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Direct methods for CSIE in weighted Zygmund spaces with uniform norm

Abstract. In this paper the authors propose two projection methods to solve CSIE having smooth or weakly singular kernels. They prove their stability and convergence in Zygmund spaces equipped with uniform norm. Some numerical examples illustrating the accuracy of the methods are given.

Keywords. Cauchy singular integral equation, projection method, Lagrange interpolation, Fourier sum.

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1 - Introduction

This paper deals with the numerical treatment of Cauchy singular integral equations of the following kind

$$af(x)v^{\alpha,\beta}(x)+\frac{b}{\pi}\int_{-1}^{1}\frac{f(y)}{y-x}v^{\alpha,\beta}(y)dy+\int_{-1}^{1}k(x,y)f(y)v^{\alpha,\beta}(y)dy=g(x), \tag{1}$$

where $|x|<1$, k and g are known functions, a, b are constant coefficients such that $a^2+b^2=1$ and $b\neq 0, f$ is the unknown and $v^{\alpha,\beta}(x)=(1-x)^{\alpha}(1+x)^{\beta}$, $-1<\alpha,\beta<1$, is a Jacobi weight. The kernel $k(x,y)$ can be a smooth or a weakly singular function in $[-1,1]^2$. The exponents of the weight $v^{\alpha,\beta}$ are related to the coeffi-

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cients a and b by

$$\alpha = M - \frac{1}{2\pi i}\log\left(\frac{a+ib}{a-ib}\right) \quad \text{and} \quad \beta = N + \frac{1}{2\pi i}\log\left(\frac{a+ib}{a-ib}\right),$$

with M and N integers chosen so that the index $\chi = -(\alpha+\beta) = -(M+N)$ is equal to 0.

For the sake of simplicity and without loss the generality, we shall consider the equation

$$(2) \qquad (D+K)f = g,$$

where

$$(Df)(x) = \cos(\alpha\pi)\, v^{\alpha,-\alpha}(x)f(x) - \frac{\sin(\alpha\pi)}{\pi}\int_{-1}^{1}\frac{f(y)}{y-x}v^{\alpha,-\alpha}(y)dy,$$

$$(Kf)(x) = \int_{-1}^{1} k(x,y)f(y)v^{\alpha,-\alpha}(y)dy$$

and $1/2 \le \alpha < 1$. In other words, we take $\beta = -\alpha$ and the constant coefficients a and b appearing in (1) as $\cos(\pi\alpha)$ and $-\sin(\pi\alpha)$, respectively. Note that the case $\alpha = 1/2$ includes the well known airfoil equation.

Denoting by $L^2_w = L^2_w([-1,1])$ the collection of all measurable functions s.t. $\|wf\|_2^2 = \int_{-1}^{1}(fw)^2(x)dx < +\infty$, it is well-known that (see, for example, [**23**]) $D : L^2_w \to L^2_{1/w}, w = v^{\frac{\alpha}{2},-\frac{\alpha}{2}}$, is a continuous and invertible map. This important property has suggested to consider equation (2) in the couple of spaces $(L^2_w, L^2_{1/w}), w = v^{\frac{\alpha}{2},-\frac{\alpha}{2}}$, and to study the stability and the convergence of the proposed numerical methods (projection and quadrature) there.

A considerable literature exists on this topic in the case where k is smooth (see, for example, the surveys [**9, 6, 7, 8, 10, 23, 24, 11, 12, 13, 27, 2**] and the references there in). Whereas, when the kernel is weakly singular, equation (2) has received less attention [**12, 17, 15**].

Some years ago, in [**19**] (see also [**3**]), the authors showed that the dominant operator D is bounded and invertible in the couple of spaces $(Z_r^\infty(v^{\alpha,0}), Z_r^\infty(v^{0,\alpha}))$ (see the definition in Section 2). This result allows to use an indirect method, i.e., to regularize equation (2) and to obtain a Fredholm equation that can be studied in a weighted space of continuous functions with uniform metric. This procedure has been shown in [**20, 4, 2**].

In the present paper we use a new idea. We consider equation (2) in a couple of Zygmund spaces with uniform norm. Without regularizing the equation, we use two projection methods in the different cases in which the kernel is smooth or weakly singular. We prove their stability and convergence and the well conditioning of the related linear systems. The error estimates are given in the Zygmund norm and we improve all the estimates available in literature [**17, 15, 11, 12**]. Numerical examples confirming the error estimates are shown.

The paper is organized as follows. In Section 2 we give some notations and preliminary results. In particular, Subsection 2.1 is devoted to the mapping properties of the operators D and K. In Section 3, we propose a numerical method that can be applied both when the kernel and the right-hand side are smooth and when the right-hand side is smooth and the kernel, although presenting weak singularity, can be written in a suitable form (see (24)). The case of weakly singular kernels is considered in Section 4. In Section 5 we prove the main results. Finally in Section 6 we show some numerical tests.

2 - Basic facts and preliminary results

In the following $\mathcal{C}$ denotes a positive constant which may have different values in different formulas. We will write $\mathcal{C} \neq \mathcal{C}(a, b, \dots)$ to say that $\mathcal{C}$ is independent of the parameters $a, b, \dots$. If $A, B \geq 0$ are quantities depending on some parameters, we write $A \sim B$, if there exists a positive constant $\mathcal{C}$ independent of the parameters of A and B, such that

$$\frac{B}{\mathcal{C}} \leq A \leq \mathcal{C}B.$$

Let L^p be the space of all measurable functions f such that

$$\|f\|_{L^p} = \left(\int_{-1}^{1} |f(x)|^p dx \right)^{\frac{1}{p}} < +\infty, \quad 1 \leq p < +\infty.$$

With the Jacobi weight $v^{\alpha,\beta}(x) = (1-x)^\alpha (1+x)^\beta, \alpha, \beta > -1/p$, we set $f \in L^p_{v^{\alpha,\beta}}$ if and only if $fv^{\alpha,\beta} \in L^p, 1 \leq p < +\infty$. We equip the space $L^p_{v^{\alpha,\beta}}$ with the norm

$$\|fv^{\alpha,\beta}\|_p := \left(\int_{-1}^{1} |f(x)v^{\alpha,\beta}(x)|^p dx \right)^{\frac{1}{p}}, \quad 1 \leq p < +\infty.$$

When $p = +\infty$ we define, for $\alpha, \beta > 0$,

$$L^\infty_{v^{\alpha,\beta}} := C_{v^{\alpha,\beta}} = \left\{ f \in C^0((-1,1)) \ : \ \lim_{|x| \to 1} (fv^{\alpha,\beta})(x) = 0 \right\},$$

where $C^0(A)$ is the collection of the continuous functions in $A \subseteq [-1,1]$. If $\alpha = 0$ (respectively, $\beta = 0$) $C_{v^{0,\beta}}$ (respectively, $C_{v^{\alpha,0}}$) consists of all continuous functions on $(-1,1]$ (respectively, $[-1,1)$) such that

$$\lim_{x\to -1}(fv^{0,\beta})(x) = 0, \quad \text{(respectively, } \lim_{x\to 1}(fv^{\alpha,0})(x) = 0).$$

In case $\alpha = \beta = 0$, we set $C_{v^{0,0}} := C^0([-1,1])$. The space $C_{v^{\alpha,\beta}}$ equipped with the norm

$$\|f\|_{C_{v^{\alpha,\beta}}} := \max_{|x|\le 1} |(fv^{\alpha,\beta})(x)| = \|fv^{\alpha,\beta}\|_\infty$$

is complete. For the sake of brevity, we will write $\|f\|_A := \max_{x\in A}|f(x)|$ and $\|fv^{\alpha,\beta}\|_{L^p(A)} := \left(\int_A |f(x)v^{\alpha,\beta}(x)|^p dx\right)^{\frac{1}{p}}, 1 \le p < +\infty$.

