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Global regularity for some MHD-α systems

Abstract. The global existence of strong solutions (for arbitrarily large initial data) to the incompressible Euler equations is a major open problem. This problem is open as well for the *ideal* MHD system, that is to say in the inviscid irresistive case, for both space dimension $n = 2$ or $n = 3$. We review some results, appeared in previous papers, concerning the global existence of regularized models for incompressible magnetofluids. In particular, we observe that a partial viscous (i.e., with positive kinematic viscosity and no magnetic resistivity) α-regularization (which yields a hyperbolic-parabolic system) is capable to provide strong global in time solvability for the ideal MHD system of equations in the 2D framework. In the more complex 3D case, we have strong global existence also for an ideal purely hyperbolic system, known as MHD-Voight model, when both the velocity and the magnetic fields are α-regularized, and when we regularize only the velocity, but the magnetic resistivity is strictly positive. If, in the latter case, we consider a double viscous model, we can get as well the existence of a unique compact global attractor and give estimates for its Hausdorff and fractal dimension. We will introduce the four different regularized magnetohydrodynamic models and motivate such a choice. In all cases, we will state the strong global existence and uniqueness result that we have obtained for the solution to the respective systems. Finally, we will give an idea of some proofs, referring to the original related papers for more details.

Keywords. Magnetohydrodynamics, MHD-α, simplified Bardina, MHD-Voight, regularizing MHD, turbulence, incompressible fluid, global existence, hyperbolic system, parabolic system.

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1 - Introduction

It is well-known, in literature, that the flow of an incompressible homogeneous magnetofluid subject to a forcing $\boldsymbol{f}$ is described by the following system (MHD),

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obtained by combining the Maxwell's equations, which rule the magnetic field, with the Navier–Stokes equations, which govern the fluid motion:

$$\boldsymbol{v}_t + (\boldsymbol{v} \cdot \nabla)\boldsymbol{v} - (\boldsymbol{B} \cdot \nabla)\boldsymbol{B} + \nabla p = \nu \Delta \boldsymbol{v} + \boldsymbol{f}\,, \tag{1a}$$

$$\boldsymbol{B}_t + (\boldsymbol{v} \cdot \nabla)\boldsymbol{B} - (\boldsymbol{B} \cdot \nabla)\boldsymbol{v} = \mu \Delta \boldsymbol{B}\,, \tag{1b}$$

$$\nabla \cdot \boldsymbol{v} = \nabla \cdot \boldsymbol{B} = 0\,, \tag{1c}$$

$$(\boldsymbol{v}, \boldsymbol{B})|_{t=0} = (\boldsymbol{v}_0, \boldsymbol{B}_0)\,, \qquad \boldsymbol{x} \in \mathbb{R}^n\,, \qquad n = 2, 3\,, \tag{1d}$$

where the fluid velocity field $\boldsymbol{v}(\boldsymbol{x}, t)$, the magnetic field $\boldsymbol{B}(\boldsymbol{x}, t)$ and the pressure $p(\boldsymbol{x}, t)$ are the unknowns, while $\nu \geq 0$ is the constant kinematic viscosity and $\mu \geq 0$ is the constant magnetic diffusivity, or resistivity (the constant density is assumed to be equal to 1), and $\boldsymbol{f}$ is a given forcing term.

This problem has been deeply studied. If $\nu > 0$ and $\mu > 0$, then there exists a unique global solution in time when $n = 2$, while for $n = 3$ the problem is still open, as discussed in [**22**], at least as to arbitrarily large initial data.

When $n = 2$, $\nu = 0$ and $\mu = 1$, local existence and small datum $\boldsymbol{B}_0$ global existence results have been established by Kozono [**17**] for bounded domains and by Casella-Secchi-Trebeschi [**6**] for unbounded domains.

When $n = 2$, $\nu = 1$ and $\mu = 0$, there is a regularity criterion for the solution $\boldsymbol{B}$ provided by Jiu-Niu [**14**]; but the problem in its generality is still open, even in the case $n = 2$.

As to global existence for arbitrarily large data, the case $n = 3$ is completely open, but there exist several regularity criteria (see for instance Zhou-Gala [**25**] and the references therein).

As pointed out in [**20**], at the moment, there is no possibility to compute the turbulent behavior of fluids neither analytically nor via direct numerical simulation (this task is prohibitively expensive and disputable as well due to sensitivity of perturbation errors in the initial data). Hence, one can try to focus only on certain statistical features of the physical phenomenon through the employment of suitable models. This is sufficient in many practical applications and, indeed, in the case of turbulence it makes more sense than pointwise evaluation. Actually, turbulent flows present a random character, which is filtered by averages.

Averaging is obtained through a filter ϕ_α, i.e. a smoothing kernel; in α-models, one special kernel is considered, the one associated to the Helmholtz operator:

$$\boldsymbol{v} = (1 - \alpha^2 \Delta)\boldsymbol{u}\,, \qquad \alpha > 0\,.$$

Then one can consider a regularized version of the previous equations, where the nonlinearity is made milder. Typically, some occurrences of $\boldsymbol{v}$ in the nonlinear terms are substituted by $\boldsymbol{u} = \phi_\alpha * \boldsymbol{v}$, so that $\boldsymbol{u} \to \boldsymbol{v}$ in a suitable sense as $\alpha \searrow 0$, thus, formally, the regularized system converges to the original standard one, up to a change in the pressure term. As a consequence, the solution becomes smoother. This phenomenon is in contrast with other approaches to regularization, such as hyperviscosity or nonlinear viscosity, which seem to have a greater impact on altering the properties of the solutions.

Moreover, this averaging suppresses any fluctuations in the flow data below $O(\alpha)$, which cause randomness and chaotic behavior, while preserves those on scales larger than $O(\alpha)$ (*large eddy simulation*), which possess a deterministic character.

Because of the remarkable success of the corresponding nonmagnetic models in producing solutions in excellent agreement with empirical data for a wide range of large Reynolds numbers (which correspond to a turbulent regime) and flow in infinite channels or pipes (see, for instance, [**5, 19**]), it is natural to consider such a kind of regularization also for magnetohydrodynamic models (in this case, also the magnetic field can be regularized, but this is in general not necessary). Several MHD-α models have been suggested and studied, for instance, in [**20, 13, 24**].

In this paper, we are interested in the global regularity of some MHD-α models that can be viewed as a method to approximate solutions to the ideal MHD model. We will consider four different cases.

