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Symmetric system of balance laws for a micromorphic continuum model of dielectrics

Abstract. An electromechanical model for dielectric solids is formulated within the microcontinuum field theory. Electric dipole and quadrupole densities are introduced consistently with the microstructure and a set of non linear balance equations for micromorphic electroelasticity are derived. Constitutive assumptions are adopted accounting for additional internal variables compatibly with the second law of thermodynamics. It is shown that the differential system of balance laws can be given in a symmetric hyperbolic form. Stability conditions on wave motion superimposed to an undeformed polarized state are obtained in the form of suitable inequalities on constitutive parameters.

Keywords. Continuum theory of electroelastic solids, continuum micromechanics, wave stability.

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1 - Introduction

The growing interest in technological applications of materials manifesting various types of electromechanical effects has motivated a noticeable theoretical effort in developing continuum theories of mechanics, capable to account for electro-elastic, magneto-elastic, thermo-electro-elastic and other couplings. In the past few decades, fundamental contributions to these theories have been

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given concerning with the derivation of balance equations and boundary conditions compatible with the lattice's structure of the solid. A suitable set of constitutive equations have been also introduced, general enough to account for the desired electromechanical interactions (among the more relevant works, see [**3, 4, 12, 13, 14, 16**]). A great part of these models can be reduced to linear theories to be applied in various specific problems. Also, non linear theories have been exploited to derive the governing equations for electromechanical fields superimposed on an initially deformed configuration. In this respect, some recent studies have been proposed on the basis of essential constitutive assumptions in order to obtain definite results on incremental motions and stability (see [**5, 17**]).

More refined theories of mechanics have been also proposed where a mechanical microstructure is attached to the continuum material element (see [**6-10**]). Although this approach, in some sense, might be viewed as a bridge between the lattice's theory of solids and the classical continuum mechanics, it has been developed considering purely mechanical microfields. Electromagnetic interactions have been accounted for via a suitable choice of constitutive equations, beside the inclusion of Maxwell equations. This approach represents a refined electromagnetoelastic theory with an increased number of degrees of freedom but, ultimately, the microfields reflect a mechanical and not electromagnetic microstructure.

In the present paper we propose a micromorphic continuum model for electroelastic solids which extends the Eringen's approach of micromechanical fields [**6**] to electric dipole and quadrupole due to charge microdensities. This approach allows a definition of electromagnetic polarization and magnetization, in a natural way, as macroscopic fields in the continuum. In the first part of the paper we summarize some fundamental kinematical and dynamical concepts of micromorphic continua giving a set of balance equations for non linear microcontinuum dielectrics (Section 2) according to a previous work [**19**]. Constitutive equations are introduced in Section 3 exploiting the second law of thermodynamics and suitable internal variables which account for dissipative effects. In Section 4 we show that balance equations, together with the entropy inequality, can be written in the form of a symmetric hyperbolic quasi-linear differential system, from which, existence and uniqueness of the solution to the Cauchy problem could be proved. Then, in Section 5, we study the stability of wave motion about a mechanically undeformed configuration where a spontaneous electric polarization exists, as occurs in ferroelectric materials. The stability condition here derived applies to both linear and non linear waves provided a strict dissipative inequality hold.

2 - Micromorphic polarizable dielectrics

According to the Eringen's approach to microcontinuum mechanics [**6**], a continuum microstructure is given in the spatial configuration by the position of the microelement center of mass $\mathbf{x} = \hat{\mathbf{x}}(\mathbf{X}, t)$ and the relative position in the microelement $\xi = \hat{\xi}(\mathbf{X}, \Xi, t)$ where $\mathbf{X}$ and Ξ are the corresponding positions in the reference (material) configuration. Gradients with respect to $\mathbf{X}$ and $\mathbf{x}$ will be denoted by $\nabla_{\mathbf{X}}$ and ∇ respectively. We assume $\xi = \chi(\mathbf{X}, t)\Xi$, where χ is the microdeformation tensor and denote by $\mathbf{F}$ the deformation tensor $(\nabla_{\mathbf{X}}\mathbf{x})^T$, letting both $\mathbf{F}$ and χ to possess continuous inverses $\mathbf{F}^{-1}$ and $\boldsymbol{\mathfrak{X}}$. We also pose $J = \det \mathbf{F}$.

A convenient choice of strain measures is given by the following material tensors (see [**6**]),

$$\mathfrak{C} = \mathbf{F}^T \boldsymbol{\mathfrak{X}}^T, \qquad \mathcal{C} = \chi^T \chi, \qquad \boldsymbol{\Gamma} = \boldsymbol{\mathfrak{X}}(\nabla_{\mathbf{X}}\chi)^T, \tag{2.1}$$

which are called, respectively, deformation strain tensor, microdeformation strain tensor and wryness tensor. Material time rates of microdeformation tensors are given in terms of the microgyration tensor $\mathbf{N}$, as $\dot{\chi} = \mathbf{N}\chi$, $\dot{\boldsymbol{\mathfrak{X}}} = -\boldsymbol{\mathfrak{X}}\mathbf{N}$, whence

$$\dot{\xi} = \mathbf{N}\xi, \tag{2.2}$$

so that the velocity of a point in the microcontinuum is $\dot{\mathbf{x}} + \dot{\xi} = \mathbf{v} + \mathbf{N}\xi$. From the previous definitions, we have

$$\dot{\mathfrak{C}} = [\boldsymbol{\mathfrak{X}}(\mathbf{L} - \mathbf{N})\mathbf{F}]^T, \qquad \dot{\mathcal{C}} = 2\chi^T(\operatorname{Sym}\mathbf{N})\chi, \qquad \dot{\boldsymbol{\Gamma}} = [\boldsymbol{\mathfrak{X}}(\nabla\mathbf{N})^T\chi]\mathbf{F}, \tag{2.3}$$

where $\mathbf{L} = (\nabla\mathbf{v})^T$.

We consider here a dielectric elastic solid where free electric charges are absent and summarize the main results obtained in [**19**] on forces, couples an power densities in the micromorphic continuum, giving the corresponding balance equations.

