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Sandro Buoncristiano and Francesco Mercuri

A brief history of the Poincaré conjecture

Abstract. In 1904 H. Poincaré proposed the problem of deciding whether a closed simply connected 3-manifold exists which is not homeomorphic to the 3-sphere. A negative answer to the question was soon labeled as the *Poincaré conjecture*, although Poincaré himself made no attempt to answer it. The question puzzled mathematicians for over 100 years and finally a positive answer to the conjecture has recently been obtained by G. Perelman. As many of the problems remained open for a long time, Poincaré conjecture or, better, the attempt at a deeper understanding of the problem produced an incalculable amount of mathematics. The aim of this essay is to give a report, short and incomplete of necessity, of some of those developments as well as brief comments on still open problems.

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Contents

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1 - Introduction

In the time elapsed between 1895 and 1904 Poincaré wrote a series of six papers on the Foundations of Topology. He introduced homology and the fundamental group. In his 1900 paper, he raised the problem of proving that a closed [1] connected 3-manifold with a vanishing first homology group is homeomorphic to the 3-sphere. In the 1904 paper though, published in the "Rendiconti del Circolo Matematico di Palermo", he described a counterexample to-day called the Poincaré sphere. This is the quotient of the standard 3-sphere by the free action of a group of isometries of order 120, which is a perfect group, i.e. it coincides with its commutator subgroup. In the same paper he also proposed the problem of studying closed 3-manifolds with a vanishing fundamental group. He did not elaborate on the problem, just commenting that "..*cette question nous entraînerait trop loin.*" He probably did not completely realize how hard the problem was and, consequently, did not have a clear idea of " how far away it would have taken us". In fact the question of whether a closed simply connected 3-dimensional manifold is homeomorphic to the 3-sphere, known as the *Poincaré Conjecture* [2] was *officially* solved only in 2006. The solution, due to G. Perelman, appeared in a series of preprints posted on the Internet in the years 2002-

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[1] i.e. compact with empty boundary.

[2] Poincaré really asked if it was possible to find a simply connected closed 3-manifold non-homeomorphic to the 3-sphere. He made no guess on the solution, i.e., he did not make any conjecture.

2003, which divided the mathematical community into two classes: the supporters of the proof and the skeptics. Finally, the 2006 meeting of the International Union of Mathematicians awarded Perelmann with the Fields medal for his work on the problem, thus making the correctness of his proof official.

There is a basic principle in Mathematics:

If you can not solve a problem, generalize it!

The first natural generalization looks at the dimension of the manifold. It is clear that simple connectedness is too weak a condition in higher dimensions. For example, the product $S^2 \times S^2$ and the complex projective plane $\mathbb{C}P^2$ are closed, simply connected 4-dimensional manifolds which are *not* homeomorphic to the 4-sphere S^4. With the development of topology, it soon became clear that the natural generalization is the following:

Generalized Poincaré Conjecture (G.P.C. for short):

Let M^n be a closed n-dimensional manifold, homotopy equivalent to the n-sphere S^n. Then "$M^n = S^n$".

Remark 1.1. It follows from general results of algebraic topology that a simply connected closed 3-manifold is homotopy equivalent to S^3. In fact, by duality such a manifold has the same homology of S^3. It follows from the Theorems of Whitehead and Hurewicz that a closed simply connected manifold with the same homology of S^n is homotopy equivalent to S^n. Therefore the classical Poincaré conjecture is equivalent to the G.P.C. for $n = 3$.

In the next section we will discuss some basic concepts in order to have a better understanding of the statement of the G.P.C. In particular, we will explain what is meant by the equality "$M^n = S^n$".

2 - Basic concepts

We recall that two maps[3] $f_0, f_1 : X \longrightarrow Y$ are *homotopic* if they can be *deformed one into the other*. More precisely, they are homotopic if there is a map:

$$H : [0,1] \times X \longrightarrow Y, \quad H(0,x) = f_0(x),\ H(1,x) = f_1(x),\ \forall x \in X.$$

If the two maps are homotopic we will write $f_0 \sim f_1$.

[3] By a map we will always mean a *continuous map*.

A map $f : X \longrightarrow Y$ is a *homotopy equivalence* if there is an inverse in the homotopy sense, i.e. a map $g : Y \longrightarrow X$ such that $g \circ f \sim 1_X,\ \ f \circ g \sim 1_Y$. When such a homotopy equivalence exists, we write $X \overset{H}{=} Y$, and say that X and Y are *homotopy equivalent.*

If M is a *closed manifold* homotopy equivalent to the sphere, we say that M is a *homotopy sphere*. Thus the hypothesis of the conjecture is that M is a homotopy sphere.

We will now discuss what "$M^n = S^n$" could mean in the statement of the G.P.C. In fact, in dimensions greater than three, there are several different meanings.

Let M be a Hausdorff, second countable topological space. An *atlas* for M is a collection of maps $\phi_\alpha : \Omega_\alpha \subseteq \mathbb{R}^n \longrightarrow M,\ \ \alpha \in \mathcal{A}$, such that ϕ_α is a homeomorphism of an open set $\Omega_\alpha \subseteq \mathbb{R}^n$ onto an open set $U_\alpha = \phi_\alpha(\Omega_\alpha) \subseteq M$ and such that $M = \bigcup_{\alpha\in\mathcal{A}} U_\alpha$. The maps $\phi_\beta^{-1} \circ \phi_\alpha$ are called *changes of coordinates*. We have the following concepts:

- M is a *topological manifold* if it admits an atlas. In this case we write $M \in TOP$.
- M is a *piecewise linear manifold* if it admits an atlas whose changes of coordinates are piecewise linear[4]. Once such an atlas has been fixed, we will write $M \in PL$.
- M is a *differentiable* (*or smooth*) *manifold* if it admits an atlas whose changes of coordinates are differentiable[5]. Once such an atlas has been fixed we will write $M \in DIFF$.

Given $M, N \in PL$ and $f : M \longrightarrow N$, we say that f is PL if the $\psi_\gamma^{-1} \circ f \circ \phi_\alpha$ are PL-maps, where ψ_γ and ψ_α run over the (fixed) PL atlas of N and M respectively. Similarly, we can define differentiable functions between manifolds in $DIFF$.

The natural equivalence relations are the following:

- $M_1, M_2 \in TOP,\ M_1 \overset{TOP}{=} M_2$ if there exists a homeomorphism $f : M_1 \longrightarrow M_2$.
- $M_1, M_2 \in PL,\ M_1 \overset{PL}{=} M_2$ if there exists a PL-map $f : M_1 \longrightarrow M_2$ with a PL inverse.
- $M_1, M_2 \in DIFF,\ M_1 \overset{DIFF}{=} M_2$ if there exists a differentiable map $f : M_1 \longrightarrow M_2$ with a differentiable inverse.

Remark 2.1. Clearly $M \in DIFF \Rightarrow M \in TOP$ and $M \in PL \Rightarrow M \in TOP$. From the triangulation theorem for differentiable manifolds, we may say

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[4] A piecewice linear map between open sets of Euclidean spaces is a map whose graph is a polyhedron, i.e a locally finite union of Euclidean simplexes.

[5] Differentiable will mean of class C^∞.

$M \in DIFF \Rightarrow M \in PL$. In general there are a lot of manifolds in *TOP* which are not in *PL* or *DIFF*, as we will see. Moreover, two manifolds in a given category may be equivalent in a larger category but not in the original one.

For the moment, let us state the following facts:

- In dimension $n \leq 3$, the three categories "coincide", as well as the corresponding equivalences. In particular, any topological manifold of dimension ≤ 3 admits a unique differentiable structure (see [**38**], [**41**]).
- The existence of a *PL* atlas on a *TOP*-manifold in dimension $n \geq 5$ is equivalent to the vanishing of a certain cohomology class

$$KS(M) \in H^4(M; \mathbb{Z}_2),$$

known as the *Kirby-Siebenmann invariant*. In particular, if $n \geq 5$ and $H^4(M; \mathbb{Z}_2) = \{0\}$, then : $M \in TOP \Leftrightarrow M \in PL$.
- If $n \leq 6$, then $PL = DIFF$, and a *PL* manifold admits a unique *DIFF* structure compatible with the given *PL* structure.

These are very deep and difficult theorems, due to various outstanding mathematicians, many of whom we will have the opportunity to quote in the course of this article.

We may consider two versions of the G.P.C. Let M be a n-dimensional homotopy sphere.

- *Strong version:* if M is in a given category, is M equivalent, *in that category* to the standard sphere?
- *Weak version:* if M is in a given category, is M equivalent to the standard sphere, *possibly in a larger category*?

We will mainly focus on the following (weak) version of the G.P.C.

$$M \in DIFF \quad \overset{?}{\Rightarrow} M \overset{TOP}{=} S^n$$

but first we will discuss differentiable structures on spheres.

The existence of different *DIFF* equivalence classes on a manifold $M^n \overset{TOP}{=} S^n$ was first investigated by J. Milnor, in his wonderful paper [**31**]. Differentiable structures on M^n that are *TOP* but not *DIFF* equivalent to the canonical structure of S^n are called *exotic structures.* The basic step is to give a procedure for constructing differentiable manifolds homeomorphic to S^n. This can be done as follows:

Consider two copies of the unit disk

$$D = D^n = \{x \in \mathbb{R}^n : \|x\| \leq 1\}$$

which we denote with the same symbol. Let $\phi : \partial D \longrightarrow \partial D$ be an orientation preserving diffeomorphism. In the disjoint union $D \coprod D$ consider the equivalence relation generated by $x \in \partial D \sim \phi(x) \in \partial D$. We will denote the quotient space by $D \cup_\phi D$.

For a better understanding of the structure of $D \cup_\phi D$, it is convenient to consider the following equivalent construction: start with two copies of $\mathbb{R}^n$ and identify a non zero point x of the first copy with the point $\phi(\frac{x}{\|x\|})/\|x\|$ of the second copy. From this definition, it is clear that $D \cup_\phi D$ has a well-defined *DIFF* structure. Moreover, by assuming the orientation induced, say, by the first chart $\mathbb{R}^n$, we obtain a well-defined oriented smooth manifold.

Proposition 2.2. *$D \cup_\phi D \overset{TOP}{=} S^n$. Moreover, if ϕ extends to a diffeomorphism of D, then $D \cup_\phi D \overset{DIFF}{=} S^n$.*

Proof. Let $S_\pm$ be the (closed) upper and lower hemispheres of the standard S^n. Identify, in the usual way, S_+ with one of the disks and map S_- onto the other disk by a homeomorphism extending ϕ.[6] This map is obviously a homeomorphism. It can be approximated by a diffeomorphism, if there exists a smooth extension of ϕ to a diffeomorphism of the disk (see [**23**], pg. 182 for details). □

Definition 2.3. Manifolds of the form $D \cup_\phi D$ are called *twisted spheres*.

We have the following important result that we will prove at the end of the next section (Theorem 3.20).

Theorem 2.4. *Let M^n be a differentiable manifold with $M^n \overset{TOP}{=} S^n$. If $n \neq 4$, M^n is a twisted sphere.*

Therefore, if $n \neq 4$, the problem of classifying exotic structures on the sphere is equivalent to the problem of how to distinguish twisted spheres up to a *DIFF* equivalence. For this purpose, we are naturally led to consider the groups $\mathit{Diff}^+(S^{n-1})$ and $\mathit{Diff}^+(D^n)$ of orientation preserving diffeomorphisms of the sphere S^{n-1} and of the disk D^n respectively, and the restriction homomorphism $r : \mathit{Diff}^+(D^n) \longrightarrow \mathit{Diff}^+(S^{n-1})$. The image is a normal subgroup and we can consider

[6] ϕ extends naturally to a homeomorphism of D by setting $\phi(tx) = t\phi(x), x \in \partial D, t \in [0,1]$. Observe that this extension is not differentiable at $0 \in D$, unless ϕ is the restriction of an orthogonal transformation. Notice, however, that the above extension will be a *PL* homeomorpfism if ϕ is *PL*. This is a particular case of the so called *cone construction* in *PL* topology, which will be referred to later on.

the quotient group:

$$\Gamma^n = Diff^+(S^{n-1})/r(Diff^+(D^n)).$$

We will shortly see that the number of oriented diffeomorphism classes of twisted n-dimensional spheres is the cardinality of Γ^n.