We will study equation (2) in Zygmund spaces. For $1 \le p \le +\infty$, these can be defined as follows:

$$Z^p_{r,k}(v^{\alpha,\beta}) = \left\{ f \in L^p_{v^{\alpha,\beta}} \ : \ \sup_{t>0} \frac{\Omega^k_\varphi(f,t)_{v^{\alpha,\beta},p}}{t^r} < \infty \ , \ k > r \in \mathbb{R}^+ \right\},$$

where

$$\Omega^k_\varphi(f,t)_{v^{\alpha,\beta},p} = \sup_{0<h\le t} \|v^{\alpha,\beta}\Delta^k_{h\varphi} f\|_{L^p(I_{kh})},$$

$k \ge 1$, $0 < t < 1$, $I_{kh} = [-1 + (2kh)^2, 1 - (2kh)^2]$, $\varphi(x) = \sqrt{1-x^2}$ and

$$\Delta^k_{h\varphi} f(x) = \sum_{i=0}^{k} (-1)^i \binom{k}{i} f\left(x + \frac{h}{2}\varphi(x)(k-2i)\right).$$

In the sequel we will briefly denote the above spaces by $Z^p_r(v^{\alpha,\beta})$, omitting the subscript $k > r$. Moreover we will set $\Omega_\varphi := \Omega^1_\varphi$, $\Omega^k_\varphi(f,t)_p := \Omega^k_\varphi(f,t)_{v^{0,0},p}$ and $Z^p_r := Z^p_r(v^{0,0})$. The norm in $Z^p_r(v^{\alpha,\beta})$ is defined as

$$\|f\|_{Z^p_r(v^{\alpha,\beta})} = \|fv^{\alpha,\beta}\|_p + \sup_{t>0} \frac{\Omega^k_\varphi(f,t)_{v^{\alpha,\beta},p}}{t^r}.$$

Now we recall some inequalities involving the error of best approximation

$$E_m(f)_{v^{\alpha,\beta},p} = \inf_{P_m\in\mathbb{P}_m} \|(f-P_m)v^{\alpha,\beta}\|_p,$$

where $\mathbb{P}_m$ is the set of all polynomials of degree at most m. For simplicity we will write $E_m(f)_\infty := E_m(f)_{v^{0,0},\infty}$. The following inequalities are well-known [5]

$$E_m(f)_{v^{\alpha,\beta},p} \le \mathcal{C} \int_0^{\frac{1}{m}} \frac{\Omega^k_\varphi(f,t)_{v^{\alpha,\beta},p}}{t} dt \tag{3}$$

and

$$\Omega_{\varphi}^{k}\left(f, \frac{1}{m}\right)_{v^{\alpha,\beta},p} \leq \frac{\mathcal{C}}{m^{k}} \sum_{i=0}^{m} (1+i)^{k-1} E_{i}(f)_{v^{\alpha,\beta},p}, \tag{4}$$

where $\mathcal{C} \neq \mathcal{C}(f, m), k<m$ and $1 \leq p \leq \infty$. For example, if $f \in Z_{s}^{p}(v^{\alpha,\beta})$ with $s>0$ and $1 \leq p \leq \infty$, we have

$$E_{m}(f)_{v^{\alpha,\beta},p} \leq \frac{\mathcal{C}}{m^{s}} \|f\|_{Z_{s}^{p}(v^{\alpha,\beta})}, \quad \mathcal{C} \neq \mathcal{C}(m,f). \tag{5}$$

Moreover, by (3) and (4) it is possible to deduce that, for $s>0$ and $1 \leq p \leq \infty$,

$$\|f\|_{Z_{s}^{p}(v^{\alpha,\beta})} \sim \|f v^{\alpha,\beta}\|_{p} + \sup_{k \geq 1} k^{s} E_{k}(f)_{v^{\alpha,\beta},p}, \tag{6}$$

where the constants in $\sim$ are independent of f.

Now, letting

$$E_{m}(f)_{Z_{r}^{p}(v^{\alpha,\beta})} = \inf_{P_{m} \in \mathbb{P}_{m}} \|f - P_{m}\|_{Z_{r}^{p}(v^{\alpha,\beta})},$$

by (6) we get

$$E_{m}(f)_{Z_{r}^{p}(v^{\alpha,\beta})} \leq \mathcal{C} \sup_{m} m^{r} E_{m}(f)_{v^{\alpha,\beta},p}, \quad 1 \leq p \leq \infty, \quad \mathcal{C} \neq \mathcal{C}(m,f). \tag{7}$$

Consequently, $E_{m}(f)_{Z_{r}^{p}(v^{\alpha,\beta})} = \mathcal{O}(m^{r-s}), s>r$, if and only if $f \in Z_{s}^{p}(v^{\alpha,\beta})$, i.e. $E_{m}(f)_{v^{\alpha,\beta},p} = \mathcal{O}(m^{-s})$. Finally, the following inequality will be useful

$$E_{m}(f)_{\infty} \leq \mathcal{C} \int_{0}^{\frac{1}{m}} \frac{\Omega_{\varphi}^{k}(f,t)_{v^{0,-\alpha},1}}{t^{3}} dt, \quad k>3. \tag{8}$$

The inequality (8), that estimates the error of best uniform approximation by means of the L^{1}- modulus of smoothness, follows by [**14**, eq. (2.15)]. As a consequence of it, we deduce $Z_{s+2}^{1}(v^{0,-\alpha}) \subset Z_{s}^{\infty}, s>0$.

Since our numerical method is based on the Lagrange interpolation, we introduce the Lagrange projection $L_{m}^{\alpha,\beta}$ based on the zeros $y_{m,j}^{\alpha,\beta}, j=1,\dots,m$, of the Jacobi polynomial $p_{m}^{\alpha,\beta}$, i.e., for every $f \in C_{v^{\alpha,\beta}}$

$$L_{m}^{\alpha,\beta}(f,x) = \sum_{j=1}^{m} l_{j}^{\alpha,\beta}(x) f(y_{m,j}^{\alpha,\beta}), \quad l_{j}^{\alpha,\beta}(x) = \frac{p_{m}^{\alpha,\beta}(x)}{[p_{m}^{\alpha,\beta}]'(y_{m,j}^{\alpha,\beta})(x - y_{m,j}^{\alpha,\beta})}.$$

We will also consider the Fourier sum of a function $f \in L_{v^{\gamma,\delta}}^{2}$ in the system $\{p_{m}^{\alpha,\beta}\}_{m}$. It is defined as follows

$$S_{m}^{\alpha,\beta}(f,y) = \sum_{v=0}^{m-1} c_{v} p_{v}^{\alpha,\beta}(y),$$

where

$$c_\nu = \int_{-1}^{1} f(t)p_\nu^{\alpha,\beta}(t)v^{\alpha,\beta}(t)dt, \quad \nu = 0,\dots,m-1,$$

are the Fourier coefficients.

The following theorem is crucial for our purposes.

Theorem 2.1. *For* $1/2 \le \alpha < 1$ *and* $|x| \le 1$, *we have*

$$|v^{\alpha,0}(x)(L_m^{\alpha,-\alpha}f)(x)| \le \mathcal{C}(\log m)\|fv^{\alpha,0}\|_\infty, \quad \forall f \in C_{v^{\alpha,0}}, \tag{9}$$

$$|v^{0,\alpha}(x)(L_m^{-\alpha,\alpha}f)(x)| \le \mathcal{C}(\log m)\|fv^{0,\alpha}\|_\infty, \quad \forall f \in C_{v^{0,\alpha}}, \tag{10}$$

$$\int_{-1}^{1} |L_m^{\alpha,-\alpha}(f,x)|v^{0,-\alpha}(x)dx \le \mathcal{C}\|f\|_\infty, \quad \forall f \in C^0([-1,1]), \tag{11}$$

and

$$\|v^{0,-\frac{\alpha}{2}}S_m^{\alpha,-\alpha}(f)\|_2 \le \mathcal{C}\|fv^{0,-\frac{\alpha}{2}}\|_2, \quad \forall f \in L^2_{v^{0,-\frac{\alpha}{2}}}, \tag{12}$$

where $\mathcal{C} \ne \mathcal{C}(m,f)$.

The bounds (9) and (10) are consequences of [**18**, Theorem 2.2], the bound (12) can be found in [**21**] and (11) is a special case of the Nevai's result in [**22**].

Since in our numerical method we will use simultaneously the projectors $L_m^{\alpha,-\alpha}$ and $L_m^{-\alpha,\alpha}$, we will denote by $t_j, j = 1,\dots,m$, the interpolation knots of $L_m^{\alpha,-\alpha}$ (zeros of $p_m^{\alpha,-\alpha}$) and by $x_i, i = 1,\dots,m$, the interpolation knots of $L_m^{-\alpha,\alpha}$ (zeros of $p_m^{-\alpha,\alpha}$). We will also denote by *im* $L_m^{\alpha,\beta}$ the range of the operator $L_m^{\alpha,\beta}$. Note that

$$\left\{ \varphi_j^{\alpha,-\alpha}(x) = \frac{l_j^{\alpha,-\alpha}(x)}{v^{\alpha,0}(t_j)}, \quad j = 1,\dots,m \right\}$$

is a basis of $\mathbb{P}_{m-1}$. Obviously, $\forall q \in \mathbb{P}_{m-1}$ one has

$$\sum_{j=1}^{m} \varphi_j^{\alpha,-\alpha}(x)(qv^{\alpha,0})(t_j) \in im\ L_m^{\alpha,-\alpha}$$

and, in virtue of (9),

$$\left\| v^{\alpha,0} \sum_{j=1}^{m} \varphi_j^{\alpha,-\alpha}(qv^{\alpha,0})(t_j) \right\|_\infty \le \mathcal{C}(\log m)\|qv^{\alpha,0}\|_\infty.$$

Analogous observations hold true for the basis

$$\left\{\varphi_i^{-\alpha,\alpha}(x)=\frac{l_i^{-\alpha,\alpha}(x)}{v^{0,\alpha}(x_i)},\quad i=1,\dots,m\right\}$$

of $\mathbb{P}_{m-1}$.