We begin by introducing the Simplified Bardina MHD model (SBMHD):

$$\boldsymbol{v}_t + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} - (\boldsymbol{B}\cdot\nabla)\boldsymbol{B} + \nabla p = \nu\Delta\boldsymbol{v} + \boldsymbol{f} \qquad \text{in } [0,T]\times\Omega\,, \tag{2a}$$

$$\boldsymbol{B}_t + (\boldsymbol{u}\cdot\nabla)\boldsymbol{B} - (\boldsymbol{B}\cdot\nabla)\boldsymbol{u} = \mu\Delta\boldsymbol{B} \qquad \text{in } [0,T]\times\Omega\,, \tag{2b}$$

$$\boldsymbol{v} = (1-\alpha^2\Delta)\boldsymbol{u}\,, \qquad \alpha > 0 \qquad \text{in } [0,T]\times\Omega\,, \tag{2c}$$

$$\nabla\cdot\boldsymbol{u} = \nabla\cdot\boldsymbol{B} = 0 \qquad \text{in } [0,T]\times\Omega\,, \tag{2d}$$

$$(\boldsymbol{v},\boldsymbol{B})|_{t=0} = (\boldsymbol{v}_0,\boldsymbol{B}_0) \qquad \boldsymbol{x}\in\Omega\,, \tag{2e}$$

where $\alpha > 0$, $\nu,\mu \geq 0$, $\Omega = [0,2\pi L]^n$, $n = 2,3$, $L > 0$, with periodic, zero-mean, divergence free initial data and forcing term $\boldsymbol{f} = \boldsymbol{f}(\boldsymbol{x})$ (so that also the solutions have the same properties with respect to the space variable $\boldsymbol{x}$). The use of a periodic domain uncouples the difficulties arising from the boundary interaction from the ones stemming from the flow itself.

The (nonmagnetic) Simplified Bardina model has been suggested by Layton-Lewandowski in [**19**] and considered by Cao-Lunasin-Titi in [**5**]. The SBMHD has been introduced in Catania-Secchi [**10**], where the following results and further details can be found.

In the first case, we consider a 3D double-viscous model ($\nu,\mu > 0$, $n = 3$).

Theorem 1.1. *If* $(\boldsymbol{v}_0,\boldsymbol{B}_0)\in \mathrm{L}^2(\Omega)\times\mathrm{H}^1(\Omega)$ *and* $\boldsymbol{f}\in\mathrm{L}^2(\Omega)$ *then, for each* $T>0$, *there exists a unique solution* $(\boldsymbol{v},\boldsymbol{B})$ *such that*

$$\boldsymbol{v}\in \mathrm{L}^\infty(0,\infty;\mathrm{L}^2(\Omega))\cap \mathrm{L}^2(0,T;\mathrm{H}^1(\Omega))\,,$$
$$\boldsymbol{B}\in \mathrm{L}^\infty(0,\infty;\mathrm{H}^1(\Omega))\cap \mathrm{L}^2(0,T;\mathrm{H}^2(\Omega))\,.$$

This result is particularly useful in order to obtain what follows. From now on, to simplify notations, we set $\|\cdot\| = \|\cdot\|_{L^2(\Omega)}$. The subscript σ will be sometimes used to put in evidence that we are considering divergence-free zero-spatial-mean space-periodic (classes of) functions.

Theorem 1.2 (Finite Dimensional Global Attractor). *There is a (unique) compact global attractor* $\mathscr{A} \subset H^1_\sigma(\Omega) \times L^2_\sigma(\Omega)$ *in terms of the solution* $(\boldsymbol{u}, \boldsymbol{B})$ *to* (2). *Moreover, we have an upper bound for the Hausdorff dimension* $d_H(\mathscr{A})$ *and the fractal dimension* $d_F(\mathscr{A})$ *of the attractor* $\mathscr{A}$*; in particular, there is a positive constant C such that*

$$d_H(\mathscr{A}) \leq d_F(\mathscr{A}) \leq CG^{6/5}\left(\frac{L}{\alpha}\right)^3 \left[\left(\frac{L}{\alpha}\right)^{\frac{3}{5}} + G^{6/5}\left(\frac{L}{\alpha}\right)^{\frac{9}{5}} + G^{3/10}\right],$$

where, set $\eta = \min\{\nu, \mu\}$,

$$G = \frac{L^{3/2}\|\boldsymbol{f}\|}{\eta^2}$$

is the modified Grashof number.

We can interpret the estimate for the attractor dimension in term of the mean rate of energy dissipation, defined by

$$\bar{\varepsilon} = \frac{1}{L^3} \sup_{(\boldsymbol{u}_0, \boldsymbol{B}_0) \in \mathscr{A}} \limsup_{T \to \infty} \frac{1}{T} \int_0^T \left(\nu\|\nabla \boldsymbol{u}(t)\|^2 + \nu\alpha^2\|\Delta \boldsymbol{u}(t)\|^2 + \mu\|\nabla \boldsymbol{B}(t)\|^2\right) \mathrm{d}t\,.$$

Moreover, in analogy with Kolmogorov dissipation length in the classical theory of turbulence, we define the dissipation length as

$$\ell_d = \left(\frac{\eta^3}{\bar{\varepsilon}}\right)^{1/4},$$

so that

$$\begin{aligned} &\sup_{(\boldsymbol{u}_0, \boldsymbol{B}_0) \in \mathscr{A}} \limsup_{T \to \infty} \frac{1}{T} \int_0^T \left(\nu\|\nabla \boldsymbol{u}(t)\|^2 + \nu\alpha^2\|\Delta \boldsymbol{u}(t)\|^2 + \mu\|\nabla \boldsymbol{B}(t)\|^2\right) \mathrm{d}t \\ &\quad = \frac{L^3\eta^3}{\ell_d^4}\,. \end{aligned} \tag{3}$$

Theorem 1.3. *The unique compact global attractor* $\mathscr{A} \subset \mathrm{H}^1_\sigma(\Omega) \times \mathrm{L}^2_\sigma(\Omega)$ *in terms of the solution* $(\boldsymbol{u}, \boldsymbol{B})$ *to* (2) *has fractal dimension* $d_{\mathrm{F}}(\mathscr{A})$ *bounded by*

$$D \doteq C \max \left\{ \left(\frac{L}{\alpha}\right)^{12/5} \left(\frac{L}{\ell_d}\right)^{12/5}, \left(\frac{L}{\alpha}\right)^{3/2} \left(\frac{L}{\ell_d}\right)^{3} \right\},$$

where C *is a positive constant.*

Identifying the dimension of the global attractor with the number of degrees of freedom of the long-time dynamics of the solution, this means that the number of degrees of freedom of problem (2) is bounded from above by a quantity which scales like D. This information is useful to establish the validity of the model as a large-eddy simulation model of turbulence. Similar results are provided in Catania [8] for different MHD-α models. Moreover, in Catania [**9**], it is proved that, in the SBMHD model, the modified Grashof number can be estimated from above by the square of the modified Reynolds number, the quantity usually considered in turbulence theory. Using this relation, an upper bound for $d_{\mathrm{F}}(\mathscr{A})$ and estimates for higher order wave numbers (which take into account intermittency), both in terms of the Reynolds number, are proved.

