable function V. Also, V has bounded total variation and hence admits at most countably many discontinuities, which correspond to either *shocks* or *contact discontinuities* of U.

We now focus on the uniqueness issue: first, we recall that in [**23**] Liu introduced a celebrated admissibility condition which can be viewed as a generalization of another condition named after Lax [**20**]. Roughly speaking, both Lax and Liu admissibility conditions are irreversibility requirements imposed on the solution at points of discontinuity. We refer to Dafermos [**9**, Chapters 8 and 9] for the rigorous definitions. The existence of a distributional solution of the Riemann problem (1), (5) satisfying these admissibility conditions was established, at increasing levels of generality, in the pioneering works by Lax [**20**] and Liu [**21, 22**].

We conclude this section by mentioning a result due to Bianchini [**2**]: there exist positive constants C and δ, δ small enough, such that, if $|U^+ - U^-| \leq \delta$, then there exists a unique distributional solution U of the Riemann problem (1), (5) satisfying the following requirements:

- **R1** U is self-similar, namely $U(t,x) = V(x/t)$ for some measurable function V.
- **R2** the function V has total variation bounded by $C\delta$.
- **R3** V is obtained by patching together at most countably many shocks, contact discontinuities and rarefactions.
- **R4** any shock and any contact discontinuity is admissible in the sense of Liu [**23**].

The analysis in Bianchini and Bressan [**3**] ensures that, in the case when $B \equiv I$, the limit of the viscous approximation (4), (7) satisfies requirements $\mathbf{R1}, \dots, \mathbf{R4}$ above and hence it coincides with the limit of the self-similar approximation (6). Also, by taking into account the analysis by Majda and Pego [**24**] ensuring that shocks and contact discontinuities obtained as limits of suitable viscous approximation satisfy Liu's admissibility condition, one expects that, if the family U^ε solving (4), (7) satisfy stability conditions and converges for $\varepsilon \to 0^+$, then the limit satisfies requirements $\mathbf{R1}, \dots, \mathbf{R4}$ above.

Consider now the limit of the self-similar viscous approximation (6) and (7). The analysis in Tzavaras [**29**, Section 9], Dafermos [**9**, Chapter 9.8] and Majda and Pego [**24**], suggests that, under reasonable assumptions, if the family V^ε solving (6) and (7) converges, then the limit satisfies requirements $\mathbf{R1}, \dots, \mathbf{R4}$. In particular, in view of the uniqueness result in [**2**], such a limit does not depend on the choice of the viscosity matrix B and coincides with the limit of the physical viscous approximation (4), (7).

3 - Viscous approximation of a boundary Riemann problem

We now assume that $x \in [0, +\infty[$ and we consider the initial-boundary value problem associated with the system of conservation laws (1). The initial-boundary value problems presents all the challenges of the Cauchy problem: classical solutions may develop discontinuities in finite time, while distributional solutions are not unique. Also, additional difficulties arise because of the presence of the boundary: first, by coupling (1) with the Dirichlet and the Cauchy data

$$U(t,0) = \bar{U}(t) \qquad U(0,x) = U_0(x) \tag{8}$$

one obtains, in general, an ill-posed problem, namely a problem that posses no solution. This can be easily seen by considering the linear, scalar case

$$\partial_t u - \partial_x u = 0$$

and observing that the solution is constant along the lines in the (x, t) plane having slope -1. Hence, the initial-boundary value problem is ill-posed unless the Dirichlet datum matches with the Cauchy datum. A notion of *admissible boundary condition* can then be introduced, see Dubois and LeFloch [**10**].

Other difficulties arise when studying the viscous approximation (4): the initial-boudary value problem obtained by coupling (4) with the data

$$U^\varepsilon(t,0) = U_b(t) \qquad U^\varepsilon(0,x) = U_0(x) \tag{9}$$

may be also ill-posed if the matrix B is singular: note that this is actually the case of the Navier-Stokes equation and of most of the physically relevant examples. To simplify notations, in the present exposition we restrict to the case when B is invertible, but the general case can be also treated, at the price of higher technicalities (see Bianchini and Spinolo [**4**, Section 2.2.1]).

