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A quadrature method for Cauchy singular integral equations with singular given functions

Abstract. Cauchy singular integral equations with index zero and with singular given functions are investigated. In order to improve the smoothness properties of the known functions, following [**29**] (see also [**31**]), a regularizing procedure is introduced and a Cauchy singular integral equation with fixed singularities of Mellin convolution type is obtained. Thus a quadrature method is proposed whose stability and convergence is shown by means of numerical tests.

Keywords. Cauchy singular integral equation.

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1 - Introduction

In this paper we consider the Cauchy singular integral equation with index zero

$$(D+K)f = g \tag{1}$$

where g is a given function on $(-1,1)$, f is the unknown, D is the dominant operator

$$(Df)(y) = \cos\pi\alpha f(y) v^{\alpha,-\alpha}(y) - \frac{\sin\pi\alpha}{\pi} \oint_{-1}^{1} \frac{f(x)}{x-y} v^{\alpha,-\alpha}(x)dx, \tag{2}$$

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and K is the perturbation operator

$$(Kf)(y) = \mu \int_{-1}^{1} k(x,y) f(x) v^{\alpha,-\alpha}(x) dx \tag{3}$$

with $\mu \in \mathbb{R}$, k a known function and $v^{\alpha,-\alpha}(x) = (1-x)^{\alpha}(1+x)^{-\alpha}$, $-1<\alpha<1$ a Jacobi weight. The symbol $\oint$ means that the integral has to be interpreted as the Cauchy principal value.

The case when the right-hand side g and the kernel k are smooth functions was extensively investigated and today there exists a wide literature about direct and indirect numerical methods (see, for instance, [3], [4], [7], [9], [14], [16], [17], [18], [21], [22], [23], [24] and more recently [8]) to approximate the solution of (1) in suitable weighted spaces.

In 1997 in [29] the generalized airfoil equation (i.e. equation (1) with $\alpha = 1/2$) was investigated in the case when the right-hand side has a jump in an inner point $-1<x_0<1$. There, the authors introduce a regularizing procedure to improve the behavior of the given functions and obtain a Cauchy singular integral equation with a smooth right-hand side and with a kernel which contains a Mellin convolution. Then the authors prove the stability and the convergence of a Galerkin method based on high-order polynomials. Moreover they also give numerical evidences which show the stability and the convergence of a collocation method. Later in [31] the author considers the same type of equation and proves the stability and the convergence of a modified collocation method.

In this paper we consider an integral equation more general than that examined in [29] and [31]. Indeed, we will investigate on equation (1) having known functions with fixed singularities of algebraic type and with a generic $-1<\alpha<1$, instead of $\alpha = 1/2$. More precisely, we will examine the case when the right-hand side and/or the kernel w.r.t. the variable y ($k_x(y)$) are of the form

$$g(y) = \frac{1}{(1+y)^{\lambda}}, \quad k_x(y) = \frac{1}{(1+y)^{\varepsilon}}, \tag{4}$$

or, more in general, may be decomposed as

$$g(y) = \frac{g_1(y)}{(1+y)^{\lambda}} + g_2(y), \quad k(x,y) = \frac{k_1(x,y)}{(1+y)^{\varepsilon}} + k_2(x,y), \tag{5}$$

where g_i and k_i are smooth functions and $\lambda, \varepsilon > -1$. In other words we will assume that the known functions have a fixed singularity in -1. However we remark that a singularity at 1 can be treated in analogous way. In this situation the low smoothness of the known functions gives a very poor theoretical order of convergence. Thus an alternative numerical approach is necessary to remove or to smooth the singularities of the kernel and/or the right-hand side in order to approximate the solution of (1) with a higher order.

In [**29**] the authors apply with success a regularizing technique to the particular case $\alpha = 1/2$ and to the case of inner singularities. In this paper we use a similar strategy to regularize the more general equation (1), whose known functions have a fixed endpoint singularity. We will give theoretical results about the boundedness of the involved operators and the smoothness properties of the known functions in suitable weighted spaces and we will show promising numerical results.

Then at first, following [**29**] (see also [**31**]), we will define a non linear transformation γ_q which is a continuous monotone function mapping $[-1,1]$ onto $[-1,1]$, depending on a parameter q arbitrarily chosen and having the first derivatives up to $q-1$ vanishing in -1. Subsequently, we will introduce a change of the variable $x = \gamma_q(t)$ and $y = \gamma_q(s)$ and we will multiply both sides by γ_q'. In this way the new given functions become smooth but, as in [**29**], also here the regularization has a price: the Cauchy singular integral equation

$$(D+K)f = g$$

is transformed in a Cauchy integral equation with a fixed singularity of Mellin convolution type

$$(D + \Sigma + \mathcal{K})\psi = \phi. \tag{6}$$

Indeed the new equation contains the operator

$$(\Sigma\psi)(s) = \int_{-1}^{1} \sigma\left(\frac{1+s}{1+t}\right)\frac{1}{1+t}\psi(t)v^{\alpha,-\alpha}(t)dt \tag{7}$$

i.e. a noncompact operator in which the kernel has a fixed singularity at $t=-1$.

At this point we will consider the new equation in suitable weighted spaces equipped with the L^2 norm. We will prove that the new given functions are smoother than the previous ones and we will give the boundedness of the involved operators by assuming that the transformed unknown function belongs to a suitable weighted space.

Then we will apply a quadrature method to the transformed equation and we will show that the numerical results obtained by this procedure are better then those obtained without the regularizing transformation, even if we have to handle with a Mellin convolution operator. The numerical procedure consists to approximate the operators D, Σ and $\mathcal{K}$ by means of a Gaussian quadrature rule and to collocate at suitable points. In this way a linear system is obtained and its unknowns allow us to construct the approximate solution of (6). Once the numerical method is introduced, the next step should be to prove its stability and convergence. Nevertheless the non compactness of the operator Σ makes difficult the theoretical study of (6) by means of standard techniques. This difficulty was emphasized in [**5**] where it was shown that, in order to solve the simplest equation $(I - K)f = g$, there exist piecewise polynomial collocation methods which converge when K is compact but which diverge when K assumes the form (7). In 2002, in [**19**] the authors proposed a polynomial collocation method based on Chebychev nodes of the second kind for Cauchy singular integral equations with fixed singularities and with non constant coefficients and they proved the stability and convergence in weighted L^2 spaces under necessary and sufficient conditions.

Here at the moment we are not able to prove a theorem of stability and convergence for the proposed quadrature method. However the numerical experiments seem to suggest that it is stable and convergent and stimulated by the higher precision of them, we will take the theoretical study of the considered problem in a future work.

The paper is structured as follows. In Section 2 we describe the spaces of functions in which we consider equation (6). In Section 3 we give the main results, in Section 4 the numerical tests are shown and in Section 5 the proofs of the main results conclude the paper.