If ΔV is the volume of a microelement and $\rho'(\mathbf{x}, \xi)$ denotes the mass density within the microelement, the zeroth and second order (in ξ) quantities

$$\frac{1}{\Delta V}\int_{\Delta V} \rho'(\mathbf{x}, \xi)\,\mathrm{d}v' := \rho(\mathbf{x}), \quad \frac{1}{\rho\Delta V}\int_{\Delta V} \rho'(\mathbf{x}, \xi)\,\xi \otimes \xi\,\mathrm{d}v' := \mathcal{I}(\mathbf{x}) \tag{2.4}$$

define, respectively, the (macroscopic) mass density and the microinertia tensor. The corresponding first order quantity vanishes identically. The momentum and moment of momentum in the continuum turn out to be

$$\rho\mathbf{v}, \qquad \mathbf{x} \times \rho\mathbf{v} + \rho\mathbf{w}, \tag{2.5}$$

where $\mathbf{w}$ has components

$$w_i = \varepsilon_{ijk} N_{kl} \mathcal{I}_{jl}.$$

In a similar way, denoting by $\sigma'(\mathbf{x},\xi)$ the electric charge density in the microelement, the first and second order quantities

$$\frac{1}{\Delta V}\int_{\Delta V}\sigma'(\mathbf{x},\xi)\,\xi\,\mathrm{d}v' := \mathbf{p}(\mathbf{x}),$$
$$\frac{1}{\Delta V}\int_{\Delta V}\sigma'(\mathbf{x},\xi)\,\xi\otimes\xi\,\mathrm{d}v' := \mathbf{Q}(\mathbf{x}), \tag{2.6}$$

are identified respectively with the electric dipole density and the electric quadrupole density. The last quantities are microcontinuum counterparts of dipole and quadrupole densities introduced in lattice's theory of polarizable crystals (see [**12**], ch.3 and references therein). Mechanical and electromagnetic force densities are given by

$$\mathbf{f}(\mathbf{x}) = \mathbf{f}'(\mathbf{x},\mathbf{0}) + \frac{1}{2}\mathcal{I}\,\nabla(\nabla\mathbf{f}')(\mathbf{x},\mathbf{0}). \tag{2.7}$$

$$\begin{aligned}\mathbf{f}^{\mathrm{em}}(\mathbf{x}) =& (\mathbf{p}\cdot\nabla)\boldsymbol{\mathcal{E}} + \frac{1}{2}(\mathbf{Q}\cdot\nabla)\nabla\boldsymbol{\mathcal{E}} + \frac{1}{c}[(\mathbf{N}-\mathbf{L})\mathbf{p}]\times\mathbf{B}\\ &+\frac{1}{c}[(\mathbf{N}-\mathbf{L})\mathbf{Q}\nabla]\times\mathbf{B} + \frac{1}{2c}\mathbf{B}\times\left[(\mathbf{Q}\cdot\nabla)\mathbf{L}^T\right],\end{aligned} \tag{2.8}$$

where $\mathbf{f}'(\mathbf{x},\xi)$ is the microelement force density and $\boldsymbol{\mathcal{E}} = \mathbf{E} + \frac{\mathbf{v}}{c}\times\mathbf{B}$. Here $\mathbf{E}$ and $\mathbf{B}$ are the local electric field and the magnetic induction within the microelement and c is the light's speed. As a consequence, the mechanical and electromagnetic force's moments are

$$\mathbf{x}\times\rho\mathbf{f} + \rho\mathbf{c}, \qquad \mathbf{x}\times\mathbf{f}^{\mathrm{em}} + \mathbf{c}^{\mathrm{em}}, \tag{2.9}$$

where, in components,

$$c_i = \varepsilon_{ijk} f'_{k,l}\,\mathcal{I}_{jl}, \qquad c_i^{\mathrm{em}} = \varepsilon_{ijk} C_{jk}^{\mathrm{em}}, \tag{2.10}$$

with

$$\mathbf{C}^{\mathrm{em}} = \mathbf{p}\otimes\boldsymbol{\mathcal{E}} + \mathbb{E}\ \mathbf{Q}. \tag{2.11}$$

Here we have introduced the tensor $\mathbb{E}$ with entries

$$E_{kl} = E_{k,l} + \frac{1}{c}\varepsilon_{kpq}(N_{pl} - v_{p,l})B_q. \tag{2.12}$$

According to [**6**] the traction on the element of surface with normal $\mathbf{n}$ and the moment of surface traction can be written in terms of an electromechanical Cauchy stress tensor $\mathbf{T}$ and a third order tensor $\mathbf{m}$ as

$$\mathbf{n}\mathbf{T}, \qquad \mathbf{x}\times(\mathbf{n}\mathbf{T}) + \mathbf{n}\mathbf{M} \tag{2.13}$$

where the second order tensor $\mathbf{M}$ is given by

$$M_{ij} = \varepsilon_{jhk} m_{ikh}.$$

The previous definitions allow us to evaluate the following mechanical and electromagnetic power of body and surface forces

$$w^{\rm me} = \rho \mathbf{f} \cdot \mathbf{v} + \rho \,\mathrm{tr}[\mathcal{I}(\nabla \mathbf{f})\mathbf{N}], \tag{2.14}$$

$$w^{\rm em} = \mathbf{E} \cdot \mathbf{N}\mathbf{p} + \mathbf{p} \cdot (\nabla \mathbf{E})\mathbf{v} + \mathrm{tr}[\mathbf{Q}(\nabla \mathbf{E})\mathbf{N}] + \mathbf{v} \cdot (\mathbf{Q} \cdot \nabla)\nabla \mathbf{E}, \tag{2.15}$$

$$w^{\mathbf{n}} = \mathbf{n}\mathbf{T} \cdot \mathbf{v} + \mathrm{tr}(\mathbf{n}\mathbf{m}\mathbf{N}^T). \tag{2.16}$$

As a fundamental consequence of the previous results, equations (2.8), (2.11) and (2.14) imply

$$w^{\rm em} - \mathbf{f}^{\rm em} \cdot \mathbf{v} - \mathrm{tr}(\mathbf{C}^{\rm em}\mathbf{N}) = 0. \tag{2.17}$$

From equations $(2.4)_2$ and (2.6) we obtain the following balance laws for microinertia, dipole density and quadrupole density,

$$\dot{\mathcal{I}} = 2\,\mathrm{Sym}(\mathbf{N}\mathcal{I}), \tag{2.18}$$

$$\begin{aligned} &\dot{\mathbf{p}} + \mathbf{p}(\nabla \cdot \mathbf{v}) = \mathbf{N}\mathbf{p}, \\ &\dot{\mathbf{Q}} + \mathbf{Q}(\nabla \cdot \mathbf{v}) = 2\,\mathrm{Sym}(\mathbf{N}\mathbf{Q}). \end{aligned} \tag{2.19}$$

From equations (2.8), (2.9), (2.13)-(2.16), we arrive at the following balance laws for momentum, spin and energy,

$$\rho \dot{\mathbf{v}} = \rho \mathbf{f} + \mathbf{f}^{\rm em} + \nabla \cdot \mathbf{T}, \tag{2.20}$$

$$\rho \boldsymbol{\sigma} = \rho (\nabla \mathbf{f})^T \mathcal{I} + \mathfrak{T}^T - \mathbf{S} + \nabla \cdot \mathbf{m}, \tag{2.21}$$

$$\rho \dot{e} = \mathrm{tr}(\mathbf{S}\mathbf{N}) + \mathrm{tr}[(\mathfrak{T} - \mathbf{C}^{\rm em})(\mathbf{L} - \mathbf{N})] + m_{ijk} N_{jk,i} + \rho h - \nabla \cdot \mathbf{q}\,, \tag{2.22}$$

where

$$\boldsymbol{\sigma} = \dot{\mathbf{N}}\mathcal{I} + \mathbf{N}\mathbf{N}\mathcal{I}, \tag{2.23}$$

is the spin inertia tensor, $\mathfrak{T} = \mathbf{T} + \mathbf{C}^{\rm em}$ and $\mathbf{S}$ is a suitable second order symmetric tensor arising from the derivation of the dual form (2.21) of the balance law for moment of momentum (see [**6**]). In addition, e, h are, respectively, the internal energy per unit mass and the heat supply per unit mass, and $\mathbf{q}$ is the heat flux. A comparison with the corresponding balance law given by Eringen in [**6**] shows that in view of equation (2.17), electromagnetic contributions to the power are here implicitly accounted for by the tensors $\mathbf{S}$ and $\mathbf{T}$, via the balance law (2.22) (see [**19**]).