Now, let us consider the 3D SBMHD with magnetic diffusivity $\mu > 0$ but no kinematic viscosity (and $\boldsymbol{f} \equiv \mathbf{0}$ for simplicity); the equations for $\boldsymbol{v}_t$ and $\boldsymbol{B}_t$ become:

(4a) $$\boldsymbol{v}_t + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} - (\boldsymbol{B} \cdot \nabla)\boldsymbol{B} + \nabla p = \mathbf{0} \qquad \text{in } [0,T] \times \Omega\,,$$

(4b) $$\boldsymbol{B}_t + (\boldsymbol{u} \cdot \nabla)\boldsymbol{B} - (\boldsymbol{B} \cdot \nabla)\boldsymbol{u} = \mu \Delta \boldsymbol{B} \qquad \text{in } [0,T] \times \Omega\,.$$

We take $\Omega = [0, 2\pi L]^3$ and consider space-periodic and zero spatial-mean initial data. Under the aforementioned conditions, we have the following result.

Theorem 1.4 (Strong Global Existence). *As to the initial data, we assume that they satisfy* $\boldsymbol{v}_0 \in \mathrm{L}^2(\Omega)$, $\boldsymbol{B}_0 \in \mathrm{H}^1(\Omega)$ *and* $\nabla \cdot \boldsymbol{v}_0 = \nabla \cdot \boldsymbol{B}_0 = 0$.

Then, problem (4) *has a unique global solution* $(\boldsymbol{v}, \boldsymbol{B})$ *such that, for each time* $T > 0$, *one has*

$$\boldsymbol{v} \in \mathrm{L}^\infty(0,T; \mathrm{L}^2(\Omega))\,, \qquad \boldsymbol{B} \in \mathrm{L}^\infty(0,T; \mathrm{H}^1(\Omega)) \cap \mathrm{L}^2(0,T; \mathrm{H}^2(\Omega))\,.$$

This result is shown in [**11**]. It says that we can get global existence for the SBMHD even in the case $\nu = 0$ (but one can not obtain the global attractor existence and related results).

In the third case, we study the irresistive 2D SBMHD (we can assume $\nu = 1$ with no loss of generality, and $\boldsymbol{f} \equiv \mathbf{0}$ for simplicity), so that $\Omega = [0, 2\pi L]^2 \subset \mathbb{R}^2$ and the equations for $\boldsymbol{v}_t$ and $\boldsymbol{B}_t$ become:

$$\boldsymbol{v}_t + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} - (\boldsymbol{B}\cdot\nabla)\boldsymbol{B} + \nabla p = \Delta\boldsymbol{v} \qquad \text{in } [0,T]\times\Omega\,, \tag{5a}$$

$$\boldsymbol{B}_t + (\boldsymbol{u}\cdot\nabla)\boldsymbol{B} - (\boldsymbol{B}\cdot\nabla)\boldsymbol{u} = \boldsymbol{0} \qquad \text{in } [0,T]\times\Omega\,. \tag{5b}$$

Theorem 1.5 (Local Existence). *Assume that the initial data satisfy*

$$\boldsymbol{v}_0 \in \mathrm{H}^m(\Omega)\,, \qquad \boldsymbol{B}_0 \in \mathrm{H}^{m+2}(\Omega)$$

and $\nabla\cdot\boldsymbol{v}_0 = \nabla\cdot\boldsymbol{B}_0 = 0$, *where* $m \geq 2$ *is an integer number.*

Then, there exists a positive time T_0 *such that problem* (5) *has a unique solution* $(\boldsymbol{v},\boldsymbol{B})$ *so that*

$$\begin{aligned}&\boldsymbol{v} \in \mathrm{L}^\infty(0,T_0;\mathrm{H}^m(\Omega)) \cap \mathrm{L}^1(0,T_0;\mathrm{H}^{m+1}(\Omega)) \cap \mathrm{L}^2(0,T_0;\mathrm{H}^2(\Omega))\,,\\ &\boldsymbol{B} \in \mathrm{L}^\infty(0,T_0;\mathrm{H}^{m+2}(\Omega))\,.\end{aligned}$$

Moreover, $\boldsymbol{v}_t \in \mathrm{L}^2(0,T_0;\mathrm{L}^2(\Omega))$, *the pressure* p *is uniquely defined up to an additive function independent of space and* $\nabla p \in \mathrm{L}^2(0,T_0;\mathrm{L}^2(\Omega))$.

Proposition 1.1 (Energy Estimate). *Assume that a solution* $(\boldsymbol{v},\boldsymbol{B})$ *of problem* (5) *is defined in the time interval* $[0,T]$.

Then, the following energy estimate holds:

$$\|\boldsymbol{u}\|^2 + \alpha^2\|\nabla\boldsymbol{u}\|^2 + \|\boldsymbol{B}\|^2 + 2\int_0^T (\|\nabla\boldsymbol{u}(t)\|^2 + \alpha^2\|\Delta\boldsymbol{u}(t)\|^2)\mathrm{d}t \leq C_0\,, \tag{6}$$

where

$$C_0 = \|\boldsymbol{u}(0)\|^2 + \alpha^2\|\nabla\boldsymbol{u}(0)\|^2 + \|\boldsymbol{B}(0)\|^2 \geq 0 \tag{7}$$

is independent of time T.

Moreover, one has

$$\int_0^T \|\boldsymbol{v}(t)\|^2\mathrm{d}t \leq C_0(2T + \alpha^2)\,. \tag{8}$$

Theorem 1.6 (Global Existence). *Assume that the initial data satisfy*

$$\boldsymbol{v}_0 \in \mathrm{H}^1(\Omega)\,, \qquad \boldsymbol{B}_0 \in \mathrm{H}^3(\Omega)\,,$$

$\nabla\cdot\boldsymbol{v}_0 = \nabla\cdot\boldsymbol{B}_0 = 0$.