2 - Spaces of functions

Let $v(x) := v^{\gamma,\delta}(x) = (1-x)^{\gamma}(1+x)^{\delta}$, $\gamma, \delta > -1$ a Jacobi weight and we denote by $L^2_v(A)$, $A \subseteq [-1,1]$ the set of all functions such that

$$\|f\|_{L^2_v(A)} := \left(\int_A f^2(x) v(x) dx \right)^{1/2} < +\infty.$$

For the sake of the simplicity if $A \equiv [-1,1]$ we will use the following notations

$$L^2_v := L^2_v([-1,1]), \quad \|fv\|_2 := \|f\|_{L^2_v([-1,1])}.$$

In order to introduce a subspace of L^2_v we define the main part modulus of smoothness as (see, for instance, [**25**])

$$\Omega^i_\varphi(f,t)_{v,2} := \sup_{0<h\le t} \|\Delta^i_{h\varphi} f\|_{L^2_v(I_{hi})} \tag{8}$$

where $\varphi(x) = \sqrt{1-x^2}$,

$$\Delta^i_{h\varphi} f(x) = \sum_{j=0}^{i} (-1)^j \binom{i}{j} f\left(x + \left(\frac{i}{2} - j\right) h\varphi(x)\right),$$

$i \in \mathbb{N}$, and $I_{hi} = [-1 + (2hi)^2, 1 - (2hi)^2]$.

Thus for a real number $r > 0$ and an integer $i > r$ we define the following norm

$$\|f\|_{Z_r(v)} := \|fv\|_2 + \sup_{\tau>0} \frac{\Omega^i_\varphi(f,\tau)_{v,2}}{\tau^r}$$

and introduce the following weighted Zygmund-type space

$$Z_r(v) = \left\{ f \in L^2_v : \|f\|_{Z_r(v)} < \infty \right\}.$$

3 - Main Results

In order to give the main results let us first introduce some notations.

We will denote by $\{p_m(v^{\rho,\theta})\}$ the system of polynomials with positive leading coefficients orthonormal with respect to the Jacobi weight $v^{\rho,\theta}$. Moreover we will denote by $x^{\rho,\theta}_{m,k} := x^{\rho,\theta}_k$ the zeros of $p_m(v^{\rho,\theta})$ $(x^{\rho,\theta}_1 < x^{\rho,\theta}_2 < \ldots < x^{\rho,\theta}_m)$ and by

$$L^{\rho,\theta}_m(f,x) = \sum_{k=1}^{m} l^{\rho,\theta}_k(x) f(x_k), \quad l^{\rho,\theta}_k(x) = \frac{p_m(v^{\rho,\theta},x)}{p'_m(v^{\rho,\theta}, x^{\rho,\theta}_k)(x - x^{\rho,\theta}_k)}$$

the Lagrange polynomial based on the zeros of $p_m(v^{\rho,\theta})$.

3.1 - *Why a regularizing procedure*

In [**21**] the authors revised in several aspects a quadrature method proposed in [**23**] to solve equation (1).

The procedure consists in using a Gaussian quadrature rule to approximate Df and Kf and collocate at suitable points. In particular, the zeros $x^{\alpha,-\alpha}_k$ of $p_m(v^{\alpha,-\alpha})$ are chosen as quadrature nodes and the zeros $x^{-\alpha,\alpha}_k$ of $p_m(v^{-\alpha,\alpha})$ are chosen as collocation knots. In this way, denoting by $\lambda^{-\alpha,\alpha}_i$ and $\lambda^{\alpha,-\alpha}_i$ the Christoffel numbers related to the weight functions $v^{\alpha,-\alpha}$ and $v^{-\alpha,\alpha}$, respectively, and setting $b_i = \sqrt{\lambda^{-\alpha,\alpha}_i} g(x^{-\alpha,\alpha}_i)$, $\forall i = 1, 2, \ldots, m$, one has the following linear system

$$\sqrt{\lambda^{-\alpha,\alpha}_i} \sum_{k=1}^{m} \sqrt{\lambda^{\alpha,-\alpha}_k} \left[-\frac{\sin \pi\alpha}{\pi(x^{\alpha,-\alpha}_k - x^{-\alpha,\alpha}_i)} + k(x^{-\alpha,\alpha}_i, x^{\alpha,-\alpha}_k) \right] \eta_k = b_i \tag{9}$$

whose unknowns η_k allow us to approximate the unknown solution f of equation (1) by means of the interpolating polynomial of degree $m-1$

$$f_m^*(x) = \sum_{k=1}^{m} \xi_k l_k^{\alpha,-\alpha}(x), \qquad \xi_k = \frac{\eta_k}{\sqrt{\lambda_k^{\alpha,-\alpha}}}. \tag{10}$$

The authors prove the stability of the method, the well conditioning of system (9), and under the assumptions

$$g \in Z_r(v^{-\alpha,\alpha}), \quad r > \frac{1}{2}$$

$$\sup_{|x|<1} \|k(x,\cdot)\|_{Z_r(v^{-\alpha,\alpha})} < \infty, \quad \sup_{|y|<1} \|k(\cdot,y)\|_{Z_r(v^{\alpha,-\alpha})} < \infty \quad r > \frac{1}{2}$$

prove that the approximate solution f_m^* tends to the exact one f^* in $L^2_{v^{\alpha,-\alpha}}$ with an error of the following type

$$\|[f_m^* - f^*]v^{\alpha,-\alpha}\|_2 \le \frac{\mathcal{C}}{m^r} \|f^*\|_{Z_r(v^{\alpha,-\alpha})} \tag{11}$$

where $\mathcal{C}$ is a positive constant independent of f^* and m.

Now if we assume that the kernel k and the right hand side g are singular in -1, for instance if we assume that

$$g(y) = k_x(y) = \frac{1}{(1+y)^\lambda}, \qquad 0 < \lambda < 1$$

the numerical procedure still works in the sense that it is stable and convergent. Nevertheless, since $g, k_x(y) \in Z_{2(\alpha-\lambda)+1}(v^{-\alpha,\alpha})$, according to (11), we have

$$\|[f_m^* - f^*]v^{\alpha,-\alpha}\|_2 = \mathcal{O}\left(\left(\frac{1}{m}\right)^{2(\alpha-\lambda)+1}\right), \quad \lambda < \frac{1}{4} + \alpha.$$

Consequently if $\lambda \sim \alpha$ the convergence estimate could be very poor.

The following example shows that the numerical results confirm our theoretical expectations.

EXAMPLE 3.1. *We consider the following integral equation*

$$(Df)(y) + \frac{1}{(1+y)^{2/3}} \int_{-1}^{1} (3x+y) f(x) v^{\frac{2}{3},-\frac{2}{3}}(x)\,dx = \frac{\sin(1+y)}{(1+y)^{2/3}}. \tag{12}$$

We solve system (9) *and we construct* f_m^* *according to* (10). *By* (11) *we have that the approximate solution* f_m^* *converges to the exact one* f^* *with an error of the order*

$\mathcal{O}\left(\frac{1}{m}\right)$. Table 1 *shows the numerical results. We take as reference solution the approximated one obtained with* $m = 600$ *and compute* $f_m^*(y)$ *with* $y \in (-1, 1)$.