The first thing we have to do is to introduce a group operation in the set of oriented diffeomorphism classes of twisted spheres, which we denote with $\overline{\Gamma}^n$ for the time being.

Let M_1, M_2 be two smooth, connected, and oriented differentiable n-dimensional manifolds. We define their connected sum as follows: choose two smooth embeddings $\phi_i : D^n \longrightarrow \phi_i(D^n) = D_i \subseteq M_i,\ i = 1,2$ with ϕ_1 orientation preserving and ϕ_2 orientation reversing. In the disjoint union $M_1 \setminus \overset{\circ}{D}_1 \coprod M_2 \setminus \overset{\circ}{D}_2$ we may consider the equivalence relation generated by $x \in \partial D_1 \sim \phi_2\phi_1^{-1}(x)$ and denote by $M_1 \sharp M_2$ the quotient space. The following result belongs to classical differential topology:

Proposition 2.5. *$M_1 \sharp M_2$ admits a structure of oriented differentiable manifold which induces the original structures on $M_i \setminus \overset{\circ}{D}_i, i = 1,2$. Moreover, such a structure is unique up to orientation preserving diffeomorphisms, and does not depend on the choice of embeddings ϕ_i.*

The proof is based on the technique of *smoothing corners* and on the Palais-Cerf Lemma that in our situation states that if ϕ_i is replaced with ϕ_i' satisfying the same assumptions, then there exists an orientation preserving diffeomorphism $h_i : M_i \longrightarrow M_i$ such that $\phi_i' = \phi_i \circ h_i, i = 1,2$. The reader is referred to [**23**], ch. 8 for all the details.

Definition 2.6. $M_1 \sharp M_2$ is called the *connected sum* of M_1 and M_2.

Let $\mathcal{M}_n$ be the set of oriented diffeomorphism classes of smooth, n-dimensional closed manifolds. It is not difficult to show that, with the connected sum operation, $\mathcal{M}_n$ is a commutative monoid with an identity element represented by the standard sphere S^n.

Remark 2.7. The decomposition Theorems of Kneser and Milnor, of which we discuss in the last section imply that $\mathcal{M}_3$ is a free monoid.

Returning to twisted spheres we have:

Proposition 2.8. *$\overline{\Gamma}^n$ is a submonoid of $\mathcal{M}_n$, in fact a commutative group.*

Proof (Sketch). Let $\Phi : Diff^{+}(S^{n-1}) \longrightarrow \mathcal{M}_n$ be the map given by $\Phi(\phi) = D \cup_{\phi} D := M(\phi)$. Clearly the image of Φ is $\overline{\Gamma}^n$. We claim that Φ is a monoid homomorphism, i.e. there is an orientation preserving diffeomorphism:

$$M(\phi \circ \phi') \cong M(\phi) \sharp M(\phi').$$

The crucial point is that the Palais-Cerf Lemma allows a certain freedom in choosing the disks to be removed in the construction of the connected sum. Relying on this, in $M(\phi) = D \cup_{\phi} D$ we remove the interior of the second copy, while in $M(\phi')$ we remove the interior of the first copy. When the attachments are performed, the resulting manifold will be exactly $M(\phi \circ \phi')$, up to diffeomorphisms which preserve the orientation. □

Theorem 2.9. *Φ induces a group isomorphism $\Phi : \Gamma^n \longrightarrow \overline{\Gamma}^n$.*

Proof (Sketch). Define Φ as above. By Proposition 2.2, $\Phi(r(Diff^{+}(D^n)) = 0$. Hence Φ induces a (surjective) homomorphism of Γ^n onto $\overline{\Gamma}^n$. On the other hand, suppose that $g : D \cup_{\phi} D \longrightarrow S^n$ is an orientation preserving diffeomorphism. Since the group of orientation preserving diffeomorphisms is transitive on the closed disks of a given connected manifold (by the Palais-Cerf Lemma quoted above), we can assume that g is the identity on the upper hemisphere. Then g is an orientation preserving diffeomorphism of the lower disk, which carries S^{n-1} onto itself and extends ϕ. Hence Φ is 1-1. □

Remark 2.10. For low values of n we have:

- $\Gamma^n = \{0\}$ if $n \leq 3$ (see Remark 2.1).
- $\Gamma^n = \{0\}$ if $n = 4$, 5 and 6 (see [**9**], [**26**]).
- Γ^n is a finite group $\forall\ n$ (see [**26**]).

For $n = 7$, we have $\Gamma^7 \cong \mathbb{Z}_{28}$. In [**15**] the authors describe this group explicitly. Consider the field of quaternions $\mathbb{H} \cong R^4$, and let $S^3 = \{p \in \mathbb{H} : \|p\| = 1\}$. Consider the sphere:

$$S^6 = \{(p, w) \in \mathbb{H} \times \mathbb{H} : Real(p) = 0, \|p\|^2 + \|w\|^2 = 1\}.$$

Define the map:

$$b : S^6 \setminus \{(p, 0)\} \longrightarrow S^3, \qquad b(p, w) = \|w\|^{-2} w \exp(\pi p) \overline{w}.$$

Here exp is the exponential map for the quaternions of norm one, i.e. $\exp v$ is the value of the geodesic $\gamma(t)$ of S^3 with $\gamma(0) = 1, \dot{\gamma}(0) = v$, at $t = 1$. Setting $b(p, 0) = -1$, the map b extends to a real analytic map of S^6 onto S^3. This map represents the generator of $\pi_6(S^3) \cong \mathbb{Z}_{12}$. Define maps:

$$\phi_n : S^6 \longrightarrow S^6, \quad \phi_n(p, w) = (\overline{b(p,q)}^n pb(p,w)^n, \overline{b(p,q)}^n wb(p,w)^n).$$

It results that ϕ_1 generates Γ^7. Observe that ϕ_{12} is homotopic to the identity (since $\pi_6(S^3) \cong \mathbb{Z}_{12}$), but not isotopic to it, and it represents a non trivial element in Γ^7.

With a similar construction we can produce exotic structures on S^{15} using octonions.

These considerations lead to the following (open) problem known as the *differentiable 4-dimensional Poincaré conjecture*:

- $M^4 \in DIFF, M^4 \overset{TOP}{=} S^4 \overset{?}{\Rightarrow} M^4 \overset{DIFF}{=} S^4$.
- $M^4 \in DIFF, M^4 \overset{TOP}{=} S^4 \overset{?}{\Rightarrow} M^4$ is a twisted sphere.

Remark 2.11. Since $\Gamma^4 = \{0\}$, the two statements are equivalent.

3 - The Smale solution of the G.P.C. in dimensions ≥ 5

There are two basic tools in the study of the topology of high dimensional manifolds, namely:

- *surgery theory*, used to prove existence of manifolds with certain topological properties;
- *the h-cobordism theorem*, used to prove uniqueness of manifolds with certain properties.

Surgery theory starts from the following basic observation:

$$\partial(S^{p-1} \times D^q) = \partial(D^p \times S^{q-1}) = S^{p-1} \times S^{q-1}.$$

For a given n-dimensional manifold M^n, $n = p + q - 1$, choose an embedding of $S^{p-1} \times D^q$ into M. Delete the interior of the image of $S^{p-1} \times D^q$ and attach a copy of $D^p \times S^{q-1}$ along the boundary. The resulting manifold depends on the choice of the embedding, and this construction can be used to simplify (or complicate) the topology of M.

Remark 3.1. The classification Theorem for closed oriented surfaces states that such a surface is obtained from the sphere S^2, by performing the following operation a suitable number of times: delete from S^2 the interior of two disjoint closed disks ($S^0 \times D^2$) and glue a cylinder ($S^1 \times D^1$) along the boundary. Therefore, any closed oriented surface is obtained from S^2 by performing surgeries. This result was essentially known at the beginning of last century, although a full proof including the non-orientable case would be dated around 1920.

It is also worthwhile observing that surgery is the basic technique used by Kervaire and Milnor to compute the groups of h-cobordism classes of homotopy spheres, which coincide with the groups Γ^n (see [**26**]).

The rest of this section will be devoted to discussion of the h-cobordism Theorem and how we can deduce the generalized Poincaré conjecture from it. We will closely follow Milnor's book ([**35**]).

3.1 - *The Morse complex*

Morse theory provides a way of computing the homology of a manifold in terms of the critical points of a suitable function and the dynamics of the gradient flow.

Let W be a smooth manifold and $f : W \longrightarrow \mathbb{R}$ be a smooth function. A *critical point* of f is a point $x \in W$, such that the differential $\mathrm{d}f_x : T_xW \longrightarrow \mathbb{R}$ vanishes. A *critical value* is the image of a critical point.

Let $x \in W$ be a critical point of f. Choose a chart $\phi : \Omega \subseteq \mathbb{R}^n \longrightarrow W$ such that $0 \in \Omega$ and $\phi(0) = x$. Then the partial derivatives of $\tilde{f} := f \circ \phi$ vanish at $0 \in \Omega$. Consider the Hessian of $\tilde{f}$ at 0,

$$H(\tilde{f}, 0) = [\frac{\partial^2 \tilde{f}}{\partial x_i \partial x_j}(0)].$$

The Hessian is a symmetric matrix, hence it is diagonalizable.

- The critical point x is *non degenerate* if $H(\tilde{f}, 0)$ is non singular.
- The *index* of f at the critical point $x \in W$ is the number of the negative eigenvalues of $H(\tilde{f}, 0)$ (counted with their multiplicity).

It is easy to see that the concepts above do not depend on the choice of chart. The behavior of f near a non degenerate critical point is described by the well known Morse Lemma:

Lemma 3.2. *If x is a non degenerate critical point of index λ, there exists a chart $\phi : \Omega \longrightarrow W$ as above, such that:*

$$\tilde{f}(x_1, \dots, x_n) = c - \sum_{i=1}^{\lambda} x_i^2 + \sum_{j=\lambda+1}^{n} x_j^2, \qquad c = f(0) \in \mathbb{R}.$$

In particular, non degenerate critical points are isolated.

A chart such as above is called a *Morse chart.*

Let W be a Riemannian manifold, which for the moment we assume to be closed, and let $f : W \longrightarrow \mathbb{R}$ be a smooth function whose critical points are non degenerate. In this situation we will say that f is a *Morse function.* The existence of (many) Morse functions on a closed manifold is a rather simple consequence of the Theorem of Sard.

The basic idea of Morse Theory is to study how the topology of the "sub levels" $W^a := \{x \in W : f(x) \leq a\}$ changes as a varies. This is done by using the gradient flow of f and the Morse Lemma.

We start by reviewing the basic properties of the gradient flow. Recall that the gradient of f, ∇f, is the vector field characterized by:

$$\langle \nabla f, \xi \rangle = \mathrm{d}f(\xi), \quad \forall \quad \text{vector field} \quad \xi.$$

For a given vector field ξ, the integral line through $x \in W$ is the solution of the initial value problem:

$$\dot{\gamma}_x(t) = \xi(\gamma(t)), \quad \gamma_x(0) = x.$$

The existence for small values of t, the uniqueness, and the smooth dependence on the initial conditions[7] are the starting-point of the theory of ordinary differential equations. Moreover, since W is assumed to be closed, the integral lines of ξ are defined for *all* $t \in \mathbb{R}$.

Lemma 3.3. *The map* $\gamma_t : W \longrightarrow W$, $\gamma_t(x) := \gamma_x(t)$ *is a diffeomorphism of* W.

Proof. The differentiability of γ_t is a consequence of the smooth dependence on the initial conditions. Moreover, it is easy to see that $\gamma_{(t+s)} = \gamma_t \circ \gamma_s$[8]. Hence $\gamma_{-t} = \gamma_t^{-1}$. □

In the absence of critical values, we have:

Theorem 3.4 (Neck principle). *If* $f^{-1}([a,b])$ *does not contain critical points, then there is a diffeomorphism of* W *taking* W^b *onto* W^a. *Moreover,* $f^{-1}([a,b])$ *is diffeomorphic to* $f^{-1}(b) \times [a,b]$.

Proof. We will essentially follow the flow of the vector field $-\nabla f$. Observe that there are no zeros in $f^{-1}([a-2\varepsilon, b+2\varepsilon])$ for ε sufficiently small. Moreover, f is decreasing along the integral lines of $-\nabla f$. In order to control how fast f decreases, we

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[7] i.e. the smoothness of the map $(x,t) \in W \times (-\varepsilon, \varepsilon) \longrightarrow \gamma_x(t) \in W$.