The previous equations must be complemented with the Maxwell equations for the electromagnetic field. Assuming that the electric polarization can be approximated by the electric dipole density, we write the electric displacement as

$$\mathbf{D} = \mathbf{E} + \mathbf{p}, \tag{2.24}$$

and, according to [**12**] and [**19**], we obtain the following expression for the magnetization vector $\boldsymbol{\mathcal{M}}$,

$$\mathcal{M}_i = \frac{1}{2c}\varepsilon_{ijk}(N_{kp} - v_{k,p})Q_{pj}. \tag{2.25}$$

Denoting by $\mathbf{H}$ the magnetic field, we have $\mathbf{B} = \mathbf{H} + \boldsymbol{\mathcal{M}}$ and, in Heaviside-Lorentz units, the following Maxwell equations must be satisfied,

$$\nabla \cdot \mathbf{D} = 0, \qquad \nabla \cdot \mathbf{B} = 0, \tag{2.26}$$

$$\nabla \times \mathbf{E} + \frac{1}{c}\dot{\mathbf{B}} = 0, \qquad \nabla \times \mathbf{H} - \frac{1}{c}\dot{\mathbf{D}} = 0. \tag{2.27}$$

3 - Constitutive assumptions

In order to formulate constitutive assumptions and require their compatibility with the second law of thermodynamics it is convenient to rewrite the governing equations of Section 2 in the material form. Concerning with the balance equations for dipole and quadrupole densities, we introduce the material quantities

$$\mathbb{P} = J\,\mathbf{F}^{-1}\mathbf{p}, \qquad \mathbb{Q} = J\mathbf{F}^{-1}\mathbf{Q}\boldsymbol{\mathfrak{X}}^T,$$

and pose

$$\mathbb{L} = \mathbf{F}^{-1}(\mathbf{L} - \mathbf{N})\mathbf{F}.$$

The balance equations (2.19) take, respectively, the following material form

$$\dot{\mathbb{P}} + \mathbb{L}\mathbb{P} = \mathbf{0}, \qquad \dot{\mathbb{Q}} + \mathbb{L}\mathbb{Q} = \mathbf{0}. \tag{3.1}$$

Then, introducing the first Piola-Kirchoff stress tensor $\mathbb{T} = J\mathbf{F}^{-1}\mathfrak{T}$, the second Piola-Kirchoff tensors

$$\mathbb{Y} = J\mathbf{F}^{-1}\mathfrak{T}\boldsymbol{\chi}, \qquad \mathbb{S} = J\boldsymbol{\mathfrak{X}}\mathbf{R}\boldsymbol{\mathfrak{X}}^T, \qquad \mathbb{M} = J\mathbf{F}^{-1}\mathbf{m}\boldsymbol{\mathfrak{X}}^T\boldsymbol{\chi},$$

and the material fields

$$\mathfrak{f} = \mathbf{f}\boldsymbol{\chi}, \quad \mathbb{N} = \boldsymbol{\chi}^T\mathbf{N}\boldsymbol{\chi}, \quad \mathfrak{I} = \boldsymbol{\mathfrak{X}}\mathcal{I}\boldsymbol{\mathfrak{X}}^T,$$

$$\mathfrak{E} = \boldsymbol{\mathcal{E}}\mathbf{F}, \qquad \mathbb{E} = \mathbf{F}^T\mathbb{E}\ \ \boldsymbol{\chi},$$

the balance equations (2.20) and (2.21) become

$$\rho_0\dot{\mathbf{v}} = \rho_0\mathbf{f} + J\mathbf{f}^{\mathrm{em}} + \nabla_{\mathbf{X}}\cdot[\mathbb{T} - (\mathbb{P}\otimes\mathfrak{E} + \mathbb{Q}\mathbb{E}^T)\mathbf{F}^{-1}], \tag{3.2}$$

$$\rho_0\dot{\mathbb{N}} = \rho_0[(\nabla_{\mathbf{X}}\mathfrak{f})^T - \mathfrak{f}\Gamma]\mathfrak{C}^{-T} + [\mathfrak{C}\mathbb{Y}^T\mathcal{C}^{-1} - \mathcal{C}\mathbb{S} + D_{\mathbf{X}}\cdot\mathbb{M} - 2\rho_0\,\mathrm{Sym}(\mathbb{N})\mathcal{C}^{-1}\mathbb{N}]\mathfrak{I}^{-1}, \tag{3.3}$$

where $\rho_0 J = \rho$ and where

$$(D_{\mathbf{X}}\cdot\mathbb{M})_{QR} = \mathbf{M}_{HQR,H} + \mathbf{M}_{HQK}\Gamma_{RKH} - \mathbf{M}_{HKR}\Gamma_{KQH}.$$

With a similar procedure, from equation (2.22) we obtain the material form of the energy balance

$$\rho_0\dot{e} = \frac{1}{2}\dot{\mathcal{C}} : \mathbb{S} + \dot{\mathbb{P}}\cdot\mathfrak{E} + \dot{\mathbb{Q}} : \mathbb{E} + \dot{\mathfrak{C}} : \mathbb{Y} + \dot{\Gamma}_{LKH}\mathbf{M}_{HLK} + \rho_0 h - \nabla_{\mathbf{X}}\cdot\mathfrak{q}, \tag{3.4}$$

where

$$\mathfrak{q} = J\mathbf{F}^{-1}\mathbf{q}.$$

Equation (3.3) is equivalent to that obtained in [**19**] which was written in terms of the spin inertia tensor. The present form is more convenient for the analysis of the next section. The Maxwell equations can be given in the material form after the introduction of the following fields,

$$\mathfrak{B} = J\mathbf{F}^{-1}\mathbf{B}, \qquad \mathfrak{D} = J\mathbf{F}^{-1}\mathbf{D}, \qquad \mathfrak{H} = \Big(\mathbf{H} - \frac{\mathbf{v}}{c}\times\mathbf{D}\Big)\mathbf{F}.$$

Thus equations (2.26) and (2.27) are replaced by (see also [**12**]),

$$\nabla_{\mathbf{X}}\cdot\mathfrak{D} = 0, \qquad \nabla_{\mathbf{X}}\cdot\mathfrak{B} = 0, \tag{3.5}$$

$$\nabla_{\mathbf{X}}\times\mathfrak{E} + \frac{1}{c}\dot{\mathfrak{B}} = 0, \qquad \nabla_{\mathbf{X}}\times\mathfrak{H} - \frac{1}{c}\dot{\mathfrak{D}} = 0. \tag{3.6}$$

Equations (3.6) can be rewritten in a form of balance law which will be useful in the analysis of the next section. Owing to equations (2.24) and (2.25) we obtain

$$\dot{\mathfrak{D}} = \nabla_{\mathbf{X}}\cdot\mathbb{X}, \tag{3.7}$$

$$\dot{\mathfrak{B}} = \nabla_{\mathbf{X}}\cdot\mathbb{Z} \tag{3.8}$$

where

$$\mathbb{X} = \frac{c}{J}\varepsilon\mathbf{C}\mathfrak{B} - 2\mathfrak{D}\otimes(\mathbf{F}^{-1}\mathbf{v}) + \mathbb{L}\mathbb{Q}\mathfrak{C}^{-T},$$

$$\mathbb{Z} = -\frac{c}{J}\varepsilon\mathbf{C}(\mathfrak{D} - \mathbb{P}) - 2(\mathbf{F}^{-1}\mathbf{v})\otimes\mathfrak{B}.$$

Here ε is a third order tensor whose entries are the permutation symbols ε_{HKL}.