Table 1.

m	$f_m^*(-0.9)$	$f_m^*(0.9)$
16	**0.1383751322128960**	**−0.1688096432661728**
32	**0.1357564274602711**	**−0.1740784457307186**
64	**0.1404195576847663**	**−0.1769422117588976**
128	**0.1414869626097728**	**−0.1765004042704331**
256	**0.1414788386196226**	**−0.1766499690881158**
512	**0.1416500039154950**	**−0.1767070774209447**

The aim of this paper is to improve the smoothness properties of the given functions in order to have the approximate solution with a satisfactory order of convergence. To this end we will use a well known technique (see, for instance, [**29**], [**31**]): introduce a regularizing procedure based on a smoothing transformation in order to remove or smooth the singularities of the kernel and/or the right-hand side.

3.2 - *A regularizing procedure*

We consider equation (1) with g and k as in (5). For the sake of the simplicity, but without loss the generality, we assume $g_2 \equiv k_2 \equiv 0$. In other words we are examining the following equation

$$(13) \qquad (Df)(y) + \frac{\mu}{(1+y)^{\varepsilon}} \int_{-1}^{1} k_1(x,y) f(x) v^{\alpha,-\alpha}(x) dx = \frac{g_1(y)}{(1+y)^{\lambda}}.$$

In order to remove or smooth the singularity of the right-hand side and of the kernel at $y = -1$, following an idea of [**29**] (see also [**31**]), we introduce a regularizing procedure.

We consider the following one-to-one map $\gamma_q : [-1,1] \to [-1,1]$

$$\gamma_q(t) = 2^{1-q}(1+t)^q - 1, \quad 1 < q \in \mathbb{N}$$

and we introduce the change of the variable $x = \gamma_q(t)$ and $y = \gamma_q(s)$ in (13).

Then, by observing that, for each Jacobi weight $v^{\alpha,\beta}$ it results

$$v^{\alpha,\beta}(\gamma_q(t)) = v^{\alpha,\beta}(t) M^{\alpha,\beta}(t)$$

with

$$M^{\alpha,\beta}(t) = 2^{(1-q)\beta}(1+t)^{q\beta-\beta}\left[\sum_{i=0}^{q-1}\left(\frac{1+t}{2}\right)^i\right]^\alpha$$

and by multiplying both sides of the new equation by $\gamma_q'(s)$ we have

$$\cos\pi\alpha\ \psi(s)v^{\alpha,-\alpha}(s) - \frac{\sin\pi\alpha}{\pi}\ \gamma_q'(s)\int_{-1}^{1}\frac{\psi(t)}{\gamma_q(t)-\gamma_q(s)}v^{\alpha,-\alpha}(t)dt$$
$$+\mu\int_{-1}^{1}\kappa(t,s)\psi(t)v^{\alpha,-\alpha}(t)dt = \phi(s)$$

where

(14) $$\psi(s) = f(\gamma_q(s))\gamma_q'(s)M^{\alpha,-\alpha}(s)$$

is the new unknown function and

(15) $$\phi(s) = q\ 2^{(1-q)(1-\lambda)}(1+s)^{q-1-q\lambda}g_1(\gamma_q(s))$$

and

(16) $$\kappa(t,s) = q\ 2^{(1-q)(1-\varepsilon)}(1+s)^{q-1-q\varepsilon}k_1(\gamma_q(t),\gamma_q(s))$$

are the new known functions.

Now we rewrite the previous equation as

$$\cos\pi\alpha\ \psi(s)v^{\alpha,-\alpha}(s) - \frac{\sin\pi\alpha}{\pi}\oint_{-1}^{1}\frac{\psi(t)}{t-s}v^{\alpha,-\alpha}(t)dt$$
$$-\frac{\sin\pi\alpha}{\pi}\int_{-1}^{1}\left\{\frac{\gamma_q'(s)}{\gamma_q(t)-\gamma_q(s)} - \frac{1}{t-s}\right\}\psi(t)v^{\alpha,-\alpha}(t)dt$$
$$+\mu\int_{-1}^{1}\kappa(t,s)\psi(t)v^{\alpha,-\alpha}(t)dt = \phi(s)$$

or equivalently as

(17) $$(D+\Sigma+\mathcal{K})\psi = \phi$$

where D is the dominant operator defined in (2),

$$(\Sigma\psi)(s) = -\frac{\sin \pi\alpha}{\pi} \int_{-1}^{1} \left\{ \frac{\gamma_q'(s)}{\gamma_q(t) - \gamma_q(s)} - \frac{1}{t-s} \right\} \psi(t) v^{\alpha,-\alpha}(t) dt \tag{18}$$

and

$$(\mathcal{K}\psi)(s) = \mu \int_{-1}^{1} \kappa(t,s) \psi(t) v^{\alpha,-\alpha}(t) dt. \tag{19}$$

We immediately remark that, by choosing $q > \dfrac{1}{1 - \max\{\varepsilon, \lambda\}}$ the original singular given functions k and g are transformed in two smooth functions κ and ϕ, respectively. We also note that the regularizing procedure produces an additional term: the operator Σ.

About this operator we remark that being

$$\frac{\gamma_q'(s)}{\gamma_q(t) - \gamma_q(s)} - \frac{1}{t-s} = \sigma\left(\frac{1+s}{1+t}\right) \frac{1}{1+t},$$

with

$$\sigma(z) = -\frac{1 + 2z + \ldots + (q-1)z^{q-2}}{1 + z + \ldots + z^{q-1}}$$

it can be rewritten as

$$(\Sigma\psi)(s) = -\frac{\sin \pi\alpha}{\pi} \int_{-1}^{1} \sigma\left(\frac{1+s}{1+t}\right) \frac{1}{1+t} \psi(t) v^{\alpha,-\alpha}(t) dt.$$

In other words Σ is an operator which has the peculiarity that its kernel contains a Mellin convolution.

In definitive the regularizing procedure removes the singularity at $y = -1$ but the Cauchy singular integral equation

$$(D + K) f = g$$

is transformed in a Cauchy singular integral equation with a fixed singularity of Mellin convolution type

$$(D + \Sigma + \mathcal{K}) \psi = \phi.$$

Moreover we mention that the regularizing technique maps

$$f \in L^2_{v^{\alpha,-\alpha}} \rightarrow \psi \in L^2_{v^{\alpha,q(\alpha-1)+1-2\alpha}}.$$

Thus, by assuming that the original equation is unisolvent in $L^2_{v^{\alpha,-\alpha}}$, since $\psi \in L^2_{v^{\alpha,q(\alpha-1)+1-2\alpha}}$, we might study the regularizing equation in $L^2_{v^{\alpha,q(\alpha-1)+1-2\alpha}}$ by assuming that it is unisolvent in this space. Nevertheless, in order to have the boundedness of the involved integral operator we need some additional assumptions on the parameters of the weight of the space. One of this is $q(\alpha-1)+1-2\alpha > -1$ which implies $q<2$. And this is impossible being $1<q \in \mathbb{N}$. Then for this reason, from now on we assume that the solution f^* of the original equation exists and is unique in $L^2_{v^{\alpha,-\alpha}}$ and, moreover, we assume that the solution ψ of the regularized equation exists in $L^2_{v^{\alpha,-\alpha}}$. Its uniqueness follows by the uniqueness of f^* (see, for instance, Lemma 4.1 in [**29**]).