[8] Both are solutions of the same ODE with the same initial condition.

"normalize" the vector field. Consider a smooth function $\lambda : W \longrightarrow \mathbb{R}$ which is 1 on $f^{-1}([a,b])$ and zero outside of $f^{-1}([a-\varepsilon, b+\varepsilon])$. Let ξ be the vector field defined by:

$$\xi = \lambda \frac{-\nabla f}{\|\nabla f\|^2} \quad \text{in} \quad f^{-1}([a-2\varepsilon, b+2\varepsilon]), \qquad \xi = 0 \quad \text{outside} \quad f^{-1}([a-2\varepsilon, b+2\varepsilon]).$$

Clearly ξ is well defined and smooth. If $x \in f^{-1}([a,b])$, then let γ_x be the integral curve of ξ through x. Then, for small values of $t \geq 0$, we have:

$$\frac{\mathrm{d}f(\gamma_x(t))}{\mathrm{d}t} = \mathrm{d}f(\dot{\gamma}_x) = \langle \nabla f, \xi \rangle = -1.$$

In particular, $f(\gamma_x(t_1)) - f(\gamma_x(t_2)) = t_2 - t_1$, and $\gamma_{(b-a)}$ is the required diffeomorphism. A diffeomorphism between $f^{-1}(b) \times [a,b]$ and $f^{-1}([a,b])$ is given by $(x,t) \longrightarrow \gamma_x(t)$. □

The preceding argument and the Morse Lemma give the following characterization of twisted spheres:

Theorem 3.5 (Reeb). *A closed manifold is a twisted sphere if and only if it admits a Morse function with only two critical points.*

Proof. Let W be a closed manifold and $f : W \longrightarrow \mathbb{R}$ a Morse function with only two critical points, $p, q \in W$. We can assume that p is the minimum, $f(p) = 0$, q is the maximum and $f(q) = 1$. According to the Morse lemma, W^{ε} is a disk for $\varepsilon > 0$ sufficiently small. By the neck principle (3.4), $W^{\frac{1}{2}}$ is a disk. A similar argument shows that $W_{\frac{1}{2}} := \{x \in W : f(x) \geq 1/2\}$ is also a disk and therefore W is a twisted sphere.

Conversely, let $\phi \in \mathit{Diff}^+(S^{n-1})$ and let W_ϕ be the twisted sphere obtained from two copies of $\mathbb{R}^n$ by identifying a non zero point x of the first copy with $y = \phi(\frac{x}{\|x\|})/\|x\|$ of the second copy. Then:

$$f = \frac{\|x\|^2}{1 + \|x\|^2} = \frac{1}{1 + \|y\|^2},$$

is a Morse function with exactly two critical points. □

Remark 3.6. Theorem 3.5 *is still true without the assumption that the critical points are non degenerate.*

Next we study what happens when we pass through a critical value. The basic operation will be *attaching cells*. We recall that the operation of attaching a λ-cell to a space X is defined as follows: given a map $\phi : \partial D^\lambda \longrightarrow X$, we consider, in the disjoint union $X \coprod D^\lambda$, the equivalence relation generated by $x \in \partial D^\lambda \sim \phi(x)$. The quotient of $X \coprod D^\lambda$ by this relation is the space *obtained from X by attaching a λ-cell by ϕ.*

Let $0 \in \mathbb{R}$ be a critical value of f and suppose, for simplicity, that we have only one critical point $p \in f^{-1}(0)$, which is non degenerate of index λ. Consider a Morse chart $\phi : \Omega \subseteq \mathbb{R}^n \longrightarrow M$. For $\varepsilon > 0$ sufficiently small, the disk of radius 2ε is contained in Ω, and 0 is the only critical point of $\tilde{f} := f \circ \phi$. The situation in the disk looks like the figure below

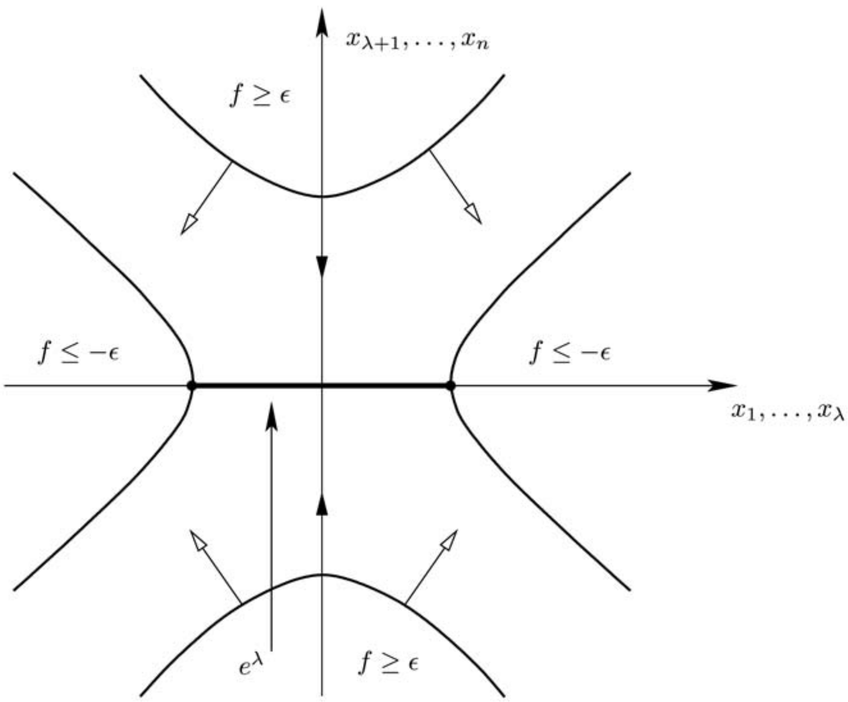

Fig. 1. The level hypersurfaces in a Morse chart.

It is reasonably clear from the picture that, in D, we can deform $\tilde{f}^{-1}((-\infty, \varepsilon])$ onto $\tilde{f}^{-1}((-\infty, -\varepsilon]) \bigcup e^\lambda$, where $e^\lambda := \{(x_1, \ldots, x_n) \in \mathbb{R}^n : x_{\lambda+1} = \cdots = x_n = 0, \sum x_i^2 \leq \varepsilon^2\}$. Thus, at least locally, "passing through a critical point of index λ" is equivalent, up to homotopy, to attaching a λ-cell.

For further reference, we must also remember that a (*finite*) *CW complex* is a space X, together with a (finite) sequence of subspaces $X^{(k)}$, such that $X^{(0)}$ is a finite set of points (with the discrete topology), and $X^{(k)}$ is obtained from $X^{(k-1)}$ by attaching (a finite number of) k-cells. $X^{(k)}$ is called the *k-skeleton*.

The *cellular chain complex* $\{C_k^{cell}, \partial_k^{cell}\}$ is associated to this complex, where:

- $C_k^{cell} = H_k(X^{(k)}, X^{(k-1)})$ (which is isomorphic, by excision, to the free abelian group generated by the k-cells).
- $\partial_k^{cell} : C_k^{cell} \longrightarrow C_{k-1}^{cell}$ is the boundary homomorphism in the exact sequence of the triple $(X^{(k)}, X^{(k-1)}, X^{(k-2)})$.

It is a standard fact that the homology of the cellular complex is isomorphic to the singular homology of X.

The local considerations above can be generalised as follows:

Theorem 3.7. *Let W be a closed differentiable manifold, and let $f : W \longrightarrow \mathbb{R}$ be a Morse function. Then, W is homotopy equivalent to a CW complex with a cell in dimension λ, for each critical point of f of index λ.*

The reader is referred to [**34**] for a proof.

The critical points of f determine the cellular chain groups. Now we will describe the boundary homomorphism. For this purpose we will assume W to be *oriented*, and consider special Morse functions.

Definition 3.8. A Morse function $f : W \longrightarrow \mathbb{R}$ is *self indexing* if, for every critical point $p \in W$, $f(p)$ is the index of f at p.

Remark 3.9. It is not difficult to show that, given a Morse function, there is a self indexing one with the same critical points (and indices).

Let $f : W \longrightarrow \mathbb{R}$ be a self indexing Morse function, $\{p_1, \ldots, p_k\}$, and $\{q_1, \ldots, q_l\}$ be the critical points of f of index $(\lambda + 1)$ and λ respectively. They are a basis of $C_{(\lambda+1)}^{cell}$ and C_λ^{cell} respectively. So:

$$\partial^{cell}(p_i) = \sum a_{ij} q_j.$$

We will now describe how to obtain the a_{ij}. Consider the integral curves of $-\nabla f$ "starting at p_i" [9]. These curves locally form a $\lambda + 1$ dimensional disk which intersects the level hypersurface $\{x \in W : f(x) = \lambda + 1/2\}$ in a λ dimensional sphere, $\Sigma_l^{p_i}$, the *lower sphere of p_i*. Since W is assumed to be oriented, the hypersurface is oriented by the choice of the (non zero) normal ∇f, and so we can choose an orientation for the lower sphere. Consider now the integral curves "ending in q_j". They form a $n - \lambda$

[9] An integral curve γ "starts" at p_i if $lim_{t \to -\infty} \gamma(t) = p_i$. In terms of a Morse chart these curves are the integral lines through the points where the last $n - \lambda - 1$ coordinates vanish. In a similar way we define the integral curves "ending" in a critical point.

dimensional disk which intersects the hypersurface in a $n - \lambda - 1$ dimensional sphere, $\Sigma_u^{q_j}$, the *upper sphere of* q_j. We want to define the *intersection number* of the two spheres. Generically the two spheres intersect each other transversally, hence for dimensional reasons, they intersect at a finite number of points. To each of these points we associate a sign ± 1 according to whether at the point the orientation of $\Sigma_l^{p_i}$, followed by the orientation of $\Sigma_u^{q_j}$, gives the orientation of the hypersurface or the opposite one. The sum of the signs is the *intersection number of the two spheres*. This number does not depend on the general-position argument or, at least up to sign, the orientations chosen on each sphere. We define the a_{ij} to be the intersection numbers as in the construction above.

The algebraic complex defined in this way is called the *Morse complex* of W associated with the function f.

Starting from different Morse functions we obtain different Morse complexes, and the basic result is the following (see [**35**] for a proof):

Theorem 3.10. *The homology of a Morse complex is isomorphic to the singular homology of W.*

3.2 - *The h-cobordism Theorem*

The Morse Theory outlined above works in a slightly more general context.

Definition 3.11. A *cobordism* is a triple (W^{n+1}, M_0, M_1), where W^{n+1} is a $(n+1)$-dimensional compact manifold, whose boundary is the disjoint union of M_0 and M_1.

We will suppose W to be oriented and consider the induced orientation on each M_i.

The simplest example of cobordism is the *product cobordism*, i.e. $W = M \times [0,1]$, $M_i = M \times \{i\},\ i = 0,1$. The point of the h-cobordism Theorem is to *give sufficient conditions for a cobordism to be diffeomorphic to a product cobordism.*

Definition 3.12. A *Morse function on a cobordism* (W^{n+1}, M_0, M_1) is a function $f : W \longrightarrow [a,b] \subseteq \mathbb{R}$ such that:

(1) $f(M_0) = a,\ \ f(M_1) = b$,
(2) f does not have critical points in a neighborhood of ∂W,
(3) all critical points of f are non degenerate.

Mutatis mutandis, we can repeat the construction of the previous subsection and define the Morse complex of a Morse function for a cobordism. The homology of this Morse complex will be isomorphic to the relative homology $H_*(W, M_0)$.

It follows from the neck principle (3.4) that, if a Morse function without critical points exists, then the cobordism is diffeomorphic to a product cobordism. Then the main idea is to find conditions under which there is a Morse function without critical points.

Definition 3.13. A cobordism (W^{n+1}, M_0, M_1) is a *h-cobordism* if the inclusions $i_j : M_j \longrightarrow W^{n+1}$ are homotopy equivalences.

Smale's h-cobordism theorem can be stated as follows:

Theorem 3.14. *Let (W^{n+1}, M_0, M_1) be a h-cobordism, W simply connected and $n \geq 5$. Then the cobordism is diffeomorphic to the product cobordism.*

The remaining part of this subsection is dedicated to sketching the main ideas behind the proof of the h-cobordism theorem. In the next subsection, we will see how the G.P.C. for $n \geq 5$ follows from this theorem, and we will describe other important results.

