Riv. Mat. Univ. Parma, Vol. 3 (2012), 1-23

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A remark on the Euler equations in dimension two

Abstract. We review some results concerning the global existence of weak solutions to the Euler equations in a two dimensional open bounded set. These results are obtained by means of a suitable vanishing viscosity approximation, through the Navier-Stokes equations equipped with Navier-type boundary conditions. Next, we prove a theorem of existence for weak solutions with a given non-zero normal velocity, slightly relaxing with respect to the time variable the known assumptions on the data of the problem.

Keywords. 2D Euler equations, Navier-Stokes equations, existence, singular limits.

Mathematics Subject Classification (2000): 35Q30, 76F65, 76D03, 76D09.

1 - Introduction

In this paper we recall an existence result for weak solutions to the Euler equations in two space dimensions and we slightly relax the known conditions on the time smoothness of the given normal velocity on the boundary of the domain. The existence of weak solutions is obtained by a vanishing viscosity approximation. The results have been announced in a seminar given from the first author during a workshop taking place in Parma, February 2010.

To better introduce the problem and to explain some of the differences with respect to the 3D case, we point out that the precise understanding of the behavior in terms of the kinematic viscosity $\nu > 0$ of solutions to the three dimensional in-

Received: October 8, 2010; accepted in revised form: June 13, 2011.

compressible Navier-Stokes equations (with constant density)

$$
\begin{aligned}
\partial_t u^\nu - \nu\Delta u^\nu + (u^\nu \cdot \nabla)\, u^\nu + \nabla\pi^\nu &= f^\nu,\\
\nabla \cdot u^\nu &= 0,\\
u^\nu(0,x) &= u_0^\nu(x),
\end{aligned}
$$

represents an outstanding open problem in presence of boundaries, while it is much better understood in the whole space case (or in the periodic setting). When $\nu \to 0^+$ the Navier-Stokes equations converge "formally" to the Euler equations:

$$
\begin{aligned}
\partial_t u^E + (u^E \cdot \nabla)\, u^E + \nabla\pi^E &= f^E,\\
\nabla \cdot u^E &= 0,\\
u^E(0,x) &= u_0^E(x).
\end{aligned}
$$

Proving that the convergence is not "only formal," but that $u^\nu \to u^E$ in appropriate (possibly strong) topologies has a long history. For the Cauchy problem we recall the results of Swann [**51**], Ebin and Marsden [**20**], Kato [**27, 29**], Beirão da Veiga [**5, 6, 8**], and Masmoudi [**45**]. The sharp convergence results (*i.e.* results of convergence in $C(0,T;X)$, where $u_0 \in X$ and there exists a unique solution which is continuous in time with values in X) are strictly linked with the continuous data dependence and with the Hadamard well-posedness, as pointed out by Kato and Lai [**29**]. See Beirão da Veiga [**4, 5, 6, 7**] for a very general approach to this problem, with a quite complete solution also to the challenging problems of singular limits for compressible fluids.

In the case of a domain Ω with smooth boundary $\Gamma := \partial\Omega$ the situation is even more complex, due to the presence of the boundary layer [**28**] created from the slip boundary conditions $(u^E \cdot n)_{|\Gamma} = 0$ which equip the Euler equations, *versus* the no-slip conditions $u^\nu{}_{|\Gamma} = 0$ supplementing the Navier-Stokes equations. This fact prevents from proving convergence unless a certain *near to the boundary* integral of the dissipation vanishes. On the other hand, recent results in presence of Navier-type boundary conditions show how the results can improve in presence of a different type of boundary conditions, see Xiao and Xin [**55**], Beirão da Veiga and Crispo [**9, 10**], Masmoudi and Rousset [**46**], and Iftimie *et al.* [**24, 25**]. A review about some aspects (especially linked with research themes of applied mathematics) of this problem can be found in [**11**].

1.1 - *Euler equations in two space dimensions*

In this paper we study the 2D problem, which is less technical and for which the understanding of the vanishing viscosity limit is much more complete. The first results for the 2D Cauchy problem date back to Golovkin [**21**] and McGrath [**47**]. In bounded domains there are the contributions of Yudovich [**56**], J. L. Lions [**37**], Bardos [**1**], Clopeau, Mikelić, and Robert [**17**], and in the stochastic context Bessaih and Flandoli [**14, 15**]. Very recent results are those of Lopes & Lopes *et al.* [**39, 40, 41**] and of Kelliher [**30, 31, 32**].

In addition, existence and uniqueness of smooth (say $C^{1,\alpha}$) solutions in two dimensions has been proved by means of a precise study of the vorticity by Lichtenstein [**36**] and Wolibner [**54**]; see also Kato [**26**]. In the spirit of linking results with the Hadamard well-posedness, we wish also to mention the endpoint results by Beirão da Veiga [**3**], obtained in the space of functions with continuous vorticity. We recalled these results because there is a very precise study of the vorticity, whose role in two dimensions will be emphasized later on.

Here, we do not address the problem of the existence and behavior of classical (smooth) solutions, but we want to study the existence of *weak solutions*, that is solutions to the Euler equations in the distributional sense, cf. (8), and at least with space derivatives in $L^\infty(0,T;L^2(\Omega))$, cf. (6). We also do not consider the case of non-smooth vorticity as in Constantin and Wu [**18**] and Marchioro [**43**], which is as well interesting.

We point out that one has to restrict to the 2D case since in three dimensions we do not know existence of satisfactory enough weak solutions. The study of weak solutions in the 3D case poses serious problems (see for instance the discussion in P.L. Lions [**38**]) and their behavior may be very wild, as emphasized in the recent work of De Lellis and Székelyhidi [**19**], which extends previous ones by Scheffer and Shnirelman. In two dimensions it is possible to handle weak solutions in a better way: One of the main tools making the 2D problem tractable is the fact that the vorticity

$$\omega^\nu = \operatorname{curl} u^\nu := \partial_1 u_2^\nu - \partial_2 u_1^\nu$$

is a scalar, which is transported by the velocity (and obviously there is also diffusion if $\nu \neq 0$). This holds because the stretching term is not present and ω^ν satisfies the following scalar equation

$$\partial_t \omega^\nu + (u^\nu \cdot \nabla)\,\omega^\nu - \nu \Delta \omega^\nu = \operatorname{curl} f^\nu. \tag{1}$$

Moreover, the fact of having a scalar equation makes possible to use the maximum principle to obtain suitable improved estimates and also to show uniqueness for weak solutions with bounded vorticity, *via* the clever ordinary differential equations-type tools introduced in [**56**].

On the other hand, the use of the vorticity equation requires *ad hoc* boundary conditions, since in general the value of ω^ν at the boundary is not known, and "generation of vorticity" appears due to the differences between the tangential velocity of the flow and that of the boundary. To this end, it is well-known that the use of slip-without-friction boundary conditions of Navier-type (also called free-boundary or curl-free)

$$(2)\qquad \begin{aligned} u^{\nu}\cdot n &= 0 \qquad \text{on } \Gamma\times\,]0,T[,\\ \omega^{\nu} &= 0 \qquad \text{on } \Gamma\times\,]0,T[, \end{aligned}$$

where n denotes the exterior normal unit vector on Γ, allows for a satisfactory control of the vorticity generation and consequently of the behavior of solutions as $\nu\to 0^+$.