We denote by η the entropy density and introduce the free energy density $\psi = e - \eta\theta$, where θ is the thermodynamic temperature. Then, the second law of thermodynamics can be written as

$$\rho\dot{\eta} \geq \rho\frac{h}{\theta} - \nabla\cdot\frac{\mathbf{q}}{\theta}. \tag{3.9}$$

In view of equation (2.22) and accounting for the previous positions, we obtain the following material form of (3.9),

$$\rho_0(\dot{\psi} + \eta\dot{\theta}) - \frac{1}{2}\dot{\mathcal{C}} : \mathbb{S} - \dot{\mathbb{P}}\cdot\mathfrak{E} - \dot{\mathbb{Q}} : \mathbb{E} - \dot{\mathfrak{C}} : \mathbb{Y} - \dot{\Gamma}_{LKH}\mathbf{M}_{HLK} + \frac{1}{\theta}\mathfrak{q}\cdot\mathbf{G} \leq 0, \tag{3.10}$$

where

$$\mathbf{G} = \nabla_{\mathbf{X}}\,\theta.$$

We account for electromechanical dissipative effects introducing a set of internal variables which satisfy suitable evolution equations (see [**18**]). Accordingly we assume the following dependence of the free energy density,

$$\psi = \widetilde{\psi}(\mathcal{C}, \mathfrak{C}, \boldsymbol{\Gamma}, \mathbb{P}, \mathbb{Q}, \theta; \boldsymbol{\Omega}, \boldsymbol{\kappa}) \tag{3.11}$$

where $\boldsymbol{\Omega}$ and $\boldsymbol{\kappa}$ are, respectively, a second order symmetric tensor and a vector which play the role of internal variables. They are supposed to satisfy the following evolution equations,

$$\dot{\boldsymbol{\Omega}} = \hat{\boldsymbol{\Omega}}(\Lambda; \hat{\Lambda}; \widetilde{\Lambda}), \qquad \dot{\boldsymbol{\kappa}} = \hat{\boldsymbol{\kappa}}(\Lambda; \hat{\Lambda}; \widetilde{\Lambda}), \tag{3.12}$$

where

$$\Lambda = \{\mathfrak{C}, \mathbb{P}, \mathbb{Q}\}, \qquad \hat{\Lambda} = \{\dot{\mathfrak{C}}, \dot{\mathbb{P}}, \dot{\mathbb{Q}}, \mathbf{G}\}, \qquad \widetilde{\Lambda} = \{\boldsymbol{\Omega}, \boldsymbol{\kappa}\}. \tag{3.13}$$

For the sake of simplicity we consider evolution equations which are linear with respect to the fields of the sets $\hat{\Lambda}$ and $\widetilde{\Lambda}$, i.e.,

$$\begin{aligned} \hat{\boldsymbol{\Omega}} &= \dot{\mathfrak{C}} + \mathbb{A}\dot{\mathbb{P}} + \mathbb{B}\dot{\mathbb{Q}} + \gamma_{\boldsymbol{\Omega}}\boldsymbol{\Omega} \\ \hat{\boldsymbol{\kappa}} &= \mathbf{R}\mathbf{G} + \gamma_{\boldsymbol{\kappa}}\boldsymbol{\kappa} \end{aligned} \tag{3.14}$$

where $\mathbb{A}, \mathbb{B}, \mathbf{R}$ are respectively third-order, fourth-order and second-order tensors and $\gamma_{\boldsymbol{\Omega}}, \gamma_{\boldsymbol{\kappa}}$ are real quantities. They are all functions of the variables in the set Λ and, in view of the symmetry of $\boldsymbol{\Omega}$, their entries comply with the following conditions

$$\mathbb{A}_{HKL} = \mathbb{A}_{KHL} \qquad \mathbb{B}_{HKLM} = \mathbb{B}_{KHLM}.$$

Substituting equations (3.11) and (3.14) into (3.10) we obtain the following constitutive equations

$$(3.15)\qquad \begin{aligned} &\mathbb{Y} = \rho_0(\widetilde{\psi}_{\mathfrak{C}} + \widetilde{\psi}_{\boldsymbol{\Omega}}), && \mathfrak{E} = \rho_0(\widetilde{\psi}_{\mathbb{P}} + \widetilde{\psi}_{\boldsymbol{\Omega}}\mathbb{A}), && \mathbb{E} = \rho_0(\widetilde{\psi}_{\mathbb{Q}} + \widetilde{\psi}_{\boldsymbol{\Omega}}\mathbb{B}),\\ &\mathbb{S} = 2\rho_0\widetilde{\psi}_{C}, && \mathbb{M} = \rho_0\widetilde{\psi}_{\boldsymbol{\Gamma}}, &&\\ &\eta = -\widetilde{\psi}_{\theta}, && \mathfrak{q} = -\rho_0\theta\widetilde{\psi}_{\boldsymbol{\kappa}}\mathbf{R} && \end{aligned}$$

together with the dissipative inequality

$$(3.16)\qquad \gamma_{\boldsymbol{\Omega}}\widetilde{\psi}_{\boldsymbol{\Omega}}\boldsymbol{\Omega} + \gamma_{\boldsymbol{\kappa}}\widetilde{\psi}_{\boldsymbol{\kappa}}\boldsymbol{\kappa} \leq 0.$$

In equations (3.15), (3.16) and in the sequel, derivatives with respect to a field variable are denoted by the pertinent subscript. In view of the identity $\dot{\mathfrak{C}} = \mathbb{L}^T\mathfrak{C}$, and exploiting equations (3.1), equation $(3.12)_1$ can be rewritten as

$$(3.17)\qquad \dot{\boldsymbol{\Omega}} = \mathbb{L}^T\mathfrak{C} - \mathbb{A}\mathbb{L}\mathbb{P} - \mathbb{B}\mathbb{L}\mathbb{Q} + \gamma_{\boldsymbol{\Omega}}\boldsymbol{\Omega}.$$

Finally, assuming that $\mathbf{R}$ depends only on θ and introducing $\mathbb{R}$ such that $\mathbf{R} = \frac{\mathrm{d}}{\mathrm{d}\theta}\mathbb{R}$, we rewrite equation $(3.12)_2$ in the form

$$(3.18)\qquad \dot{\boldsymbol{\kappa}} = \nabla_{\mathbf{X}} \cdot \mathbb{R}^T + \gamma_{\boldsymbol{\kappa}}\boldsymbol{\kappa}.$$

We observe that the choice of the sets of variables $\hat{\Lambda}$ and $\widetilde{\Lambda}$ is essential to account for thermoelectromechanical dissipative effects. More general choices are possible which include dependence on microdeformation strain and wryness (see [**19**]).