Then, problem (5) *has a unique global solution* $(\boldsymbol{v},\boldsymbol{B})$ *such that, for each time* $T > 0$, *one has*

$$\boldsymbol{v} \in \mathrm{L}^\infty(0,T;\mathrm{H}^1(\Omega)) \cap \mathrm{L}^2(0,T;\mathrm{H}^2(\Omega))\,, \quad \boldsymbol{B} \in \mathrm{L}^\infty(0,T;\mathrm{H}^3(\Omega))\,.$$

In other words, in two space-dimension, we can prove global existence and uniqueness for the SBMHD even if $\mu = 0$.

Remark 1.1. 1. *These results are shown in* [**7**], *where the case* $\Omega = \mathbb{R}^2$ *is considered; nevertheless, all results can be obtained in the same* (*if not easier*) *way for the torus, just resorting to results for bounded domains that we will cite when needed during the proof for global existence* (*see, in particular, the estimates* (14) *and* (16)).

2. *Actually, the proof for global existence shows as well that we can assume just* $\boldsymbol{B}_0 \in \mathrm{H}^2(\Omega)$ *and conclude* $\boldsymbol{B} \in \mathrm{L}^\infty(0,T;\mathrm{H}^2(\Omega))$.

3. *The local existence and uniqueness result for* $\boldsymbol{v} \in \mathrm{L}^\infty\mathrm{H}^1 \cap \mathrm{L}^2\mathrm{H}^2$ *and* $\boldsymbol{B} \in \mathrm{L}^\infty\mathrm{H}^3$ (*or* $\boldsymbol{B} \in \mathrm{L}^\infty\mathrm{H}^2$) *can be obtained from Theorem* 1.5 *by a density argument used to approximate the initial data in the required spaces.*

4. *Similar results can be proved for the inviscid resistive* (*dissipation for* $\boldsymbol{B}$ *but not for* $\boldsymbol{v}$) *2D SBMHD, however we find that the case that we have considered* (*irresistive*) *is the most interesting one, since for the 2D inviscid case we have the results of Kozono* [**17**] *and Casella-Secchi-Trebeschi* [**6**] (*even if in those papers* $\boldsymbol{B}_0$ *is small*).

Eventually, we consider the case without viscosity nor diffusivity, but with regularizations both in the velocity $\boldsymbol{v}$ and the magnetic field $\boldsymbol{B}$; the following model is known as MHD-Voight:

$$\boldsymbol{v}_t + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} - (\boldsymbol{b}\cdot\nabla)\boldsymbol{b} + \nabla p = \boldsymbol{0} \qquad \text{in } [0,T]\times\Omega\,, \tag{9a}$$

$$\boldsymbol{B}_t + (\boldsymbol{u}\cdot\nabla)\boldsymbol{b} - (\boldsymbol{b}\cdot\nabla)\boldsymbol{u} = \boldsymbol{0} \qquad \text{in } [0,T]\times\Omega\,, \tag{9b}$$

$$\boldsymbol{v} = (1-\alpha^2\Delta)\boldsymbol{u}\,,\ \boldsymbol{B} = (1-\beta^2\Delta)\boldsymbol{b}\,, \qquad \alpha,\beta>0 \qquad \text{in } [0,T]\times\Omega\,, \tag{9c}$$

$$\nabla\cdot\boldsymbol{u} = \nabla\cdot\boldsymbol{b} = 0 \qquad \text{in } [0,T]\times\Omega\,, \tag{9d}$$

$$(\boldsymbol{v},\boldsymbol{B})|_{t=0} = (\boldsymbol{v}_0,\boldsymbol{B}_0) \qquad \boldsymbol{x}\in\Omega\,. \tag{9e}$$

Let us note that a pressure-gradient term can be included in the equation for $\boldsymbol{B}_t$ as well. We assume that $\Omega = [0,2\pi L]^3 \subset \mathbb{R}^3$ and consider space-periodic and zero spatial-mean initial data. Under these conditions, we have the following global existence results.

Theorem 1.7 (Weak Global Existence for the MHD-Voight). *Let us set*

$$\boldsymbol{u}_0 = (1-\alpha^2\Delta)^{-1}\boldsymbol{v}_0\,, \qquad \boldsymbol{b}_0 = (1-\beta^2\Delta)^{-1}\boldsymbol{B}_0\,,$$

and assume that $\boldsymbol{u}_0$, $\boldsymbol{b}_0 \in \mathrm{H}^1(\Omega)$ *and* $\nabla\cdot\boldsymbol{u}_0 = \nabla\cdot\boldsymbol{b}_0 = 0$.

Then, problem (9) *has a unique global solution* $(\boldsymbol{u}, \boldsymbol{b})$ *such that*

$$\boldsymbol{u}\,, \boldsymbol{b} \in \mathrm{L}^{\infty}(0, \infty; \mathrm{H}^1(\Omega))\,.$$

The couple $(\boldsymbol{u}, \boldsymbol{b})$ *is a* weak solution *of* (9) *in the sense of* (21) (*see next section*).

Theorem 1.8 (Strong Global Existence for the MHD-Voight). *If we assume initial data satisfying* $\boldsymbol{v}_0, \boldsymbol{B}_0 \in \mathrm{L}^2(\Omega)$ *and* $\nabla \cdot \boldsymbol{v}_0 = \nabla \cdot \boldsymbol{B}_0 = 0$, *then problem* (9) *has a unique global solution* $(\boldsymbol{v}, \boldsymbol{B})$ *such that, for each time* $T > 0$, *one has*

$$\boldsymbol{v}\,, \boldsymbol{B} \in \mathrm{L}^{\infty}(0, T; \mathrm{L}^2(\Omega))\,.$$

These results have been obtained in [**11**] ([1]).

