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Stefano Spirito

Solutions of the Navier-Stokes equations constructed by artificial compressibility approximation are suitable

Abstract. In this paper we prove that weak solution constructed by artificial compressibility method are suitable in the sense of Scheffer, [**18**], [**19**]. Using Hilbertian setting and Fourier transform with respect to the time we obtain non-trivial estimates for the pressure and the time derivative of the velocity field which allow us to pass into the limit.

Keywords. Navier-Stokes equations, suitable weak solutions.

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1 - Introduction

In this paper we investigate if the weak solutions of the Navier-Stokes equations are suitable in the sense of Scheffer [**18**], [**19**]. The incompressible Navier-Stokes equations in three spatial dimensions with unit viscosity and zero external force are given by the following system

$$\begin{cases} \partial_t u - \Delta u + (u \cdot \nabla) u + \nabla p = 0 \\ \operatorname{div} u = 0, \end{cases} \tag{1}$$

where $(x, t) \in \mathbb{R}^3 \times [0, T]$, $u \in \mathbb{R}^3$ denotes the velocity vector field and $p \in \mathbb{R}$ the pressure of the fluid. Let us recall the notion of Leray weak solution.

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Definition 1.1. *We say that* $u \in L^\infty((0,T);L^2(\mathbb{R}^3)) \cap L^2((0,T);\dot{H}^1(\mathbb{R}^3))$ *is a* Leray weak solution *of the Navier-Stokes equations if it satisfies* (1) *in the sense of distribution for all* $\psi \in C_0^\infty(\mathbb{R}^3 \times \mathbb{R})$ *with* $\operatorname{div}\psi = 0$ *and moreover the following energy inequality holds for every* $t \in [0,T]$

$$\int_{\mathbb{R}^3} dx|u(x,t)|^2 + 2\int_0^t ds \int_{\mathbb{R}^3} dx|\nabla u(x,s)|^2 \leq \int_{\mathbb{R}^3} dx|u(x,0)|^2. \tag{2}$$

In the mathematical literature there exist several proofs of the global existence of Leray weak solution for divergence-free initial data in $L^2(\mathbb{R}^3)$, see for example the book P. L. Lions [**14**] or the monograph of Témam [**27**]. Several problems about the Leray weak solutions are still open, for example it is not know whether or not the solutions are unique or develop singularities in a finite time also for smooth initial data. In fact the regularity requirements for proving uniqueness and global in time regularity are not available at the present time for Leray weak solutions. Scheffer in [**18**], [**19**] introduced the notion of suitable weak solutions that we recall here.

Definition 1.2. *Let* (u,p), $u \in L^2((0,T);H^1(\mathbb{R}^3)) \cap L^\infty((0,T);\, L^2(\mathbb{R}^3))$, $p \in \mathcal{D}'((0,T);\, L^2(\mathbb{R}^3))$, *be a weak solution to the Navier-Stokes equation* (1). *The pair* (u,p) *is said a* suitable weak solutions *if the following local energy balance*

$$\partial_t\left(\frac{1}{2}|u|^2\right) + \nabla\cdot\left(\left(\frac{1}{2}|u|^2 + p\right)u\right) - \Delta\left(\frac{1}{2}|u|^2\right) + |\nabla u|^2 - f\cdot u \leq 0 \tag{3}$$

holds in the distributional sense.

In mathematical literature (3) is called *generalized energy inequality*.
It is important to remark that weak solutions constructed by Leray are suitable. At the moment the best regularity result for the weak solution of the Navier-Stokes equations is a partial regularity result, i.e. the so called Caffarelli-Kohn-Nirenberg theorem, [**2**]. This theorem asserts that the one-dimensional parabolic Hausdorff measure of the singular set is zero and holds only for suitable weak solutions. If the class of suitable weak solutions is a proper class of Leray weak solution is an open problem since Scheffer's works. The method usually used to construct suitable weak solutions are regularization of the non-linear term, [**2**], adding hyper viscosity, [**1**]. Recently Guermond, [**9**] proved that some type of Faedo-Galerkin approximation lead to a suitable weak solution. He used classical Lions [**12**] method of fractional derivative in order to obtain maximal regularity in negative Sobolev space. In particular Guermond proved that Faedo-Galerkin weak solutions of the three dimen-

sional Navier-Stokes equations with Dirichlet boundary condition are suitable provided they are constructed using finite-dimensional approximation spaces having a discrete commutator property. In this paper we prove that Leray weak solutions constructed by the artificial compressibility method are suitable in the sense of the Definition 1.2. The artificial compressibility approximation was introduced by Chorin [**3, 4**], Oskolkov [**15**] and Témam [**25, 26**], in order to deal with the difficulty induced by the incompressibility constraints in the numerical approximation. The approximation system reads as follows

$$\begin{cases} \partial_t u^\varepsilon + \nabla p^\varepsilon = \Delta u^\varepsilon - (u^\varepsilon \cdot \nabla) u^\varepsilon - \dfrac{1}{2}(\operatorname{div} u^\varepsilon) u^\varepsilon \\ \varepsilon \partial_t p^\varepsilon + \operatorname{div} u^\varepsilon = 0, \end{cases} \tag{4}$$

where $(x,t) \in \mathbb{R}^3 \times [0,T]$, $u^\varepsilon = u^\varepsilon(x,t) \in \mathbb{R}^3$ and $p^\varepsilon = p^\varepsilon(x,t) \in \mathbb{R}$, $f^\varepsilon = f^\varepsilon(x,t) \in \mathbb{R}^3$.