4 - Quasi-linear system of balance laws

In order to obtain a symmetric form of the system of balance laws, we reduce the previous analysis to the case in which the non-local dependence on micro and macro fields are neglected. To this end we introduce the quantity

$$(4.1)\qquad \varepsilon = e + \frac{1}{2}v^2 + \frac{1}{2}\mathbb{N} : \mathfrak{I}\mathbb{N} + \frac{1}{\rho_0}(\mathfrak{E}\cdot\mathfrak{D} + \mathfrak{H}\cdot\mathfrak{B}),$$

and replace the governing equations derived in Section 3 by a set of balance equations for the field

$$\mathbf{z} = (\mathbf{F}, \mathbf{v}, \boldsymbol{\chi}, \mathbb{N}, \varepsilon, \mathbb{P}, \mathbb{Q}, \mathfrak{D}, \mathfrak{B}, \boldsymbol{\Omega}, \boldsymbol{\kappa}),$$

assuming that these equations hold for null spatial gradients of $\mathbf{z}$. We also discard the body force density $\mathbf{f}$ and, owing to the material form of the microinertia balance (2.18),

$$\boldsymbol{\chi}\dot{\mathfrak{I}}\boldsymbol{\chi}^T = \mathbf{0},$$

we pose $\mathfrak{I} = \mathfrak{I}_0$, where $\mathfrak{I}_0$ is a constant tensor. Accordingly, equations (3.2) and (3.3) reduce to

$$\rho_0 \dot{\mathbf{v}} = \nabla_{\mathbf{X}} \cdot [\mathbb{T} - (\mathbb{P} \otimes \mathfrak{E} + \mathbb{Q}\mathbb{E}^T)\mathbf{F}^{-1}] + J\mathbf{f}^{\text{em}}, \tag{4.2}$$

$$\rho_0 \dot{\mathbb{N}} = \nabla_{\mathbf{X}} \cdot (\mathbb{M}\mathfrak{I}_0^{-1}) + \boldsymbol{\Sigma}, \tag{4.3}$$

where

$$\boldsymbol{\Sigma} = [\mathfrak{C}\mathbb{Y}^T \mathcal{C}^{-1} - \mathcal{C}\mathbb{S} - 2\rho_0 \,\text{Sym}\,(\mathbb{N})\mathcal{C}^{-1}\mathbb{N}]\mathfrak{I}_0^{-1}.$$

Owing to (4.2) and (4.3) we can rewrite the energy equation (3.4) as a balance law for ε in the following form

$$\rho_0 \dot{\varepsilon} = -\nabla_{\mathbf{X}} \cdot \mathfrak{q} + J\mathbf{v} \cdot \mathbf{f}^{\text{em}} + \boldsymbol{\Sigma}\mathfrak{I}_0 : \mathbb{N} + \Sigma + (\dot{\mathfrak{E}} \cdot \mathfrak{D} + \dot{\mathfrak{H}} \cdot \mathfrak{B}) + \rho_0 h \tag{4.4}$$

where

$$\Sigma = \text{Sym}\,(\mathbb{N}) : \mathbb{S} + \mathbb{L}^T \mathfrak{C} : \mathbb{Y} - \mathbb{L}\mathbb{P} \cdot \mathfrak{E} - \mathbb{L}\mathbb{Q} : \mathbb{E},$$

and where we exploited equations (3.7), (3.8) and the independence on the gradients of $\mathbf{z}$. The deformation tensor and the microdeformation tensor satisfy the equations

$$\dot{\mathbf{F}} = (\nabla_{\mathbf{X}} \mathbf{v})^T, \qquad \dot{\boldsymbol{\chi}} = \boldsymbol{\mathfrak{X}}^T \mathbb{N}, \tag{4.5}$$

hence the whole set of balance equations for the field $\mathbf{z}$ is given by equations (4.1)-(4.4), (3.1), (3.7), (3.8), (3.17) and (3.18). It can be cast in the following form

$$\partial_\alpha \mathbf{K}_\alpha(\mathbf{z}) = \mathbf{h}(\mathbf{z}), \qquad \alpha = 0, 1, 2, 3, \tag{4.6}$$

where $\partial_0 = \dfrac{\partial}{\partial t}$, $\partial_K = \dfrac{\partial}{\partial X_K}$, $K = 1, 2, 3$, $\mathbf{K}_0 = \mathbf{z}$ and $(\mathbf{K}_1, \mathbf{K}_2, \mathbf{K}_3) = \mathbb{K}$, with

$$\mathbb{K} = -\begin{pmatrix} \mathbf{v}\mathbb{I} \\ \dfrac{1}{\rho_0}[\mathbb{T} - (\mathbb{P} \otimes \mathfrak{E} + \mathbb{Q}\mathbb{E}^T)\mathbf{F}^{-1}] \\ \mathbf{0} \\ \dfrac{1}{\rho_0}\mathbb{M}\mathfrak{I}_0^{-1} \\ -\dfrac{1}{\rho_0}\mathfrak{q} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbb{X} \\ \mathbb{Z} \\ \mathbf{0} \\ \mathbb{R}^T \end{pmatrix}, \tag{4.7}$$

$$\mathbf{h} = \begin{pmatrix} \mathbf{0} \\ \dfrac{J}{\rho_0}\mathbf{f}^{\mathrm{em}} \\ \boldsymbol{\mathfrak{X}}^T\mathbb{N} \\ \dfrac{1}{\rho_0}\boldsymbol{\Sigma} \\ \dfrac{1}{\rho_0}[J\mathbf{v}\cdot\mathbf{f}^{\mathrm{em}} + \boldsymbol{\Sigma}\mathfrak{I}_0 : \mathbb{N} + \Sigma + (\dot{\mathfrak{E}}\cdot\mathfrak{D} + \dot{\mathfrak{H}}\cdot\mathfrak{B})] + h \\ -\mathbb{L}\mathbb{P} \\ -\mathbb{L}\mathbb{Q} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbb{L}^T\mathfrak{C} - \mathbb{A}\mathbb{L}\mathbb{P} - \mathbb{B}\mathbb{L}\mathbb{Q} + \gamma_{\boldsymbol{\Omega}}\boldsymbol{\Omega} \\ \gamma_{\boldsymbol{\kappa}}\boldsymbol{\kappa} \end{pmatrix}, \tag{4.8}$$

where $\mathbb{I}$ has entries δ_{HK}. Again, it is understood that the components of $\mathbb{K}$ and $\mathbf{h}$ are valued for null gradients of the fields $\mathbf{z}$.