Let $f : W \longrightarrow \mathbb{R}$ be a self indexing Morse function on a cobordism (W, M_0, M_1). Suppose $p, q \in W$ are critical points of indexes $\lambda + 1$ and λ respectively, and that there is a unique integral curve γ of ∇f starting at p and ending in q. In this case we say that the points p, q are *complementary*. It is not difficult to see that complementary critical points do not contribute to the homology of W. The first question to ask is the following: is it possible to "cancel" two complementary critical points, i.e. to find a Morse function $\tilde{f} : W \longrightarrow \mathbb{R}$ which coincides with f outside a neighborhood U of the two points, and which does not have any critical points in U?

The following picture illustrates the situation

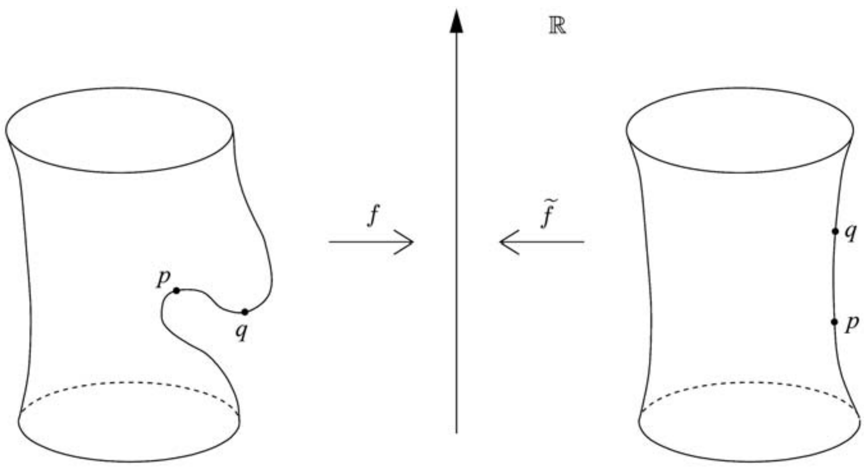

Fig. 2. Modifying the function.

In the situation described above it is, in fact, possible to cancel the two critical points. This fact, essentially due to Morse, is non trivial but by no means the most difficult step of the proof of the entire theorem. The main idea is to modify the gradient field in order to obtain a new vector field, which coincides with $-\nabla f$ outside U, has no singularities in U, and is, essentially,[10] the gradient of a new function $\tilde{f}$. The situation is illustrated in the next figure

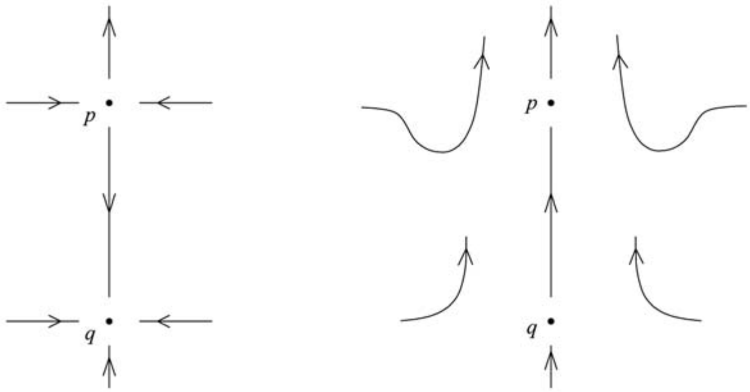

Fig. 3. Modifying the gradient.

We need a more general result. First we can observe that, if $p,\ q$ are complementary critical points, the corresponding coefficient a_{pq} in $\partial^{cell}(p)$ is ± 1. Then there is the following question: assuming only the algebraic condition $a_{pq} = \pm 1$, is it still possible to cancel the two critical points p and q? This can certainly be done if we can modify the function in such a way that the lower and upper spheres intersect just at one point. This is the most delicate part of the proof of the Theorem and it is here that the condition $n + 1 = \dim W > 5$ plays an essential role. We will give a rough sketch of the argument.

First of all it is not too difficult to see that, without loss of generality, we can make the following assumptions: p and q are the only critical points of index $\lambda + 1$ and λ respectively, $2 \leq \lambda \leq n - 3$, and the level hypersurface $S = f^{-1}(\lambda + 1/2)$ is simply connected.

Consider the two spheres Σ_l^p and Σ_u^q transversal in S and let $x,\ y$ be two points of their intersection having opposite signs. We join x and y by two arcs, one in Σ_l^p and the other in Σ_u^q, containing no other points of intersection. Let γ be the closed loop on S given by the two arcs. Since S is simply connected, then γ is the boundary of a

[10] i.e. it is what is called a *pseudo gradient*. See [35] for details.

singular disc D. This disc, by a general position argument, may be deformed into an *embedded* disk, which, in turn, is in general position with the two spheres [11]. At this point it is not difficult to use the disk or, better, a regular neighborhood of it, as a support of an isotopy that takes Σ_u^q onto a sphere Σ' with $\Sigma' \cap \Sigma_l^p = \Sigma_l^p \cap \Sigma_u^q \setminus \{p,\ q\}$. Thus, using the Whitney Lemma, we can limit ourselves to the case where the two spheres intersect in exactly one point. Then we may apply the result of Morse on complementary critical points.

By suitable rearrangements it is possible to use the argument above to cancel critical points as long as the *incidence number* a_{ij} *is nonzero*.

Finally, the condition that (W^{n+1}, M_0, M_1) is a h-cobordism, in particular that $H_*(W^{n+1}, M_0) = \{0\}$ implies that the critical points occur in pairs of “cancelable ones” in the sense of the constructions above, and this concludes the proof.

Remark 3.15. We observe that the restriction on the dimension of W, which allows one to use the general position arguments, *does not* represent a technical problem, but a fact of nature! In fact the *DIFF* version of the h-cobordism Theorem does not hold in lower dimensions as we will see in the next section.

3.3 - *The Poincaré conjecture in dimensions* ≥ 5

We will see now how the Poincaré conjecture follows from the h-cobordism Theorem. We may start from a characterization of the disks.

Theorem 3.16. *Let W be a compact, contractible smooth manifold of dimension $n \geq 6$, with a (non empty) simply connected boundary. Then, W is diffeomorphic to a disk.*

Proof. Let p be an interior point of W and D a small disc centered at p. Consider $W \setminus \mathring{D}$. Then $(W \setminus \mathring{D}, \partial W, \partial D)$ is a cobordism. Since W is contractible, it follows from standard algebraic topology that such a cobordism is a h-cobordism (see [**35**] for example). From the h-cobordism theorem, we can say that it is diffeomorphic to $S^{n-1} \times [0, 1]$. Hence W is diffeomorphic to a disk with a collar attached, which is still a disk. □

Regarding the G.P.C. in dimension $n \geq 6$, we can say:

---

[11] The disk D is called a *Whitney disk* and the procedure of cancelation of the intersection–points is part of a more general result known as the Whitney Lemma.

Theorem 3.17. *Let M^n be a n-dimensional closed smooth manifold, homotopy equivalent to S^n. If $n \geq 6$, M is a twisted sphere S^n.*

Proof. Let D be a n-disc in M. Now $M \setminus \overset{\circ}{D}$ satisfies the hypothesis of Theorem 3.16, hence it is diffeomorphic to a disk. Then M is the union of two disks attached by a diffeomorphism of their boundaries, i.e. a twisted sphere. □

The case $n = 5$ is exceptional. We quote the following result of Kervaire and Milnor (see [**26**]):

Theorem 3.18. *Let M be a closed smooth manifold homotopy equivalent to S^n. If $n = 4, 5, 6$, M is the boundary of a compact contractible smooth manifold.*

From Theorems 3.18, 3.14 and 3.16 we deduce the following:

Corollary 3.19. *A n-dimensional smooth homotopy sphere is diffeomorphic to S^n, if $n = 5, 6$.*

At this point, we can prove Theorem 2.4, i.e:

Theorem 3.20. *Let M^n be a n-dimensional closed smooth manifold homeomorphic to S^n, $n \neq 4$. Then M^n is a twisted sphere.*

Proof. The case $n \geq 5$ immediately follows from Theorems 3.17 and 3.19. For $n \leq 3$, the result is a consequence of the already mentioned equivalence $TOP = DIFF$. □

Remark 3.21. We observe that the above results imply $\Gamma^n = \{0\}$, for $n = 5, 6$.

3.4 - *The strong G.P.C. for the PL case*

The PL h-cobordism Theorem is stated, with the obvious modifications, exactly as its $DIFF$ counterpart:

Theorem 3.22. *Let (W^{n+1}, M_0, M_1) be a simply connected PL h-cobordism, $n \geq 5$. Then W is PL isomorphic to the product PL cobordism $M_0 \times [0, 1]$. In particular, $M_0 \overset{PL}{=} M_1$.*

The method of proof is the same as that used in the differentiable case, only that the concept of Morse function is now replaced by the equivalent concept of attaching handles to a *PL* or *DIFF* manifold. In this context the procedure of canceling critical points, in order to simplify a Morse function, is translated into the procedure of canceling handles of a given handle-decomposition W. Here it is worth mentioning that handlebodies were introduced by Smale to prove his h-cobordism Theorem in the *DIFF* category. Today the main reference for the fundamentals of *PL* topology and the *PL* h-cobordism theorem is [**55**].

Here is the first important consequence of Theorem 3.22, which is proved exactly as its *DIFF* counterpart:

Theorem 3.23. *Let W be a contractible PL manifold of dimension $n \geq 6$, with a non-empty, and simply-connected boundary. Then $W \overset{PL}{=} D^n$.*

Let S^n denote the n-sphere with its canonical *PL* structure. Then, regarding the *PL* G.P.C. in dimension $n \geq 6$, as a pleasant surprise, we can say that the *strong* version holds:

Theorem 3.24. *Let M^n be a n-dimensional, closed PL manifold, homotopy equivalent to S^n. If $n \geq 6$, $M \overset{PL}{=} S^n$.*

Proof. Similarly to the *DIFF* case, one shows that M^n is the union of two *PL* disks attached by a *PL* isomorphism between their boundaries. But such a manifold is a *PL* sphere by the cone construction in *PL* topology (see Footnote 6). □

The 5-dimensional case is also true but its proof is even more recondite than the higher dimensional case:

Theorem 3.25. *A 5-dimensional PL homotopy sphere M^5 is PL isomorphic to S^5.*

Proof. The proof will only be roughly outlined. By the Hirsch-Munkres theory of the smoothings of *PL* manifolds, the only obstruction to the existence of a smooth structure on M^5 lies in $H^5(M, \Gamma^4)$. Since $\Gamma^4 = \{0\}$, there exists a smooth structure. Let M_{DIFF} be the manifold M with this differentiable structure. This structure is compatible (according to Whitehead) with the given *PL* structure on M. By Theorem 3.19, M_{DIFF} is diffeomorphic to S^5. It follows that on M_{DIFF} there are two *PL* structures, imported respectively from the original *PL* structure and from the standard *PL* structure of S^5. By Whitehead's Theorem on uniqueness of triangulations, those two structures must be *PL* equivalent, and so $M^5 \overset{PL}{=} S^5$. □

Therefore, in the PL case the *strong* G.P.C. is true, for $n \geq 5$. The difference from the smooth case is exactly that, by the cone construction, there are no nontrivial twisted spheres in PL (see Footnote 6).

Remark 3.26. For $n = 4$ the strong PL G.P.C. is still an open problem. It is equivalent to its smooth counterpart, because $PL = DIFF$ up to dimension 6.

Finally we observe that the G.P.C. is true in TOP. In fact, if $n \geq 5$ a topological homotopy sphere has a PL structure, since the only obstruction is the Kirby-Siebenmann class that lives in $H^4(M, \mathbb{Z}_2) = \{0\}$ (see Remark 2.1). Then the positive answer to the TOP conjecture follows from the positive answer to the PL one. The cases $n = 3, 4$ will be discussed in the next sections.

Remark 3.27. A more direct treatment of the TOP case, in dimensions at least five, is due to Newmann (1966). The weak PL Poincaré conjecture obviously follows from the strong one. However, a direct proof was given by Stallings and Zeeman (1960-62) using sophisticated techniques of PL topology (see the Appendix for a sketch of the proof).