Let us note that this model is particularly interesting since it preserves three physical quantities, that is to say the energy $E^{\alpha,\beta} = 1/2 \int_{\Omega} \big(\boldsymbol{v}(\boldsymbol{x}) \cdot \boldsymbol{u}(\boldsymbol{x}) + \boldsymbol{B}(\boldsymbol{x}) \cdot \boldsymbol{b}(\boldsymbol{x})\big) \mathrm{d}\boldsymbol{x}$, the cross helicity $H_{\mathrm{C}}^{\alpha,\alpha} = 1/2 \int_{\Omega} \big(\boldsymbol{u}(\boldsymbol{x}) \cdot \boldsymbol{b}(\boldsymbol{x}) + \alpha^2 \nabla \boldsymbol{u}(\boldsymbol{x}) \cdot \nabla \boldsymbol{b}(\boldsymbol{x})\big) \mathrm{d}\boldsymbol{x}$ (here we are assuming $\beta = \alpha$, which is absolutely reasonable) and the magnetic helicity $H_{\mathrm{M}}^{\alpha,\beta} = 1/2 \int_{\Omega} \big(\boldsymbol{a}(\boldsymbol{x}) \cdot \boldsymbol{b}(\boldsymbol{x}) + \beta^2 \nabla \boldsymbol{a}(\boldsymbol{x}) \cdot \nabla \boldsymbol{b}(\boldsymbol{x})\big) \mathrm{d}\boldsymbol{x}$, where $\boldsymbol{a}$ is a vector potential, so that $\boldsymbol{b} = \nabla \times \boldsymbol{a}$. Moreover, as $\alpha, \beta \to 0$, these quantities reduce to the corresponding conserved ideal quadratic invariants of the MHD equations. Note that the ideal version of the SBMHD (2) conserves the energy and the magnetic helicity, but at the moment we are unable to find an invariant quantity corresponding to cross helicity.

Moreover, system (9) is damped hyperbolic (with no viscosities), and it can be considered as a first example of ideal model useful to approximate solutions to the ideal MHD for $\alpha, \beta \to 0$ (this is different from large eddy simulation).

2 - Some proofs

In this section, we will prove the global existence result relative to the 2D irresistive SBMHD (for the local existence and other details, see Catania [**7**]). Afterwards, the proofs concerning the 3D MHD-Voight model are given in detail (as done in [**11**]), while we will not give any information (besides what said in the introduction) related to the 3D inviscid SBMHD (whose results are proved in [**11**] as well) or to the 3D double viscous SBMHD (whose results are proved in [**10**]).

([1]) After the submission of the paper [**11**], we were informed of the work by Larios-Titi [**18**], whose preprint appeared just two days **after** ours, and that contains higher order regularity results as well.

We will make use of the following identities, which hold provided $\nabla \cdot \boldsymbol{f} = 0$:

$$\int (\boldsymbol{f} \cdot \nabla)\boldsymbol{g} \cdot \boldsymbol{h} \mathrm{d}\boldsymbol{x} = -\int (\boldsymbol{f} \cdot \nabla)\boldsymbol{h} \cdot \boldsymbol{g} \mathrm{d}\boldsymbol{x}\,, \tag{10}$$

$$\int (\boldsymbol{f} \cdot \nabla)\boldsymbol{g} \cdot \boldsymbol{g} \mathrm{d}\boldsymbol{x} = 0\,. \tag{11}$$

2.1 - *2D irresistive SBMHD*

In order to get global existence, we recall that, for each two-dimensional vector $\boldsymbol{w} = (w_1, w_2)$, we can define the scalar quantity

$$\operatorname{rot} \boldsymbol{w} = \partial_{x_1} w_2 - \partial_{x_2} w_1.$$

Setting $\omega = \operatorname{rot} \boldsymbol{v}$ and applying rot to equation (5a), we get

$$\omega_t + (\boldsymbol{u} \cdot \nabla) \operatorname{rot} \boldsymbol{u} - (\boldsymbol{B} \cdot \nabla) \operatorname{rot} \boldsymbol{B} = \varDelta\omega\,. \tag{12}$$

Taking the scalar product with ω and integrating in space, we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int \omega^2 \mathrm{d}\boldsymbol{x} + \int |\nabla\omega|^2 \mathrm{d}\boldsymbol{x} = \int [(\boldsymbol{B} \cdot \nabla) \operatorname{rot} \boldsymbol{B}]\omega \mathrm{d}\boldsymbol{x} - \int [(\boldsymbol{u} \cdot \nabla) \operatorname{rot} \boldsymbol{u}]\omega \mathrm{d}\boldsymbol{x}\,. \tag{13}$$

Let us recall the following estimates in [**12**] (see [**1, 2**] for the case of bounded domains), where $\mathcal{H}^1$ is the Hardy space, BMO denotes the bounded mean-oscillation function space (dual to $\mathcal{H}^1$), while $\nabla \cdot \boldsymbol{f} = 0$ and $\operatorname{rot} \boldsymbol{g} = 0$:

$$\|\boldsymbol{f} \cdot \boldsymbol{g}\|_{\mathcal{H}^1} \le C\|\boldsymbol{f}\|\,\|\boldsymbol{g}\|\,, \tag{14}$$

$$\|\omega\|_{\mathrm{BMO}} \le C\|\nabla\omega\|\,; \tag{15}$$

from (6) and (13), we deduce

$$\begin{aligned}
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int \omega^2 \mathrm{d}\boldsymbol{x} + \int |\nabla\omega|^2 \mathrm{d}\boldsymbol{x} &\le (\|(\boldsymbol{B} \cdot \nabla) \operatorname{rot} \boldsymbol{B}\|_{\mathcal{H}^1} + \|(\boldsymbol{u} \cdot \nabla) \operatorname{rot} \boldsymbol{u}\|_{\mathcal{H}^1})\|\omega\|_{\mathrm{BMO}} \\
&\le C(\|\boldsymbol{B}\|\,\|\nabla \operatorname{rot} \boldsymbol{B}\|\,\|\nabla\omega\| + \|\boldsymbol{u}\|\,\|\nabla \operatorname{rot} \boldsymbol{u}\|\,\|\nabla\omega\|) \\
&\le C(\|\varDelta\boldsymbol{B}\|\,\|\nabla\omega\| + \|\varDelta\boldsymbol{u}\|\,\|\nabla\omega\|) \\
&\le \frac{1}{4}\|\nabla\omega\|^2 + C(\|\varDelta\boldsymbol{B}\|^2 + \|\varDelta\boldsymbol{u}\|^2)\,.
\end{aligned}$$

We combine this result with the estimate proved in [**13**]

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int |\varDelta\boldsymbol{B}|^2 \mathrm{d}\boldsymbol{x} \\
&\quad\le C\Big[\|\boldsymbol{v}\| \ln(\mathrm{e} + \|\omega\| + \|\varDelta\boldsymbol{B}\|) + \|\boldsymbol{v}\|^{6/7} + \|\boldsymbol{v}\|^{2/3}\Big]\,\|\varDelta\boldsymbol{B}\|^2 + \frac{1}{2}\|\nabla\omega\|^2\,.
\end{aligned}$$