Using the last two equations in (3.15), the entropy inequality (3.9) can be rewritten as

$$\dot{\widetilde{\psi}}_\theta \le -\nabla_{\mathbf{X}}\cdot(\widetilde{\psi}_{\boldsymbol{\kappa}}\mathbb{R}') - \frac{h}{\theta}.$$

Since the free energy introduced in the constitutive theory of Section 3 can be expressed as a function of $\mathbf{z}$, we rewrite that inequality in the following form

$$\partial_\alpha \pi_\alpha(\mathbf{z}) \le g(\mathbf{z}), \tag{4.9}$$

where

$$\pi_0 = \widetilde{\psi}_\theta, \qquad \pi_K = (\widetilde{\psi}_{\boldsymbol{\kappa}})_H \mathbb{R}'_{HK}, \qquad g = -\frac{h}{\theta}.$$

Equation (4.6) represents a system of first-order quasi-linear partial differential equations for the fields $\mathbf{z}$. Equation (4.9) is an additional scalar inequality which is required to be satisfied by the solutions of system (4.6). This inequality can be exploited to derive a symmetrizability condition by the use of the entropy theorem (see [**1, 11, 20, 23**]). According to this theorem there exist a privileged field $\mathbf{z}'$ and four potentials π'_α such that

$$\mathrm{d}\pi_\alpha = \mathbf{z}'\cdot\mathrm{d}\mathbf{K}_\alpha, \qquad \mathbf{K}_\alpha = (\pi'_\alpha)_{\mathbf{z}'}. \tag{4.10}$$

Substitution of $(4.10)_2$ into (4.6) yields the equivalent system

$$(\pi'_\alpha)_{\mathbf{z}'\mathbf{z}'}\partial_\alpha\mathbf{z}' = \mathbf{h}. \tag{4.11}$$

If π'_0 is a strictly convex function of $\mathbf{z}'$, equation (4.11) represents a symmetric hyperbolic system of balance laws. Well posedness of the local Cauchy problem with smooth initial data can be proved for such a class of differential system. From equations (4.10) we get

$$\pi_\alpha = \mathbf{z}' \cdot \mathbf{K}_\alpha - \pi'_\alpha, \tag{4.12}$$

whence, for $\alpha = 0$,

$$\pi_0 = \mathbf{z}' \cdot \mathbf{z} - \pi'_0, \qquad \mathbf{z}' = (\pi_0)_{\mathbf{z}}.$$

This means that π'_0 is the Legendre transform of π_0 and the convexity of π_0 implies the convexity of π'_0. From (4.1) we have

$$\pi_0 = \frac{1}{\theta}\left[\widetilde{\psi} - \varepsilon + \frac{1}{2}v^2 + \frac{1}{2}\mathbb{N} : \mathfrak{I}\mathbb{N} + \frac{1}{\rho_0}(\mathfrak{E}\cdot\mathfrak{D} + \mathfrak{H}\cdot\mathfrak{B})\right],$$

and accounting for the constitutive assumptions on $\widetilde{\psi}$ and the dependence of $\mathfrak{H}$ on $\mathbf{F}, \mathbf{v}, \boldsymbol{\chi}, \mathbb{N}, \mathbb{Q}, \mathfrak{D}, \mathfrak{B}$, we obtain

$$\mathbf{z}' = \frac{1}{\theta}\begin{pmatrix} \boldsymbol{\mathfrak{X}}^T\widetilde{\psi}_{\mathbb{G}} + \dfrac{1}{\rho_0}(\mathfrak{E}_{\mathbf{F}}\cdot\mathfrak{D} + \mathfrak{H}_{\mathbf{F}}\cdot\mathfrak{B}) \\ \mathbf{v} + \dfrac{1}{\rho_0}\mathfrak{H}_{\mathbf{v}}\cdot\mathfrak{B} \\ \boldsymbol{\chi}^T\widetilde{\psi}_{\mathcal{C}} + \widetilde{\psi}_{\mathcal{C}}\boldsymbol{\chi} + \dfrac{1}{\rho_0}\left(\mathfrak{E}_{\boldsymbol{\chi}}\cdot\mathfrak{D} + \mathfrak{H}_{\boldsymbol{\chi}}\cdot\mathfrak{B}\right) \\ \mathbb{N} + \dfrac{1}{\rho_0}\mathfrak{H}_{\mathbb{N}}\cdot\mathcal{B} \\ -1 \\ \widetilde{\psi}_{\mathbb{P}} + \dfrac{1}{\rho_0}\mathfrak{E}_{\mathbb{P}}\cdot\mathfrak{D} \\ \widetilde{\psi}_{\mathbb{Q}} + \dfrac{1}{\rho_0}(\mathfrak{E}_{\mathbb{Q}}\cdot\mathfrak{D} + \mathfrak{H}_{\mathbb{Q}}\cdot\mathfrak{B}) \\ \dfrac{1}{\rho_0}(\mathfrak{E} + \mathfrak{H}_{\mathfrak{D}}\cdot\mathfrak{B}) \\ \dfrac{1}{\rho_0}(\mathfrak{H} + \mathfrak{H}_{\mathfrak{B}}\cdot\mathfrak{B}) \\ \widetilde{\psi}_{\boldsymbol{\Omega}} + \dfrac{1}{\rho_0}\mathfrak{E}_{\boldsymbol{\Omega}}\cdot\mathfrak{D} \\ \widetilde{\psi}_{\boldsymbol{\kappa}} + \dfrac{1}{\rho_0}\mathfrak{E}_{\boldsymbol{\kappa}}\cdot\mathfrak{D} \end{pmatrix}. \tag{4.13}$$

We observe that, substituting equation (4.12) into inequality (4.9) we obtain

$$\mathbf{z}' \cdot \mathbf{h} \leq g,$$

which, in view of equations (4.8) and (4.13), coincides with the dissipative inequality (3.16). Denoting by $\mathcal{H}(\hat{\mathbf{z}})$ the hessian matrix of the restriction of π_0 to the field $\hat{\mathbf{z}}$, the convexity requirement amounts to

$$\begin{aligned} &\mathcal{H}(\mathbf{F}, \boldsymbol{\chi}, \mathbb{P}, \mathbb{Q}, \mathfrak{D}, \mathfrak{B}, \boldsymbol{\Omega}, \boldsymbol{\kappa}) \qquad \text{is positive definite,} \\ &\widetilde{\psi}_{\theta\theta} < 0, \end{aligned} \tag{4.14}$$

and, in particular, this implies the condition

$$\begin{pmatrix} \widetilde{\psi}_{\boldsymbol{\Omega}\boldsymbol{\Omega}} & \widetilde{\psi}_{\boldsymbol{\Omega}\boldsymbol{\kappa}} \\ \widetilde{\psi}_{\boldsymbol{\kappa}\boldsymbol{\Omega}} & \widetilde{\psi}_{\boldsymbol{\kappa}\boldsymbol{\kappa}} \end{pmatrix} \quad \text{is positive definite.} \tag{4.15}$$

Together with inequality (3.16), the constraint (4.15) yields

$$\gamma_{\boldsymbol{\Omega}} \leq 0, \qquad \gamma_{\boldsymbol{\kappa}} \leq 0. \tag{4.16}$$

We note that the question about the existence of global solutions for the Cauchy problem requires, in general, a more detailed analysis of system (4.11), in order to check specific constraints on characteristic eigenvectors (a general view on this point can be found, for example, in [**22**] and references therein). In the next section we restrict the problem of smooth global solutions to a more specific setting.