1.2 - *Setting of the problem*

We consider the 2D Euler equations in a smooth and bounded domain $\Omega\subset\mathbb{R}^2$, with non-homogeneous conditions on the normal component of the velocity

$$(3)\qquad \begin{aligned} \partial_t u^E + (u^E\cdot\nabla)\,u^E + \nabla p^E &= f &&\text{in } \Omega\times\,]0,T],\\ \nabla\cdot u^E &= 0 &&\text{in } \Omega\times\,]0,T],\\ u^E\cdot \underline{\mathrm{n}} &= g(t,x) &&\text{on } \Gamma\times\,]0,T],\\ u^E(0,x) &= u_0(x) &&\text{in } \Omega, \end{aligned}$$

and we impose on g the compatibility condition (due to incompressibility)

$$(4)\qquad \int_{\Gamma} g(t,x)\,dS = 0 \qquad t\in[0,T].$$

We construct weak solutions by approximation through the Navier-Stokes with the boundary conditions (2)

$$(5)\qquad \begin{aligned} \partial_t u^{\nu} - \nu\Delta u^{\nu} + (u^{\nu}\cdot\nabla)\,u^{\nu} + \nabla p^{\nu} &= f &&\text{in } \Omega\times\,]0,T],\\ \nabla\cdot u^{\nu} &= 0 &&\text{in } \Omega\times\,]0,T],\\ u^{\nu}\cdot\underline{\mathrm{n}} &= g(t,x) &&\text{on } \Gamma\times\,]0,T],\\ \omega^{\nu} &= 0 &&\text{on } \Gamma\times\,]0,T],\\ u^{\nu}(0,x) &= u_0(x) &&\text{in } \Omega, \end{aligned}$$

with the same external force, and initial/boundary data.

The reasons for the study of a problem with non-zero normal velocity have been explained in detail in the previous work [**13**], which was based on an approach similar to the so-called "*vorticity seeding*" method for modeling turbulent flows. The presence of a non zero (hopefully fast time-oscillating) normal component of the velocity has been proposed by Layton [**34**] to simulate triggering separation and detachment from the boundary. The need for an understanding of these non-stationary phenomena, which are out from the conventional time-averaged theory of boundary layers (see *e.g.* Schlichting [**49**]), motivates the use of time-dependent quantities.

In this respect one would like to identify the weakest (with respect to time) conditions ensuring existence of weak solutions in the usual Leray-Hopf class. In particular, observe that the equations (5) are not the standard Navier-Stokes equations with Dirichlet boundary conditions, but the choice of the boundary conditions makes possible to control the vorticity at the boundary.

To conclude this short introduction, we observe that the use of the vorticity equation is in general troublesome in domains with boundary. This because we do not know the boundary conditions for the vorticity itself, when Dirichlet conditions are imposed on the velocity, see also the recent studies of Rautmann [**48**].

2 - On existence of weak solutions

In two space dimensions the theory of weak solutions is rather satisfactory. Some available reviews (also on results for the 3D case) are those of Majda and Bertozzi [**42**], Marchioro and Pulvirenti [**44**], and Bardos and Titi [**2**]. In particular the 2D non-stationary case has been treated by Bardos [**1**] by means of a "viscous approximation" with the system (5). Here, we briefly explain this technique and next we will show how to combine it with the results obtained in [**13**], in order to relax some of the assumptions on the time-derivative of g. In the sequel we will use the classical Lebesgue spaces $(L^p(\Omega), \|\,.\,\|_p)$, the Sobolev spaces $(H^k(\Omega), \|\,.\,\|_{H^k})$ for $k \in \mathbb{N}$, and we do not distinguish between scalar and vector valued functions. As usual in the study of the Navier-Stokes equations, we define

$$H := \{v \in (L^2(\Omega))^2 : \ \nabla \cdot v = 0 \text{ and } (v \cdot \underline{\mathrm{n}})_{|\Gamma} = 0\},$$

and

$$V := \{v \in (H^1(\Omega))^2 : \ \nabla \cdot v = 0 \text{ and } (v \cdot \underline{\mathrm{n}})_{|\Gamma} = 0\}.$$

Moreover A is the Stokes operator associated with curl-free functions at the boundary and with domain

$$D(A) = \{v \in V \cap (H^2(\Omega))^2 : \ (\operatorname{curl} v)_{|\Gamma} = 0\}.$$

We will denote by $(H^s(\Gamma), \|\,.\,\|_{s,\Gamma})$ the standard trace spaces on the boundary Γ and we will also use the usual spaces $H^s(0, T; X)$ employed in the study of evolution equations. In some explicit calculations we will also use Einstein's convention of summation over repeated indices.

The main result we prove is the following modest improvement, concerned with relaxing to $g \in H^{1/2+\varepsilon}(0, T; H^{3/2}(\Gamma))$, for any $0 < \varepsilon < 1/2$, the condition $g \in H^1(0, T; H^{3/2}(\Gamma))$ required in the previous references, cf. [**1, 50**].

Theorem 2.1. *Assume that* $\Omega \subset \mathbb{R}^2$ *is smooth and bounded, that* $f = 0$, *and that* $g \in H^{\frac{1}{2}+\varepsilon}(0, T; H^{\frac{3}{2}}(\Gamma))$, *for some* $\varepsilon \in]0, 1/2[$. *Assume that the compatibility condition* (4) *is satisfied and let the divergence-free* $u_0 \in H^1(\Omega)$ *satisfy* $(u_0 \cdot n)_{|\Gamma} = g(0)$. *Then, there exists a weak solution*

$$u^E \in L^\infty(0, T; H^1(\Omega)), \tag{6}$$

of the Euler system (3).

Remark 2.1. *The introduction of an external force* $f \in L^2(0, T; H^1(\Omega))$ *can be handled without difficulties and we leave it for the interested reader.*

Remark 2.2. *The same conclusions of Theorem* 2.1 *can be proved also if we set the problem in a domain with flat boundary, as for instance* $Q =]-1, 1[^2$ *and we impose the Navier-type boundary conditions* (2) *on* ∂Q, *where*

$$\partial Q := \{x \in \mathbb{R}^2 : |x_1| < 1,\ x_2 = -1\} \cup \{x \in \mathbb{R}^2 : |x_1| < 1,\ x_2 = 1\},$$

while the problem is assumed periodic (*with period* 2) *in the* x_1*-direction. Some minor modifications of the functional setting* (*with restriction on the mean value in the horizontal variable*) *are also required. In this setting the proof becomes simpler since many of the boundary terms involved in the integration by parts are identically vanishing. For instance, under the boundary conditions* $u_2 = v_2 = 0$ *and* $\mathrm{curl}\, u = \mathrm{curl}\, v = 0$ *on* ∂Q, *by direct calculation one obtains*

$$-\int_Q \Delta u\, v\, dx = \int_Q \nabla u\, \nabla v\, dx,$$

instead of the formula (7) *below, involving a surface integral.*

2.1 - *A review about the existence result*

We start by recalling the main lines of the existence result for the (boundary) homogeneous problem, *i.e.* that with $g \equiv 0$ and with $f \in L^2(0, T; V)$. We follow Bardos [**1**] and the case of a smooth non-zero g is also considered in [**1**], by constructing a suitable extension and by treating the non-homogeneous problem in a standard way. The weak formulation for the homogeneous Euler equations is: Find $u^E(t) \in V$ a.e. such that (in the sense of $\mathcal{D}'(]0, T[)$)

$$\frac{d}{dt}\int_\Omega u^E(t)\, v\, dx + \int_\Omega (u^E(t) \cdot \nabla)\, u^E(t)\, v\, dx = \int_\Omega f(t)\, v\, dx \qquad \forall\, v \in V.$$