2.1 - *Mapping properties of the operators D and K*

Regarding the operator D the following theorem proved in [**19**] (see also [**3**]) holds true.

Theorem 2.2. *For all* $r>0$

$$D: Z_r^\infty(v^{\alpha,0})\to Z_r^\infty(v^{0,\alpha})$$

is a continuous and invertible operator.

Moreover, with the notation $k(x,y)=k_x(y)$, concerning the operator K, we state the following

Lemma 2.1. *Letting*

$$\Gamma=\sup_{|x|\le 1} v^{0,\alpha}(x)\|k_x v^{0,-\alpha}\|_1 \quad \textit{and} \quad \Gamma_m=\sup_{|x|\le 1} v^{0,\alpha}(x)E_m(k_x)_{v^{0,-\alpha},1},$$

we have

$$\|v^{0,\alpha}Kf\|_\infty\le \Gamma\|fv^{\alpha,0}\|_\infty \tag{13}$$

and

$$E_m(Kf)_{C_{v^{0,\alpha}}}\le \Gamma_m\|fv^{\alpha,0}\|_\infty. \tag{14}$$

Moreover, if for some $s>0$ *it results* $\sup_m m^s\Gamma_m\le \mathcal{C}<+\infty$ *then, for any* $r<s$,

$$K: Z_r^\infty(v^{\alpha,0})\to Z_r^\infty(v^{0,\alpha})$$

is a compact operator. Consequently, assuming that $Ker(D+K)=\{0\}$,

$$D+K: Z_r^\infty(v^{\alpha,0})\to Z_r^\infty(v^{0,\alpha})$$

is a continuous and invertible linear operator.

We remark that the assumption $\sup_m m^s\Gamma_m\le \mathcal{C}<+\infty$ is fulfilled by any kernel k satisfying $\sup_{|x|\le 1} v^{0,\alpha}(x)\|k_x\|_{Z_s^1(v^{0,-\alpha})}<+\infty$, i.e., $\Gamma_m\sim m^s$. For example, the kernel

$k(x,y) = (1+x)^{-1/4}|x-y|^{3/2}$ satisfies the previous equivalence with $s > 5/2$ and for any $\alpha \geq 1/2$.

3 - Numerical method for continuous kernels and right-hand sides

Coming back to the equation (2), in this section we assume a certain smoothness of the known term g and of the kernel k. More precisely, we assume

$$g \in Z_s^\infty(v^{0,\alpha}) \quad \text{and} \quad \sup_{|x|\leq 1} v^{0,\alpha}(x)\|k_x\|_{Z_s^\infty} < +\infty, \quad s > 0. \tag{15}$$

Note that, under the assumptions (15), Lemma 2.1 holds true and then K is a compact operator. Moreover, by Theorem 2.2, D is invertible. It follows that if $Ker(D+K) = \{0\}$ the equation $(D+K)f = g$ admits a unique solution in $Z_r^\infty(v^{\alpha,0}),\ r<s$.

The quadrature method is the following. Recalling the projectors $L_m^{\alpha,-\alpha}$ and $L_m^{-\alpha,\alpha}$, we set

$$f_m(x) = \sum_{j=1}^m \varphi_j^{\alpha,-\alpha}(x)a_j \in im\ L_m^{\alpha,-\alpha}, \quad a_j = (f_m v^{\alpha,0})(t_j), \tag{16}$$

$$g_m(x) = \sum_{i=1}^m \varphi_i^{-\alpha,\alpha}(x)b_i \in im\ L_m^{-\alpha,\alpha}, \quad b_i = (gv^{0,\alpha})(x_i), \tag{17}$$

and

$$(K_m f_m)(x) = L_m^{-\alpha,\alpha}(\tilde{K}_m f_m, x), \tag{18}$$

where

$$(\tilde{K}_m f_m)(x) = \int_{-1}^{1} L_m^{\alpha,-\alpha}(k_x, y) f_m(y) v^{\alpha,-\alpha}(y)dy.$$

Then we solve the finite dimensional equation $(D+K_m)f_m = g_m$, being [**9**]

$$Dp_m^{\alpha,-\alpha} = p_m^{-\alpha,\alpha}, \quad m = 0,1,2,\ldots.$$

Since it can be also written as

$$L_m^{-\alpha,\alpha}(Df_m + \tilde{K}_m f_m, x) = L_m^{-\alpha,\alpha}(g, x),$$

comparing the coefficients to both sides, we get

$$(v^{0,\alpha}Df_m)(x_i) + (v^{0,\alpha}\tilde{K}_m f_m)(x_i) = (v^{0,\alpha}g)(x_i), \quad i = 1,\ldots,m.$$

Moreover, since (see, for example, [**11**, Lemma 1.15])

$$(Df_m)(x_i) = \frac{\sin(\alpha\pi)}{\pi} \sum_{j=1}^{m} \frac{\lambda_j^{\alpha,-\alpha}}{v^{\alpha,0}(t_j)(x_i - t_j)} a_j \tag{19}$$

and

$$(\tilde{K}_m f_m)(x_i) = \sum_{j=1}^{m} k(x_i, t_j) \frac{\lambda_j^{\alpha,-\alpha}}{v^{\alpha,0}(t_j)} a_j,$$

we get the linear system

$$v^{0,\alpha}(x_i) \sum_{j=1}^{m} \frac{\lambda_j^{\alpha,-\alpha}}{v^{\alpha,0}(t_j)} \left[\frac{\sin(\alpha\pi)}{\pi} \frac{1}{(x_i - t_j)} + k(x_i, t_j) \right] a_j = b_i, \quad i = 1, \dots, m, \tag{20}$$

that is equivalent to the equation $(D + K_m)f_m = g_m$.

Note that the above system is well-defined, in fact, letting $x_{m,i} = \cos \tau_{m,i}$ and $t_{m,j} = \cos \theta_{m,j}$, $i, j = 1, \dots, m$, in [**16**] the authors proved that

$$\min_{i,j=1,\dots,m} |\tau_{m,i} - \theta_{m,j}| \geq \frac{\mathcal{C}}{m}, \tag{21}$$

from which it is easy to deduce that there exists a positive constant $\mathcal{C} \neq \mathcal{C}(m, j, i)$ s.t.

$$\frac{v^{0,\alpha}(x_i)\lambda_j^{\alpha,-\alpha}}{v^{\alpha,0}(t_j)|t_j - x_i|} > \mathcal{C}.$$

If $\mathbf{a} = (a_1, \dots, a_m)^T$ is the unique solution of the linear system (20) then f_m, defined in (16), is the unique solution of the equation $(D + K_m)f_m = g_m$.

Now we prove the stability and the convergence of the method.

Theorem 3.1. *Let* $1/2 \leq \alpha < 1$. *Assume that* $Ker(D + K) = \{0\}$ *in* $Z_r^\infty(v^{\alpha,0})$ *and that*

$$\sup_{|x|\leq 1} v^{0,\alpha}(x) \|k_x\|_{Z_s^\infty} < +\infty \quad \textit{and} \quad g \in Z_s^\infty(v^{0,\alpha}), \quad 0 < r < s.$$

Then system (20), *for a sufficiently large* m (*say* $m > m_0$), *is unisolvent and, denoting by* A_m *its matrix and by* $\operatorname{cond}(A_m)_\infty$ *its condition number in uniform norm, we have*

$$\sup_m \frac{\operatorname{cond}(A_m)_\infty}{\log^2 m} < +\infty. \tag{22}$$

Moreover the following error estimate

$$\|f - f_m\|_{Z_r^\infty(v^{\alpha,0})} \leq \mathcal{C} \frac{\log m}{m^{s-r}} \|g\|_{Z_s^\infty(v^{0,\alpha})} \sup_{|x|\leq 1} v^{0,\alpha}(x) \|k_x\|_{Z_s^\infty} \tag{23}$$

holds true, where $\mathcal{C} \neq \mathcal{C}(m, k, g)$.

3.1 - *Special case*

The previous procedure can be used when the kernel $k(x,y)$, although presenting weak singularity, can be written as follows

$$k(x,y)=\frac{h(x,y)-h(x,x)}{y-x}. \tag{24}$$

Note that if k is a weakly singular kernel of the form

$$k(x,y)=|x-y|^{\mu},\quad -1<\mu\le 0,$$

then the representation (24) holds true with $h(x,y)=(y-x)|x-y|^{\mu}$.