Roughly speaking, the problem concerning the viscous approximation is the following: consider the family of initial-boundary value problems (4) and (9) and assume that as $\varepsilon \to 0^+$ it converges (in a suitable topology) to a limit function U such that the trace $\lim_{x \to 0^+} U(t,x)$ is well-defined. These assumptions can be rigorously established in the case when the matrix B is the identity, see Ancona and Bianchini [**1**]. In general, one has that the limits $x \to 0^+$ and $\varepsilon \to 0^+$ do not commute, in other words

$$\lim_{x \to 0^+} U(t,x) \neq U_b(t).$$

Also, the limit U, in general, depends on B, namely if one fixes the flux function F and the data U_b and U_0 and let B vary, then the limit U varies. This was pointed out by Gisclon and Serre in [**11**] and [**12**].

To figure out why the limit depends on B, let us focus on the case of the so-called *boundary Riemann problem* and assume that the data imposed on the viscous approximation are

$$U^\varepsilon(t,0) = U_b \qquad U^\varepsilon(0,x) = U_0, \tag{10}$$

where U_0 and U_b are two given constant values in $\mathbb{R}^N$. Roughly speaking, the boundary Riemann problem is important for the same reasons why the Riemann problem is relevant: in particular, it constitutes the building block for the construction of approximation schemes that are used to establish existence and uniqueness results for global in time, admissible distributional solutions. Unless one imposes additional conditions on the flux function F, these results require that the total variation of the data is sufficiently small. In the present exposition, we impose on F only $\mathcal{C}^2$ regularity and strict hyperbolicity (3) and we assume that the data U_0 and U_b are sufficiently close, $|U_b - U_0| \leq \delta$ for some small constant $\delta > 0$. Also, we first focus on the so-called *non characteristic boundary* case i.e. all the eigenvalues of the Jacobian matrix $DF(U)$ are bounded away from 0.

Assume that the family of functions U^ε solving (4), (10) satisfies suitable stability requirements and converges as $\varepsilon \to 0^+$ to a limit U (see e.g. Bianchini and Spinolo [**4**, Section 2.3] for the precise requirements). Then the following holds: first, U satisfies **R1**, …, **R4** above. Being U a self-similar function with bounded total variation, then the trace $\bar U = \lim_{x \to 0^+} U(t,x)$ is well-defined and does not depend on t. Second, there exists a function $W(y)$ taking values in $\mathbb{R}^N$ and solving the system

$$\begin{cases} B(W)W' = F(W) - F(\bar U) \\ W(0) = U_b \qquad \lim_{y \to +\infty} W(y) = \bar U. \end{cases} \tag{11}$$

The function W is a *boundary layer* for the viscous approximation (4) connecting U_b and $\bar U$. Given U_b and $\bar U$, whether or not system (11) admits a solution depends on both the function F and the matrix B. Since $\bar U$ is the trace of the hyperbolic limit U, this explains why in general the limit of (4), (10) depends on B. Explicit examples of different viscosity matrices leading to different limits can be found in Gisclon [**11**].

We now consider (4), (10) in the case when the boundary is *characteristic*, namely an eigenvalue of the Jacobian matrix $DF(U)$ can attain the value 0. One expects that, if the family U^ε converges, then the limit U satisfies $\mathbf{R1}, \dots, \mathbf{R4}$ above, so that in particular the trace $\bar{U}$ is again well defined. Also, for some constant $C > 0$ depending only on U_0, F and B the following holds:

R5 There exists a value $\underline{U}$, $|U_0 - \underline{U}| \leq C\delta$, such that the following conditions are satisfied:

a. $F(\bar{U}) = F(\underline{U})$;
b. the shock or the contact discontinuity between $\bar{U}$ (on the right) and $\underline{U}$ (on the left) is admissible in the sense of Liu;
c. there is a boundary layer $W(y)$ such that

$$\begin{cases} B(W)W' = F(W) - F(\bar{U}) \\ W(0) = U_b \qquad \lim_{y \to +\infty} W(y) = \underline{U}. \end{cases} \tag{12}$$

and $|W'(y)| \leq C\delta$, $|W(y) - \underline{U}| \leq C\delta$ for every y.