4 - Freedman's classification of simply connected closed 4-manifolds

The proof of the $DIFF$ h-cobordism Theorem fails hopelessly in dimension 4, as shown by Donaldson (see, for example, [**12**], [**13**]). Working on a TOP version of the h-cobordism theorem in dimension 4, Freedman, at the end of the 70's, obtained a major breakthrough in low dimensional topology: the classification of closed, simply connected [12] 4-manifolds. The following is a brief summary of Freedman's work.

4.1 - *The intersection form*

Compact and connected surfaces can be described in terms of the first homology group with $\mathbb{Z}_2$ coefficients as follows: $H_1(M^2; \mathbb{Z}_2)$ is the $\mathbb{Z}_2$ vector space spanned by equivalence classes of maps $\alpha : S^1 \longrightarrow M^2$, where two such maps are equivalent if there exists a compact oriented 2-manifold W, with a boundary two copies of S^1, and a map $F : W \longrightarrow M^2$ which coincides with the two given maps on the boundary. Given two such classes, we can choose representatives which are transversal immersions.

[12] We can not classify closed 4-manifolds in general, since any finitely presented group is the fundamental group of a 4-manifold and those groups can not be classified.

In particular they intersect in a finite number of points. Whether the cardinality of the set $\mu_{M^2}([\alpha_1], [\alpha_2])$ of intersection points is even or odd it does not depend on the given representatives. So we can define a bilinear form called the *intersection form*:

$$\mu_{M^2} : H_1(M^2; \mathbb{Z}_2) \oplus H_1(M^2; \mathbb{Z}_2) \longrightarrow \mathbb{Z}_2 \cong H_2(M^2, \mathbb{Z}_2).$$

The intersection form is clearly symmetric and, according to Poincaré duality, non singular.

Here are some examples:

- For the 2-sphere S^2 we have $H_1(S^2; \mathbb{Z}_2) = \{0\}$.
- For the torus $T = S^1 \times S^1$ we have that $H_1(T; \mathbb{Z}_2)$ is 2-dimensional, as a $\mathbb{Z}_2$ vector space, and it is generated by the inclusions of S^1 as $S^1 \times \{p\}$ and $\{q\} \times S^1$ respectively. The intersection form μ is the *hyperbolic form*, given by the matrix:

$$\mathbb{H} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

- For the projective plane $\mathbb{R}P^2$ we know that $H_1(\mathbb{R}P^2; \mathbb{Z}_2)$ is generated by the inclusion of a projective line $\mathbb{R}P^1 = S^1 \longrightarrow \mathbb{R}P^2$. Two such lines, if transversal, intersect at exactly one point, hence the intersection number is 1.
- A simple application of the Mayer-Vietoris sequence shows that the intersection form of a connected sum of surfaces is the direct sum of their respective intersection forms.

From an algebraic point of view, finite-dimensional symmetric, non singular bilinear forms over $\mathbb{Z}_2$ are direct sums of a diagonal form and hyperbolic forms. Since a closed surface is a sphere or a connected sum of tori and projective planes, it follows that:

Theorem 4.1. *The map that associates to a given closed surface M^2 its intersection form μ_{M^2} is a bijection. Moreover, M^2 is orientable if and only if $\mu(x, x) = 0, \ \forall \ x \in H_1(M^2, \mathbb{Z}_2)$.*

Four-dimensional closed, simply connected manifolds can be dealt with in a similar way using homology with coefficients in $\mathbb{Z}$ (we will omit the reference to the group of coefficients in the notation).

If M is such a manifold, its second homology group with integer coefficients $H_2(M)$ is a finitely generated free abelian group, hence isomorphic to the direct sum of $b = b_2(M)$ copies of $\mathbb{Z}$, where $b_2(M)$ is the *second Betti number* of M. Moreover, an evaluation of a cohomology class on a homology class gives a canonical isomorphism between 2-dimensional homology and 2-dimensional cohomology. The intersection form of M is the bilinear map given by the *cup product*:

$$\mu_M : H^2(M) \times H^2(M) \longrightarrow H^4(M), \qquad \mu_M([\alpha]_1, [\alpha]_2) = [\alpha]_1 \cup [\alpha]_2.$$

If M is a smooth manifold, we can view the intersection form in different ways. We may mention a couple of them.

Fix an orientation on M, i.e. an isomorphism $H^4(M) \longrightarrow \mathbb{Z}$.

- Using the de-Rham cohomology of M, let α_i be closed 2-forms which represent cohomology classes $[\alpha_i]$, $i = 1, 2$. Then, up to a fixed constant

$$\mu_M([\alpha_1], [\alpha_2]) = \int_M \alpha_1 \wedge \alpha_2.$$

- Since M is simply connected, by the Hurewicz Theorem any homology class in $H_2(M)$ is the image of the fundamental class of the 2-sphere S^2 through the map induced in homology by a continuous function $f : S^2 \longrightarrow M$. Such a function may be assumed to be a smooth immersion. Thus, roughly speaking, this class is $f(S^2)$. We can put two such maps transversal to each other without changing the homology classes, so that their images will intersect at a finite number of points, $\{p_1, \ldots, p_k\}$. To each p_j we can associate a sign $\varepsilon(p_j) = \pm 1$ according to whether the orientation of the tangent space to the first submanifold, followed by the one on the tangent space to the second, is the orientation of $T_{p_j}M$ or the opposite one. The *intersection number* $\sum_1^k \varepsilon(p_j)$ is well defined, i.e. it only depends on the homology classes. This defines a map:

$$\tilde{\mu}_M : H_2(M) \times H_2(M) \longrightarrow \mathbb{Z},$$

which is the intersection form, up to canonical isomorphisms.

Remark 4.2. A result of Quinn (see [**52**]) guarantees that a *TOP* manifold admits a *DIFF* structure in the complement of a point. This allows us to carry the above constructions to the *TOP* category and, more in general, to use *DIFF* methods in many proofs of purely *TOP* results.

It follows from basic algebraic topology that the intersection form is bilinear, symmetric and unimodular. In particular, with respect to a given basis, it is represented by a symmetric matrix of determinant ± 1. Here are some examples that are very similar to the ones given in the two–dimensional case:

- $M = S^4$. The intersection form is the empty form.
- $M = S^2 \times S^2$. The second homology group is generated by the canonical inclusions of S^2 as $S^2 \times \{p\}$ and as $\{q\} \times S^2$. These classes do not depend on the particular choices of $p, q \in S^2$. Clearly the two submanifolds intersect at exactly one point with a positive intersection number. On the other hand, if we choose two different basepoints, two immersions of the first type (or of the second) do not intersect,

hence the generators have self–intersection number equal to zero. The intersection form is then the *hyperbolic form*, that, in this basis, is given by the matrix:

$$\mathbb{H} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

• $M = \mathbb{C}P^2$, the complex projective plane. The 2-dimensional homology is generated by the inclusion of a projective line $S^2 = \mathbb{C}P^1 \subseteq \mathbb{C}P^2$. Two such lines intersect at exactly one point, with a positive intersection number. Hence the intersection form is $\mu = [1]$.

• $M = \overline{\mathbb{C}P}^2$, the complex projective plane with the opposite orientation. In this case the intersection form is $\mu = [-1]$. In general, inverting the orientation of the 4-manifold the intersection form changes its sign.

• If M is the connected sum of two manifolds, then its intersection form is the direct sum [13] of the intersection forms.

4.2 - *Integral bilinear forms*

If $\mathbb{E}$ is a real vector space, a symmetric bi-linear non singular map on $\mathbb{E}$ is represented, in a suitable basis, by a diagonal matrix with entries ± 1. It follows that real symmetric automorphisms are classified by the dimension of $\mathbb{E}$ and by the signature (the number of positive eigenvalues minus the number of negative ones). For integral symmetric, unimodular bi-linear forms the situation is quite different. In reality, these forms are not fully classified.

In order to study isomorphism classes of symmetric unimodular integral bilinear forms it is convenient to consider various cases. Let

$$\mu : \mathbb{Z}^b \oplus \mathbb{Z}^b \longrightarrow \mathbb{Z},$$

be such a bilinear form. We will say that:

- μ is of *even type* if $\mu(x,x) \equiv 0$ $(mod\ 2)$, $\forall x \in \mathbb{Z}^b$.
- μ is of *odd type* if it is not of even type.
- μ is *definite* if $\mu(x,x)$ is positive (or negative), $\forall\, x \in \mathbb{Z}^b$.
- μ is *indefinite* if is not definite.

An important example is the form $\mathbb{E}_8$, which is the bilinear form on $\mathbb{Z}^8$ whose matrix in the canonical basis is:

[13] The direct sum of two bilinear forms is defined in a standard way, in the direct sum of the corresponding lattices.

$$\mathbb{E}_8 = \begin{bmatrix} 2 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & -1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 \end{bmatrix}.$$

This is the Cartan form of the exceptional Lie group E_8 and is characterized as the unique positive form of the even type and rank 8.

An algebraic feature of the forms of even type, that is not difficult to prove (see [**24**]), is the following:

Lemma 4.3. *A symmetric unimodular form of even type has a signature which is a multiple of* 8*.*

Indefinite bilinear forms are thus classified:

Theorem 4.4. *If μ is an indefinite form, then:*

- $\mu = m[1] \oplus n[-1], \quad m, n \neq 0$ *if μ is of odd type.*
- $\mu = m\mathbb{H} \oplus n\mathbb{E}_8, \quad m \neq 0$ *if μ is of even type.*

For the definite forms, the situation is completely different. We know that, for a given rank, the set of isomorphism classes of such forms is finite, but its cardinality grows very fast with the rank. For example, there are at least 10^{51} classes of rank 40. A useful reference for this is [**24**].

4.3 - *Freedman's Theorem*

It was already known before Freedman's work that the intersection form is a basic invariant of 4-dimensional simply connected closed *TOP* manifolds. For example, Whitehead proved the following Theorem (see [**24**] for a simple proof):

Theorem 4.5. *Two closed, simply connected* 4*-manifolds are homotopy equivalent if and only if their intersection forms are isomorphic.*

At this point, two natural questions arise, closely analogous to the 2-dimensional case (see Theorem 4.1):

- Given a symmetric bilinear, unimodular form, does a manifold (in *TOP* or $PL = DIFF$), whose intersection form is the given one exist?
- If two manifolds (in *TOP* or $PL = DIFF$) have isomorphic intersection forms, are they equivalent in the corresponding category?

For a better understanding of the first question, it is important to produce examples of manifolds with a given intersection form. For instance, does a differentiable manifold with intersection form $\mathbb{E}_8$ or $2\mathbb{E}_8$ exist? [14] A possible strategy to find such a manifold is to start with the *Kümmer surface*. This is the quartic surface of $\mathbb{C}P^3$:

$$K = \{[z_0, z_1, z_2, z_3] \in \mathbb{C}P^3 : \sum_0^3 z_i^4 = 0\}.$$

Using the theory of characteristic classes, we may compute the intersection form, and we find that

$$\mu_K = 3\mathbb{H} \oplus 2\mathbb{E}_8.$$

The idea now is to decompose K as the connected sum of two manifolds, one with intersection form $3\mathbb{H}$ and the other one with intersection form $2\mathbb{E}_8$. For this purpose, Casson found open submanifolds $CH_i, i = 1, 2, 3$, properly homotopy equivalent to the complement of a 4-disc in $S^2 \times S^2$, the so called *Casson handles*, that realize the summands $\mathbb{H}$ in the intersection form of K. Then the natural strategy would be to perform a *DIFF* surgery in order to eliminate the CH_i regions, and fill up the holes with disks. In this way we would obtain a manifold with intersection form $2\mathbb{E}_8$. Since such a manifold does not exist in *DIFF* (see Theorem 4.10), this strategy is bound to fail.

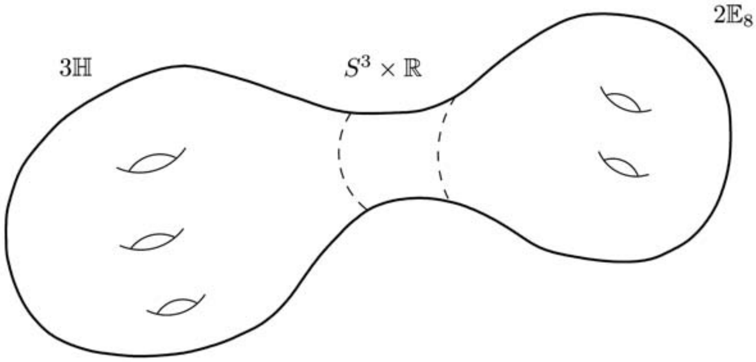

Fig. 4. The Kümmer surface and its Casson handles.