Let us remark that this inequality relies in particular on the estimates due to Kato-Ponce [15] and Kenig-Ponce-Vega [16]

$$\begin{aligned}
&\|(-\Delta)^{a/2}(\boldsymbol{f}\cdot\boldsymbol{g}) - \boldsymbol{f}\cdot(-\Delta)^{a/2}\boldsymbol{g}\|_{\mathrm{L}^p}\\
&\qquad \le C\big(\|\nabla\boldsymbol{f}\|_{\mathrm{L}^{p_1}}\|(-\Delta)^{(a-1)/2}\boldsymbol{g}\|_{\mathrm{L}^{q_1}} + \|(-\Delta)^{a/2}\boldsymbol{f}\|_{\mathrm{L}^{p_2}}\|\boldsymbol{g}\|_{\mathrm{L}^{q_2}}\big)\,,\\
&\|(-\Delta)^{a/2}(\boldsymbol{f}\cdot\boldsymbol{g})\|_{\mathrm{L}^p} \le C\big(\|\boldsymbol{f}\|_{\mathrm{L}^{p_1}}\|(-\Delta)^{a/2}\boldsymbol{g}\|_{\mathrm{L}^{q_1}} + \|(-\Delta)^{a/2}\boldsymbol{f}\|_{\mathrm{L}^{p_2}}\|\boldsymbol{g}\|_{\mathrm{L}^{q_2}}\big)\,,\\
&\text{for each}\qquad a>0\,,\qquad \frac{1}{p}=\frac{1}{p_i}+\frac{1}{q_i}\,,\qquad i=1,2\,,
\end{aligned}$$

and on the logarithmic Sobolev inequality due to Brézis-Gallouet [3] and Brézis-Wainger [4] (see [21] for a remark on bounded domains)

$$\|\boldsymbol{f}\|_{\mathrm{L}^\infty} \le C\|\boldsymbol{f}\|_{\mathrm{H}^1}\ln(\mathrm{e}+\|\boldsymbol{f}\|_{\mathrm{H}^2})\,. \tag{16}$$

Combining the two previous estimates, we have

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\mathrm{e}+\|\omega\|^2+\|\Delta\boldsymbol{B}\|^2)+\frac{1}{4}\|\nabla\omega\|^2\\
&\qquad \le C(1+\|u\|^2+\|v\|^2)\ln(\mathrm{e}+\|\omega\|+\|\Delta\boldsymbol{B}\|)\big[\mathrm{e}+\|\omega\|^2+\|\Delta\boldsymbol{B}\|^2\big]\,,
\end{aligned}$$

since $\|\Delta\boldsymbol{u}\|^2 \le 2\alpha^{-4}(\|u\|^2+\|v\|^2)$.

The above inequality has form

$$y'(t)+\|\nabla\omega\|^2 \le C(1+\|\boldsymbol{u}\|^2+\|\boldsymbol{v}\|^2)y(t)\ln y(t)\,, \tag{17}$$

with $y=\mathrm{e}+\|\omega\|^2+\|\Delta\boldsymbol{B}\|^2$. This implies

$$\int_{y_0}^{y}\frac{\mathrm{d}\bar{y}}{\bar{y}\ln\bar{y}} \le C\int_0^t(1+\|\boldsymbol{u}\|^2+\|\boldsymbol{v}\|^2)\mathrm{d}\bar{t}\,,$$

that is to say

$$\ln\,\ln y - \ln\ln y_0 \le C(t+1)$$

because of (6) and (8), and finally

$$y(t) \le \mathrm{e}^{C\mathrm{e}^{C(T+1)}}$$

for each $t\in[0,T]$; thus we can conclude that

$$\|\omega\|_{\mathrm{L}^\infty(0,T;\mathrm{L}^2)}+\|\Delta\boldsymbol{B}\|_{\mathrm{L}^\infty(0,T;\mathrm{L}^2)} \le C(T)\,.$$

Now, using the Gagliardo-Nirenberg inequality $\|\nabla\boldsymbol{B}\| \le C\|\boldsymbol{B}\|^{1/2}\|\Delta\boldsymbol{B}\|^{1/2}$, we deduce immediately

$$\|\boldsymbol{B}\|_{\mathrm{L}^\infty(0,T;\mathrm{H}^2)} \le C(T)\,.$$

Moreover, from (17) we get $\|\nabla\omega\|_{L^2(0,T;L^2)} \leq C(T)$ and therefore

$$\|\omega\|_{L^\infty(0,T;L^2)} + \|\omega\|_{L^2(0,T;H^1)} \leq C(T)\,,$$

which implies

$$\|\boldsymbol{v}\|_{L^\infty(0,T;H^1)} + \|\boldsymbol{v}\|_{L^2(0,T;H^2)} \leq C(T)\,, \tag{18}$$

$$\|\boldsymbol{u}\|_{L^\infty(0,T;H^3)} + \|\boldsymbol{u}\|_{L^2(0,T;H^4)} \leq C(T)\,. \tag{19}$$

Again as in [**13**], we get the inequality

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\partial^3\boldsymbol{B}\|^2 \leq C(T)[\|\partial^3\boldsymbol{B}\|^2 + (1+\|\partial^3\boldsymbol{B}\|)\|\partial^3\boldsymbol{B}\| + \|\partial^4\boldsymbol{u}\|\,\|\partial^3\boldsymbol{B}\|]\,,$$

where ∂^3 denotes a generical partial derivative in space of order 3. Setting $z(t) = (1+\|\partial^3\boldsymbol{B}(t)\|^2)$, we deduce

$$z'(t) \leq C(T)z(t)$$

thanks to (19). We obtain that

$$(1+\|\partial^3\boldsymbol{B}(t)\|)^2 \leq (1+\|\partial^3\boldsymbol{B}_0\|)^2 \mathrm{e}^{TC(T)}$$

for each $t \in [0,T]$, thus

$$\|\boldsymbol{B}\|_{L^\infty(0,T;H^3)} \leq C(T)\,. \tag{20}$$

The completion of the proof of Theorem 1.6 follows by a standard argument based on the Local Existence Theorem 1.5, the above a priori estimates, and the energy inequality.