5 - Wave stability with respect to an undeformed state

The symmetric system of balance laws derived in the previous section is exploited here to derive a stability condition for wave propagation. The present derivation parallels that performed in [**18**] concerning a continuum dielectric model where polarization gradients are accounted for. The basic point underlying the present derivation is that the asymptotic stability condition for non-linear waves, with respect to a fixed configuration $\mathbf{z}_0$, is equivalent to the stability condition of high frequency linear waves propagating about $\mathbf{z}_0$ (see [**15, 21**]).

Here we have in mind a ferroelectric solid, for example, of dipole type with θ_0 below the Curie temperature. We assume in the following that the heat supply h be zero and that an undeformed configuration exists at $\theta = \theta_0$ in which macro and microdeformations are absent and, possibly, a natural electric polarization is allowed in such a way that system (4.6) be satisfied with $\mathbf{h} = \mathbf{0}$. Incidentally, we observe that a linearization of the governing system, and, in particular, of equations (3.1) implies the trivial vanishing of dipole and quadrupole densities if they are zero in the initial unperturbed configuration. In this sense, the present model is a genuinely non linear theory for polarizable electromagnetic solids.

In terms of the field $\mathbf{z}$ the unperturbed state is

$$\mathbf{z}_0 = (\mathbf{I}, \mathbf{0}, \mathbf{I}, \mathbf{0}, 0, \mathbb{P}_0, \mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{0}). \tag{5.1}$$

The corresponding privileged field $\mathbf{z}'$ turns out to be

$$\mathbf{z}'_0 = \frac{1}{\theta_0}(\mathbf{I}\widetilde{\psi}^{(0)}_{\mathfrak{C}}, \mathbf{0}, 2\mathbf{I}\widetilde{\psi}^{(0)}_{\mathcal{C}}, \mathbf{0}, -1, \widetilde{\psi}^{(0)}_{\mathbb{P}}, \widetilde{\psi}^{(0)}_{\mathbb{Q}}, \widetilde{\psi}^{(0)}_{\mathbb{P}} + \widetilde{\psi}^{(0)}_{\boldsymbol{\Omega}}\mathbb{A}, \mathbf{0}, \widetilde{\psi}^{(0)}_{\boldsymbol{\Omega}}, \widetilde{\psi}^{(0)}_{\boldsymbol{\kappa}}) \tag{5.2}$$

where $\mathbf{I}$ has entries δ_{iK} and where all the derivatives of $\widetilde{\psi}$ are valued at $\mathcal{C} = \mathfrak{C} = \mathbb{I}$, $\boldsymbol{\Gamma} = \mathbf{0}$, $\mathbb{P} = \mathbb{P}_0$, $\mathbb{Q} = \mathbf{0}$, $\theta = \theta_0$ and $\boldsymbol{\Omega} = \boldsymbol{\kappa} = \mathbf{0}$. Posing $\tilde{\mathbf{z}}' = \mathbf{z}' - \mathbf{z}'_0$, we can linearize system (4.11) about the undeformed state obtaining

$$\mathbf{A}^{(0)}_0 \partial_t \tilde{\mathbf{z}}' + \mathbf{A}^{(0)}_K \partial_K \tilde{\mathbf{z}}' = \mathbf{B}^{(0)} \tilde{\mathbf{z}}', \tag{5.3}$$

where the matrices

$$\mathbf{A}^{(0)}_0 = \left[(\pi'_0)_{\mathbf{z}'\mathbf{z}'}\right]_0, \qquad \mathbf{A}^{(0)}_K = \left[(\pi'_K)_{\mathbf{z}'\mathbf{z}'}\right]_0, \qquad \mathbf{B}^{(0)} = [\mathbf{h}_{\mathbf{z}'}]_0, \tag{5.4}$$

are valued at $\mathbf{z}' = \mathbf{z}'_0$. Since the derivatives of $\mathbf{h}$ with respect to the second and the ninth field in $\mathbf{z}'$ are null at $\mathbf{z}'_0$, we reduce the previous system to a simpler one for the corresponding nine fields variable $\hat{\mathbf{z}}'$,

$$\hat{\mathbf{A}}^{(0)}_0 \partial_t \hat{\mathbf{z}}' + \hat{\mathbf{A}}^{(0)}_K \partial_K \hat{\mathbf{z}}' = \hat{\mathbf{B}}^{(0)} \hat{\mathbf{z}}'. \tag{5.5}$$

Accounting for (4.14), the matrix $\hat{\mathbf{A}}^{(0)}_0$ is positive definite. Introducing the matrix $\mathbf{U} = [\hat{\mathbf{A}}^{(0)}_0]^{1/2}$, system (5.5) can be rewritten in the form

$$\partial_t \mathbf{w} + \mathbf{A}_K \partial_K \mathbf{w} = \mathbf{B}\mathbf{w} \tag{5.6}$$

where

$$\mathbf{w} = \mathbf{U}\hat{\mathbf{z}}', \qquad \mathbf{A}_K = \mathbf{U}^{-1}\hat{\mathbf{A}}^{(0)}_K \mathbf{U}^{-1}, \qquad \mathbf{B} = \mathbf{U}^{-1}\hat{\mathbf{B}}^{(0)}\mathbf{U}^{-1}.$$

Now we consider wave propagation along one coordinate axis, restricting equation (5.6) to the single value $K = 1$. This hypothesis does not affect the generality of the results. Denoting by $\mathbf{w}_1$ an eigenvector of $\mathbf{A}_1$ corresponding to the eigenvalue λ_1, the stability condition for plane waves propagating along X_1 (at high frequencies) requires (see [**15**])

$$\mathbf{w}_1 \cdot \mathbf{B}\mathbf{w}_1 < 0, \tag{5.7}$$

for any amplitude $\mathbf{w}_1$ in the eigenspace of the phase speed λ_1. Owing to the last of equations (6.4) the symmetric part of $\hat{\mathbf{B}}^{(0)}$ is explicitly

$$
(5.8)\qquad \hat{\mathbf{B}}_s^{(0)} = \theta_0 \begin{pmatrix}
\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbb{W}_{\mathbf{C}}^T \\
\mathbf{0} & \mathbf{0} & \mathbb{G}^T & \frac{1}{2}\mathbb{P}_0^T \otimes \mathbb{I} & \mathbf{0} & \frac{1}{2}(\mathbb{I} - \mathbb{A}\mathbb{P}_0)^T \otimes \mathbb{I} \\
\mathbf{0} & \mathbb{G} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & \frac{1}{2}\mathbb{I} \otimes \mathbb{P}_0 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbb{W}_{\mathbb{P}}^T \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbb{W}_{\mathbb{E}}^T \\
\mathbb{W}_{\mathbf{C}} & \frac{1}{2}\mathbb{I} \otimes (\mathbb{I} - \mathbb{A}\mathbb{P}_0) & \mathbf{0} & \mathbb{W}_{\mathbb{P}} & \mathbb{W}_{\mathbb{E}} & \mathbb{W}
\end{pmatrix},
$$