We will also consider the viscous approximation and we observe that the initial-boundary value problem can be studied by the usual variational techniques. To this end we recall the following equality, which holds for two dimensional smooth vector fields u, v, such that $(u \cdot n)_{|\Gamma} = (v \cdot n)_{|\Gamma} = 0$ and $(\operatorname{curl} u)_{|\Gamma} = 0$:

$$0 \equiv v_j \partial_j (u_i n_i) = v_j (\partial_j u_i) n_i + v_j (\partial_j n_i) u_i = v_j (\partial_i u_j) n_i + v_j (\partial_j n_i) u_i \qquad \text{on } \Gamma.$$

Hence, under the same assumptions, the following formula for integration by parts holds true

$$\begin{aligned} -\int_\Omega \Delta u\, v\, dx &= -\int_\Omega (\partial_i^2 u_j) v_j \, dx \\ &= \int_\Omega (\partial_i u_j)(\partial_i v_j)\, dx - \int_\Gamma n_i (\partial_i u_j) v_j \, dS \\ &= \int_\Omega (\partial_i u_j)(\partial_i v_j)\, dx + \int_\Gamma u_i (\partial_j n_i) v_j \, dS \\ &= \int_\Omega \nabla u\, \nabla v\, dx + \int_\Gamma u \cdot (\nabla n)^T \cdot v \, dS. \end{aligned} \tag{7}$$

By defining the bilinear form

$$a_\nu(u, v) := \nu \int_\Omega \nabla u\, \nabla v\, dx + \nu \int_\Gamma u \cdot (\nabla n)^T \cdot v\, dS,$$

the weak formulation for the homogeneous Navier-Stokes equations with boundary conditions (2) is then: Find $u(t) \in V$ a.e., such that

$$\frac{d}{dt}\int_\Omega u^\nu(t)\, v\, dx + a_\nu(u^\nu(t), v) + \int_\Omega (u^\nu(t) \cdot \nabla)\, u^\nu(t)\, v\, dx = \int_\Omega f(t)\, v\, dx \quad \forall\, v \in V. \tag{8}$$

By using the trace inequality and the smoothness of Γ it follows that for each $\varepsilon > 0$ there exists $C_\varepsilon = C_\varepsilon(\Omega) > 0$ such that

$$\left| \int_\Gamma u \cdot (\nabla n)^T \cdot u\, dS \right| \leq \varepsilon \|\nabla u\|_2^2 + C_\varepsilon \|u\|_2^2 \qquad \forall\, u \in H^1(\Omega). \tag{9}$$

In particular, we are using the fact that the curvature of the domain is bounded and this makes possible to formally write for the Navier-Stokes equations the following energy balance (obtained by using u^ν as test function)

$$\frac{d}{dt}\|u^\nu(t)\|_2^2 + \nu \|\nabla u^\nu(t)\|_2^2 \leq C(1+\nu)\|u^\nu(t)\|_2^2 + \|f(t)\|_2^2.$$

(Calculations are formal and can be justified for example by means of Galerkin approximate functions. The existence of a smooth basis of eigenfunctions of the Stokes operator with the prescribed boundary conditions is proved for instance in [**17**].) The above argument proves the following bounds, uniformly in $\nu > 0$,

$$u^\nu \in L^\infty(0,T;H) \qquad \text{and} \qquad \sqrt{\nu}\, u^\nu \in L^2(0,T;V).$$

Next, by taking the curl of the Navier-Stokes equation one gets the scalar equation (1), since the two dimensional equations do not have the vortex stretching term. Multiplying (1) by ω^ν and performing standard integrations by parts one obtains (since $\omega^\nu_{|\Gamma} = 0$)

$$\frac{d}{dt}\|\omega^\nu\|_2^2 + 2\nu\|\nabla\omega^\nu\|_2^2 \leq \|\mathrm{curl}\, f\|_2^2 + \|\omega\|_2^2.$$

In this way it is possible to prove the following crucial estimate

$$\omega^\nu \in L^\infty(0,T;L^2(\Omega)),$$

again with a bound independent of $\nu > 0$.

The next step is to show that the bound on ω^ν implies the same on ∇u^ν. As observed in [**1, 56**], the system

$$\begin{cases} -\Delta u^\nu = \nabla^\perp \omega^\nu & \text{in } \Omega, \\ u^\nu \cdot n = 0 & \text{on } \Gamma, \\ \quad \omega^\nu = 0 & \text{on } \Gamma, \end{cases}$$

where $\nabla^\perp := (\partial_2, -\partial_1)$, is elliptic. If we multiply the above equation by u^ν and integrate by parts over Ω we obtain

$$\exists\, C = C(\Omega) > 0 : \qquad \|\nabla u^\nu\|_2^2 \leq C\big(\|u^\nu\|_2^2 + \|\omega^\nu\|_2^2\big).$$

This finally proves that (under the above assumptions on the data of the problem) the bound on ω^ν, together with that on u^ν coming from the energy balance, imply

$$u^\nu \in L^\infty(0,T;V), \tag{10}$$

uniformly in $\nu > 0$.

By comparison it follows (again uniformly in $\nu > 0$) that $\partial_t u^\nu \in L^2(0,T;V')$, hence we can extract (with the Aubin-Lions argument, see [**37**] or the Friederichs inequality, see Hopf [**23**]) a sub-sequence $\{\nu_n\}_{n\in\mathbb{N}}$, with $\nu_n \to 0^+$, such that $\{u^{\nu_n}\}$ converges weakly$*$ in $L^\infty(0,T;V)$ and converges strongly in $L^2((0,T)\times\Omega)$ to a function that we call u^E, in such a way that

$$\int_0^T\!\!\int_\Omega (u^{\nu_n}\cdot\nabla)\, u^{\nu_n}\, v\,\psi\, dx\, dt \overset{n\to+\infty}{\longrightarrow} \int_0^T\!\!\int_\Omega (u^E\cdot\nabla)\, u^E\, v\,\psi\, dx\, dt,$$

for all $v \in V$ and $\psi \in L^2(0, T)$. Moreover, specifically from (10) we also obtain that

$$a_{\nu_n}(u^{\nu_n}, v) \stackrel{n\to+\infty}{\longrightarrow} 0 \qquad \forall\, v \in V,$$

hence the function $u^E \in L^\infty(0, T; V)$ turns out to be a (possibly non-unique) weak solution of the 2D Euler equations. With a slightly more precise argument, one can show indeed that $u^E \in L^\infty(0, T; V) \cap C(0, T; H)$, with convergence $u^{\nu_n} \to u^E$ in $C(0, T; H)$, cf. [**1**, Rem. I].