Concerning the function $h(x,y)$, we will assume

$$\sup_{|x|\le 1}\|h_x\|_{Z_s^\infty}<+\infty.$$

We approximate the equation (2) by the equation $(D+K_m)f_m=g_m$ where f_m, g_m and K_m are defined in (16), (17) and (18), respectively. Then we get the linear system

$$v^{0,\alpha}(x_i)\sum_{j=1}^{m}\frac{\lambda_j^{\alpha,-\alpha}}{v^{\alpha,0}(t_j)}\left[\frac{\sin\alpha\pi}{\pi}\frac{1}{(x_i-t_j)}+\frac{h(x_i,t_j)-h(x_i,x_i)}{t_j-x_i}\right]a_j=b_i, \tag{25}$$

$$i=1,\dots,m,$$

in the unknowns $a_1,\dots,a_m$ that is system (20) with $k(x,y)$ having the form (24).

Theorem 3.1 becomes

Theorem 3.2. *Let* $1/2\le\alpha<1$ *and assume* $Ker(D+K)=\{0\}$ *in* $Z_r^\infty(v^{\alpha,0})$ *and*

$$\sup_{|x|\le 1}\|h_x\|_{Z_s^\infty}<+\infty\quad and\quad g\in Z_s^\infty(v^{0,\alpha}),\quad 0<r<s.$$

Then equation (2) *is unisolvent in* $Z_r^\infty(v^{\alpha,0}), r<s$, *system* (25), *for a sufficiently large* m (*say* $m>m_0$), *admits a unique solution and its matrix* A_m *satisfies*

$$\sup_m\frac{\text{cond}(A_m)_\infty}{\log^2 m}<+\infty.$$

Moreover the error is estimated as follows

$$\|(f-f_m)v^{\alpha,0}\|_{Z_r^\infty(v^{\alpha,0})}\le \mathcal{C}\,\frac{\log^2 m}{m^{s-r}}\,\|g\|_{Z_s^\infty(v^{0,\alpha})}\sup_{|x|\le1}\|h_x\|_{Z_s^\infty},\quad r<s,$$

where $\mathcal{C}\ne\mathcal{C}(m,h,g)$.

4 - General case

When the kernel k cannot be represented in the form (24), for example it has weak singularities along stretches of curves contained in $[-1,1]^2$, we go to solve the finite dimensional equations

$$(D + K_m^*) f_m = g_m, \tag{26}$$

where f_m and g_m are the same defined in (16) and (17), respectively, and

$$(K_m^* f_m)(x) = L_m^{-\alpha,\alpha}(\widehat{K}_m f_m, x)$$

with

$$(\widehat{K}_m f_m)(x) = \int_{-1}^{1} S_m^{\alpha,-\alpha}(k_x, y) f_m(y) v^{\alpha,-\alpha}(y) dy,$$

where $S_m^{\alpha,-\alpha}(k_x, y)$ is the Fourier sum of the function k_x in the system $\{p_m^{\alpha,-\alpha}\}_m$. By (19) and taking into account that

$$(\widehat{K}_m f_m)(x_i) = \sum_{j=1}^{m} S_m^{\alpha,-\alpha}(k_{x_i}, t_j) \frac{\lambda_j^{\alpha,-\alpha}}{v^{\alpha,0}(t_j)} a_j, \quad a_j = (f_m v^{\alpha,0})(t_j),$$

the finite dimensional equation (26) is equivalent to the system

$$v^{0,\alpha}(x_i) \sum_{j=1}^{m} \frac{\lambda_j^{\alpha,-\alpha}}{v^{\alpha,0}(t_j)} \left[\frac{\sin \alpha\pi}{\pi} \frac{1}{(x_i - t_j)} + S_m^{\alpha,-\alpha}(k_{x_i}, t_j) \right] a_j = b_i, \tag{27}$$

$$i = 1, \ldots, m,$$

that is system (20) with $k(x_i, t_j)$ (that may not exist) replaced by $S_m^{\alpha,-\alpha}(k_{x_i}, t_j)$. Of course the computation of the Fourier coefficients c_ν, $\nu = 0, \ldots, m-1$, requires the main computational effort.

If $\mathbf{a} = (a_1, \ldots, a_m)^T$ is the unique solution of the linear system (27) then f_m, defined in (16), is the unique solution of the equation $(D + K_m^*) f_m = g_m$.

Now, making an L^2-assumption on the kernel k, we prove the stability and the convergence of the method.

Theorem 4.1. *Let $1/2 \le \alpha < 1$ and assume that $Ker(D + K) = \{0\}$ and that, for some $s > 0$, it results*

$$\sup_{|x| \le 1} v^{0,\alpha}(x) \|k_x\|_{Z_s^2(v^{0,-\frac{\alpha}{2}})} < +\infty \quad \textit{and} \quad g \in Z_s^\infty(v^{0,\alpha}).$$

Then equation (2) admits a unique solution in $Z_r^\infty(v^{\alpha,0}), r < s$, system (27), for a sufficiently large m (say $m > m_0$), is unisolvent and the condition number of the *matrix A_m satisfies*

$$\sup_m \frac{\mathrm{cond}(A_m)_\infty}{\log^2 m} < +\infty.$$

Moreover the following error estimate

$$\|(f-f_m)v^{\alpha,0}\|_{Z_r^\infty(v^{\alpha,0})} \le \mathcal{C}\frac{\log m}{m^{s-r}}\|g\|_{Z_s^\infty(v^{0,\alpha})} \sup_{|x|\le 1} v^{0,\alpha}(x)\|k_x\|_{Z_s^2(v^{0,-\frac{\alpha}{2}})} \tag{28}$$

holds true, where $\mathcal{C} \ne \mathcal{C}(m,k,g)$.

As already announced in the introduction, equation (2) has been previously considered in weighted L^2 spaces. In particular, in [**15**, Theorem 3.2], assuming

$$\sup_{|x|\le 1}\|k_x\|_{Z_s^2(v^{\alpha,-\alpha})} < +\infty, \quad \sup_{|y|\le 1}\|k_y\|_{Z_s^2(v^{-\alpha,\alpha})} < +\infty, \quad g \in Z_s^2(v^{-\alpha,\alpha}),$$

the following estimate

$$\|(f-f_m)v^{\frac{\alpha}{2},-\frac{\alpha}{2}}\|_2 = \mathcal{O}(m^{-s}), \quad s > \frac{1}{2}, \tag{29}$$

has been proved. It improves an analogous estimate proved in [**17**].

Using [**15**, Theorem 3.5], the L^∞ version of (29) is

$$\|(f-f_m)v^{\alpha,0}\|_\infty = \mathcal{O}(m^{-s+\frac{1}{2}})$$

that, in $Z_r^\infty(v^{\alpha,0})$, $0<r<s$, becomes

$$\|f-f_m\|_{Z_r^\infty(v^{\alpha,0})} = \mathcal{O}(m^{-s+r+\frac{1}{2}}).$$

It implies the convergence of the error for $s > r + 1/2$, while, by (23), the convergence follows for $s > r$.

5 - Proofs

Proof of Lemma 2.1. We first prove (13). We have

$$\begin{aligned} v^{0,\alpha}(x)|(Kf)(x)| &= v^{0,\alpha}(x)\left|\int_{-1}^{1} k(x,y)(fv^{\alpha,-\alpha})(y)dy\right| \\ &\le \|fv^{\alpha,0}\|_\infty v^{0,\alpha}(x)\int_{-1}^{1}|k(x,y)|v^{0,-\alpha}(y)dy \\ &\le \|fv^{\alpha,0}\|_\infty \sup_{|x|\le 1} v^{0,\alpha}(x)\|k_x v^{0,-\alpha}\|_1. \end{aligned}$$

Now we prove (14). We preliminary observe that a polynomial $q_m(x,y)$ of degree m with respect to both the variables separately can be represented as follows

$$q_m(x,y) = \sum_{i=0}^{m} p_{i,m}(x)y^{m-i}, \quad p_{i,m} \in \mathbb{P}_m,$$

and

$$\int_{-1}^{1} q_m(x,y)f(y)v^{\alpha,-\alpha}(y)dy$$

is a polynomial in x of degree m. Thus

$$\begin{aligned} E_m(Kf)_{v^{0,\alpha},\infty} &\leq \sup_{|x|\leq 1} v^{0,\alpha}(x) \int_{-1}^{1} |k(x,y) - q_m(x,y)|(fv^{\alpha,-\alpha})(y)dy \\ &\leq \|fv^{\alpha,0}\|_\infty \sup_{|x|\leq 1} v^{0,\alpha}(x) \int_{-1}^{1} |k(x,y) - q_m(x,y)|v^{0,-\alpha}(y)dy. \end{aligned}$$