[14] We now know that such a manifold does not exist (see the next subsections).

The main techniques of differential topology, namely the surgery theory and the h-cobordism theorem, work well in *DIFF* for dimensions ≥ 5, but not for dimension 4. In fact we will see, the "neck" in figure 4 is homeomorphic but *not* diffeomorphic to $S^3 \times \mathbb{R}$. In order to construct a manifold with intersection form $2\mathbb{E}_8$ and, generally, to answer the two questions stated above, the effort was concentrated to find a *TOP* version of surgery and h-cobordism. Freedman proved that the Casson handles are *TOP*-equivalent to the complement of a 4-disc in $S^2 \times S^2$, a very difficult result indeed. He found an open submanifold, containing the Casson handles, that realizes the $3\mathbb{H}$ summand in the intersection form and is homeomorphic to $S^3 \times \mathbb{R}$ outside a compact set. So he was able perform a *TOP*-surgery to obtain K as a connected sum of two *TOP*-manifolds, with intersection forms $2\mathbb{E}_8$ and $3\mathbb{H}$ respectively. Using this kind of techniques, Freedman proved the existence of a *TOP*-manifold with any given intersection form.

In order to answer the second question, Freedman proved a *TOP* version of the h-cobordism Theorem. In the previous section, we noticed that the hypothesis on the dimension in the h-cobordism Theorem is used essentially to find an *embedded* 2-disc on a level hypersurface. One of the main achievements of Freedman's version of the h-cobordism Theorem is the proof that such a disk *exists* in *TOP*. Freedman's remarkable results may be summarized as follows:

Theorem 4.6 (Freedman). *Let $\mathcal{M}_1$, $\mathcal{M}_2$ be the homeomorphism classes of simply-connected topological manifolds with intersection form of odd and even type respectively. Then:*

(1) *If μ is a symmetric unimodular bi-linear form of odd type, then there are exactly two elements in $\mathcal{M}_1$ having μ as their intersection form, and they are very different: the product of one of them with S^1 is a triangulable manifold, while the other is a non-triangulable manifold.*

(2) *If μ is a symmetric unimodular bi-linear form of even type, then there is exactly one element in $\mathcal{M}_2$ having μ as its intersection form.*

The 4-dimensional Poincaré conjecture follows by applying the Theorem to the empty form (which is even!):

Theorem 4.7 (4-dimensional topological Poincaré conjecture). *A closed, simply connected topological 4-manifold, homotopy equivalent to S^4, is homeomorphic to S^4.*

Another interesting consequence of Theorem 4.6 is the existence of a 4-manifold, homotopy equivalent to $\mathbb{C}P^2$, which is not triangulable.

For later use we state a *TOP* characterization of $\mathbb{R}^4$ which is a consequence of the *TOP* version of the 4-dimensional proper h-cobordism Theorem.

Theorem 4.8 (Freedman). *Let $M \in TOP$ be a 4-dimensional simply connected manifold with $H_2(M) = \{0\}$, and with one end (complement of a compact set) homeomorphic to $S^3 \times \mathbb{R}$. Then M is homeomorphic to $\mathbb{R}^4$.*

4.4 - *The DIFF-case and exotic structures on* $\mathbb{R}^4$

As far as differentiable manifolds are concerned, the Theorem of Freedman leads to the following questions:

• Given a symmetric unimodular bi-linear form μ, does a *smooth* manifold having μ as its intersection form exist?

• Given two smooth manifolds with isomorphic intersection forms, are they diffeomorphic?

The answer to the first question is negative in general, but we know two important necessary conditions. The first of them, due to Rohlin, has been known for a long time and may be considered as the *DIFF* version of Lemma 4.3 (see [**53**]):

Theorem 4.9. *If a smooth, closed, simply connected 4-dimensional manifold has an intersection form of even type, then its signature is a multiple of* 16.

The second, very striking, was obtained by Donaldson (see [**12**]).

Theorem 4.10. *If M is a simply connected closed smooth 4-manifold with a definite intersection form μ, then, up to orientation, $\mu = n[1]$.*

Note that such an intersection form is realized in *DIFF* by the connected sum of n copies of $\mathbb{C}P^2$ (or $\overline{\mathbb{C}P^2}$).

Once the question for definite forms is settled, we examine the case of the indefinite forms. We will consider the two cases mentioned explicitly in the classification Theorem 4.4.

• If μ is odd, then $\mu = n[1] \oplus m[-1]$. These intersection forms are realized in *DIFF* by the connected sum of n copies of $\mathbb{C}P^2$ and m copies of $\overline{\mathbb{C}P^2}$ (observe that the other manifold with the same intersection form is not in *DIFF* because it is not triangulable).

• If μ is even, then $\mu = n\mathbb{H} \oplus k\mathbb{E}_8$. If $M \in DIFF$, the signature of μ is a multiple of 16 by 4.9, so $k = 2m$. If $n = 3m + h, h \geq 0$, such a form is realized in *DIFF* by a connected sum of m copies of the Kummer surface and h copies of $S^2 \times S^2$.

The latter consideration leads to the following conjecture:

Conjecture. *There are no smooth manifolds with intersection form* $n\mathbb{H} \oplus 2m\mathbb{E}_8$ *and* $n < 3m$.

Remark 4.11. Observe that, for $\mu = n\mathbb{H} \oplus 2m\mathbb{E}_8$, the rank of μ is $b_2 = 2n + 16m$ (b_2 is the second Betti number of the manifold) and the signature is $\sigma = 16m$. The condition $n < 3m$ is therefore equivalent to $b_2 < \frac{11}{8}\sigma$. For this reason the conjecture takes the name of *the* $\frac{11}{8}$*-conjecture*. A positive answer to this conjecture has been given by Donaldson for $n = 1, 2$. Note that a positive answer to the full conjecture would imply that *every smooth, closed, simply connected 4-manifold is homeomorphic to the connected sum of algebraic surfaces.*

Remark 4.12. We notice the interesting similarity between the results quoted above and the classification Theorem 4.1 for 2-dimensional manifolds. The role of orientable surfaces is played by the *spin* manifolds (μ of even type), and the role of the non orientable surfaces is played by the non-spin manifolds (μ of odd type).

As far as the second question is concerned, very little is known. However, the techniques and results we mentioned above lead to proving the existence of exotic structures on $\mathbb{R}^4$. This is a very surprising fact because, if $n \neq 4$, $\mathbb{R}^n$ has long been known to admit a unique *PL* and *DIFF* structure (see the Appendix).

To give an idea of the construction of an exotic structure on $\mathbb{R}^4$ we will go back to the example of the Kümmer surface. As we stated earlier, this surface contains an open submanifold X, homeomorphic to the complement of a 4-disc in the connected sum of three copies of $S^2 \times S^2$. It follows that X has one end, $End(X)$, homeomorphic to $S^3 \times \mathbb{R}$. This end is an open set of K, in particular, it admits a *DIFF*-structure. However this structure *is not* diffeomorphic to $S^3 \times \mathbb{R}$ since, otherwise, we could perform a *DIFF*-surgery, thus obtaining a manifold in *DIFF* with intersection form $2\mathbb{E}_8$, which contradicts Donaldson's Theorem 4.10.

There is a smooth embedding of X into the connected sum $\sharp_1^3 S^2 \times S^2$ that represent the 2-dimensional homology of $\sharp_1^3 S^2 \times S^2$. Identify X with its image, and let X_0 be X with its end truncated at level 0. Then, let M be the complement of X_0 in $\sharp_1^3 S^2 \times S^2$. Using the Van Kampen Theorem applied to $M \cup X$, and the Mayer-Vietoris sequence, we see that M is simply connected and the second homology group vanishes. Moreover, by construction, M has one end, that is homeomorphic to $S^3 \times \mathbb{R}$. Then, by Freedman's Theorem 4.8, M is homeomorphic to $\mathbb{R}^4$.

However, *M can not be diffeomorphic to* $\mathbb{R}^4$. In fact, the compact set $M \setminus M \cap X$ can not be contained in any smooth disc, or again, by a *DIFF*-surgery, we would then obtain a smooth manifold with intersection form $2\mathbb{E}_8$. But in $\mathbb{R}^4$ all compact sets are contained in a smooth disc!

Even more is known:

Theorem 4.13 (De Michelis-Freedman). *There is an uncountable family of non-diffeomorphic differentiable structures on* $\mathbb{R}^4$.

5 - The 3-dimensional case

While the higher dimensional cases of the G.P.C. only involve (sophisticated) methods of differential topology, the known solution of the classical case, $n = 3$, involves (hard) analysis on manifolds [15]. The method, introduced by Hamilton in the 80's and known as the *Ricci flow method*, consists of modifying a given metric in order to obtain a "better" one, and using differential geometry to deduce topological properties of the manifold. Again, the results give a positive answer to a much more general conjecture, the *geometrization conjecture*, stated by Thurston in the 70's.

5.1 - *The geometrization conjecture*

Since the beginning of last century it was clear that Riemannian geometry, which had recently been born, was to play an important role in the study of the topology of surfaces, a topic that was still in its infancy. Particulary remarkable in this sense is the *Uniformization Theorem*, coming from the work of Poincaré, Klein and others:

Theorem 5.1. *Any* 2*-dimensional manifold admits a complete metric of constant curvature* [16].

Simply connected, complete surfaces of constant curvature k are, up to normalization of the metric, the unit sphere, the Euclidean plane and the hyperbolic plane [17], with $k = 1, 0, -1$ respectively. In particular, compact surfaces have com-

[15] In reality, also Donaldson's work, discussed in the previous section, is based on hard analysis and differential geometric methods.

[16] Such a metric exists in any conformal class.

[17] A model is the upper half plane $\mathbb{H}^2 := \{(x, y) \in \mathbb{R}^2 : y > 0\}$ with the metric $ds^2 = (dx^2 + dy^2)/y^2$.

plete, simply connected, constant-curvature surfaces as universal coverings, and therefore, they are quotients of such spaces by properly discontinuous groups of isometries. This would eventually lead to the topological classification of compact surfaces.

A similar program was suggested by Thurston in the 70's for the case of 3-dimensional manifolds. Not surprisingly, one can not expect the existence of a constant-curvature metric, but at the most the existence of a "nice" metric. Let us explain what is meant by "nice".

We recall that a "geometry" in the sense of Klein is essentially a manifold M together with a transitive group of diffeomorphisms G, and geometry is the study of properties of the subsets of M which are invariant under the action of G. If the isotropy groups $G_x := \{g \in G : gx = x\}$ are compact, we can define a Riemannian metric on M for which G acts as a group of isometries. Since G is transitive, the metric is homogeneous, hence it is complete. This leads to the following:

Definition 5.2. A *geometry* on a 3-dimensional manifold M is a locally homogeneous complete Riemannian metric with compact quotients [18]. If the manifold admits such a metric, we will say that it is *geometrizable*.

Of course we will consider equivalent two geometries which differ by a diffeomorphism commuting with the actions. Moreover, since there may be several transitive groups acting on M with the same orbits, we will assume that the group is *maximal*.

Thurston gives a classification of simply connected (maximal) geometries (see [**62**]). They are:

- The spaces of constant curvature, S^3, $\mathbb{R}^3$, $\mathbb{H}^3$, with a 6-dimensional isometry group.
- The products $S^2 \times \mathbb{R}$, $\mathbb{H}^2 \times \mathbb{R}$, with a 4-dimensional isometry group.
- The (nilpotent) 3-dimensional Heisenberg group:

$$Nil_3 = \{H = \begin{bmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{bmatrix} : x, y, z \in \mathbb{R}\}$$

with a left invariant metric. The isometry group is 4-dimensional.

[18] The requirement for compact quotients is due to a desire to model compact manifolds with simply connected geometries and to simplify the classification of such geometries.

• The universal covering group $\widetilde{SL(2,\mathbb{R})}$ of the special linear group, with a left invariant metric and a 4-dimensional isometry group.

• The solvable 3-dimensional group, Sol_3 consisting of isometries of the Lorentzian metric $\mathrm{d}x^2 - \mathrm{d}y^2$ on $\mathbb{R}^2$, with a left invariant metric and a 3-dimensional isometry group.

Remark 5.3. The above classification is a special case of Bianchi's classification of homogeneous Lorentzian metrics. The classification of 3-dimensional geometries is important in General Relativity and Cosmology (see [**1**]).