2.2 - *Some remarks*

In the case of a simply connected domain one can introduce (up to an additive constant) a stream function Φ^ν such that

$$\frac{\partial \Phi^\nu}{\partial x_1} = -u_2^\nu \qquad \text{and} \qquad \frac{\partial \Phi^\nu}{\partial x_2} = u_1^\nu,$$

in such a way that $\omega^\nu = -\Delta\Phi^\nu$. The vorticity equation (1) becomes

$$\partial_t(-\Delta\Phi^\nu) + \mathcal{R}(\Phi^\nu) + \nu\Delta^2\Phi^\nu = \operatorname{curl} f,$$

where

$$\mathcal{R}(\Phi^\nu) = \frac{\partial}{\partial x_2}\left(\frac{\partial \Phi^\nu}{\partial x_1}\Delta\Phi_\nu\right) - \frac{\partial}{\partial x_1}\left(\frac{\partial \Phi^\nu}{\partial x_2}\Delta\Phi_\nu\right),$$

and with the boundary conditions

$$\begin{aligned} \Phi^\nu &= \text{const.} && \text{on } \Gamma \times]0, T[, \\ \Delta\Phi^\nu &= 0 && \text{on } \Gamma \times]0, T[. \end{aligned}$$

This approach has been used in Yudovich [**56**] and J. L. Lions [**37**, § 6.9] to study the vanishing viscosity limit in simply connected domains.

Uniqueness of weak solutions can be proved if the initial datum and the external force have bounded vorticity, see Yudovich [**56**]. Improved results in spaces with unbounded vorticity are those obtained by Yudovich [**57**] and Vishik [**53**]. Related results in Besov spaces are those by Vishik [**52**] and Hmidi and Keraani [**22**] and we observe that there is big activity along this path, also to study the existence and uniqueness of strong solutions in critical spaces.

The argument explained to obtain the *a priori* estimates is just formal and one needs to justify the calculations, by means of suitable approximations. In particular, in [**1**] this is not addressed by the Galerkin approach, but by considering the "modified" Navier-Stokes equations

$$\begin{aligned} \partial_t v_\varepsilon^\nu - \nu\Delta v_\varepsilon^\nu + (u_\varepsilon^\nu \cdot \nabla)\, v_\varepsilon^\nu + \nabla p_\varepsilon^\nu &= f && \text{in } \Omega \times]0, T], \\ \nabla \cdot v_\varepsilon &= 0 && \text{in } \Omega \times]0, T], \end{aligned} \tag{11}$$

with the same initial and boundary data and where $u^\nu_\varepsilon = (\mathrm{I}+\varepsilon A)^{-1} v^\nu$ (In particular also $\nabla \cdot u_\varepsilon = 0$). The weak formulation for the approximate velocity v^ν_ε is then: Find $v^\nu_\varepsilon(t) \in V$ a.e. such that

$$\frac{d}{dt}\int_\Omega v^\nu_\varepsilon(t)\, v\, dx + a_\nu(v^\nu_\varepsilon(t), v) + \int_\Omega (u^\nu_\varepsilon(t)\cdot\nabla)\, v^\nu_\varepsilon(t)\, v\, dx = \int_\Omega f(t)\, v\, dx \quad \forall\, v\in V.$$

The approximation is made by replacing the transport term by a smoother one. This is the original idea of Leray [**35**] to solve the Navier-Stokes equations, even if he employed it for the Cauchy problem, with mollification by a smooth and compactly supported kernel. The role of Leray's approach has been recently (re)-analyzed and emphasized in the LES community by Cheskidov *et al.* [**16**]. The system (11) is known nowadays as the Leray-α model (in this case since the regularization parameter is ε, it should be called Leray-ε). For a link with filtering and classical LES models see also [**11, 12**]. We also observe that this approach requires some care to handle the curl of the convective term. In fact, in general, for 2D divergence-free vector fields a and b

$$\mathrm{curl}\,\big[(a\cdot\nabla)\, b\big] = (a\cdot\nabla)\,\mathrm{curl}\, b + \big((\partial_1 a_i)(\partial_i b_2) - (\partial_2 a_i)(\partial_i b_1)\big),$$

and the term between parentheses does not vanish if $a \neq b$. This explains why slightly different estimates are needed in [**1**] to handle the term

$$\int_\Omega \mathrm{curl}\;\big[(u^\nu_\varepsilon\cdot\nabla)\, v^\nu_\varepsilon\big]\,\mathrm{curl}\, v^\nu_\varepsilon\, dx = -\int_\Omega (u^\nu_\varepsilon\cdot\nabla)\, v^\nu_\varepsilon\, \Delta v^\nu_\varepsilon\, dx,$$

where the boundary term in the formula of integration by parts vanishes since $\mathrm{curl}\, v_\varepsilon = 0$ on Γ.

2.3 - *Proof of the main result*

We can give now the proof of the main result, which follows by a combination of the above techniques with the *a priori* estimate for fractional time-derivative introduced in [**13**].

Proof (of Theorem 2.1). To prove the existence of u^E, a solution of (3), we consider the vanishing viscosity approximation. First, we have to prove the existence of weak solutions for the initial-boundary value problem (5) and this will be done in three steps, by using the same approach of [**13**]. Next, we prove the estimates for the vorticity allowing to pass to the limit as $\nu \to 0^+$.

a) Existence of weak solutions of Navier-Stokes equations

We can smoothly extend g into a function defined on the domain Ω by solving a boundary value problem. We recall that $g \in H^{\frac{1}{2}+\varepsilon}(0,T;H^{3/2}(\Gamma))$ and satisfies the compatibility condition (4). Let us consider the following problem, which is elliptic for $\lambda \in \mathbb{R}^+$ large enough and depending on the curvature of Γ (cf. [**1**, Prop. 1])

$$\begin{aligned} -\Delta G + \lambda G + \nabla \Pi &= 0 && \text{in } \Omega \times]0,T], \\ \nabla \cdot G &= 0 && \text{in } \Omega \times]0,T], \\ G \cdot \underline{\mathrm{n}} &= g(t,x) && \text{on } \Gamma \times]0,T], \\ \operatorname{curl} G &= 0 && \text{on } \Gamma \times]0,T], \end{aligned} \tag{12}$$

where the time-variable in system (12) is just a parameter. The standard theory allows us to use Lax-Milgram's lemma and show that there exists a unique solution to (12) such that

$$G(t,x) \in H^{1/2+\varepsilon}(0,T;H^1(\Omega)).$$

Moreover, further regularity holds true (cf. [**17**]) and there is a constant C_0, depending only on Ω, such that

$$\|G\|_{H^{1/2+\varepsilon}(0,T;H^2(\Omega))} + \|\Pi\|_{H^{1/2+\varepsilon}(0,T;H^1(\Omega))} \leq C_0 \|g\|_{H^{1/2+\varepsilon}(0,T;H^{3/2}(\Gamma))}.$$