Taking the infimum on all polynomials (in the variable y) of degree m, we deduce

$$E_m(Kf)_{v^{0,\alpha},\infty} \leq \|fv^{\alpha,0}\|_\infty \sup_{|x|\leq 1} v^{0,\alpha}(x)E_m(k_x)_{v^{0,-\alpha},1} = \varGamma_m\|fv^{\alpha,0}\|_\infty. \tag{30}$$

In order to prove the second part of the lemma, let $r > 0$ and $r < s$. Then applying (7) with Kf in place of f, by (30), we get

$$\begin{aligned} E_m(Kf)_{Z_r^\infty(v^{0,\alpha})} &\leq \mathcal{C} \sup_m m^r E_m(Kf)_{v^{0,\alpha},\infty} \\ &\leq \mathcal{C}\|fv^{\alpha,0}\|_\infty \sup_m m^r \varGamma_m \\ &\leq \mathcal{C}\|f\|_{Z_r^\infty(v^{\alpha,0})} \sup_m m^r \varGamma_m. \end{aligned}$$

Since we assume $\sup_m m^s \varGamma_m \leq A < +\infty, s > r$, we obtain

$$\frac{E_m(Kf)_{Z_r^\infty(v^{0,\alpha})}}{\|f\|_{Z_r^\infty(v^{\alpha,0})}} \leq \frac{\mathcal{C}A}{m^{s-r}}, \quad s > r,$$

and, using [**26**, p. 44], the compactness of the operator $K : Z_r^\infty(v^{\alpha,0}) \to Z_r^\infty(v^{0,\alpha})$ follows.

The last part of the lemma is a consequence of Theorem 2.2. □

Lemma 5.1. *Let $1/2 \leq \alpha < 1$. If, for some $s > 0$,*

$$\sup_{|x|\leq 1} v^{0,\alpha}(x)\|k_x\|_{Z_s^\infty} < +\infty,$$

then, for every $0<r<s$,

$$\|(K-K_m)f\|_{Z_r^\infty(v^{0,\alpha})} \le \mathcal{C}\|f\|_{Z_r^\infty(v^{\alpha,0})} \sup_{|x|\le 1} v^{0,\alpha}(x)\|k_x\|_{Z_s^\infty} \frac{\log m}{m^{s-r}},$$

where $\mathcal{C} \neq \mathcal{C}(m,f,k)$.

Proof. By definitions of K and K_m we have

$$(K-K_m)f(x) = \int_{-1}^{1} k(x,y)f(y)v^{\alpha,-\alpha}(y)dy$$
$$-L_m^{-\alpha,\alpha}\left(\int_{-1}^{1} L_m^{\alpha,-\alpha}(k_x,y)(fv^{\alpha,-\alpha})(y)dy, x\right).$$

The following notation will be useful. With $a:[-1,1]^2 \to \mathbb{R}$, we set

$$(K^a f)(x) = \int_{-1}^{1} a(x,y)(fv^{\alpha,-\alpha})(y)dy$$

and

$$(K_m^a f)(x) = L_m^{-\alpha,\alpha}\left(\int_{-1}^{1} L_m^{\alpha,-\alpha}(a(x,\cdot),y)(fv^{\alpha,-\alpha})(y)dy, x\right).$$

Then if $a(x,y)=P_m(x,y)$, being $P_m(x,y)$ a polynomial of degree m with respect to both the variables separately, we have

$$K^{P_m} - K_m^{P_m} = 0$$

and then, with $R(x,y)=k(x,y)-P_m(x,y)$, we get

$$\|v^{0,\alpha}(K-K_m)f\|_\infty \le \|v^{0,\alpha}K^R f\|_\infty + \|v^{0,\alpha}K_m^R f\|_\infty.$$

Regarding the first addendum, we use (13) and obtain

$$\|v^{0,\alpha}K^R f\|_\infty \le \|fv^{\alpha,0}\|_\infty \sup_{|x|\le 1} v^{0,\alpha}(x)\|R_x v^{0,-\alpha}\|_1$$
$$\le \mathcal{C}\|fv^{\alpha,0}\|_\infty \sup_{|x|\le 1} v^{0,\alpha}(x)\|k_x - P_{m,x}\|_\infty,$$

being $R_x(y)=k_x(y)-P_{m,x}(y)$. For the second addendum, we use (10) first and then (11). We deduce

$$\|v^{0,\alpha}K_m^R f\|_\infty \le \mathcal{C}(\log m)\sup_{|x|\le 1} v^{0,\alpha}(x)\left|\int_{-1}^{1} L_m^{\alpha,-\alpha}(R_x,y)(fv^{\alpha,-\alpha})(y)dy\right|$$

$$\le \mathcal{C}(\log m)\|fv^{\alpha,0}\|_\infty \sup_{|x|\le 1} v^{0,\alpha}(x)\int_{-1}^{1}|L_m^{\alpha,-\alpha}(R_x,y)|v^{0,-\alpha}(y)dy$$

$$\le \mathcal{C}(\log m)\|fv^{\alpha,0}\|_\infty \sup_{|x|\le 1} v^{0,\alpha}(x)\|k_x - P_{m,x}\|_\infty .$$

Then we obtain

$$\|v^{0,\alpha}(K-K_m)f\|_\infty \le \mathcal{C}(\log m)\|fv^{\alpha,0}\|_\infty \sup_{|x|\le 1} v^{0,\alpha}(x)\|k_x - P_{m,x}\|_\infty$$

and, taking the infimum on all polynomials (in the variable y) of degree m, we get

$$\|v^{0,\alpha}(K-K_m)f\|_\infty \le \mathcal{C}(\log m)\|fv^{\alpha,0}\|_\infty \sup_{|x|\le 1} v^{0,\alpha}(x)E_m(k_x)_\infty .$$

Finally, in virtue of the assumption on k, we estimate $E_m(k_x)_\infty$ by (5), and obtain

(31) $$\|v^{0,\alpha}(K-K_m)f\|_\infty \le \mathcal{C}\|f\|_{Z_r^\infty(v^{\alpha,0})} \sup_{|x|\le 1} v^{0,\alpha}(x)\|k_x\|_{Z_s^\infty}\frac{\log m}{m^s}.$$

Now, taking into account the equivalence (6), we get

$$\|(K-K_m)f\|_{Z_r^\infty(v^{0,\alpha})} \le \mathcal{C}\sup_m m^r\|v^{0,\alpha}(K-K_m)f\|_\infty .$$

Therefore, applying (31), the thesis follows. □

Proof of Theorem 3.1. By the identity

$$(D+K)f_m = (K-K_m)f_m + (D+K_m)f_m$$

we get

$$\|f_m\|_{Z_r(v^{\alpha,0})} \le \|(D+K)^{-1}\|_{Z_r(v^{0,\alpha})\to Z_r(v^{\alpha,0})}[\|(K-K_m)f_m\|_{Z_r(v^{0,\alpha})} + \|(D+K_m)f_m\|_{Z_r(v^{0,\alpha})}]$$

and, taking into account Lemma , it follows that

$$\mathcal{C}\|f_m\|_{Z_r(v^{\alpha,0})} \le \|(D+K_m)f_m\|_{Z_r(v^{0,\alpha})},$$

where

$$\mathcal{C} \leq \|(D+K)^{-1}\|^{-1}_{Z_r(v^{0,\alpha})\to Z_r(v^{\alpha,0})} - \mathcal{O}\left(\frac{\log m}{m^{s-r}}\right), \quad s > r.$$

Consequently $D + K_m : im\ L_m^{\alpha,-\alpha} \to im\ L_m^{-\alpha,\alpha}$ is invertible and, for a sufficiently large m (say $m > m_0$), $(D + K_m)f_m = g_m$ has a unique solution $f_m \in im\ L_m^{\alpha,-\alpha}$.