Some remarks are here in order: first, the equality $F(\bar{U}) = F(\underline{U})$ implies that $\bar{U}$ and $\underline{U}$ are connected by a zero-speed shock (or contact discontinuity) because it ensures that the so-called *Rankine-Hugoniot conditions* are satisfied (see Dafermos [**9**, Chapter 3] for a discussion about these conditions). Hence condition **R5**b makes sense. Second, in the non characteristic boundary case the equality $F(\bar{U}) = F(\underline{U})$ implies via the local invertibility theorem that $\bar{U} = \underline{U}$ and therefore **R5** is equivalent to the existence of a boundary layer W satisfying (11). Hence, in the following, we use the formulation **R5** in both cases of a characteristic and a non characteristic boundary.

The existence of a distributional solution of (1) defined on the domain $t \in [0, +\infty[$, $x \in [0, +\infty[$ and satisfying requirements $\mathbf{R1}, \dots, \mathbf{R5}$ was established in Bianchini and Spinolo [4] by relying on the hypotheses that $|U_b - U_0|$ is sufficiently small, that the system is strictly hyperbolic (3) and that the matrix B satisfies suitable conditions. In particular, the case when the matrix B is singular is taken into account and the analysis applies to the Navier-Stokes equation written in Lagrangian coordinates.

The uniqueness of solutions satisfying $\mathbf{R1}, \dots, \mathbf{R5}$ was obtained in Christoforou and Spinolo [**6**] and is discussed in the next section.

4 - A uniqueness criterion for viscous limits of boundary Riemann problems

Before introducing the precise statement, we recall that an entropy-entropy flux pair for system (1) is a pair (η, q), with $\eta,\ q : \mathbb{R}^N \to \mathbb{R}$ satisfying

$$\nabla\eta(U)\cdot DF(U) = \nabla q(U) \quad \text{for every } U \in \mathbb{R}^N.$$

We can now state our theorem.

Theorem 4.1. *Let* $F : \mathbb{R}^N \to \mathbb{R}^N$ *be a given* C^2 *function satisfying strict hyperbolicity* (3) *and let* U_0 *be a given state in* $\mathbb{R}^N$. *Also, assume that*

(i) *system* (1) *admits an entropy-entropy flux pair* (η, q) *with* η *strictly convex;*
(ii) *for any given compact set* $K \subseteq \mathbb{R}^n$, *there exists a constant* $\alpha_K > 0$ *such that, for every* $U \in K$,

$$D^2\eta(U)\vec{v}\cdot B(U)\vec{v} \geq \alpha_K|\vec{v}|^2 \qquad \forall\, \vec{v} \in \mathbb{R}^n. \tag{13}$$

(iii) *the transversality condition provided by Hypothesis* 4 *in* [**6**] *is satisfied.*

Then there exist constants C *and* δ, δ *small enough, such that, for every* U_b *such that* $|U_0 - U_b| \leq \delta$, *the distributional solution of* (1) *satisfying the initial condition* $U(0,x) = U_0$ *and requirements* $\mathbf{R1}, \dots, \mathbf{R5}$ *is unique.*

Some remarks are here in order: first, conditions (i) and (ii) in the statement of Theorem 4.1 are quite classical in this context and they imply that the matrix B is strictly stable in the sense of Majda and Pego [**24**]. Condition (ii) implies that the matrix B is invertible. The extension of Theorem 4.1 to the case of a singular viscosity matrix, which is more interesting in view of physical applications, does not pose any apparent difficulty provided that the assumptions introduced by Kawashima and Shizuta in [**19**] and a condition of so-called *block linear degeneracy* defined in Bianchini and Spinolo [4] are all satisfied.

Second, it should be noted that Theorem 4.1 covers both cases of characteristic and non characteristic boundary. Also, the only assumption we impose on the flux function F is strict hyperbolicity and neither genuine nonlinearity nor linear degeneracy of the characteristic fields is required.

Third, conditions $\mathbf{R1}, \dots, \mathbf{R5}$ are not sharp, because one can impose slightly weaker requirement and then use that U is a distributional solution of (1) having bounded total variation. However, here for simplicity we employ conditions $\mathbf{R1}, \dots, \mathbf{R5}$. We refer to Theorem 1.1 in Christoforou and Spinolo [**6**] for a sharper (and more technical) formulation.