A "Uniformization type Theorem", in this context, should assert the existence of a geometry on a 3-dimensional manifold. But this is not the case. However Thurston proved (see [**63**]) that if the fundamental group is "sufficiently large" then the manifold can be decomposed into open sets, each of which admits a geometry (for a more precise statement see 5.11). He also asked whether the hypothesis on the fundamental group was necessary. This question took the name of *The Geometrization Conjecture*. It can be roughly summarized as follows:

Conjecture. *Any closed 3-dimensional manifold can be decomposed into parts admitting a geometry.*

Remark 5.4. In general the geometries on the pieces will not nicely glue on the intersections, so they do not give a (global) geometry for the manifold. For instance it can be seen that, in general, the connected sum of two geometries does not admit a geometry.

Nice references for the Geometrization Conjecture (and the Ricci flow, that we will discuss later) are, between others, [**30**], [**10**] and the recent book [**3**].

We shall be more precise on what the term "decomposition" means. For simplicity we will deal with manifolds that are *compact and orientable*.

It is clear that if two manifolds are geometrizable, the geometrization conjecture has a positive answer for their connected sum. So we can focus on manifolds which do not decompose into connected sums of non trivial elements.

Definition 5.5. A manifold M is *prime* if, for every connected–sum decomposition $M = M_1 \sharp M_2$, either M_1 or M_2 is a sphere.

Remark 5.6. A similar concept is that of an *irreducible* manifold. A n-dimensional manifold is irreducible if every embedded $(n-1)$ dimensional sphere bounds a n–dimensional disc. In dimension 3, a manifold is prime if and only if it is either irreducible or diffeomorphic to $S^1 \times S^2$.

As it happens with integers with respect to multiplication, we have a unique prime decomposition, due to Kneser (existence, see [**28**]) and Milnor (uniqueness, see [**33**]):

Theorem 5.7. *A closed 3-dimensional manifold M admits a connected sum decomposition:*

$$M = (\sharp_1^p\, K_i) \quad \sharp \quad (\sharp_1^q\, L_j) \quad \sharp \quad (\sharp_i^l\, S^1 \times S^2),$$

where:

- *The K_i and L_j summands are closed irreducible 3-dimensional manifolds.*
- *The K_i summands have finite fundamental group and a homotopy sphere as universal covering.*
- *The L_j summands have infinite fundamental group and contractible universal covering.*

Moreover, assuming that none of the summands is homeomorphic to S^3, unless $M = S^3$, the decomposition is unique up to the order[19].

At this point, the problem is reduced to that of deciding whether a prime manifold admits a geometry. The answer is again negative. We need a finer decomposition.

Definition 5.8. A surface in a 3-dimensional manifold is called *incompressible*, if the inclusion induces a monomorphism between the fundamental groups.

The positive answer to the Geometrization Conjecture takes the following form:

Theorem 5.9. *Let M be a closed prime 3-dimensional manifold. Then there is a finite collection of incompressible tori, such that the complement is a union of geometrizable submanifolds.*

Hence any closed 3-dimensional manifold is geometrizable away from a finite number of spheres and incompressible tori.

---

[19] Because $M \sharp S^3 = M$.

Remark 5.10. Jaco-Shalen and Johannsen (see [**25**]) had already shown that there is a collection of finitely many incompressible tori which separate the manifold into compact submanifolds (with boundary), which are either torus irreducible or Seifert fibred spaces [20]. The latter were known to be geometrizable (see [**57**]).

Thurston's result quoted above may be stated precisely as follows:

Theorem 5.11. *If a 3-dimensional closed manifold M contains an incompressible torus, then the Geometrization Conjecture is true for M.*

By virtue of Thurston's result we may concentrate on the case where there are no incompressible tori. So the decomposition in (5.9) is the prime decomposition (5.7). Since in such decomposition the $S^1 \times S^2$ summands admit a geometry, we are left with the following two cases:

• Elliptic case: if M is an irreducible closed 3-dimensional manifold with a finite fundamental group, then M admits a geometry, necessarily a S^3 geometry, and therefore, it is diffeomorphic to the quotient of S^3 by a properly discontinuous action of a subgroup of $O(4)$ [21].
• Hyperbolic case: If M is an irreducible closed 3-dimensional manifold with an infinite fundamental group containing no subgroups isomorphic to $\mathbb{Z} \oplus \mathbb{Z}$, then M admits a complete metric of constant curvature -1 and compact quotients.

In particular, *a positive answer to the first question implies a positive answer to the 3-dimensional Poincaré conjecture.*

Remark 5.12. The reduction to the hyperbolic case follows from the fact that if the fundamental group of M contains a subgroup isomorphic to $\mathbb{Z} \oplus \mathbb{Z}$, then either it has a non-trivial center, or M contains an incompressible torus (see [**56**]). In the first case M is a Seifert fibred space (see [**7**]), for which the positive solution of the geometrization conjecture was known (see [**57**]), while in the second case the geometrization follows from Theorem 5.11.

[20] A compact 3-dimensional manifold is *torus irreducible* if every embedded incompressible torus can be deformed into a torus that lies in the boundary. A *Seifert fibred space* is a manifold foliated by circles.

[21] Since such a manifold has a homotopy sphere as its universal covering, the only geometry that it can support is the constant-curvature 1 geometry.

Remark 5.13. A well known result (by Preissman) states that every (non-trivial) abelian subgroup of the fundamental group of a closed manifold, with negative sectional curvature, is isomorphic to $\mathbb{Z}$. Therefore, the existence of a subgroup isomorphic to $\mathbb{Z} \oplus \mathbb{Z}$ in the fundamental group of a closed manifold is an obstruction to the existence of a hyperbolic geometry.

Remark 5.14. The above-mentioned reduction is not as effective as it seems. In fact, as we will see, in the course of the proof we may need to perform surgeries, which "change the manifold".

5.2 - *The Ricci flow*

At the beginning of the 80's Hamilton started a program to attack the Geometrization Conjecture. The basic idea was to use an evolution type equation on the metric. The classical evolution equation in the theory of P.D.E. is the heat equation:

$$\frac{\partial}{\partial t} u(t, x) = \Delta u, \qquad u(0, x) = u_0(x).$$

This equation describes the heat distribution, at time t, in a body with initial heat distribution u_0. It is intuitively clear that the distribution tends to be uniform as time passes. So, starting with a similar equation on a Riemannian metric, the hope is that such a metric tends to be "uniform" as time evolves. But which object should we put on the right hand side of the equation? We would like a symmetric bilinear form that depends only on the space derivatives of the metric of order at the most two. Up to multiples of the metric, there is essentially only one simple and natural such form, namely *Ricc*, the Ricci tensor [22].

Before continuing with our discussion, let us recall some few basic facts from Riemannian geometry. Let M be a n-dimensional Riemannian manifold and $\mathcal{H}(M)$ the space of smooth vector fields on M. We have the Levi-Civita connection

$$\nabla : \mathcal{H}(M) \times \mathcal{H}(M) \longrightarrow \mathcal{H}(M), \qquad (X, Y) \rightsquigarrow \nabla_Y X,$$

which, if M is isometrically embedded in some Euclidean space [23], is just the projection of the derivative of X in the direction Y on the tangent space of M. Now the

[22] This is essentially the same reason that led to the Einstein field equations in general relativity.

[23] By a celebrated Theorem of Nash, a Riemannian manifolds may be isometrically embedded is some Eucledean space.

geometry at a point $x \in M^n$ is the geometry of $\mathbb{R}^n$ in the sense that T_xM^n is isometric to $\mathbb{R}^n$. But this is not the case in a neighborhood of x, and the "curvature" measures how much the two geometries differ. The curvature is measured by the *Riemann curvature tensor*

$$R(X,Y)Z := \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z,$$

which is tri-linear with respect to smooth functions, i.e. a tensor[24]. We will be interested in various "contractions" of the curvature tensor.

• The *sectional curvature*. If $\pi \subseteq T_xM$ is a 2-dimensional subspace, the sectional curvature of π is given by

$$K(\pi) = \langle R(X,Y)Y, X\rangle,$$

where $X, Y \in \pi$ are orthonormal. So the sectional curvature is a function on the bundle of 2-planes tangent to M.

• The *Ricci tensor*. For $X, Y \in T_xM$, the Ricci tensor is the bi-linear form

$$Ricc(X,Y) = \sum_1^n \langle R(X_i, X)Y, X_i\rangle$$

where $\{X_1, \ldots, X_n\}$ is an orthonormal basis for T_xM. For a unit vector X, the quadratic form $Ricci(X) := Ricc(X,X)$ is called the *Ricci curvature* of X. Then, the Ricci curvature is a function on the unit tangent bundle.

• *The scalar curvature*. The scalar curvature S is a function on M, defined as the trace of the Ricci curvature, i.e.

$$S(x) = \sum_1^n Ricci(X_i).$$

Here are some geometric interpretations.

• The sectional curvature $K(\pi)$ is the Gaussian curvature, at x, of the (local) surface M_π spanned by the geodesics with origin in x and tangent vector in π. It measures the ratio between the length of "circles" in M and the length of Eucledian circles. More precisely, if $C = \{y \in M_\pi : d(x,y) = r\}$, then, for r small,

$$L(C) = 2\pi r(1 - \frac{K(\pi)}{6}r^2 + o(r^2)).$$

• The Ricci curvature measures the change of the Riemannian volume form. More precisely, if we move from x to a nearby point y, at distance r in the direction of

[24] Observe that ∇ is not linear in the first variable with respect to functions, hence it is not a tensor.

a unit vector $X \in T_x M^n$, we have

$$\mathrm{d}v(y) = (1 - \frac{r^2}{6} Ricci(X) + o(r^2))\mathrm{d}v_{Eucl}$$

where $\mathrm{d}v(y)$ is the volume form of M^n at y and $\mathrm{d}v_{Eucl}$ is the volume form of $\mathbb{R}^n$.

• The scalar curvature measures the ratio between the volume of a disc in M and the Euclidean volume. More precisely, if $D = \{y \in M : d(x, y) \leq r\}$, then, for r small,

$$vol(D) = \frac{4\pi}{3}(r^3 - \frac{1}{30}S(x)r^5 + o(r^5)).$$

We go back to our previous discussion. The natural equation to study is

$$\frac{\partial}{\partial t} g(t) = -2Ricc(g(t)) + \lambda(t)g(t), \qquad g(0) = g_0, \tag{1}$$

where g_0 is a given Riemannian metric on the manifold.

Hamilton also considered the equation (1) for $\lambda(t) = 0$:

$$\frac{\partial}{\partial t} g(t) = -2Ricc(g(t)), \quad g(0) = g_0. \tag{2}$$

The two equations are essentially equivalent in the sense that (2) can be transformed into (1), by rescaling the metric and the time variable. For example, by rescaling the metric in such a way that the volume is constant, we get equation (1) with $\lambda(t) = \int_M S(t)\mathrm{d}v(t)$ where $S(t)$ is the scalar curvature of $g(t)$ and $\mathrm{d}v(t)$ is the volume form.

Remark 5.15. The curve of metrics $g(t)$ may be thought of as an integral curve of the vector field $-2Ricc(g(t)) + \lambda(t)g(t)$ in the space of metrics on the manifold. Unfortunately, this space is not a nice space, even though it is a cone in the linear space of sections of the bundle of symmetric bilinear forms; a priori, there is no associated flow.

Hamilton proved that there is a solution of the equation (1), defined on a maximal interval $[0, T) \subseteq \mathbb{R}$, and that such a solution is unique once we fix the initial data. This fact follows from the classical existence and uniqueness theory for such equations, once written in suitable local coordinates[25]. The "singular time" T has to be understood as the time when the solution tends to a bilinear form which is not positive definite. Observe that the sign minus is essential. In general, there are no solutions

[25] The equation looks parabolic but it is not. However the proof of existence and uniqueness may be reduced to the parabolic case by an argument known as the De Turck trick. This simplifies the original proof of Hamilton.

for negative times, while the factor 2 is introduced just for convenience. Note also that, as we will see in the examples below, the interval is not the same for (2) and (1).