Since we will treat the nonlinear problem as a perturbation of the linear one, we need to establish a quite precise existence theorem for the following time-dependent linear Stokes system:

$$\begin{aligned} \partial_t u^L - \nu \Delta u^L + \nabla p^L &= 0 && \text{in } \Omega \times]0,T], \\ \nabla \cdot u^L &= 0 && \text{in } \Omega \times]0,T], \\ u^L \cdot n &= g && \text{on } \Gamma \times]0,T], \\ \operatorname{curl} u^L &= 0 && \text{on } \Gamma \times]0,T], \\ u^L(0,x) &= G(0,x) && \text{in } \Omega. \end{aligned} \tag{13}$$

To better treat the contribution of boundary terms we introduce the new unknown

$$Z(t,x) := u^L(t,x) - G(t,x)$$

in such a way that $Z(t,x)$ satisfies the following homogeneous problem

$$\begin{aligned} \partial_t Z - \nu \Delta Z + \nabla q^L &= -\partial_t G + \nu \Delta G && \text{in } \Omega \times]0,T], \\ \nabla \cdot Z &= 0 && \text{in } \Omega \times]0,T], \\ Z \cdot \underline{\mathrm{n}} &= 0 && \text{on } \Gamma \times]0,T], \\ \operatorname{curl} Z &= 0 && \text{on } \Gamma \times]0,T], \\ Z(0,x) &= 0 && \text{in } \Omega. \end{aligned} \tag{14}$$

In the right-hand side there is the term $\nu \Delta G \in L^\infty(0,T;L^2(\Omega))$ which does not cause any problem, while $-\partial_t G$ has low regularity (it does not belong to any Lebesgue space with respect to the time variable) and we cannot apply standard methods to prove existence results. However, we are able to prove existence of weak solutions by using a method employing both the Faedo-Galerkin approximation and the estimates for fractional derivatives.

Since we know at least that $\partial_t G \in H^{-\frac{1}{2}+\varepsilon}(0,T;H^2(\Omega))$, we introduce a sequence $\{G^N\}_{N\in\mathbb{N}} \subset H^1(\mathbb{R};H^2(\Omega))$ of approximate functions such that

(a) $G^N{}_{|_{[0,T]}} \to G$ in $H^{\frac{1}{2}+\varepsilon}(0,T;H^2(\Omega))$, as $N \to \infty$,
(b) $\|\partial_t G^N\|_{L^2(0,T;L^2(\Omega))} = N$.

Let now $\{\phi_k\}_{k\in\mathbb{N}}$ be an Hilbert basis of the space V, made of smooth functions, such that $\operatorname{curl}\phi_n = 0$ on Γ and let $Z_{n,N}(t,x) = \sum_{k=1}^{n} \zeta^N_{n,k}(t)\phi_k(x)$ be the solution of the following (finite-dimensional) linear system of ordinary differential equations for $\zeta^N_{n,k}$:

$$\frac{d}{dt}\int_\Omega Z_{n,N}\,\phi_j\,dx + a_\nu(Z_{n,N},\phi_j) = -\frac{d}{dt}\int_\Omega G^N\,\phi_j\,dx + \nu\int_\Omega \Delta G^N\,\phi_j\,dx,$$

for $t \in (0,T), j = 1,\dots,n$, and with $\int_\Omega Z_{n,N}(0,x)\,\phi_j(x)\,dx = 0$. By using a very standard argument, this system of ordinary differential equations has a unique solution and by using $\zeta^N_{n,j}(t)\phi_j(x)$ as test function and summing over j one easily obtains the following estimate

$$\sup_{0\le t\le T}\|Z_{n,N}(t)\|_2^2 + \nu\int_0^T \|\nabla Z_{n,N}(\tau)\|_2^2\,d\tau$$
$$\le C\big(\|G^N\|^2_{H^1(0,T;L^2)} + \|G^N\|^2_{L^2(0,T;H^2)}\big),$$

with a constant C, depending only on Ω (for $0<\nu<\nu_0$, with ν_0 given). The derivation of the energy balance follows the same lines of that for the Navier-Stokes equations from Section 2.1.

Unfortunately, these estimates are not uniform in N, due to the property (b) of the approximate sequence $\{G^N\}_{N\in\mathbb{N}}$. To overcome this difficulty we estimate (after multiplication by $Z_{n,N}$) the first term from the right-hand side of (14) in this way:

$$\left|\int_0^T\int_\Omega \partial_t G^n\cdot Z_{n,N}\,dx\,ds\right| \le \|\partial_t G^N\|_{H^{-\frac{1}{2}+\varepsilon}(0,T;L^2)}\|Z_{n,N}\|_{H^{\frac{1}{2}-\varepsilon}(0,T;L^2)} \tag{15}$$
$$\le \|G^N\|_{H^{\frac{1}{2}+\varepsilon}(0,T;L^2)}\|Z_{n,N}\|_{H^{\frac{1}{2}-\varepsilon}(0,T;L^2)}.$$

We need now an uniform estimate (with respect to both n and N) of $Z_{n,N}$ in the space $H^{\frac{1}{2}-\varepsilon}(0,T;L^2(\Omega))$. We shall use the Fourier transform characterization of the norm in fractional Sobolev spaces and if

$$\widetilde{Z}_{n,N} := \begin{cases} Z_{n,N} & \text{for } t\in[0,T],\\ 0 & \text{elsewhere,}\end{cases}$$

we can write the following equality:

$$\frac{d}{dt}\int_\Omega \widetilde{Z}_{n,N}\phi_k\,dx + a_\nu(\widetilde{Z}_{n,N},\phi_k) = -\frac{d}{dt}\int_\Omega \widetilde{G}^N\phi_k\,dx$$
$$+\nu\int_\Omega \Delta\widetilde{G}^N\phi_k\,dx + \delta(t)\int_\Omega G^N(0)\phi_k\,dx - \delta(t-T)\int_\Omega (Z_{n,N}(T)+G^N(T))\,\phi_k\,dx,$$

for each $k=1,\dots,n$, where $\delta(\,\cdot\,)$ is the usual Dirac's *delta function*. By passing to the Fourier variable ξ, (and $\widehat{Z}_{n,N}$ denotes the Fourier transform of $\widetilde{Z}_{n,N}$) the above equation reads as follows:

$$-\mathrm{i}\xi\int_\Omega \widehat{Z}_{n,N}\,\phi_k\,dx + a_\nu(\widetilde{Z}_{n,N},\phi_k) = \mathrm{i}\xi\int_\Omega \widehat{G}^N\phi_k dx + \nu\int_\Omega \Delta\widehat{G}^N\phi_k dx$$
$$+\int_\Omega G^N(0)\phi_k dx - \mathrm{e}^{-\mathrm{i}\xi T}\int_\Omega (Z_{n,N}(T)+G^N(T))\phi_k dx.$$