By comparing Theorem 4.1 with the uniqueness result by Bianchini [**2**] mentioned at the end of Section 2, one immediately gets that the novelty is the presence of requirement **R5**. It should be noted that in the case of initial-boundary value problems **R1**, ..., **R4** are no more sufficient to single out a unique solution: this can be seen by recalling that, under reasonable stability assumptions, the limit of any viscous approximation (4), (10) satisfies **R1**, ..., **R4**, but the limit varies when B varies.

From the technical point of view, requirement **R5** is taken into account by a careful analysis of system (12) and by using some techniques coming from the study of ordinary differential equations (ODEs), like center manifold techniques and center-stable manifold techniques. In Section 6, we outline the proof of Theorem 4.1.

Finally, Theorem 4.1 has applications to the analysis of the self-similar viscous approximation of a boundary Riemann problem. We discuss these in Section 5.

5 - Comparison between the physical and the self-similar viscous approximation of a boundary Riemann problem

In [**15**], Joseph and LeFloch studied the self-similar viscous approximation of a boundary Riemann problem. More precisely, they considered the family of equations (6) defined on the domain $t \in [0, +\infty[$, $x \in [0, +\infty[$ and coupled them with the initial and boundary data

$$U^{\varepsilon}(t, 0) = U_b \qquad U^{\varepsilon}(0, x) = U_0, \tag{14}$$

with $|U_b - U_0|$ sufficiently small. They focused on the case $B(U) \equiv I$ and they obtained compactness results. Also, they provided a careful analysis of the limit in both cases of a non characteristic and (under some further technical assumptions) of a characteristic boundary. The analysis was extended in Joseph and LeFloch [**16**] to other classes of viscosity matrices B.

In the general case, one expects that, if U^{ε} converges to a limit U, then the limit satisfies **R1**, ..., **R4**. However, as mentioned before, in the case of initial-boundary value problems these requirements are no more sufficient to single out a unique solution and hence it is not a priori clear that the limits of the self-similar and of the physical viscous approximation, if any, coincide.

However, the analysis in Joseph and LeFloch [**15**] ensures that, if the total variation of the family U^{ε} is uniformly bounded in ε (which is the case when $B \equiv I$, see [**15**]), then condition **R5** is satisfied.

As a direct application of the uniqueness result provided by Theorem 4.1, one gets that the limits of the physical and the self-similar viscous approximation are expected to coincide. See also Christoforou and Spinolo [**7**] for further results is this direction.

6 - Main ideas in the proof of Theorem 4.1

The main novelty of Theorem 4.1 is that it deals with the initial-boundary value problem and hence one needs to take into account possible boundary layers by introducing condition **R5**. In the following, we sketch how this requirement is handled, and we refer to [**6**] for the technical analysis and for the complete proof.

The key point to take into account condition **R5** is the analysis of system (12), for which we use techniques coming from the study of ordinary differential equations, like center manifold, stable manifold and center-stable manifold techniques.

In Section 6.1, we briefly go over the main results concerning these tools, and we refer to the books by Perko [**25**] and by Katok and Hasselblatt [**18**] for extended discussions. In Section 6.2, we outline how these techniques can serve the analysis of (12).

6.1 - *Invariant manifolds for ordinary differential equations*

Consider the ordinary differential equation (ODE)

$$\frac{dW}{dy} = G(W), \tag{15}$$

where $W(y) \in \mathbb{R}^d$, $y \in \mathbb{R}$ and $G : \mathbb{R}^d \to \mathbb{R}^d$ is a smooth function attaining the value zero at some point. With no loss of generality, we can assume that $\vec{0}$ is an equilibrium for the ODE (15), i.e. $G(\vec{0}) = \vec{0}$. Also, to avoid trivial cases we assume that the Jacobian matrix $DG(\vec{0})$ admits some eigenvalues with strictly negative real part and other eingevalues with zero real part.