In the following proposition we show that the new given functions are smoother than the original ones.

Proposition 3.1. *Assume that*

$$g_1 \in Z_r(v^{-\alpha,\alpha})$$

$$\sup_{|x|<1} \|k_1(x,\cdot)\|_{Z_r(v^{-\alpha,\alpha})} < \infty, \qquad \sup_{|y|<1} \|k_1(\cdot,y)\|_{Z_r(v^{\alpha,-\alpha})} < \infty. \tag{20}$$

Then if $q > \dfrac{1}{1-\max\{\varepsilon,\lambda\}}$, *the functions defined by* (15) *and* (16) *are such that*

$$\phi \in Z_\eta(v^{-\alpha,\alpha}), \qquad \eta = \min\{r, 2(q-1-q\lambda)+1\}, \tag{21}$$

$$\sup_{|s|<1} \|\kappa(\cdot,s)\|_{Z_\vartheta(v^{\alpha,-\alpha})} < \infty, \quad \vartheta = \min\{r, 2(q-1-q\varepsilon)+1\}, \tag{22}$$

$$\sup_{|t|<1} \|\kappa(t,\cdot)\|_{Z_r(v^{-\alpha,\alpha})} < \infty. \tag{23}$$

To give an example let us consider the right-hand side of equation (12) $g(y) = \dfrac{\sin(1+y)}{(1+y)^{2/3}} \in Z_1(v^{-\frac{2}{3},\frac{2}{3}})$. After the transformation it becomes

$$\phi(s) = q2^{\frac{1}{3}(1-q)}(1+s)^{\frac{1}{3}q-1}\sin(2^{1-q}(1+s)^q) \in Z_{2(\frac{q}{3}-1)+1}(v^{-\frac{2}{3},\frac{2}{3}}).$$

The following result gives the properties of the operators involved in equation (17).

Proposition 3.2. *Let* D, Σ *and* $\mathcal{K}$ *be the operators defined by* (2), (18) *and* (19), *respectively with* $-1<\alpha<1$. *Then*

(a) *$D : L^2_{v^{\alpha,-\alpha}} \to L^2_{v^{-\alpha,\alpha}}$ is a linear bounded invertible operator such that*

$$\|(D\psi)v^{-\alpha,\alpha}\|_2 = \|\psi v^{\alpha,-\alpha}\|_2$$

and its inverse $\widehat{D} : L^2_{v^{-\alpha,\alpha}} \to L^2_{v^{\alpha,-\alpha}}$ defined as

$$(\widehat{D}\psi)(s) = \cos \pi\alpha \psi(s) v^{\alpha,-\alpha}(s) + \frac{\sin \pi\alpha}{\pi} \oint_{-1}^{1} \frac{\psi(t)}{t-s} v^{\alpha,-\alpha}(t)dt,$$

is a linear bounded operator.

(b) *$\Sigma : L^2_{v^{\alpha,-\alpha}} \to L^2_{v^{-\alpha,\alpha}}$ is a linear bounded operator.*

(c) *If* (20) *is satisfied, $\mathcal{K} : L^2_{v^{\alpha,-\alpha}} \to L^2_{v^{-\alpha,\alpha}}$ is a compact operator.*

3.3 - *A quadrature method*

In this subsection we propose a direct method to solve equation (17).

The numerical method consists in approximating the unknown solution ψ of (17) by means of the polynomial of degree $m-1$

$$\psi_m(x) = \sum_{k=1}^{m} a_k l_k^{\alpha,-\alpha}(x). \tag{24}$$

To this end we approximate the integrals Σ and $\mathcal{K}$ by means of the Gaussian rule based on the zeros of $p_m(v^{\alpha,-\alpha})$. Hence, we define

$$(\mathcal{K}_m\psi)(s) = \mu \sum_{k=1}^{m} \lambda_k^{\alpha,-\alpha} \kappa(x_k^{\alpha,-\alpha}, s)\psi(x_k^{\alpha,-\alpha}) \tag{25}$$

$$(\Sigma_m\psi)(s) = \sum_{k=1}^{m} \lambda_k^{\alpha,-\alpha} \sigma\left(\frac{1+s}{1+x_k^{\alpha,-\alpha}}\right) \frac{\psi(x_k^{\alpha,-\alpha})}{1+x_k^{\alpha,-\alpha}} \tag{26}$$

where $\lambda_k^{\alpha,-\alpha}$ denotes the k-th coefficient of the Gaussian rule.

At this point we project the equation

$$(D + \Sigma_m + \mathcal{K}_m)\psi_m = \phi$$

on the set of polynomials of degree at most $m-1$ by means of the operator $L_m^{-\alpha,\alpha}$. Hence we consider the finite dimensional equation

$$L_m^{-\alpha,\alpha}(D + \Sigma_m + \mathcal{K}_m)\psi_m = L_m^{-\alpha,\alpha}\phi$$

or equivalently

$$(D + L_m^{-\alpha,\alpha}\Sigma_m + L_m^{-\alpha,\alpha}K_m)\psi_m = L_m^{-\alpha,\alpha}\phi \tag{27}$$

in virtue of the well known property of the dominant operator (see, for instance, [**30**])

$$Dp_m(v^{\alpha,-\alpha}) = p_m(v^{-\alpha,\alpha}).$$

Then by using the linear property of $L_m^{\alpha,-\alpha}$, by applying the Gaussian rule based on the zeros of $p_m(v^{\alpha,-\alpha})$ and taking into account that (see, for instance, [**26**, p. 448])

$$(D\psi_m)(x_j^{-\alpha,\alpha}) = -\frac{\sin \pi\alpha}{\pi} \sum_{k=1}^{m} \lambda_k^{\alpha,-\alpha} \frac{\psi_m(x_k^{\alpha,-\alpha})}{(x_k^{\alpha,-\alpha} - x_j^{-\alpha,\alpha})},$$

we get the following linear system

$$\sqrt{\lambda_j^{-\alpha,\alpha}} \sum_{k=1}^{m} \sqrt{\lambda_k^{\alpha,-\alpha}} \left[-\frac{\sin \pi\alpha}{\pi} \left\{ \frac{1}{x_k^{\alpha,-\alpha} - x_j^{-\alpha,\alpha}} + \sigma \left(\frac{1 + x_j^{-\alpha,\alpha}}{1 + x_k^{\alpha,-\alpha}} \right) \frac{1}{1 + x_k^{\alpha,-\alpha}} \right\} \right.$$
$$\left. + \mu\ \kappa(x_k^{\alpha,-\alpha}, x_j^{-\alpha,\alpha}) \right] \eta_k = \sqrt{\lambda_j^{-\alpha,\alpha}} \phi(x_j^{-\alpha,\alpha}) \quad j = 1, 2, \ldots, m \tag{28}$$

where $\eta_k = a_k(\lambda_k^{\alpha,-\alpha})^{1/2} = \psi_m(x_k^{-\alpha,\alpha})(\lambda_k^{\alpha,-\alpha})^{1/2}$. In this way we have proved the following result.