Consequently, by multiplying by $\overline{\widehat{Z}_{n,N}}$ (the complex conjugate of $\widehat{Z}_{n,N}$) we get – with some integration by parts –

$$-\mathrm{i}\xi\|\widehat{Z}_{n,N}(\xi)\|_2^2 + \nu\|\nabla\widehat{Z}_{n,N}(\xi)\|_2^2 = \mathrm{i}\xi\int_\Omega \widehat{G}^N\,\overline{\widehat{Z}_{n,N}}\,dx + \nu\int_\Omega \Delta\widehat{G}^N\,\overline{\widehat{Z}}_{n,N}dx$$
$$+\int_\Omega G^N(0)\,\overline{\widehat{Z}_{n,N}}\,dx - \mathrm{e}^{-\mathrm{i}\xi T}\int_\Omega (Z_{n,N}(T)+G^N(T))\,\overline{\widehat{Z}_{n,N}}\,dx.$$

We take the imaginary part and multiply both sides of the previous formula by $|\xi|^{2\lambda-1}$, with $\lambda<1/2$ so that (by using Young's inequality) one gets

$$|\xi|^{2\lambda}\|\widehat{Z}_{n,N}(\xi)\|_2^2 \le C|\xi|^{2\lambda}\|\widehat{G}^N\|_2^2$$
$$+C|\xi|^{2\lambda-2}(\nu\|\Delta\widehat{G}^N(\xi)\|_2 + \|G^N(T)\|_2 + \|Z_{n,N}(T)\|_2 + \|G^N(0)\|_2)^2.$$

In order to estimate the integral $\int_{\mathbb{R}} |\xi|^{2\lambda}\|\widehat{Z}_{n,N}(\xi)\|_2^2\,d\xi$, we split it into two parts: the "inner" integral and the "outer" one. By the above estimate, we can show that

$$\int\limits_{|\xi|>1} |\xi|^{2\lambda}\|\widehat{Z}_{n,N}(\xi)\|_2^2\,d\xi \le C\int\limits_{\mathbb{R}} |\xi|^{2\lambda}\|\widehat{G}^N\|_2^2 + \nu \int\limits_{|\xi|>1} |\xi|^{2\lambda-2}\|\Delta\widehat{G}^N\|_2^2\,d\xi$$
$$+ C(\|G^N(T)\|_2 + \|Z_{n,N}(T)\|_2 + \|G^N(0)\|_2)^2 \int\limits_{|\xi|>1} |\xi|^{2\lambda-2}d\xi.$$

The first term on the right-hand side is controlled by $C\|G^N\|^2_{H^{\frac{1}{2}+\varepsilon}(0,T;L^2)}$. Moreover, for $|\xi|>1$ we also have

$$\int\limits_{|\xi|>1} |\xi|^{2\lambda-2}\|\Delta\widehat{G}^N\|_2^2\,d\xi \le \int\limits_{|\xi|>1} |\xi|^{2\lambda}\|\Delta\widehat{G}^N\|_2^2\,d\xi \le C\|G^N\|^2_{H^{\frac{1}{2}+\varepsilon}(0,T;H^2)},$$

while (15) implies that

$$\|Z_{n,N}(T)\|_2^2 \le C\|G^N\|_{H^{\frac{1}{2}+\varepsilon}(0,T;L^2)}\|Z_{n,N}\|_{H^{\frac{1}{2}-\varepsilon}(0,T;L^2)}.$$

Next, $\|G^N(0)\|_2$ is bounded by $\|G(0)\|_2$ and finally, by using the Morrey inequality $H^{1/2+\varepsilon}(0,T)\subset C([0,T])$, we get

$$\|G^N(T)\|_2 \le \|G^N\|_{H^{\frac{1}{2}+\varepsilon}(0,T;L^2(\Omega))}.$$

Observe that: a) for the validity of the Morrey inequality it is essential that $\varepsilon>0$; b) the integral of $|\xi|^{2\lambda-2}$ is finite due to $\lambda<1/2$.

The inner part is estimated as follows, by using Parseval's theorem, Poincaré inequality (which is still valid in V), and estimate (15):

$$\int\limits_{|\xi|\le 1} |\xi|^{2\lambda}\|\widehat{Z}_{n,N}\|_2^2\,d\xi \le \int\limits_{\mathbb{R}} \|\widehat{Z}_{n,N}\|_2^2\,d\xi = \int\limits_0^T \|Z_{n,N}(t)\|_2^2\,dt$$
$$\le C\int\limits_0^T \|\nabla Z_{n,N}(t)\|_2^2\,dt \le C\|G^N\|_{H^{\frac{1}{2}+\varepsilon}(0,T;L^2)}\|Z_{n,N}\|_{H^{\frac{1}{2}-\varepsilon}(0,T;L^2)}.$$

In conclusion, by collecting the above estimates we get that, for each $\varepsilon\in(0,1/2)$, there exists a constant C, depending only on Ω and ε, such that

$$\|Z_{n,N}\|_{H^{\frac{1}{2}-\varepsilon}(0,T;L^2(\Omega))} \le C\|G^N\|_{H^{\frac{1}{2}+\varepsilon}(0,T;H^2(\Omega))},$$

which, together with (15), shows that $\{Z_{n,N}\}$ is bounded, uniformly in n and N, in the spaces $H^{\frac{1}{2}-\varepsilon}(0,T;H)$, $L^\infty(0,T;H)$, and $L^2(0,T;V)$. As usual, it is possible to extract a (diagonal) sub-sequence converging weakly in $L^2(0,T;V)$, and weakly$*$ in

$L^\infty(0,T;H)$ to the unique solution Z of problem (14). Next, u^L is obtained adding together G and Z; this is the solution with the required estimates in terms of the data, that is

$$\|u^L\|^2_{L^\infty(0,T;L^2(\Omega))\cap H^{1/2-\varepsilon}(0,T;L^2(\Omega))}+\nu\|u^L\|^2_{L^2(0,T;H^1(\Omega))} \leq C\|g\|^2_{H^{1/2+\varepsilon}(0,T;H^{3/2}(\Gamma))}.$$

Remark 2.3. *We observe that under the same assumptions it can be proved that the solution u^L is smoother, but we postpone it to the last part, showing now existence of weak solutions for the viscous problem.*

We study now the nonlinear problem and to prove existence of weak solutions for the Navier-Stokes equation we introduce two new unknowns. Let (u^L, p^L) be a solution of the system (13) and define

$$U := u^\nu - u^L \qquad \text{and} \qquad P := p^\nu - p^L.$$

Then, the couple (U,P) solves the following homogeneous problem

$$(16)\qquad \begin{aligned} \partial_t U - \nu\Delta U + [(U+u^L)\cdot\nabla](U+u^L) + \nabla P &= 0 && \text{in } \Omega\times\,]0,T],\\ \nabla\cdot U &= 0 && \text{in } \Omega\times\,]0,T],\\ U\cdot n &= 0 && \text{on } \Gamma\times\,]0,T],\\ \operatorname{curl} U &= 0 && \text{on } \Gamma\times\,]0,T],\\ U(0,x)+G(0,x) &= u_0(x) && \text{in } \Omega, \end{aligned}$$

with weak formulation: Find $U(t)\in V$ a.e such that

$$\frac{d}{dt}\int_\Omega U(t)\,v\,dx + a_\nu(U(t),v) + \int_\Omega [(U(t)+u^L)\cdot\nabla](U(t)+u^L)\,v\,dx = 0 \quad \forall\, v\in V.$$