The Stable Manifold Theorem, the Center Manifold Theorem and the Center-Stable Manifold Theorem respectively state the existence of manifolds $\mathcal{M}^s \subseteq \mathbb{R}^d$, $\mathcal{M}^c \subseteq \mathbb{R}^d$ and $\mathcal{M}^{cs} \subseteq \mathbb{R}^d$ that satisfy the following properties:

1. the dimension of a center manifold $\mathcal{M}^c$ is equal to the number of eigenvalues of $DG(\vec{0})$ having zero real part, the dimension of the stable manifold $\mathcal{M}^s$ is equal to the number of eigenvalues of $DG(\vec{0})$ having strictly negative real part and the dimension of a center-stable manifold $\mathcal{M}^{cs}$ is the sum of the previous two;
2. the three manifolds are all locally invariant for (15), meaning that, if $W_0 \in \mathcal{M}^c$ (respectively $W_0 \in \mathcal{M}^s$, $W_0 \in \mathcal{M}^{cs}$), then the solution of the Cauchy problem

$$\begin{cases} dW/dy = G(W) \\ W(0) = W_0 \end{cases} \tag{16}$$

satisfies $W(y) \in \mathcal{M}^c$ (respectively $W(y) \in \mathcal{M}^s$, $W(y) \in \mathcal{M}^{cs}$) if $|y|$ is small enough;
3. there exists a small enough ball, $B_r(\vec{0})$, centered at $\vec{0}$, such that, if $W(y) \in B_r(\vec{0})$ for every y, then $W(y) \in \mathcal{M}^c$ for every y;

4. there exists $c > 0$ such that, if $W_0 \in \mathcal{M}^s$, then the solution of the Cauchy problem (16) satisfies
$$\lim_{y \to +\infty} |W(y)| e^{cy} = 0;$$
5. there exists a small enough ball, $B_r(\vec{0})$, centered at $\vec{0}$, such that, if $W(y) \in B_r(\vec{0})$ for every $y > 0$, then $W(y) \in \mathcal{M}^{cs}$ for every y.

6.2 - *A decomposition result for boundary layers with small amplitude*

We now outline how condition **R5** is taken into account. We first address the non characteristic boundary case, which is technically easier and occurs when all the eigenvalues of the Jacobian matrix $DF(U)$ are bounded away from 0. In this case, **R5** is equivalent to the requirement that there exists a function W satisfying (11). By relying on some ODE analysis one can show that, being the Jacobian matrix $DF(\bar{U})$ a non singular matrix, any solution of the above system actually satisfies
$$\lim_{y \to +\infty} |W(y) - \bar{U}| e^{cy} = 0 \qquad \lim_{y \to +\infty} |W'(y)| e^{cy} = 0,$$
where $c > 0$ is a suitable constant depending on $DF(\bar{U})$. By applying the Stable Manifold Theorem, one can then show that system (11) admits a solution if and only if U_b lies on a suitable manifold having dimension equal to the number of eigenvalues of $DF(\bar{U})$ with strictly negative real part. It turns out that, in the non characteristic case, this is enough to conclude that the solution satisfying **R1**, ..., **R5** coincides with the one described in [4].

The boundary characteristic case occurs when an eingenvalue of the Jacobian matrix DF can attain the value zero. In this case the key point of the analysis is the following: given a value $\underline{U} \in \mathbb{R}^N$, we want to determine all the values U_b such that the system

$$\begin{cases} B(W)W'' = F(W) - F(\bar{U}) \\ U(0) = U_b \qquad \lim_{y \to +\infty} W(y) = \underline{U} \end{cases} \tag{17}$$

admits a solution with

$$|W(y) - \underline{U}| \leq C\delta \qquad |W'(y)| \leq C\delta, \tag{18}$$

where C and δ are the same constants as in **R5**. The difficulty in the analysis of (17) is taking into account the possibility that $W(y)$ converges to the equilibrium $\bar{U}$, but $|W(y) - \bar{U}|$ does not decay exponentially fast to $\bar{U}$. This behavior is ruled out in the non characteristic boundary case as a consequence of the fact that all the eigenvalues of DF are bounded away from 0.

As a first step in the analysis, we write (17) as a system of $2N$ first order ODEs. Then, we observe that without loss of generality we can focus on solutions lying on a suitable center-stable manifold because we look for solutions that both converge and satisfy bounds (18).