Now we prove (22). By (20) we have

$$\|A_m\|_\infty \leq \mathcal{C} \max_{i=1,\ldots,m} v^{0,\alpha}(x_i)\left[\sum_{j=1}^m \frac{\lambda_j^{\alpha,-\alpha}}{v^{\alpha,0}(t_j)|x_i - t_j|} + \sum_{j=1}^m \frac{\lambda_j^{\alpha,-\alpha}}{v^{\alpha,0}(t_j)}|k(x_i, t_j)|\right].$$

Recalling that [**25**, p. 353, eq. 15.3.10] $\lambda_j^{\alpha,-\alpha} \sim \Delta t_j v^{\alpha,-\alpha}(t_j)$, $\Delta t_j = t_{j+1} - t_j$, and that in virtue of (21) we have (see, for instance, [**1**, (5.16)])

$$\sum_{j=1}^m \frac{\Delta t_j}{|x_i - t_j|} v^{0,-\alpha}(t_j) \leq \mathcal{C} v^{0,-\alpha}(x_i)\log m, \tag{32}$$

we get

$$\|A_m\|_\infty \leq \mathcal{C} \max_{i=1,\ldots,m} v^{0,\alpha}(x_i)\left[\sum_{j=1}^m \frac{\Delta t_j}{|x_i - t_j|} v^{0,-\alpha}(t_j) + \sum_{j=1}^m \Delta t_j v^{0,-\alpha}(t_j)|k(x_i, t_j)|\right]$$

$$\leq \mathcal{C}\log m + \mathcal{C}\left(\int_{-1}^1 v^{0,-\alpha}(x)dx\right)\sup_{|x|\leq 1} v^{0,\alpha}(x)\|k_x\|_\infty.$$

Then, by the assumption on k, we obtain

$$\|A_m\|_\infty \leq \mathcal{C}\log m. \tag{33}$$

It remain to estimate $\|A_m^{-1}\|_\infty$. In virtue of the equivalence of the system (20) with the equation $(D + K_m)f_m = g_m$, for every $\theta = (\theta_1, \ldots, \theta_m)$ there exists a unique $\xi = (\xi_1, \ldots, \xi_m)$ such that $A_m^{-1}\theta = \xi$ if and only if $(D + K_m)^{-1}\tilde{\theta}(x) = \tilde{\xi}(x)$, where

$$\tilde{\theta}(x) = \sum_{j=1}^m \varphi_j^{-\alpha,\alpha}(x)\theta_j, \quad \theta_j = (\tilde{\theta}v^{0,\alpha})(x_j)$$

and

$$\tilde{\xi}(x) = \sum_{j=1}^m \varphi_j^{\alpha,-\alpha}(x)\xi_j, \quad \xi_j = (\tilde{\xi}v^{\alpha,0})(t_j).$$

Then, for all θ, we get

$$\|A_m^{-1}\theta\|_{l^\infty} = \|\xi\|_{l^\infty} \leq \|\tilde{\xi}v^{\alpha,0}\|_\infty = \|(D + K_m)^{-1}\tilde{\theta}v^{\alpha,0}\|_\infty$$

$$\leq \|(D + K_m)^{-1}_{|\mathbb{P}_{m-1}}\|_{C_{v^{0,\alpha}}\to C_{v^{\alpha,0}}}\ \|\theta\|_{l^\infty}\ \|L_m^{-\alpha,\alpha}\|_{C_{v^{0,\alpha}}\to C_{v^{\alpha,0}}}.$$

Using (10), we obtain

$$\|A_m^{-1}\|_\infty \le \mathcal{C} \log m. \tag{34}$$

Now, combining (33) with (34), (22) follows.

In order to prove (23), we use the following identity

$$(f - f_m) = (D + K)^{-1}[(g - g_m) - (K - K_m)f_m].$$

Then, since by (6), (10) and the assumptions on g, we get

$$\|g - g_m\|_{Z_r^\infty(v^{0,\alpha})} \le \mathcal{C}\,\frac{\log m}{m^{s-r}}\,\|g\|_{Z_s^\infty(v^{0,\alpha})},$$

using Lemma 5.1, (23) follows. □

In order to prove Theorem 3.2 we need the following notations and results.

We denote by ω_φ^k the complete $\varphi-$ modulus of smoothness [**5**]:

$$\begin{aligned}\omega_\varphi^k(f,t)_{v^{\alpha,\beta},p} =& \Omega_\varphi^k(f,t)_{v^{\alpha,\beta},p}\\ &+ \inf_{q\in\mathbb{P}_{k-1}} \|(f-q)v^{\alpha,\beta}\|_{L^p([-1,-1+(2kt)^2])}\\ &+ \inf_{q\in\mathbb{P}_{k-1}} \|(f-q)v^{\alpha,\beta}\|_{L^p([1-(2kt)^2,1])}.\end{aligned}$$

We will set $\omega_\varphi := \omega_\varphi^1$ and $\omega_\varphi^k(f,t)_\infty := \omega_\varphi^k(f,t)_{v^{0,0},\infty}$.

Lemma 5.2. *Let* $1/2 \le \alpha < 1$. *If the kernel* k *has the form* (24), *where, for some* $s > 0$,

$$\sup_{|x|\le 1} \|h_x\|_{Z_s^\infty} < +\infty,$$

then, for every $0 < r < s$,

$$E_m(Kf)_{Z_r(v^{0,\alpha})} \le \mathcal{C}\|fv^{\alpha,0}\|_\infty \sup_{|x|\le 1}\|h_x\|_{Z_s^\infty}\,\frac{\log m}{m^{s-r}},$$

where $\mathcal{C} \ne \mathcal{C}(m,f,h)$.

Proof. We can write

$$\begin{aligned} v^{0,\alpha}(x)(Kf)(x) &= (1+x)^\alpha \int_{-1}^{1} \frac{h(x,y)-h(x,x)}{y-x}(fv^{\alpha,-\alpha})(y)dy\\ &= (1+x)^\alpha \left\{ \int_{|x-y|>\frac{1+x}{4}} + \int_{x-\frac{1+x}{4}}^{x+\frac{1+x}{4}} \right\} \frac{h(x,y)-h(x,x)}{y-x}(fv^{\alpha,-\alpha})(y)dy\\ &:= I_1 + I_2. \end{aligned} \tag{35}$$

We have

$$|I_1| \le \mathcal{C}\|fv^{\alpha,0}\| \sup_{|x|\le 1} \|h_x\|_\infty. \tag{36}$$

Concerning I_2, we have

$$\begin{aligned} |I_2| &\le \mathcal{C}\|fv^{\alpha,0}\|_\infty \int_{x-\frac{1+x}{4}}^{x+\frac{1+x}{4}} \left|\frac{h(x,y)-h(x,x)}{y-x}\right| dy \\ &= \mathcal{C}\|fv^{\alpha,0}\|_\infty \int_{-\frac{\sqrt{1+x}}{4}}^{\frac{\sqrt{1+x}}{4}} \left|\frac{h_x(x+\sqrt{1+x}u)-h_x(x)}{u}\right| du \\ &\le \mathcal{C}\|fv^{\alpha,0}\|_\infty \int_0^{\frac{\sqrt{1+x}}{4}} \left\{\frac{|\overrightarrow{\Delta}_{u\varphi} h_x(x)|}{u} + \frac{|\overleftarrow{\Delta}_{u\varphi} h_x(x)|}{u}\right\} du, \end{aligned}$$

where $\overrightarrow{\Delta}_{u\varphi}$ and $\overleftarrow{\Delta}_{u\varphi}$ denote the forward and the backward finite differences operators of step $u\varphi$, respectively, and, recalling that [**5**, p. 26]

$$\omega_\varphi(h_x,t)_\infty \sim \sup_{0<u\le t} \|\Delta_{u\varphi} h_x\|_\infty \sim \sup_{0<u\le t} \|\overleftarrow{\Delta}_{u\varphi} h_x\|_\infty \sim \sup_{0<u\le t} \|\overrightarrow{\Delta}_{u\varphi} h_x\|_\infty,$$

we deduce

$$|I_2| \le \mathcal{C}\|fv^{\alpha,0}\|_\infty \int_0^1 \frac{\omega_\varphi(h_x,u)_\infty}{u} du. \tag{37}$$

Combining (36) and (37) with (35), we get

$$\|v^{0,\alpha}(Kf)\|_\infty \le \mathcal{C}\|fv^{\alpha,0}\|_\infty \sup_{|x|\le 1}\left\{\|h_x\|_\infty + \int_0^1 \frac{\omega_\varphi(h_x,t)_\infty}{t} dt\right\}.$$

Let $p_N(x,y)$ be a polynomial of degree $N = \left\lfloor \frac{m+1}{2} \right\rfloor$ with respect to both the variables x and y separately. Then $q(x,y) = \dfrac{p_N(x,y)-p_N(x,x)}{y-x}$ is a polynomial of degree $N-1$ with respect the variables y and of degree m with respect to the variable x and

$$P_m(x) = \int_{-1}^1 q(x,y) f(y) v^{\alpha,-\alpha}(y) dy$$

is a polynomial of degree m. We have

$$E_m(Kf)_{v^{0,\alpha},\infty} \le \sup_{|x|\le 1}\left| v^{0,\alpha}(x)\int_{-1}^{1}[k(x,y)-q(x,y)](fv^{\alpha,-\alpha})(y)dy\right|$$

$$= \sup_{|x|\le 1}\left| v^{0,\alpha}(x)\int_{-1}^{1}\frac{[h_x(y)-p_{N,x}(y)]-[h_x(x)-p_{N,x}(x)]}{y-x}(fv^{\alpha,-\alpha})(y)dy\right|$$

and, proceeding as done for (35), we get

$$E_m(Kf)_{v^{0,\alpha},\infty} \le \mathcal{C}\|fv^{\alpha,0}\|_\infty \sup_{|x|\le 1}\left\{\|h_x-p_{N,x}\|_\infty + \int_0^1 \frac{\omega_\varphi(h_x-p_{N,x},t)_\infty}{t}\,dt\right\}.$$