where zeroes represent null block matrices of suitable order and where

$$
\mathbb{G} = \widetilde{\psi}_{\mathbb{C}}^{(0)} + \widetilde{\psi}_{\boldsymbol{\Omega}}^{(0)} + \frac{1}{2}\theta_0 \Big[\widetilde{\psi}_{\theta\mathbb{C}}^{(0)} + \widetilde{\psi}_{\theta\boldsymbol{\Omega}}^{(0)} - 2\widetilde{\psi}_{\theta C}^{(0)}\Big] - \frac{1}{2}\Big[\widetilde{\psi}_{\mathbb{P}}^{(0)} \otimes \mathbb{P}_0 + \widetilde{\psi}_{\boldsymbol{\Omega}}^{(0)} \mathbb{A}\mathbb{P}_0\Big],
$$

$$
\mathbb{W}_{\mathbf{C}} = \frac{1}{2}\begin{pmatrix}
\gamma_{\boldsymbol{\Omega}}\mathbf{I}\Big[\widetilde{\psi}_{\mathbb{C}\boldsymbol{\Omega}}^{(0)}\Big]^{-1} & \gamma_{\boldsymbol{\kappa}}\mathbf{I}\Big[\widetilde{\psi}_{\mathbb{C}\boldsymbol{\kappa}}^{(0)}\Big]^{-1} \\
\gamma_{\boldsymbol{\Omega}}\mathbf{I}\Big[\widetilde{\psi}_{C\boldsymbol{\Omega}}^{(0)}\Big]^{-1} & \gamma_{\boldsymbol{\kappa}}\mathbf{I}\Big[\widetilde{\psi}_{C\boldsymbol{\kappa}}^{(0)}\Big]^{-1}
\end{pmatrix}
$$

$$
\mathbb{W}_{\mathbb{P}} = \frac{1}{2}\begin{pmatrix}
\gamma_{\boldsymbol{\Omega}}\Big[\widetilde{\psi}_{\mathbb{P}\boldsymbol{\Omega}}^{(0)}\Big]^{-1} \\
\gamma_{\boldsymbol{\kappa}}\Big[\widetilde{\psi}_{\mathbb{P}\boldsymbol{\kappa}}^{(0)}\Big]^{-1}
\end{pmatrix}
\qquad
\mathbb{W}_{\mathbb{E}} = \frac{1}{2}\begin{pmatrix}
\gamma_{\boldsymbol{\Omega}}\Big[\widetilde{\psi}_{\mathbb{Q}\boldsymbol{\Omega}}^{(0)}\Big]^{-1} & \gamma_{\boldsymbol{\Omega}}\Big[\widetilde{\psi}_{\mathbb{P}\boldsymbol{\Omega}}^{(0)} + \widetilde{\psi}_{\boldsymbol{\Omega}\boldsymbol{\Omega}}^{(0)}\mathbb{A}\Big]^{-1} \\
\gamma_{\boldsymbol{\kappa}}\Big[\widetilde{\psi}_{\mathbb{Q}\boldsymbol{\kappa}}^{(0)}\Big]^{-1} & \gamma_{\boldsymbol{\kappa}}\Big[\widetilde{\psi}_{\mathbb{P}\boldsymbol{\kappa}}^{(0)} + \widetilde{\psi}_{\boldsymbol{\Omega}\boldsymbol{\kappa}}^{(0)}\mathbb{A}\Big]^{-1}
\end{pmatrix}
$$

$$
\mathbb{W} = \begin{pmatrix}
\gamma_{\boldsymbol{\Omega}}\Big[\widetilde{\psi}_{\boldsymbol{\Omega}\boldsymbol{\Omega}}^{(0)}\Big]^{-1} & \frac{1}{2}\Big\{\gamma_{\boldsymbol{\Omega}}\Big[\widetilde{\psi}_{\boldsymbol{\Omega}\boldsymbol{\kappa}}^{(0)}\Big]^{-1} + \gamma_{\boldsymbol{\kappa}}\Big[\widetilde{\psi}_{\boldsymbol{\kappa}\boldsymbol{\Omega}}^{(0)}\Big]^{-1}\Big\} \\
\frac{1}{2}\Big\{\gamma_{\boldsymbol{\kappa}}\Big[\widetilde{\psi}_{\boldsymbol{\kappa}\boldsymbol{\Omega}}^{(0)}\Big]^{-1} + \gamma_{\boldsymbol{\Omega}}\Big[\widetilde{\psi}_{\boldsymbol{\Omega}\boldsymbol{\kappa}}^{(0)}\Big]^{-1}\Big\} & \gamma_{\boldsymbol{\kappa}}\Big[\widetilde{\psi}_{\boldsymbol{\kappa}\boldsymbol{\kappa}}^{(0)}\Big]^{-1}
\end{pmatrix}.
$$

In view of the previous positions,

$$
\mathbf{w}_1 \cdot \mathbf{B}\mathbf{w}_1 = (\mathbf{U}^{-1}\mathbf{w}_1) \cdot \hat{\mathbf{B}}_s^{(0)}(\mathbf{U}^{-1}\mathbf{w}_1),
$$

then, from (5.7) we conclude that the condition for stability of linear waves as well as for asymptotic stability of non linear waves is

$$
(5.9)\qquad \hat{\mathbf{B}}_s^{(0)} \quad \text{is negative definite.}
$$

In particular, owing to (5.8), the expression of $\mathbb{W}$ and equation (4.15), a necessary condition for stability is that inequalities (4.16) hold strictly. This result is in accordance with a corresponding one obtained in [**18**]. In terms of the theory of weak discontinuity waves in continuum mechanics, the condition (5.9) is also sufficient to say that waves propagating on the configuration $\mathbf{z}_0$ admit a superior bound for the initial amplitude above which no critical time occurs (see [**2, 21**]). Of course, the same result holds for configurations in which a natural polarization is absent.

6 - Conclusions

In this paper we have considered some mathematical properties of a micro-continuum model for a dielectric solid. The idea underlying this model is that, in contrast to the standard approach, the continuum microstructure intrinsically accounts for both mechanical and electromagnetic quantities via the mass and charge microdensities. Polarization and quadrupole densities are then introduced in a natural way and the balance equations for both mechanical and electromagnetic fields are derived coherently. We have shown that the differential system of balance laws can be written in a symmetric hyperbolic form for a set of field variables. To this end, some inequalities are required by the thermodynamic constraints. This system is then exploited to study the stability of linear and non linear waves about an undeformed configuration which involves a spontaneous polarization. The stability condition is obtained in terms of definiteness of a suitable matrix of constitutive quantities.

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