Note that, in general, a center-stable manifold may contain orbits that do not converge for $y \to +\infty$. On the other hand, in (17) we require the existence of the limit $\underline{U}$: to implement this condition we proceed as follows. In [**1**] and then [**4**] the authors constructed a converging solution W, lying on the center-stable manifold and having the following structure:

$$W(y) = W_s(y) + W_c(y) + W_p(y), \tag{19}$$

where W_s is the stable component and is exponentially decaying, W_c is the center component and converges to $\bar{U}$, but $|W(y) - \bar{U}|$, in general, does not decay exponentially fast to 0. Finally, W_p is a perturbation term, meaning that W_p is identically zero if either $W_s(y) \equiv 0$ or $W_c(y) \equiv \bar{U}$. Also, W_p is small with respect to W_s and W_c, namely

$$|W_p(y)| \leq \tilde{C}\delta^2 e^{-c/4y}.$$

Here, δ is the same constant as in **R5** and in (18) and $\tilde{C} > 0$ and $c > 0$ are positive constants independent of W_s and W_c. By relying on (19), one can then show that system (17) has a solution if U_b lies on a suitable set, depending on $\underline{U}$, which in the following is denoted by $\mathcal{D}(\bar{U})$.

Lemma 3.1 in Christoforou and Spinolo [**6**] states that actually *any* converging solution that satisfies (18) admits decomposition (19) and hence (17) has a solution *if and only if* $U_b \in \mathcal{D}(\bar{U})$. By relying on this result, one can eventually prove with some additional work that any distributional solution of (1) satisfying requirements $\mathbf{R1}, \dots, \mathbf{R5}$ coincides with the one described in [**4**]. This concludes the proof of Theorem 4.1.

Acknowledgments. This work was written when Spinolo was affiliated to the Institute of Mathematics of the University of Zurich, which she thanks for the nice hospitality. The authors kindly thank the organizers of the "Intensive Research Month on Hyperbolic Conservation Laws and Fluid Dynamics" held at the Department of Mathematics of the University of Parma, Italy, for the opportunity of presenting the results discussed in the present work in a talk given by the second author and for supporting her participation to the conference. Spinolo also thanks the Centro De Giorgi of Scuola Normale Superiore of Pisa, Italy, for partial support.

References

[**1**] F. Ancona and S. Bianchini, *Vanishing viscosity solutions of hyperbolic systems of conservation laws with boundary*, in "WASCOM 2005"—13th Conference on Waves and Stability in Continuous Media, World Sci. Publ., Hackensack, NJ 2006, 13-21.

[2] S. BIANCHINI, *On the Riemann problem for non-conservative hyperbolic systems*, Arch. Ration. Mech. Anal. **166** (2003), no. 1, 1-26.

[3] S. BIANCHINI and A. BRESSAN, *Vanishing viscosity solutions of nonlinear hyperbolic systems*, Ann. of Math. (2) **161** (2005), no. 1, 223-342.

[4] S. BIANCHINI and L. SPINOLO, *The boundary Riemann solver coming from the real vanishing viscosity approximation*, Arch. Ration. Mech. Anal. **191** (2009), no. 1, 1-96.

[5] A. BRESSAN, *Hyperbolic systems of conservation laws. The one-dimensional Cauchy problem*, Vol. 20 of "Oxford Lecture Series in Mathematics and its Applications", Oxford University Press, Oxford 2000.

[6] C. CHRISTOFOROU and L. SPINOLO, *A uniqueness criterion for viscous limits of boundary Riemann problems*, J. Hyperbolic Differ. Equ. 8 (2011), no. 3, 507-544.

[7] C. CHRISTOFOROU and L. SPINOLO, *Boundary layers for self-similar viscous approximations of nonlinear hyperbolic systems*, Quart. Appl. Math., to appear. Also arXiv:1106.5603.

[8] C. M. DAFERMOS, *Solution of the Riemann problem for a class of hyperbolic systems of conservation laws by the viscosity method*, Arch. Rational Mech. Anal. **52** (1973), 1-9.

[9] C. M. DAFERMOS, *Hyperbolic conservation laws in continuum physics*, 2nd ed., Vol. 325 of "Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]", Springer-Verlag, Berlin 2005.

[10] F. DUBOIS and P. LEFLOCH, *Boundary conditions for nonlinear hyperbolic systems of conservation laws*, J. Differential Equations **71** (1988), no. 1, 93-122.