Taking $p_{N,x}$, for every fixed x, as the polynomial of best approximation of the function h_x, recalling that [**20**, Lemma 2.1]

$$\int_0^{\frac{1}{m}}\frac{\omega_\varphi(f-P_m,t)_\infty}{t}\,dt \le \mathcal{C}\int_0^{\frac{1}{m}}\frac{\omega_\varphi^k(f,t)_\infty}{t}\,dt$$

and applying the Jackson theorem [**5**], we deduce

$$E_m(Kf)_{v^{0,\alpha},\infty} \le \mathcal{C}\|fv^{\alpha,0}\|_\infty \sup_{|x|\le 1}\left\{\omega_\varphi^k\left(h_x,\frac{1}{m}\right)_\infty \log m + \int_0^{\frac{1}{m}}\frac{\omega_\varphi^k(h_x,t)_\infty}{t}\,dt\right\}.$$

Finally, in virtue of our assumption on the function $h(x,y)$, by (5), we obtain

$$E_m(Kf)_{v^{0,\alpha},\infty} \le \mathcal{C}\sup_{|x|\le 1}\|h_x\|_{Z_s^\infty}\|fv^{\alpha,0}\|_\infty\frac{\log m}{m^s}. \tag{38}$$

Now, using (7) with Kf in place of f and (38) we get

$$\begin{aligned} E_m(Kf)_{Z_r^\infty(v^{0,\alpha})} &\le \mathcal{C}\sup_m m^r E_m(Kf)_{v^{0,\alpha},\infty} \\ &\le \mathcal{C}\|fv^{\alpha,0}\|_\infty \sup_{|x|\le 1}\|h_x\|_{Z_s^\infty}\frac{\log m}{m^{s-r}}, \end{aligned}$$

i.e. the thesis. □

Lemma 5.3. *Let* $1/2 \le \alpha < 1$. *If the kernel* k *has the form* (24)*, where, for some* $s>0$,

$$\sup_{|x|\le 1}\|h_x\|_{Z_s^\infty}<+\infty,$$

then, for every $0<r<s$,

$$\|(K-K_m)f_m\|_{Z_r^\infty(v^{0,\alpha})}\le \mathcal{C}\|f_m\|_{Z_r^\infty(v^{\alpha,0})}\sup_{|x|\le 1}\|h_x\|_{Z_s^\infty}\frac{\log^2 m}{m^{s-r}},$$

where $\mathcal{C}\neq\mathcal{C}(m,f_m,h)$.

Proof. Let $p_N(x,y)$ be the polynomial of degree $N=\left\lfloor\frac{m}{2}\right\rfloor$ with respect to both the variables x and y separately. Then $q(x,y)=\dfrac{p_N(x,y)-p_N(x,x)}{y-x}$ is a polynomial of degree $N-1$ with respect the variable y and of degree $m-1$ with respect to the variable x and

$$\begin{aligned}&\int_{-1}^{1}\frac{p_N(x,y)-p_N(x,x)}{y-x}(f_m v^{\alpha,-\alpha})(y)dy\\&=L_m^{-\alpha,\alpha}\left(\int_{-1}^{1}L_m^{\alpha,-\alpha}\left(\frac{p_N(x,\cdot)-p_N(x,x)}{\cdot-x},y\right)(f_m v^{\alpha,-\alpha})(y)dy,x\right).\end{aligned}$$

Then, setting $R(x,y)=h(x,y)-p_N(x,y)$, we can write

$$\begin{aligned}(K-K_m)f(x)=&\int_{-1}^{1}\frac{R(x,y)-R(x,x)}{y-x}(f_m v^{\alpha,-\alpha})(y)dy\\&-L_m^{-\alpha,\alpha}\left(\int_{-1}^{1}L_m^{\alpha,-\alpha}\left(\frac{R(x,\cdot)-R(x,x)}{\cdot-x},y\right)(f_m v^{\alpha,-\alpha})(y)dy,x\right)\end{aligned}$$

and, letting

$$F(x)=\int_{-1}^{1}L_m^{\alpha,-\alpha}\left(\frac{R(x,\cdot)-R(x,x)}{\cdot-x},y\right)(f_m v^{\alpha,-\alpha})(y)dy,$$

$$\begin{aligned}v^{0,\alpha}(x)|(K-K_m)f(x)|\le&\left|v^{0,\alpha}(x)\int_{-1}^{1}\frac{R(x,y)-R(x,x)}{y-x}(f_m v^{\alpha,-\alpha})(y)dy\right|\\&+|v^{0,\alpha}(x)L_m^{-\alpha,\alpha}(F,x)|\\&=:A+B.\end{aligned}\tag{39}$$

Now, using (9), we have

$$\begin{aligned} B &\le \left(\max_{|x|\le 1} v^{0,\alpha}(x) \sum_{k=1}^{m} \frac{|l_k^{-\alpha,\alpha}(x)|}{v^{0,\alpha}(x_k)} \right) \max_{k=1,\dots,m} v^{0,\alpha}(x_k)|F(x_k)| \\ &\le \mathcal{C}(\log m) \max_{k=1,\dots,m} v^{0,\alpha}(x_k)|F(x_k)|. \end{aligned}$$

Moreover, applying the gaussian rule and (32), we get

$$\begin{aligned} B &\le \mathcal{C}(\log m) \max_{k=1,\dots,m} v^{0,\alpha}(x_k) \sum_{i=1}^{m} \frac{|R(x_k,t_i) - R(x_k,x_k)|}{|t_i - x_k|} |f_m(t_i)| \lambda_i^{\alpha,-\alpha} \\ &\le \mathcal{C}(\log m)\ \|f_m v^{\alpha,0}\|_\infty \max_{k=1,\dots,m} v^{0,\alpha}(x_k) \|R_x\|_\infty \sum_{i=1}^{m} \frac{\Delta t_i}{|t_i - x_k|} v^{0,-\alpha}(t_i) \\ &\le \mathcal{C}(\log^2 m)\ \|f_m v^{\alpha,0}\|_\infty \sup_{|x|\le 1} \|h_x - p_{N,x}\|_\infty. \end{aligned}$$

Now, taking the infimum on all polynomials (with respect to y) of degree N, we have

$$B \le \mathcal{C}(\log^2 m)\ \|f_m v^{\alpha,0}\|_\infty \sup_{|x|\le 1} E_N(h_x)_\infty$$

and, in virtue of the assumption on h_x, by (5), we obtain

$$B \le \mathcal{C} \|f_m v^{\alpha,0}\|_\infty \sup_{|x|\le 1} \|h_x\|_{Z_s^\infty} \frac{\log^2 m}{m^s}. \tag{40}$$

On the other hand, proceeding as done for the proof of Lemma 5.2, we obtain

$$A \le \mathcal{C} \|f_m v^{\alpha,0}\|_\infty \sup_{|x|\le 1} \|h_x\|_{Z_s^\infty} \frac{\log m}{m^s}. \tag{41}$$

Therefore, combining (39) with (41) and (40), we get

$$\|v^{0,\alpha}(K - K_m) f_m\|_\infty \le \mathcal{C} \|f_m v^{\alpha,0}\|_\infty \sup_{|x|\le 1} \|h_x\|_{Z_s^\infty} \frac{\log^2 m}{m^s}$$

and then, taking into account equivalence (6), the thesis follows. □

Proof of Theorem 3.2. By Lemma 5.2 and using [**26**, p. 44], the compactness of the operator $K : Z_r^\infty(v^{\alpha,0}) \to Z_r^\infty(v^{0,\alpha})$ follows. Moreover, using Theorem 2.2 and the assumption $Ker(D + K) = \{0\}$, equation (2) admits a unique solution in $Z_r^\infty(v^{\alpha,0})$.

Taking into account Lemma 5.3, the other part of the proof is similar to the proof of Theorem 3.1. □

In order to prove Theorem 4.1 we need the following lemma.