Proposition 3.3. *Let* ψ_m *as in* (24). *Then it is solution of equation* (27) *if and only if its coefficients are given by* $a_k = \eta_k(\lambda_k^{\alpha,-\alpha})^{-1/2}$ *where the vector* $[\eta_1, \eta_2, \ldots, \eta_m]^T$ *is the solution of system* (28).

At this point the next steps should be to prove that

1 equation (27) has a unique solution and i.e. that there exists the inverse operator $(D + L_m^{-\alpha,\alpha}\Sigma_m + L_m^{-\alpha,\alpha}\mathcal{K}_m)^{-1}$;

2 the operator $(D + L_m^{-\alpha,\alpha}\Sigma_m + L_m^{-\alpha,\alpha}\mathcal{K}_m)^{-1}$ is uniformly bounded with respect to m. This assures the stability of the method;

3 denoted by f^* the unique solution of the original equation, it results

$$\|[\tilde{f}_m^q - f^*]v^{\alpha,-\alpha}\|_2 \to 0$$

where, according to (14),

$$\tilde{f}_m^q(y) = \frac{\psi_m(\gamma_q'^{-1}(y))}{\gamma_q'^{-1}(y)M(\gamma_q^{-1}(y))}, \tag{29}$$

with ψ_m defined in (24). This assures the convergence of the method.

Now in order to prove **1-3**, because of the non compactness of the operator Σ, it is not possible to use the same technique, employed in [**21**] and then it is necessary to proceed in a different way. At the moment we are not able to give this proof but the numerical results seem to suggest the stability of the method. Moreover numerical tests have shown that the approximate solution $\tilde{f}_m^q$ converges to a value which is the exact one. This was observed by comparing $\tilde{f}_m^q$ with the approximate solution defined in (10) f_m^* which converges to the exact solution f^*, according to (11).

Moreover, if we compare the numerical results obtained by the regularizing procedure with those we have without this technique, we can note that the former are better, even if one has a kernel with a fixed singularity. Furthermore it seems that the order of convergence increases as q does. Intuitively, it is possible to understand this, because it was proved in several papers that regularizing procedures applied to Fredholm integral equations (see, for instance, [**28**], [**13**]) and to equations with a Mellin convolution type (see, for instance, [**10**], [**27**]) give a good order of convergence which improves as q increases.

Obviously it is natural the following question: can the regularizing parameter be taken large as we want? At this point we cannot give a precise answer because the error estimate is not known. Of course in this estimate will appear a constant which will depend on q and could be very large if q increases (see, for instance, [**11**], [**12**]). Consequently, the numerical results can be compromised. In any case our numerical results underline that also by taking q large (for instance $q = 7$) the speed of convergence does not slow down. This denotes that the constant does not become very large.

In the next section we will give some numerical results which confirm all the observations made here while the theoretical study of the stability and the convergence will be subject of a future work.

Finally we underline that, in the case when we have a Cauchy singular integral equation in which the given functions are singular at inner points, it is possible to reduce it in a system of Cauchy singular integral equations or to apply an appropriate regularizing transformation to smooth the inner singularities. Nevertheless, we mention that in this second case, all the involved operators and all the known functions have to be studied in a weighted space L_u^2 with a generalized Jacobi weight u. On the contrary, by proceeding as suggested in the first case, the inner singularities become singularities at the endpoints and then the procedure shown in this paper can be applied and the theoretical results still hold true. Further investigations are however needed.

4 - Numerical Tests

In this section we show some numerical experiments. To this end we proceed in the following way. By applying the procedure given in Subsection 3.2, we first transform the considered equation in (27). Then we solve system (28) and we construct ψ_m according to (24). Thus, in virtue of the adopted procedure, we compute the approximate solution $\tilde{f}_m^q$ of the original equation defined in (29).

In each example we take as reference solution that obtained with $m = 600$ points and with the larger $\overline{q}$, and we give $\tilde{e}_m^q(y) = |(\tilde{f}_{600}^{\overline{q}} - \tilde{f}_m^q)(y)|$ with $y \in (-1, 1)$. Moreover in each test we compare the obtained results with those we have without regularization i.e. solving system (9). In this case we compute $e_m^*(y) = |(f_{600}^{\overline{q}} - f_m^*)(y)|$ with $y \in (-1, 1)$ where f_m^* is constructed according to (10). In the sequel for the sake of the simplicity when we fix the value of q or $\overline{q}$, we will write $\tilde{f}_m^{q=\text{the fixed value}}$ instead of $\tilde{f}_m^q$ or $\tilde{f}_m^{\overline{q}}$. All the computations were performed in 16-digits arithmetic.

EXAMPLE 4.1. *We consider equation* (12). *By applying the numerical procedure shown in* Section 3 *we obtain equation* (27) *with*

$$\phi(s) = q 2^{\frac{1}{3}(1-q)} (1+s)^{\frac{q}{3}-1} \sin(2^{1-q}(1+s)^q)$$

and

$$\kappa(t,s) = q 2^{\frac{1}{3}(1-q)} (1+s)^{\frac{q}{3}-1} (2^{1-q}[3(1+t)^q + (1+s)^q] - 4).$$

We immediately note that if q is a multiple of three the given functions are analytic. Table 2 *shows the numerical results obtained without regularization while* Table 3 *and* Table 4 *give the errors $\tilde{e}_m^q(y)$ with $q = 3$ and $q = 6$, respectively in the points $y = 0.9$ and $y = -0.9$. In the other points the absolute errors are very similar. We take as reference solution $\tilde{f}_m^{\overline{q}}$ with $\overline{q} = 6$ and we mention that* $\tilde{f}_{600}^{q=6}(-0.9) = 0.14163738783797730$, $\tilde{f}_{600}^{q=6}(0.9) = -0.1766939981739069$.