[11] M. GISCLON, *Étude des conditions aux limites pour un système strictement hyperbolique, via l'approximation parabolique*, J. Math. Pures Appl. (9) **75** (1996), no. 5, 485-508.

[12] M. GISCLON and D. SERRE, *Étude des conditions aux limites pour un système strictement hyberbolique via l'approximation parabolique*, C. R. Acad. Sci. Paris Sér. I Math. **319** (1994), no. 4, 377-382.

[13] J. GLIMM, *Solutions in the large for nonlinear hyperbolic systems of equations*, Comm. Pure Appl. Math. **18** (1965), 697-715.

[14] H. K. JENSSEN, *Blowup for systems of conservation laws*, SIAM J. Math. Anal. **31** (2000), no. 4, 894-908 (electronic).

[15] K. T. JOSEPH and P. G. LEFLOCH, *Boundary layers in weak solutions of hyperbolic conservation laws. II. Self-similar vanishing diffusion limits*, Commun. Pure Appl. Anal. **1** (2002), no. 1, 51-76.

[16] K. T. JOSEPH and P. G. LEFLOCH, *Singular limits for the Riemann problem: general diffusion, relaxation, and boundary conditions*, in "Analytical approaches to multidimensional balance laws", O. S. Rozanova, ed., Nova Sci. Publ., New York 2006, 143-171.

[17] A. S. KALAŠNIKOV, *Construction of generalized solutions of quasi-linear equations of first order without convexity conditions as limits of solutions of parabolic equations with a small parameter*, Dokl. Akad. Nauk SSSR **127** (1959), 27-30.

[18] A. KATOK and B. HASSELBLATT, *Introduction to the modern theory of dynamical systems*, Vol. 54 of "Encyclopedia of Mathematics and its Applications", with a supplementary chapter by Katok and Leonardo Mendoza, Cambridge University Press, Cambridge 1995.

[19] S. KAWASHIMA and Y. SHIZUTA, *On the normal form of the symmetric hyperbolic-parabolic systems associated with the conservation laws*, Tohoku Math. J. (2) **40** (1988), 449-464.

[20] P. D. LAX, *Hyperbolic systems of conservation laws. II*, Comm. Pure Appl. Math. **10** (1957), 537-566.

[21] T. P. LIU, *The Riemann problem for general* 2×2 *conservation laws*, Trans. Amer. Math. Soc. **199** (1974), 89-112.

[22] T. P. LIU, *The Riemann problem for general systems of conservation laws*, J. Differential Equations **18** (1975), 218-234.

[23] T. P. LIU, *The entropy condition and the admissibility of shocks*, J. Math. Anal. Appl. **53** (1976), no. 1, 78-88.

[24] A. MAJDA and R. L. PEGO, *Stable viscosity matrices for systems of conservation laws*, J. Differential Equations **56** (1985), no. 2, 229-262.

[25] L. PERKO, *Differential equations and dynamical systems*, 3rd ed., Vol. 7 of "Texts in Applied Mathematics", Springer-Verlag, New York 2001.

[26] D. SERRE, *Systems of conservation laws. 1* and *2*, translated from the 1996 French original by I. N. Sneddon, Cambridge University Press, Cambridge, 1999 and 2000.

[27] V. A. TUPČIEV, *The problem of decomposition of an arbitrary discontinuity for a system of quasi-linear equations without the convexity condition*, Ž. Vyčisl. Mat. i Mat. Fiz. **6** (1966), 527-547.

[28] A. E. TZAVARAS, *Elastic as limit of viscoelastic response, in a context of self-similar viscous limits*, J. Differential Equations **123** (1995), no. 1, 305-341.

[29] A. E. TZAVARAS, *Wave interactions and variation estimates for self-similar zero-viscosity limits in systems of conservation laws*, Arch. Rational Mech. Anal. **135** (1996), no. 1, 1-60.

CLEOPATRA CHRISTOFOROU
Department of Mathematics and Statistics
University of Cyprus
1678 Nicosia, Cyprus
e-mail: Christoforou.Cleopatra@ucy.ac.cy
http://194.42.1.1/~kleopatr

LAURA V. SPINOLO
IMATI-CNR
via Ferrata 1
27100 Pavia, Italy
e-mail: spinolo@imati.cnr.it
http://www.imati.cnr.it/~spinolo/