The proof of existence is quite standard since is again based on the Galerkin method. We show only the *a priori* estimate, which can be turned into a rigorous proof working with Galerkin approximate function $\{U_n\}$. To this end we multiply the momentum equation in (16) by U itself and integrating by parts we get

$$(17)\qquad \frac{1}{2}\frac{d}{dt}\|U\|_2^2 + \nu\|\nabla U\|_2^2 \leq \left|\int_\Omega [(U+u^L)\cdot\nabla]\,(U+u^L)\,U\right| + \nu\, C\int_\Gamma |U|^2\,dS.$$

We handle in the usual way the boundary integral with (9). It remains only to estimate the nonlinear term and we observe that since $\nabla\cdot U = 0$ and $U\cdot n = 0$, then

$$\int_\Omega (U\cdot\nabla)\,U\,U\,dx = 0 \qquad \text{and} \qquad \int_\Omega (U\cdot\nabla)\,u^L\,U\,dx = -\int_\Omega (U\cdot\nabla)\,U\,u^L\,dx.$$

Hence, we have

$$\begin{aligned}\int_\Omega [(U+u^L)\cdot\nabla]\,(U+u^L)\,U\,dx \\ = \int_\Omega (u^L\cdot\nabla U)\,U\,dx - \int_\Omega (U\cdot\nabla U)\,u^L\,dx + \int_\Omega (u^L\cdot\nabla u^L)\,U\,dx.\end{aligned} \tag{18}$$

We estimate the terms from the right-hand side by using the Hölder inequality, the Young inequality, and also the following Gagliardo-Nirenberg inequality (cf. [**1**, Eq. 41])

$$\|u\|_{L^4} \le C(\Omega)\|u\|_2^{1/2}\|\nabla u\|_2^{1/2} \qquad \forall\, u \in H^1(\Omega).$$

Note that in the case of functions vanishing on Γ one can find a constant C not depending on Ω for the same inequality: For any open set Ω (cf. Ladyženskaya [**33**, §1]) it holds $\|u\|_{L^4} \le 2^{1/4}\|u\|_2^{1/2}\|\nabla u\|_2^{1/2}$, for all $u \in H^1_0(\Omega)$.

We then obtain

$$\begin{aligned}&\left|\int_\Omega [(U+u^L)\cdot\nabla]\,(U+u^L)\,U\,dx\right| \\ &\le 2\|u^L\|_4\|U\|_4\|\nabla U\|_2 + \|u^L\|_4\|\nabla u^L\|_2\|U\|_4 \\ &\le C\|u^L\|_2^{1/2}\|\nabla u^L\|_2^{1/2}\|U\|_2^{1/2}\|\nabla U\|_2^{3/2} + C\|u^L\|_2^{1/2}\|\nabla u^L\|_2^{3/2}\|U\|_2^{1/2}\|\nabla U\|_2^{1/2},\end{aligned}$$

and consequently

$$\begin{aligned}&\left|\int_\Omega [(U+u^L)\cdot\nabla]\,(U+u^L)\,U\,dx\right| \\ &\le \frac{\nu}{4}\|\nabla U\|_2^2 + \frac{C}{\nu^3}\|u^L\|_2^2\|\nabla u^L\|_2^2\|U\|_2^2 + \frac{\nu}{4}\|\nabla U\|_2^2 + \frac{C}{\nu^{1/3}}\|u^L\|_2^{2/3}\|\nabla u^L\|_2^2\|U\|_2^{2/3}.\end{aligned}$$

We then get the ν-dependent *a priori* estimate

$$\frac{d}{dt}\|U\|_2^2 + \nu\|\nabla U\|_2^2 \le \frac{C}{\nu^3}\|u^L\|_2^2\|\nabla u^L\|_2^2\|U\|_2^2 + \frac{C}{\nu^{1/3}}\|u^L\|_2^{2/3}\|\nabla u^L\|_2^2\|U\|_2^{2/3}$$

and since

$$u^L \in L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;H^1(\Omega)),$$

by using the Gronwall lemma we can infer that, non uniformly in $\nu > 0$,

$$U \in L^\infty(0,T;H) \cap L^2(0,T;V).$$

The passage to the limit (in the Galerkin parameter) can be done by employing the usual compactness tools: By comparison and by using the same regularity of u^L as above one can bound $\partial_t U$ in $L^2(0,T;V')$; then one can apply Aubin-Lions compactness theorem to pass to the limit in the nonlinear term. This finally proves that, under the assumptions on g of Theorem 2.1 there exists a solution u^ν to (5) such that

$$u^\nu \in L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;H^1(\Omega)).$$

It is clear that the above result is enough to prove existence for each *fixed* positive ν but, contrary to the homogeneous case, some care is needed to prove estimates uniform with respect to ν.

The last part concerns in fact with the estimates independent on the viscosity and needed to pass to the limit as $\nu \to 0^+$.

b) Vorticity estimate and vanishing viscosity limit

As a preliminary step we observe that the same argument employed as before to study the linear problem (14) can be also applied to the linear problem for $\operatorname{curl} Z$, which is obtained by taking the curl of (14):

$$\begin{aligned} \partial_t \operatorname{curl} Z - \nu\Delta \operatorname{curl} Z = -\partial_t \operatorname{curl} G + \nu\Delta \operatorname{curl} G \qquad & \text{in } \Omega \times]0,T], \\ \operatorname{curl} Z = 0 \qquad & \text{on } \Gamma \times]0,T], \\ \operatorname{curl} Z(0,x) = 0 \qquad & \text{in } \Omega. \end{aligned}$$

In particular, observe that if we take the curl of (12) we get that

$$-\Delta \operatorname{curl} G + \lambda \operatorname{curl} G = 0,$$

hence we can rewrite the equation satisfied by Z as follows:

$$\partial_t \operatorname{curl} Z - \nu\Delta \operatorname{curl} Z = -\partial_t \operatorname{curl} G - \nu\lambda \operatorname{curl} G.$$

Under the same assumptions as before on g, the elliptic regularity implies that $\operatorname{curl} G \in H^{\frac{1}{2}+\varepsilon}(0,T;H^1(\Omega))$. Hence, we can apply the same argument as before to show that the unique solution of the linear problem for $\operatorname{curl} Z$ satisfies

$$\operatorname{curl} Z \in H^{\frac{1}{2}-\varepsilon}(0,T;L^2(\Omega)) \cap L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;H^1_0(\Omega)).$$

This proves that Z is more regular and consequently we have also

$$u^L \in H^{\frac{1}{2}-\varepsilon}(0,T;H^1(\Omega)) \cap L^\infty(0,T;H^1(\Omega)) \cap L^2(0,T;H^2(\Omega)).$$