2.2 - *3D MHD-Voight*

As to the local existence and uniqueness of a weak solution, let us note that we can restate system (9) in the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix}\boldsymbol{u}\\ \boldsymbol{b}\end{pmatrix} = F(\boldsymbol{u},\boldsymbol{b}) \doteq \begin{pmatrix}(1-\alpha^2\Delta)^{-1}[\mathscr{B}(\boldsymbol{b},\boldsymbol{b})-\mathscr{B}(\boldsymbol{u},\boldsymbol{u})]\\ (1-\beta^2\Delta)^{-1}[\mathscr{B}(\boldsymbol{b},\boldsymbol{u})-\mathscr{B}(\boldsymbol{u},\boldsymbol{b})]\end{pmatrix}, \tag{21}$$

where $\mathscr{B}(\boldsymbol{f},\boldsymbol{g}) = P[\boldsymbol{f}\cdot\nabla\boldsymbol{g}]$, P denoting the Helmholtz-Leray projection over the divergence free functions of L^2. Then $(\boldsymbol{u},\boldsymbol{b})$ will be a weak solution of (9) provided that it is a solution of (21).

We want to prove that the operator F is locally Lipschitz in H^1 equipped with the scalar product

$$\left\langle\begin{pmatrix}\boldsymbol{u}\\ \boldsymbol{b}\end{pmatrix},\begin{pmatrix}\boldsymbol{w}\\ \boldsymbol{d}\end{pmatrix}\right\rangle = \Big((1-\alpha^2\Delta)^{1/2}\boldsymbol{u},(1-\alpha^2\Delta)^{1/2}\boldsymbol{w}\Big)_{L^2} + \Big((1-\beta^2\Delta)^{1/2}\boldsymbol{b},(1-\beta^2\Delta)^{1/2}\boldsymbol{d}\Big)_{L^2}.$$

We have

$$F_1 \doteq \|\mathscr{B}(\boldsymbol{b}_1,\boldsymbol{b}_1) - \mathscr{B}(\boldsymbol{u}_1,\boldsymbol{u}_1) - \mathscr{B}(\boldsymbol{b}_2,\boldsymbol{b}_2) + \mathscr{B}(\boldsymbol{u}_2,\boldsymbol{u}_2)\|_{\mathrm{H}^{-1}}$$
$$= \|\mathscr{B}(\boldsymbol{b}_1,\boldsymbol{b}_1-\boldsymbol{b}_2) + \mathscr{B}(\boldsymbol{b}_1-\boldsymbol{b}_2,\boldsymbol{b}_2) - \mathscr{B}(\boldsymbol{u}_1,\boldsymbol{u}_1-\boldsymbol{u}_2) - \mathscr{B}(\boldsymbol{u}_1-\boldsymbol{u}_2,\boldsymbol{u}_2)\|_{\mathrm{H}^{-1}}$$

and also

$$\|\mathscr{B}(\boldsymbol{f},\boldsymbol{g})\|_{\mathrm{H}^{-1}} \le \sup_{\|\nabla \boldsymbol{h}\|=1}\left|\int (\boldsymbol{f}\cdot\nabla\boldsymbol{g})\cdot\boldsymbol{h}\right| = \sup_{\|\nabla \boldsymbol{h}\|=1}\left|\int (\boldsymbol{f}\cdot\nabla\boldsymbol{h})\cdot\boldsymbol{g}\right|$$
$$\le \|\nabla\boldsymbol{h}\|\,\|\boldsymbol{f}\|_{\mathrm{L}^6}\|\boldsymbol{g}\|_{\mathrm{L}^3} \le C\|\nabla\boldsymbol{f}\|\,\|\boldsymbol{g}\|^{1/2}\|\nabla\boldsymbol{g}\|^{1/2}$$
$$\le C\|\nabla\boldsymbol{f}\|\,\|\nabla\boldsymbol{g}\|\,,$$

having used the Hölder inequality,

$$\|\boldsymbol{f}\|_{\mathrm{L}^6} \le C\|\nabla\boldsymbol{f}\|\,, \qquad \|\boldsymbol{g}\|_{\mathrm{L}^3} \le C\|\boldsymbol{g}\|^{1/2}\|\nabla\boldsymbol{g}\|^{1/2}\,,$$

and the Poincaré inequality. Thus we easily get

$$F_1 \le C(\|\nabla\boldsymbol{u}_1\| + \|\nabla\boldsymbol{u}_2\| + \|\nabla\boldsymbol{b}_1\| + \|\nabla\boldsymbol{b}_2\|)(\|\nabla(\boldsymbol{u}_1-\boldsymbol{u}_2)\| + \|\nabla(\boldsymbol{b}_1-\boldsymbol{b}_2)\|)\,.$$

Similarly,

$$F_2 \doteq \|\mathscr{B}(\boldsymbol{b}_1,\boldsymbol{u}_1) - \mathscr{B}(\boldsymbol{u}_1,\boldsymbol{b}_1) - \mathscr{B}(\boldsymbol{b}_2,\boldsymbol{u}_2) + \mathscr{B}(\boldsymbol{u}_2,\boldsymbol{b}_2)\|_{\mathrm{H}^{-1}}$$
$$= \|\mathscr{B}(\boldsymbol{b}_1,\boldsymbol{u}_1-\boldsymbol{u}_2) + \mathscr{B}(\boldsymbol{b}_1-\boldsymbol{b}_2,\boldsymbol{u}_2) - \mathscr{B}(\boldsymbol{u}_1,\boldsymbol{b}_1-\boldsymbol{b}_2) - \mathscr{B}(\boldsymbol{u}_1-\boldsymbol{u}_2,\boldsymbol{b}_2)\|_{\mathrm{H}^{-1}}$$
$$\le C(\|\nabla\boldsymbol{u}_1\| + \|\nabla\boldsymbol{u}_2\| + \|\nabla\boldsymbol{b}_1\| + \|\nabla\boldsymbol{b}_2\|)(\|\nabla(\boldsymbol{u}_1-\boldsymbol{u}_2)\| + \|\nabla(\boldsymbol{b}_1-\boldsymbol{b}_2)\|)\,.$$

Hence F is locally Lipschitz, using that $(1-\alpha^2\Delta)^{-1}$ is an isomorphism from H^{-1} onto H^1, and consequently we get the local existence and uniqueness of a weak solution through the Cauchy-Lipschitz theorem.