Finally, we underline that system (28) *is well conditioned. Indeed, denoted by $\boldsymbol{A}_m$ its matrix of coefficients and by cond$(\boldsymbol{A}_m)$ its condition number, it results for each m cond$(\boldsymbol{A}_m) \leq 21$ if $q = 3$ and cond$(\boldsymbol{A}_m) \leq 37$ if $q = 6$.*

Table 2.

m	$\lvert(\tilde{f}_{600}^{q=6} - f_m^*)(-0.9)\rvert$	$\lvert(\tilde{f}_{600}^{q=6} - f_m^*)(0.9)\rvert$
16	3.26e-003	7.88e-003
32	5.88e-003	2.61e-003
64	1.21e-003	2.48e-004
128	1.50e-004	1.93e-004
256	1.58e-004	4.40e-005
512	1.26e-005	1.30e-005

Table 3.

m	$\lvert(\tilde{f}_{600}^{q=6} - \tilde{f}_m^{q=3})(-0.9)\rvert$	$\lvert(\tilde{f}_{600}^{q=6} - \tilde{f}_m^{q=3})(0.9)\rvert$
16	5.07e-004	4.68e-004
32	1.83e-005	7.79e-005
64	9.32e-006	8.78e-006
128	2.51e-007	1.41e-006
256	4.67e-008	1.29e-007
512	2.00e-008	2.37e-008

Table 4.

m	$\lvert(\tilde{f}_{600}^{q=6} - \tilde{f}_m^{q=6})(-0.9)\rvert$	$\lvert(\tilde{f}_{600}^{q=6} - \tilde{f}_m^{q=6})(0.9)\rvert$
16	6.78e-007	6.13e-006
32	5.85e-008	2.16e-007
64	7.37e-009	1.15e-008
128	2.95e-010	4.82e-010
256	1.02e-011	3.09e-013
512	1.52e-014	1.26e-013

In Table 5 *the values of* $\tilde{f}_m^{q=6}(y)$, *in* $y = 0.9$ *and* $y = 0.9$ *are given.*

Table 5.

m	$\tilde{f}_m^{q=6}(-0.9)$	$\tilde{f}_m^{q=6}(0.9)$
16	**0.1416**367093338771	**−0.1766**878650149057
32	**0.14163**73292861998	**−0.17669**42147830024
64	**0.1416373**804621628	**−0.176694**0097614645
128	**0.14163738**75425943	**−0.17669399**86563052
256	**0.14163738784**82631	**−0.176693998173**6002
512	**0.14163738783**79556	**−0.176693998173**7810

EXAMPLE 4.2. *We consider*

$$(Df)(y) + \frac{1}{9}\int_{-1}^{1} (2+y)\left|x - \frac{1}{2}\right|^{11/2} f(x)v^{\frac{3}{5},-\frac{3}{5}}(x)dx = \frac{e^y}{(1+y)^{3/7}}.$$

By applying the numerical procedure showed in Section 3 *we obtain equation* (17) *with*

$$\phi(s) = q\ 2^{(1-q)\frac{4}{7}}\ (1+s)^{\frac{11}{7}q-1}\ e^{2^{1-q}(1+s)^q-1},$$

and

$$\kappa(t,s) = (1 + 2^{1-q}(1+s)^q)\left|2^{1-q}(1+s)^q - \frac{3}{2}\right|^{11/2}.$$

The numerical results we have in the case when $q = 3$ *and* $q = 7$ *are shown in* Table 6 *and* Table 7, *respectively. We take as reference solution* $\tilde{f}_m^{\overline{q}}$ *with* $\overline{q} = 7$ *and we remark that* $\tilde{f}_{600}^{q=7}(-0.8) = 0.1872792001336907$ *and* $\tilde{f}_{600}^{q=7}(0.5) = 0.8286337335141194$.

Table 6.

m	$\lvert(\tilde{f}_{600}^{q=7} - \tilde{f}_m^{q=3})(-0.8)\rvert$	$\lvert(\tilde{f}_{600}^{q=7} - f_m^{q=3})(0.5)\rvert$
32	4.63e-007	1.28e-006
64	1.92e-007	1.34e-007
128	8.12e-009	3.03e-009
256	3.14e-010	3.72e-010
512	4.57e-011	5.13e-011

Table 7.

m	$\lvert(\tilde{f}_{600}^{q=7} - \tilde{f}_m^{q=7})(-0.8)\rvert$	$\lvert(\tilde{f}_{600}^{q=7} - f_m^{q=7})(0.5)\rvert$
32	5.94e-010	2.31e-009
64	7.81e-012	1.29e-011
128	5.03e-013	5.81e-013
256	1.02e-013	4.72e-013
512	1.28e-013	4.39e-014

By comparing these results with those obtained without the regularizing procedure (*see* Table 8), *we note that the former are better. Indeed, by solving system* (28) *with* $m = 32$ *and* $q = 3$ *we have an error of the order* 10^{-7} *in* $y = -0.8$. *The same order was obtained by solving system* (9) *with* $m = 128$. *Moreover we remark that if* q *is large the order of convergence increases and the system we solve is well conditioned. In fact it results for each* m *cond*$(\boldsymbol{A}_m) \leq 40$ *if* $q = 3$ *and cond*$(\boldsymbol{A}_m) \leq 91$ *if* $q = 7$.

Table 8.

m	$\lvert(\tilde{f}_{600}^{q=7} - f_m^*)(-0.8)\rvert$	$\lvert(\tilde{f}_{600}^{q=7} - f_m^*)(0.5)\rvert$
32	3.96e-004	7.64e-005
64	8.94e-005	5.43e-005
128	4.13e-007	2.60e-006
256	2.34e-006	2.28e-006
512	1.05e-006	8.25e-008

Table 9 *and* Table 10 *show the values of* f_m^* *and* $\tilde{f}_m^{q=7}$ *in the points* $y = -0.8$ *and* $y = 0.5$.

Table 9.

m	$f_m^*(-0.8)$	$\tilde{f}_m^{q=3}(-0.8)$	$\tilde{f}_m^{q=7}(-0.8)$
32	**0.1876755663312321**	**0.1872796638463117**	**0.1872792007285498**
64	**0.1873686715211495**	**0.1872790073133861**	**0.1872792001258752**
128	**0.1872787864734236**	**0.1872791920044783**	**0.1872792001331868**
256	**0.1872815497301738**	**0.1872792004485168**	**0.1872792001337930**
512	**0.1872802525324761**	**0.1872792000879684**	**0.1872792001338190**

Table 10.

m	$f_m^*(0.5)$	$\tilde{f}_m^{q=3}(0.5)$	$\tilde{f}_m^{q=7}(0.5)$
32	**0.8287101391766663**	**0.8286350225621202**	**0.8286337358315067**
64	**0.8286881305050452**	**0.8286338681002020**	**0.8286337335012025**
128	**0.8286363393311549**	**0.8286337304801502**	**0.8286337335135402**
256	**0.8286360200441175**	**0.8286337338861416**	**0.8286337335145958**
512	**0.8286338160378816**	**0.8286337335654589**	**0.8286337335141653**

EXAMPLE 4.3. *We consider equation*

$$(Df)(y) + \frac{1}{6}\int_{-1}^{1} \cos(x + y + 3) f(x) v^{-\frac{1}{8},\frac{1}{8}}(x)dx = \frac{e^{(1+y)}}{(1+y)^{1/9}} + \log(1 + y).$$

By applying a quadrature method directly to the given equation we have that for $m = 256$ *in* $y = -0.9$ *we have an error of the order* 10^{-5}, *as we can see in* Table 11.