This allows us to estimate in a better way the integrals in (18). In fact, $(u^L \cdot \nabla)\, u^L \in L^2(0,T;L^2(\Omega))$ and as in [**1**, Eq. 105] we obtain (with the same tools as before) the following estimate

$$\Big|\int_\Omega [(U+u^L)\cdot\nabla](U+u^L)\,U\,dx\Big| \leq 2\|u^L\|_\infty\|U\|_2\|\nabla U\|_2+\|u^L\|_\infty\|\nabla u^L\|_2\|U\|_2$$
$$\leq C\big(\|u^L\|^2_{H^2}\|U\|_2^2+\|\nabla U\|_2^2+\|\nabla u^L\|_2^2\big),$$

for some $C=C(\Omega)$ depending also on the Sobolev embedding $H^2(\Omega)\hookrightarrow L^\infty(\Omega)$. We finally obtain

$$\frac{d}{dt}\|U\|_2^2+\nu\|\nabla U\|_2^2\leq C\big(\|u^L\|^2_{H^2}\|U\|_2^2+\|\nabla U\|_2^2+\|\nabla u^L\|_2^2\big). \tag{19}$$

The quantities $\|u^L\|^2_{H^2}$ and $\|\nabla u^L\|_2^2$ both belong to $L^1(0,T)$, but to handle the term $\|\nabla U\|$ from the right-hand side – without absorbing it into the left-hand side – we need another differential inequality for first order derivatives. To this end, let us consider the initial-boundary value problem satisfied by the vorticity of solutions of the Navier-Stokes equations (5)

$$\begin{aligned} \omega^\nu_t-\nu\Delta\omega^\nu+(u^\nu\cdot\nabla)\,\omega^\nu&=0 \qquad \text{in } \Omega\times\,]0,T],\\ \omega^\nu&=0 \qquad \text{on } \Gamma\times\,]0,T],\\ \omega^\nu(0,x)&=\operatorname{curl}u_0^\nu \qquad \text{in } \Omega. \end{aligned} \tag{20}$$

We multiply (20) by ω^ν and we integrate by parts (calculations can be justified by a suitable smoothing or approximation of the problem). By using the divergence-free constraint and the fact that $\omega^\nu_{|\Gamma}=0$ we get that

$$\int_\Omega (u^\nu\cdot\nabla)\,\omega^\nu\,\omega^\nu\,dx=\int_\Gamma (u^\nu\cdot n)\,\frac{|\omega^\nu|^2}{2}\,dS-\int_\Gamma(\nabla\cdot u^\nu)\frac{|\omega^\nu|^2}{2}\,dx=0.$$

We have then

$$\frac{1}{2}\frac{d}{dt}\|\omega^\nu\|_2^2+\nu\|\nabla\omega^\nu\|_2^2\leq 0 \tag{21}$$

and consequently, the following bound independent on ν,

$$\omega^\nu\in L^\infty(0,T;L^2(\Omega)).$$

The last technical step is to show that the bound on the vorticity implies a bound on the full gradient of u^ν, in order to use the information contained in the estimates for U. Differently from the homogeneous case (cf. the end of Section 2.1) we have now the following system

$$\begin{cases} -\Delta u^\nu=\nabla^\perp\omega^\nu & \text{in } \Omega,\\ u^\nu\cdot n=g & \text{on } \Gamma,\\ \omega^\nu=0 & \text{on } \Gamma. \end{cases}$$

By multiplying this system by u^ν and integrating by parts over Ω we get

$$-\int_\Omega \Delta u^\nu \cdot u^\nu = \|\nabla u^\nu\|_2^2 - \int_\Gamma n \cdot \nabla u^\nu \cdot u^\nu \, dS = \|\omega^\nu\|_2^2.$$

By direct calculations and by using that $\omega^\nu = 0$ on Γ we obtain

$$\begin{aligned}(u^\nu \cdot \nabla)\, g &= u_j^\nu \partial_j (u_i^\nu n_i)\\ &= u_j^\nu (\partial_j u_i^\nu) n_i + u_j^\nu (\partial_j n_i) u_i^\nu = u_j^\nu (\partial_i u_j^\nu) n_i + u_j^\nu (\partial_j n_i) u_i^\nu \qquad \text{on } \Gamma.\end{aligned}$$

By using the above equality and the trace theorems, the boundary integral of $n \cdot \nabla u^\nu \cdot u^\nu$ can be estimated as follows

$$\begin{aligned}\left|\int_\Gamma n \cdot \nabla u^\nu \cdot u^\nu \, dS\right| &\le \int_\Gamma \left|u^\nu \cdot (\nabla n)^T \cdot u^\nu\right| + \left|u^\nu \cdot \nabla g\right| dS\\ &\le C \int_\Gamma |u^\nu|^2 \, dS + \int_\Gamma |u^\nu|\,|\nabla g|\, dS\\ &\le C\big(\|u^\nu\|_{0,\Gamma}^2 + \|g\|_{1,\Gamma}\|u^\nu\|_{0,\Gamma}\big)\\ &\le \varepsilon \|\nabla u^\nu\|_2^2 + c_\varepsilon\big(\|u^\nu\|_2^2 + \|g\|_{H^{3/2}}^2\big).\end{aligned}$$

We finally obtain the following estimate

$$\|\nabla u^\nu\|_2^2 \le C\big(\|\omega^\nu\|_2^2 + \|u^\nu\|_2^2 + \|g\|_{H^{3/2}}^2\big). \tag{22}$$

Next, we add (19) together with (21) and by using repeatedly $u^\nu = u^\nu - u^L + u^L = U + u^L$ and also the inequality (22) we obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\big(\|U\|_2^2 + \|\omega^\nu\|_2^2\big) + \nu\big(\|\nabla U\|_2^2 + \|\nabla \omega^\nu\|_2^2\big)\\ &\quad \le C\big(\|u^L\|_{H^2}^2 \|U\|_2^2 + \|\nabla U\|_2^2 + \|\nabla u^L\|_2^2\big),\\ &\quad \le C\big(\|u^L\|_{H^2}^2 \|U\|_2^2 + \|\nabla u^\nu\|_2^2 + \|\nabla u^L\|_2^2\big),\\ &\quad \le C\big(\|u^L\|_{H^2}^2 \|U\|_2^2 + \|\omega^\nu\|_2^2 + \|u^\nu\|_2^2 + \|g\|_{H^{3/2}}^2 + \|\nabla u^L\|_2^2\big),\\ &\quad \le C\big(\|u^L\|_{H^2}^2 \|U\|_2^2 + \|\omega^\nu\|_2^2 + \|U\|_2^2 + \|u^L\|_2^2 + \|g\|_{H^{3/2}}^2 + \|\nabla u^L\|_2^2\big),\end{aligned}$$

for some constant C depending on the domain but not on the viscosity. Gronwall's lemma and the known regularity of g and of u^L give the uniform estimate

$$\sup_{t \in]0,T[} \|u^\nu(t)\|_{H^1} \le C,$$

which is sufficient to pass to the limit in the viscosity parameter. We finally obtained that (up to a sub-sequence) u^ν converges, as $\nu \to 0^+$, to a weak solution $u^E \in L^\infty(0, T; H^1(\Omega))$ of the Euler equations (3). □

Acknowledgments. L. C. Berselli thanks the organizers of the workshop held within the "Intensive Research Month on Hyperbolic Conservation Laws and Fluid Dynamics", Parma, February 2010, for the kind invitation.

S. Spirito acknowledges support and hospitality offered by the Centro De Giorgi (Scuola Normale Superiore di Pisa) and by the Department of Applied Mathematics "U. Dini" (University of Pisa) during the preparation of the paper.

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