Second, in order to get an energy identity, we take the scalar product in H^1 (previously defined) of (21) with $(\boldsymbol{u},\boldsymbol{b})$. Using (11), (10) and integrating by parts when needed, we deduce the energy equality

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\|\boldsymbol{u}\|^2 + \alpha^2\|\nabla\boldsymbol{u}\|^2 + \|\boldsymbol{b}\|^2 + \beta^2\|\nabla\boldsymbol{b}\|^2\right) = 0\,,$$

or

$$\|\boldsymbol{u}\|^2 + \alpha^2\|\nabla\boldsymbol{u}\|^2 + \|\boldsymbol{b}\|^2 + \beta^2\|\nabla\boldsymbol{b}\|^2 = C_1\,, \tag{22}$$

where

$$C_1 \doteq \|\boldsymbol{u}_0\|^2 + \alpha^2\|\nabla\boldsymbol{u}_0\|^2 + \|\boldsymbol{b}_0\|^2 + \beta^2\|\nabla\boldsymbol{b}_0\|^2\,.$$

Now, using the bound for the H^1 norm of the solution provided by the energy identity (22), we deduce that such a solution can be extended for all positive time (indeed, the time interval of local existence has a lower bound depending only on the initial data).

Hence, we have the global existence of a unique weak solution

$$\boldsymbol{u}, \boldsymbol{b} \in \mathrm{L}^\infty(0,\infty;\mathrm{H}^1(\Omega)). \tag{23}$$

This concludes the proof of Theorem 1.7.

In order to prove Theorem 1.8 for strong solutions, we can proceed similarly. We only need an upper bound for higher derivatives. With this aim, we take the scalar product with $\boldsymbol{v}$ and $\boldsymbol{B}$, and integrate over Ω, getting

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\boldsymbol{v}\|^2 + \int (\boldsymbol{u}\cdot\nabla)\boldsymbol{u}\cdot\boldsymbol{v} - \int (\boldsymbol{b}\cdot\nabla)\boldsymbol{b}\cdot\boldsymbol{v} = 0\,, \tag{24}$$

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\boldsymbol{B}\|^2 + \int (\boldsymbol{u}\cdot\nabla)\boldsymbol{b}\cdot\boldsymbol{B} - \int (\boldsymbol{b}\cdot\nabla)\boldsymbol{u}\cdot\boldsymbol{B} = 0\,. \tag{25}$$

Using Gagliardo-Nirenberg inequality

$$\|\boldsymbol{u}\|_{\mathrm{L}^\infty} \le C\|\Delta\boldsymbol{u}\|^{3/4}\|\boldsymbol{u}\|^{1/4} + C\|\boldsymbol{u}\| \tag{26}$$

and Poincaré inequality $\|\boldsymbol{u}\| \le C\|\Delta\boldsymbol{u}\|$, we have

$$\|\boldsymbol{u}\|_{\mathrm{L}^\infty} \le C\|\Delta\boldsymbol{u}\|^{3/4}\|\boldsymbol{u}\|^{1/4}\,,$$

and therefore

$$\begin{aligned}\left|\int (\boldsymbol{u}\cdot\nabla)\boldsymbol{u}\cdot\boldsymbol{v}\right| &\le \|\boldsymbol{u}\|_{\mathrm{L}^\infty}\|\nabla\boldsymbol{u}\|\,\|\boldsymbol{v}\| \\ &\le C\|\Delta\boldsymbol{u}\|^{3/4}\|\boldsymbol{u}\|^{1/4}\|\nabla\boldsymbol{u}\|\,\|\boldsymbol{v}\| \\ &\le C\|\boldsymbol{v}\|^{7/4}\|\boldsymbol{u}\|^{1/4}\|\nabla\boldsymbol{u}\|\,.\end{aligned} \tag{27}$$

Proceeding similarly for the other terms in (24) and (25), we deduce

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\|\boldsymbol{v}\|^2 + \|\boldsymbol{B}\|^2) \le\,& C\big(\|\boldsymbol{u}\|^{1/4}\|\nabla\boldsymbol{u}\|\,\|\boldsymbol{v}\|^{7/4} + \|\boldsymbol{b}\|^{1/4}\|\nabla\boldsymbol{b}\|\,\|\boldsymbol{B}\|^{3/4}\|\boldsymbol{v}\| \\ &+ \|\boldsymbol{u}\|^{1/4}\|\nabla\boldsymbol{b}\|\,\|\boldsymbol{v}\|^{3/4}\|\boldsymbol{B}\| + \|\boldsymbol{b}\|^{1/4}\|\nabla\boldsymbol{u}\|\,\|\boldsymbol{B}\|^{7/4}\big) \\ \le\,& C\big(\|\boldsymbol{v}\|^{7/4} + \|\boldsymbol{B}\|^{3/4}\|\boldsymbol{v}\| + \|\boldsymbol{v}\|^{3/4}\|\boldsymbol{B}\| + \|\boldsymbol{B}\|^{7/4}\big)\,,\end{aligned}$$

having used (23). Applying Young's inequality with exponents $7/3$ and $7/4$ to the middle terms, we get

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}(1 + \|\boldsymbol{v}\|^2 + \|\boldsymbol{B}\|^2) &\le C(\|\boldsymbol{v}\|^{7/4} + \|\boldsymbol{B}\|^{7/4}) \le C(\|\boldsymbol{v}\|^2 + \|\boldsymbol{B}\|^2)^{7/8} \\ &\le C(1 + \|\boldsymbol{v}\|^2 + \|\boldsymbol{B}\|^2)\,;\end{aligned}$$

the differential form of Gronwall lemma implies

$$1+\|\boldsymbol{v}(t)\|^2+\|\boldsymbol{B}(t)\|^2 \le (1+\|\boldsymbol{v}_0\|^2+\|\boldsymbol{B}_0\|^2)\mathrm{e}^{Ct} \qquad \forall t>0\,,$$

and finally

$$\boldsymbol{v},\boldsymbol{B}\in \mathrm{L}^\infty(0,T;\mathrm{L}^2(\Omega)) \qquad \forall T>0\,,$$

or

$$\boldsymbol{u},\boldsymbol{b}\in \mathrm{L}^\infty(0,T;\mathrm{H}^2(\Omega)) \qquad \forall T>0\,.$$

Remark 2.1. *Let us note that the same estimates hold also in the case* $\Omega=\mathbb{R}^3$, *with no need of periodicity hypotheses. The proof is indeed slightly simplified, since Gagliardo-Nirenberg estimate* (26) *is straightforwardly*

$$\|\boldsymbol{u}\|_{\mathrm{L}^\infty}\le C\|\Delta\boldsymbol{u}\|^{3/4}\|\boldsymbol{u}\|^{1/4}\,.$$

Nevertheless, in this case one needs a different approach to prove local existence.

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