Table 11.

m	$\lvert(\tilde f_{600}^{q=3} - f_m^*)(-0.9)\rvert$	$\lvert(\tilde f_{600}^{q=3} - f_m^*)(0.3)\rvert$
32	6.67e-003	1.78e-004
64	7.89e-004	2.04e-004
128	4.96e-004	4.47e-005
256	5.76e-005	9.06e-006
512	8.97e-006	2.98e-006

Now we apply the procedure suggested in Section 3. *Then we have equation* (17) *with*

$$\phi(s) = q\ 2^{\frac{8}{9}(1-q)}\ (1+s)^{\frac{8}{9}q-1}\ e^{2^{1-q}(1+s)^q} + q\ 2^{1-q}\ (1+s)^{q-1}\ \log(2^{1-q}(1+s)^q)$$

and

$$\kappa(t,s) = q\ 2^{1-q}\ (1+s)^q\ \cos(2^{1-q}((1+s)^q + (1+t)^q) + 1).$$

The numerical results are given in Table 12 *and* Table 13. *In this example we take as reference solution* $\tilde f_m^{\overline{q}}$ *with* $\overline{q} = 3$ *and we note that* $\tilde f_{600}^{q=3}(-0.9) = -1.447979122000331$ *and* $\tilde f_{600}^{q=3}(0.3) = 3.745425426376415$.

Table 12.

m	$\lvert(\tilde f_{600}^{q=3} - \tilde f_m^{q=2})(-0.9)\rvert$	$\lvert(\tilde f_{600}^{q=3} - \tilde f_m^{q=2})(0.3)\rvert$
32	2.01e-005	7.00e-007
64	9.26e-007	8.23e-008
128	2.95e-008	2.76e-009
256	1.34e-009	2.16e-010
512	4.58e-010	6.62e-011

Table 13.

m	$\lvert(\tilde f_{600}^{q=3} - \tilde f_m^{q=3})(-0.9)\rvert$	$\lvert(\tilde f_{600}^{q=3} - \tilde f_m^{q=3})(0.3)\rvert$
32	2.14e-008	5.71e-009
64	5.88e-010	1.03e-010
128	2.37e-011	1.65e-012
256	2.56e-012	1.60e-012
512	1.42e-012	3.51e-012

We mention that the system we solve is well conditioned. Indeed for each m *it results cond*$(\boldsymbol{A}_m) \le 1.8$ *if* $q = 2$ *and cond*$(\boldsymbol{A}_m) \le 2.6$ *if* $q = 3$.

Table 14 shows the values of $f_m^*, \tilde{f}_m^{q=2}$ and $\tilde{f}_m^{q=3}$ in $y = -0.9$.

Table 14.

m	$f_m^*(-0.9)$	$\tilde{f}_m^{q=2}(-0.9)$	$\tilde{f}_m^{q=3}(-0.9)$
32	−**1.44**1300158921933	−**1.447**999239495348	−**1.447979**143406848
64	−**1.44**8768220755139	−**1.4479**80048448261	−**1.447979121**412321
128	−**1.44**7482626607250	−**1.447979**151536435	−**1.447979121**976605
256	−**1.4479**21446106832	−**1.4479791**20651159	−**1.447979121**997767
512	−**1.4479**70144847015	−**1.447979121**541527	−**1.447979121**998908

5 - Proofs

In order to prove the main results stated in Section 3.2 we introduce the best polynomial approximation of $f \in L^2_{v^{\gamma,\delta}}$, by means of polynomials of degree at most m ($P \in \mathbb{P}_m$) as

$$E_m(f)_{v^{\gamma,\delta}} = \inf_{P\in\mathbb{P}_m} \|[f-P]v^{\gamma,\delta}\|_2.$$

Then we recall that $\forall f \in Z_r(v^{\gamma,\delta})$ it results

$$E_m(f)_{v^{\gamma,\delta}} \leq \frac{\mathcal{C}}{m^r}\|f\|_{Z_r(v^{\gamma,\delta})} \quad \mathcal{C} \neq \mathcal{C}(m,f) \tag{30}$$

and $\forall f_1 \in L^2_{v^{\gamma,\delta}}$ and $\forall f_2 \in L^2_{v^{0,0}}$ there holds (see, for instance, [**25**])

$$E_{2m}(f_1 f_2)_{v^{\gamma,\delta}} \leq \mathcal{C}\ [\ \|f_1 v^{\gamma,\delta}\|_2\ E_m(f_2) + \|f_2\|_2\ E_m(f_1)_{v^{\gamma,\delta}}] \qquad \mathcal{C} \neq \mathcal{C}(m, f_1, f_2). \tag{31}$$

Here and in the sequel $\mathcal{C}$ denotes a positive constant which may take different values in different formulae and we write $\mathcal{C} \neq \mathcal{C}(a, b, c, \dots)$ if such constant is independent of the parameters $a, b, c, \dots$.

Proof of Proposition 3.1. We begin by proving (21). By definition (15) and by applying (31) we have

$$E_m(\phi)_{v^{-\alpha,\alpha}} \leq \mathcal{C}[\ \|g_1(\phi_q)v^{-\alpha,\alpha}\|_2 E_{[\frac{m}{2}]}((1+\cdot)^{q-1-q\lambda}) + E_{[\frac{m}{2}]}(g_1(\phi_q))_{v^{-\alpha,\alpha}}\|(1+\cdot)^{q-1-q\lambda}\|_2].$$

Now by the assumption $g_1 \in Z_r(v^{-\alpha,\alpha})$ and since $g_1(\gamma_q)$ is the composition of g_1 with a polynomial then $g_1(\gamma_q) \in Z_r(v^{-\alpha,\alpha})$, too. Thus, taking also into account that

$q-1-q\lambda \geq 0$ and $(1+s)^{q-1-q\lambda} \in Z_{2(q-1-q\lambda)+1}(v^{0,0})$, by applying (30) we have

$$E_m(\phi)_{v^{-\alpha,\alpha}} \leq \mathcal{C}\left[\frac{1}{m^r}+\frac{1}{m^{2(q-1-q\lambda)+1}}\right].$$

Hence, setting $\eta=\min\{r, 2(q-1-q\lambda)+1\}$, by the well known Stechkin-type inequality (see, for instance [**25**])

$$\Omega_\varphi^k(\phi,\tau)_{v^{-\alpha,\alpha},2} \leq \mathcal{C}\tau^k \sum_{i=0}^{[\frac{1}{\tau}]}(1+i)^{k-1}E_i(\phi)_{v^{-\alpha,\alpha}}$$

we get

$$\Omega_\varphi^k(\phi,\tau)_{v^{-\alpha,\alpha},2} \leq \mathcal{C}\tau^\eta$$

and then (21) is proved. In the same way it is possible to deduce (22). Finally (23) follows taking into account that $\kappa(t,\cdot)=k_1(\gamma_q(t),\gamma_q(\,\cdot\,))$. Thus it is the composition of a function $k_1(x,\cdot) \in Z_r(v^{-\alpha,\alpha})$ with a polynomial and then it belongs to $Z_r(v^{-\alpha,\alpha})$, too.