Here are some examples:

• Let us suppose that the initial metric has a constant Ricci curvature, i.e. $Ricc(g(0)) = kg(0),\ k \in \mathbb{R}$. Then the solutions of (2) are given by $g(t) = (1 - 2kt)g(0)$. Observe that the solution is a (positively defined) metric for $t \in [0, 1/2k)$, if $k > 0$. If we rescale the metric in such a way that the volume is constant, we get the constant solution $g(t) = g(0)$, which is a positive definite metric for all $t \in [0, \infty)$.

• For 2-dimensional manifolds, it is possible to show that the Ricci flow "converges" to a constant-curvature metric. In this way, we obtain an alternative proof of the Uniformization Theorem 5.1.

• Let Σ be a surface of constant curvature k. Consider the product metric on $S^1 \times \Sigma$. Then, the Ricci flow preserves the product structure, keeps the metric on S^1 invariant, and modifies the metric on Σ according to the formula of the first example.

• In general, the fixed points of (2) are the Ricci flat metrics, while for the rescaled flow with constant volume, they are the Einstein metrics, i.e. metrics of constant Ricci curvature.

Therefore the task is to study the flow, and try to prove that the solution converges to a "nice solution", i.e. to a homogeneous metric. While this is generally not the case, it works under additional conditions. A first step is to study the curvature of the solution.

The scalar curvature satisfies the following equation

$$\frac{\partial S}{\partial t} = \Delta S + 2\|Ricc\|^2, \tag{3}$$

where Δ and $Ricc$ are (space) Laplacian and the Ricci tensor for the metric $g(t)$. A basic fact has to be observed. Suppose that S has a minimum, S_m at $x \in M$. Then $\Delta S(x)$ is non negative, and so is $\frac{\partial S}{\partial t}$. Hence, heuristically, S_m is non-decreasing with t. This can be made rigorous, and it gives what is called the *scalar maximum principle*.

The Ricci tensor satisfies the following equation

$$\frac{\partial Ricc}{\partial t} = \Delta Ricc + Q(Ricc), \tag{4}$$

where Q is a quadratic expression in $Ricc$.

Hamilton proved an analogous maximal principle for the solution of the equation (4) that implies, in particular, that if the Ricci curvature of the initial data is positive, then it stays positive under the flow. Using this principle and estimates on the gradient of S, Hamilton proved a seminal result in the theory (see [**20**]).

Theorem 5.16. *If the manifold is compact and the initial metric has a positive Ricci curvature, then the rescaled solution is defined for all positive times and converges, as* $t \longrightarrow \infty$, *to a solution of constant curvature.*

In particular we have the following "Riemannian version of the Poincaré conjecture":

Corollary 5.17. *A compact simply connected Riemannian 3-dimensional manifold with positive Ricci curvature is diffeomorphic to* S^3.

5.3 - *The Ricci flow with surgery*

Let us consider the Ricci flow with an arbitrary initial condition. If the curvature of the metric $g(t)$ goes to infinity (somewhere) as $t \longrightarrow T < \infty$, then the solution can not be defined for $t \geq T$. In fact, the converse is also true: if the curvature of $g(t)$ is bounded for $t < T$, then the solution can be extended to $[0, T + \varepsilon)$ for $\varepsilon > 0$, sufficiently small. In particular, if the curvature remains bounded, the solution is defined in the interval $[0, \infty)$. In this case Hamilton proved the following result (see [**22**]):

Theorem 5.18. *Suppose that the Ricci flow is defined for all positive times. Then we have one of the following possibilities:*

(1) *The manifold admits a flat metric.*

(2) *There is a family of disjoint submanifolds* H_i, *admitting hyperbolic metrics, whose complement is the union of Seifert fibred spaces and* Sol_3 *manifolds.*

In particular, in both cases, the geometrization conjecture holds true.

We now analyze the case where the curvature becomes unbounded as t approaches $T < \infty$. One of the main achievements obtained by Perelman is the so called *canonical neighborhood Theorem*. The precise statement is rather technical, then we will just give a very inaccurate statement (see [**43**] for the precise statement).

Theorem 5.19. *There is a universal constant* r_0 *such that, if we start with a suitable normalized metric, any point with scalar curvature greater than* r_0^{-2} *has a neighborhood that looks like:*

(1) *cylinders, with a metric close to the standard one* (*called necks*),

(2) *balls or complement of a ball in the projective space, that are close to a cylinder outside a small set* (*called caps*)

or the manifold is diffeomorphic to a finite quotient of S^3.

If the curvature becomes big as $t \longrightarrow T < \infty$, then it may be unbounded everywhere, or on a proper subset. In the first case we will say that the manifold *becomes extinct* (big curvature, "small" manifold). If the manifold becomes extinct, it is, eventually, totally covered by canonical neighborhoods. Pasting these canonical neighborhoods together, whose topology is known, Perelman proved in [**44**].

Theorem 5.20. *If the manifold becomes extinct, then it is either a finite quotient of S^3 or $S^1 \times S^2$ or $\mathbb{R}P^3 \sharp \mathbb{R}P^3$. In all cases the geometrization conjecture holds true.*

In general, there will be a region where the curvature remains bounded, the *thick* part, and a region where it explodes, the *thin* part. Perelman took over an idea of Hamilton considering the *Ricci flow with surgery*. Let (x_i, t_i) be a sequence such that the curvature of $g(t_i)$ at x_i tends to infinity, as $t_i \longrightarrow T$. Then, near a limit point, we have a canonical neighborhood U, as in 5.19. We cut U off and fill up the boundary with caps obtaining a new manifold M_T, possibly not connected, with a Riemannian metric g_T. It was noted that, if M_T is geometrizable, so is M. Then we start a new Ricci flow on M_T with initial condition g_T. This procedure should be seen as a "non continuous" version of the classical Ricci flow (not even the manifold is the same!). The figures below give a very rough idea of what is happening.

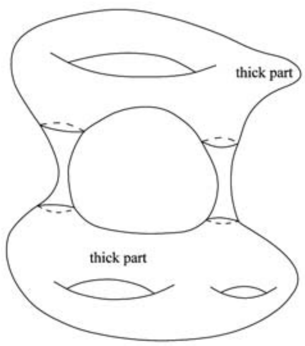

Fig. 5. The situation before surgery.

Then Perelman proved a key result: *the singular times do not accumulate*, i.e. in a finite interval we only need to perform a finite number of surgeries. The proof is quite involved and is based on beautiful arguments of Riemannian geometry. But not only this, again based on ideas of Hamilton, he proved that, if we start with a simply connected manifold, then the flow with surgery becomes extinct at a finite time, i.e the connected components of the modified manifold are covered by canonical neighborhoods. Then, arguing as in Theorem 5.20, we have the positive answer to the Poincaré conjecture.

Theorem 5.21. *A compact, simply connected 3-manifold is diffeomorphic to S^3.*

Remark 5.22. An alternative proof, based on harmonic maps theory, is given in [**11**].

If the fundamental group is finite, the argument implies that the manifold is a finite quotient of S^3. If the fundamental group in infinite, we have to allow the Ricci flow with surgery to be defined for all times. In this case, in the thin-thick decomposition, hyperbolic pieces emerge from the thick part, while the thin part will collapse, producing, after rescaling, what is called a *graph manifold*, essentially Seifert fibrations glued along boundaries. Further comments on the situation "*nous entraînerait trop loin*" as Poincaré said, and we may stop our discussion here.

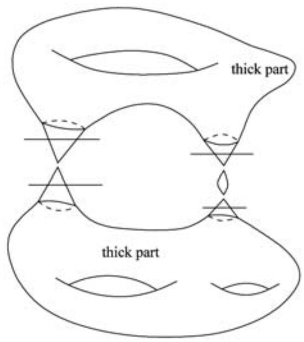

Fig. 6. The situation after surgery.

5.4 - *A remark on the Ricci flow in higher dimensions*

One of the reasons why the Ricci flow method works so well for 3-dimensional manifolds is that, in this dimension, the Ricci curvature completely determines the Riemannian curvature tensor. In particular, positive Ricci curvature is equivalent to positive sectional curvature. This is not the case in higher dimensions. However, suitable conditions on curvature imply, even in higher dimensions, the convergence of the Ricci flow. This is a very recent and fruitful field of research. The main point is to show that the positivity of certain curvatures is preserved by the flow and, in fact, improved. An example of such a result is the *differentiable pinching Theorem*. The classical pinching Theorem, proved in the 60's (Berger and Klingenberg), states that a compact simply connected Riemannian manifold, whose sectional curvature is in the interval $(1/4, 1]$, is homeomorphic to a sphere. A natural question, unanswered for fifty years, is whether such a manifold is *diffeomorphic* to a sphere. A positive answer to this question (with slightly weaker hypothesis) has been recently given by Brendle and Schoen. In [**5**], they prove that for those manifolds the (normalized) Ricci flow converges to a constant-curvature metric.

6 - Appendix

In this appendix, we briefly describe some interesting results related to the Poincaré conjecture that we did not include in the main part of the paper, so as not to deviate from the main focus of the points presented.

Definition 6.1. Let $M \in TOP$. A *triangulation* of M is a homeomorphism $f : |K| \longrightarrow M$, where $|K|$ is the support of a locally finite simplicial complex K; f is said to be *combinatorial* if $|K|$ is a PL manifold or, equivalently, if the link of any vertex is a PL-sphere.

Intuition may lead us to think that any triangulation of a topological manifold is combinatorial. Surprisingly, Edwards showed, in 1974, the existence of non combinatorial triangulations of some manifolds, even very simple ones like spheres and Euclidean spaces of suitable dimensions. For example, the double suspension of the Poincaré sphere gives a non combinatorial triangulation of S^5. It is important to observe that combinatorial triangulations of S^n or $\mathbb{R}^n$ are PL equivalent, if $n \neq 4$. More precisely:

Theorem 6.2. *If $n \neq 4$, $\mathbb{R}^n$ admits a unique PL structure as well as a unique DIFF structure.*

Proof (Sketch). Since $\mathbb{R}^n$ has vanishing cohomology in positive dimensions, Munkres' smoothing-theory ensures that $DIFF$-uniqueness follows from PL-uniqueness. The latter was proved by Moise for $n = 3$ (see also 2.1), and by Stallings for $n \geq 5$ (see [**61**]). If M^n_{PL} is a PL manifold, with $M^n_{PL} \overset{TOP}{=} \mathbb{R}^n$ and $n \geq 5$, Stallings proves, with a very delicate argument, that each compact subset of M^n_{PL} is contained in the interior of a PL disk. From this, the equality $M^n_{PL} \overset{PL}{=} \mathbb{R}^n$ follows by applying the PL version of the Palais-Cerf Lemma, which is due to Newman and Gugenheim (see [**55**], pg. 44). □

Remark 6.3. The only properties of M^n_{PL}, apart from $n \geq 5$, used in Stallings' argument are that M^n_{PL} is contractible and simply connected at infinity, i.e. each compact subset lies in a compact subset whose complement is simply connected. Clearly $\mathbb{R}^n$ enjoys these properties for $n > 2$.

As far as spheres are concerned we have:

Theorem 6.4. *If $n \neq 4$, S^n has a unique PL structure.*

Proof. This is a classical result for $n \leq 2$, and it is due to Moise for $n = 3$. For $n \geq 5$ the result follows from Smale's strong version of the *PL* Poincaré conjecture. □

In conclusion, spheres and Euclidean spaces in general admit a unique *PL* structure, but *not* a unique polyhedral structure.

We now briefly go back to the weak G.P.C:

Theorem 6.5. *Let M^n be a PL homotopy sphere, $n \geq 5$. Then M is homeomorphic to S^n.*

Proof. Let $p \in M$. A simple argument shows that $M \setminus \{p\}$ is contractible, and simply connected at infinity. Then, by Theorem 6.2 (and Remark 6.3), M^n is homeomorphic to the one point compactification of $\mathbb{R}^n$, i.e. to S^n. □

Finally, we may quote an interesting consequence of the 3-dimensional Poincaré conjecture:

Theorem 6.6. *If $M^4 \in TOP$, any triangulation of M^4 is combinatorial.*

Proof (Sketch). One can show (see [**19**]) that the link of any vertex in a given triangulation is a simply-connected 3-manifold, hence such a link is a sphere. □

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SANDRO BUONCRISTIANO
Dipartimento di Matematica
Università di Roma Tor Vergata
Roma, Italy
e-mail: buoncris@mat.uniroma2.it

FRANCESCO MERCURI
Departamento de Matemática
Universidada Estadual de Campinas
Campinas, S. P., Brazil
e-mail: mercuri@ime.unicamp.br