Lemma 5.4. *Let* $1/2 \le \alpha < 1$. *If, for some* $s > 0$,

$$\sup_{|x|\le 1} v^{0,\alpha}(x)\|k_x\|_{Z_s^2(v^{0,-\frac{\alpha}{2}})} < +\infty,$$

then, for every $0 < r < s$,

$$\|(K - K_m^*)f\|_{Z_r^\infty(v^{0,\alpha})} \le \mathcal{C}\|f\|_{Z_r^\infty(v^{\alpha,0})} \max_{|x|\le 1} v^{0,\alpha}(x)\|k_x\|_{Z_s^2(v^{0,-\frac{\alpha}{2}})} \frac{\log m}{m^{s-r}},$$

where $\mathcal{C} \ne \mathcal{C}(m, f, k)$.

Proof. Proceeding as done for the proof of Lemma 5.1, we get

$$\|v^{0,\alpha}(K - K_m^*)f\|_\infty \le \|v^{0,\alpha}K^R f\|_\infty + \|v^{0,\alpha}K_m^{*R} f\|_\infty, \tag{42}$$

being $R(x,y) = k(x,y) - P_m(x,y)$. For the first addendum, by (13) and the Cauchy inequality, we get

$$\begin{aligned} \|v^{0,\alpha}K^R f\|_\infty &\le \|fv^{\alpha,0}\|_\infty \sup_{|x|\le 1} v^{0,\alpha}(x)\|R_x v^{0,-\alpha}\|_1 \\ &\le \mathcal{C}\|fv^{\alpha,0}\|_\infty \sup_{|x|\le 1} v^{0,\alpha}(x)\|(k_x - P_{m,x})v^{0,-\frac{\alpha}{2}}\|_2. \end{aligned} \tag{43}$$

While for the second addendum, by (10), we have

$$\|v^{0,\alpha}K_m^{*R} f\|_\infty \le \mathcal{C}\|fv^{\alpha,0}\|_\infty (\log m) \max_{|x|\le 1} v^{0,\alpha}(x) \int_{-1}^{1} |S_m^{\alpha,-\alpha}(R_x, y)| v^{0,-\alpha}(y) dy$$

and, using the Cauchy inequality and (12), we get

$$\begin{aligned} \|v^{0,\alpha}K_m^{*r} f\|_\infty &\le \mathcal{C}\|fv^{\alpha,0}\|_\infty (\log m) \max_{|x|\le 1} v^{0,\alpha}(x)\|S_m^{\alpha,-\alpha}(R_x)v^{0,-\frac{\alpha}{2}}\|_2 \\ &\le \mathcal{C}\|fv^{\alpha,0}\|_\infty (\log m) \max_{|x|\le 1} v^{0,\alpha}(x)\|(k_x - P_{m,x})v^{0,-\frac{\alpha}{2}}\|_2. \end{aligned} \tag{44}$$

Combining (42) with (43) and (44), we deduce

$$\|v^{0,\alpha}(K - K_m^*)f\|_\infty \le \mathcal{C}\|fv^{\alpha,0}\|_\infty (\log m) \max_{|x|\le 1} v^{0,\alpha}(x)\|(k_x - P_{m,x})v^{0,-\frac{\alpha}{2}}\|_2$$

and, taking the infimum on all polynomials (in the variable y) of degree m, we get

$$\|v^{0,\alpha}(K - K_m^*)f\|_\infty \le \mathcal{C}\|fv^{\alpha,0}\|_\infty (\log m) \max_{|x|\le 1} v^{0,\alpha}(x) E_m(k_x)_{v^{0,-\frac{\alpha}{2}},2}.$$

Thus, under the assumption on k, using (5) we get

$$\|v^{0,\alpha}(K-K_m^*)f\|_\infty \le \mathcal{C}\|f\|_{Z_r^\infty(v^{\alpha,0})} \max_{|x|\le 1} v^{0,\alpha}(x)\|k_x\|_{Z_s^2(v^{0,-\frac{\alpha}{2}})} \frac{\log m}{m^s}$$

and then, using the equivalence (6), (42) follows. □

P r o o f o f T h e o r e m 4.1. Since

$$\Gamma_m = \sup_{|x|\le 1} v^{\alpha,0}(x)E_m(k_x)_{v^{0,\alpha},1} \le \mathcal{C} \sup_{|x|\le 1} v^{\alpha,0}(x)E_m(k_x)_{v^{0,-\frac{\alpha}{2}},2},$$

in virtue of the assumption on k_x, by (5), we get

$$\Gamma_m \le \frac{\mathcal{C}}{m^s} \sup_{|x|\le 1} v^{\alpha,0}(x)\|k_x\|_{Z_s^2(v^{0,-\frac{\alpha}{2}})}.$$

Then, proceeding as done for the proof of Lemma 2.1, under the assumption $Ker(D+K)=\{0\}$, the equation (2) admits a unique solution in $Z_r^\infty(v^{\alpha,0}), r<s$.

Moreover, taking into account Lemma 5.4, the remainder part of the proof is similar to the proof of Theorem 3.1. □

6 - Numerical Examples

Now we give some numerical tests. Of course, according to the theoretical estimates, the convergence order of the errors depends on the smoothness of the kernels and the right-hand sides, while the condition numbers depend on the construction of the systems which are equivalent to the finite dimensional equations.

In all the tables that follow we show the values of the weighed approximate solutions $v^{\alpha,0}f_m$ in two different points. When we do not know the exact solutions, we will think as exact their values obtained for $m=512$ and we will report only the digits which are correct according to them.

EXAMPLE 1. We first consider the following equation

$$-\frac{1}{\pi}\int_{-1}^{1} \frac{f(y)}{y-x}\sqrt{\frac{1-y}{1+y}}dy + \frac{\sqrt{2}}{2}\int_{-1}^{1} y^2\cos(x)\, f(y)\sqrt{\frac{1-y}{1+y}}dy = 1+\frac{\sqrt{2}\pi\cos(x)}{4},$$

whose exact solution is the function $f(x)=1$.

Since both the kernel and the right-hand side are very smooth, we compute the weighted approximate solutions $v^{\frac{1}{2},0}f_m$ by solving system (20). The convergence is very fast, in fact, it is sufficient to take $m=8$ to get approximations with 15 exact decimal digits. The condition number in uniform norm of the matrix A_m of the solved linear systems is less than 28.

EXAMPLE 2. Now we take the integral equation

$$\cos\left(\frac{7}{10}\pi\right) v^{\frac{7}{10},-\frac{7}{10}}(x)f(x) - \frac{\sin\left(\frac{7}{10}\pi\right)}{\pi}\int_{-1}^{1}\frac{f(y)}{y-x}v^{\frac{7}{10},-\frac{7}{10}}dy + \frac{1}{4}\int_{-1}^{1}\frac{|x-y|^{\frac{7}{2}}}{(5+x^2+y^2)^2}f(y)v^{\frac{7}{10},-\frac{7}{10}}(y)dy = \sin(1+x).$$

Its exact solution is unknown.

Also in this case we compute the weighted approximate solutions $v^{\frac{7}{10},0}f_m$ by solving system (20), but, since the kernel is less smooth than in the previous example, as shown in Table 1 and according to estimate (23), we need to increase m to get exact decimal digits. The condition number of the matrix A_m is less than 22.

Table 1.

m	$(v^{\frac{7}{10},0}f_m)(0.1)$	$(v^{\frac{7}{10},0}f_m)(-0.8)$
8	0.83449	1.23694
16	0.8344928	1.2369480
32	0.83449289	1.236948044
64	0.8344928964	1.2369480441
128	0.83449289646	1.23694804410
256	0.83449289646	1.23694804410

EXAMPLE 3. Finally, we consider the equation

$$\cos\left(\frac{2}{3}\pi\right) v^{\frac{2}{3},-\frac{2}{3}}(x)f(x) - \frac{\sin\left(\frac{2}{3}\pi\right)}{\pi}\int_{-1}^{1}\frac{f(y)}{y-x}v^{\frac{2}{3},-\frac{2}{3}}dy + \int_{-1}^{1}|x-y|^{-\frac{1}{4}}f(y)v^{\frac{2}{3},-\frac{2}{3}}(y)dy = (x^2+x)\cos(x),$$

whose exact solution is unknown.

Since here the kernel is weakly singular, we compute the weighted approximate solutions $v^{\frac{2}{3},0}f_m$ by solving system (27). In order to compute the Fourier coefficients we use the recurrence relation showed in [**20**]. In Table 2 we show that, according to estimate (28), it is necessary to take $m = 256$ to get approximations of the solution with 8 exact decimal digit. The condition number of the matrices of the solved linear systems is less than 49.

Table 2.

m	$(v^{\frac{2}{3},0}f_m)(0.1)$	$(v^{\frac{1}{10},0}f_m)(-0.5)$
8	0.7581	−0.046
16	0.7581	−0.04660
32	0.75810	−0.04660
64	0.7581070	−0.046605
128	0.7581070	−0.046605
256	0.758107081	−0.04660563

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