Proof of Proposition 3.2. The properties of the dominant operator D are well known. The reader can consult [**1, 2, 15, 20, 23, 30**] and the related references. In order to prove **(b)** we first write $\Sigma=(\Sigma_0 \circ v^{\alpha,-\alpha}I): L^2_{v^{\alpha,-\alpha}} \to L^2_{v^{-\alpha,\alpha}}$ where $\Sigma_0 : L^2_{v^{-\alpha,\alpha}} \to L^2_{v^{-\alpha,\alpha}}$ is defined as

$$(\Sigma_0\Psi)(s)=\int_{-1}^{1}\sigma\left(\frac{1+s}{1+t}\right)\frac{1}{1+t}\psi(t)dt$$

and

$$v^{\alpha,-\alpha}I : L^2_{v^{\alpha,-\alpha}} \to L^2_{v^{-\alpha,\alpha}}$$

is an isomorphism. Now we compute the Mellin symbol

$$\widehat{\sigma}_0(z) := \int_0^\infty y^{z-1}\sigma(y)dy$$

of the kernel

$$\sigma(y)=-\frac{\sum_{\ell=0}^{q-2}(\ell+1)y^\ell}{\sum_{\ell=0}^{q-1}y^\ell} := -\frac{h'(y)}{h(y)}.$$

Since $(y-1)h(y) = y^q - 1 = \prod_{j=0}^{q-1}(y - e^{\omega_j i})$, $\omega_j = \dfrac{2\pi j}{q}$ we have

$$h(y) = \prod_{j=1}^{q-1}(y - e^{\omega_j i}), \quad h'(y) = \sum_{\ell=1}^{q-1} \prod_{\substack{j=1\\ j\neq \ell}}^{q-1}(y - e^{\omega_j i}),$$

from which we deduce

$$\sigma(y) = -\sum_{\ell=1}^{q-1} \frac{1}{y - e^{\omega_\ell i}}.$$

Thus taking into account that for $0<\omega<2\pi$ we have (see, for instance, [**6**])

$$\int_0^\infty \frac{y^{z-1}}{y - e^{\omega i}} dy = \frac{-e^{-i\omega} e^{i(\omega-\pi)z}}{\sin \pi z}$$

we deduce that the Mellin symbol $\widehat{\sigma}$ is analytic in the strip $\{z \in \mathbb{C} : 0 < \Re z < 1\}$. Moreover since

$$\sup_{z: c_0 < \Re z < c_1} \left| \frac{d^k}{dz^k} \widehat{\sigma_0}(z)(1+|z|)^{1+k} \right| < \infty, \quad k = 0, 1, 2, \dots . \tag{32}$$

in virtue of a well known result (see, for instance, [**19**]) we deduce that Σ_0 is bounded in $L^2_{v^{-\alpha,\alpha}}$ if $0 < \dfrac{1}{2} - \dfrac{\alpha}{2} < 1$ from which (**b**) follows being $-1<\alpha<1$. Now we prove (**c**). To this end we have to prove that (see, for instance [**32**])

$$\lim_m \left(\sup_{\|\psi v^{\alpha,-\alpha}\|_2 \le 1} E_m(\mathcal{K}\psi)_{v^{-\alpha,\alpha}} \right) = 0.$$

By applying the Schwarz's inequality we have

$$\begin{aligned}\|(\mathcal{K}\psi)v^{-\alpha,\alpha}\|_2 &\le \|\psi v^{\alpha,-\alpha}\|_2 \left(\int_{-1}^{1}\int_{-1}^{1} \kappa(t,s)^2 v^{\alpha,-\alpha}(t) dt v^{-\alpha,\alpha}(s) ds \right)^{1/2} \\ &\le \mathcal{C}\|\psi v^{\alpha,-\alpha}\|_2.\end{aligned}$$

Moreover

$$\begin{aligned}&\|\Delta^i_{h\varphi}(\mathcal{K}\psi)v^{-\alpha,\alpha}\|_{L^2_{(I_{hi})}} \\ &= \left(\int_{-1+(2hi)^2}^{1-(2hi)^2} [\Delta^i_{h\varphi}(\mathcal{K}\psi)(s)]^2 v^{-\alpha,\alpha}(s) ds \right)^{1/2}\end{aligned}$$

$$= \left(\int_{-1+(2hi)^2}^{1-(2hi)^2} \left[\Delta^i_{h\varphi} \int_{-1}^{1} \kappa(t,s)\psi(t)v^{\alpha,-\alpha}(t)dt \right]^2 v^{-\alpha,\alpha}(s)ds \right)^{1/2}$$

$$= \left(\int_{-1+(2hi)^2}^{1-(2hi)^2} \left[\int_{-1}^{1} \Delta^i_{h\varphi}\kappa(t,\cdot)\psi(t)v^{\alpha,-\alpha}(t)dt \right]^2 v^{-\alpha,\alpha}(s)ds \right)^{1/2}$$

$$\leq \|\psi v^{\alpha,-\alpha}\|_2 \left(\int_{-1}^{1} \int_{-1+(2hi)^2}^{1-(2hi)^2} [\Delta^i_{h\varphi}\kappa(t,\cdot)]^2 v^{-\alpha,\alpha}(s)\ ds\ v^{\alpha,-\alpha}(t)dt \right)^{1/2}$$

$$= \|\psi v^{\alpha,-\alpha}\|_2 \left(\int_{-1}^{1} \|\Delta^i_{h\varphi}\kappa(t,\cdot)v^{-\alpha,\alpha}\|_{L^2_{I_{ih}}} v^{\alpha,-\alpha}(t)dt \right)^{1/2}.$$

Thus taking the supremum on $0<h\leq\tau$ by (8) we have

$$\begin{aligned} \Omega^i_\varphi(\mathcal{K}\psi,\tau)_{v^{-\alpha,\alpha}} &\leq \|\psi v^{\alpha,-\alpha}\|_2 \sup_{|t|<1} \Omega^i_\varphi(\kappa(t,\cdot),\tau)_{v^{-\alpha,\alpha}} \left(\int_{-1}^{1} v^{\alpha,-\alpha}(t)dt \right)^{1/2} \\ &\leq \mathcal{C}\|\psi v^{\alpha,-\alpha}\|_2 \tau^\theta \sup_{|t|<1} \sup_{\delta>0} \frac{\Omega^i_\varphi(\kappa(t,\cdot),\delta)_{v^{-\alpha,\alpha}}}{\delta^\theta} \\ &\leq \mathcal{C}\tau^\theta \end{aligned}$$

according to Proposition 3.1.

Thus by the well-known weak Jackson's inequality according to which

$$E_n(f)_v \leq \mathcal{C} \int_0^{1/n} \frac{\Omega^i_\varphi(f,u)_v}{u} du, \quad \forall f \in L^2_v$$

we have

$$E_n(\mathcal{K}\psi)_{v^{-\alpha,\alpha}} \leq \mathcal{C}\ \|\psi v^{\alpha,-\alpha}\|_2 \left(\int_0^{\frac{1}{m}} u^{\theta-1}du \right) \leq \frac{\mathcal{C}}{m^\theta}\|\psi v^{\alpha,-\alpha}\|_2$$

from which **(c)** follows.

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