Témam showed the convergence of this approximation on bounded domain. Recently in [**6**] and [**7**] the result was extended in the case of the whole space and exterior domain, respectively. In [**6**] the converge towards Leray weak solutions of the Navier-Stokes equations is achieved by using the dispersive structure of the system. Here we will consider the system (4) endowed with the following initial conditions

$$u^\varepsilon(x,0) = u_0^\varepsilon(x) \qquad p^\varepsilon(x,0) = p_0^\varepsilon(x), \tag{5}$$

such that

$$u_0^\varepsilon \to u_0 \text{ in } L^2(\mathbb{R}^3) \qquad \text{as } \varepsilon \to 0, \tag{6}$$

$$\sqrt{\varepsilon} p_0^\varepsilon \to 0 \quad \text{in } L^2(\mathbb{R}) \qquad \text{as } \varepsilon \to 0. \tag{7}$$

As we will see later on in Section 3, in order to get some a priori estimate on u^ε and p^ε we need to assume that

$$u_0^\varepsilon \in \dot{H}^1(\mathbb{R}^3). \tag{8}$$

We will be able to prove the following theorem

Theorem 1.1. *Let* $(u^\varepsilon, p^\varepsilon)$ *be a weak solution of the system* (4) *with initial data* (5) *and such that* (6), (7) *and* (8) *are satisfied. Then* $(u^\varepsilon, p^\varepsilon)$ *converges to a suitable weak solution of the Navier-Stokes system as* ε *goes to zero.*

For proving this theorem we have to estimate carefully the pressure term p^ε and $\sqrt{\varepsilon}\, p^\varepsilon$. Since we haven't the incompressibility constraint we cannot use classical method based on the elliptic equation associated to the pressure. Also the dispersive approach, as in [**6**], cannot be used. We will use the method of Lions of the fractional derivative in order to get the necessary estimates. The paper is organized as follows. In the Section 2 we recall some basic fact about Navier-Stokes equations and the artificial compressibility method and fix some notations. In Section 3 we obtain the *a priori* estimates which allow us to pass into the limit. In Section 4 we give the proof of the main result. We want to point out that the estimates of the Lemma 3.3 are the core of this paper.

2 - Notations and preliminary results

For convenience of the reader we establish some notations and recall some basic fact about the artificial compressibility approximation.

2.1 - *Notations*

We will denote by $\mathcal{D}(\mathbb{R}^d \times \mathbb{R}_+)$ the space of test function $C_0^\infty(\mathbb{R}^d \times \mathbb{R}_+)$, by $\mathcal{D}'(\mathbb{R}^d \times \mathbb{R}_+)$ the space of Schwartz distributions and $\langle \cdot, \cdot \rangle$ the duality bracket between $\mathcal{D}'$ and $\mathcal{D}$ and by $\mathcal{M}_t X'$ the space $C_c^0([0,T];X)'$. The inner product in $L^2(\mathbb{R}^d)$ will be denoted by parentheses, namely $(u,v) := \int_{\mathbb{R}^d} u(x)\bar{v(x)}dx$. Moreover $W^{k,p}(\mathbb{R}^d) = (I - \Delta)^{-\frac{k}{2}} L^p(\mathbb{R}^d)$ and $H^k(\mathbb{R}^d) = W^{k,2}(\mathbb{R}^d)$ denote the non homogeneous Sobolev spaces for any $1 \le p \le \infty$ and $k \in \mathbb{R}$. $\dot{W}^{k,p}(\mathbb{R}^d) = (-\Delta)^{\frac{k}{2}} L^p(\mathbb{R}^d)$ and $\dot{H}^k(\mathbb{R}^d) = W^{k,2}(\mathbb{R}^d)$ denote the homogeneous Sobolev spaces.

Let H be a Hilbert space, we define $H^\gamma(\mathbb{R}; H)$, $\gamma \in \mathbb{R}$ the space of tempered distributions $v \in S'(\mathbb{R}; H)$ such that

$$\int_{\mathbb{R}} (1 + |k|)^{2\gamma} \|\tilde{v}\|_H^2 dk < +\infty$$

where $\tilde{v}$ is the Fourier transform with respect to the time of v. The space $H^\gamma((0,T); H)$ is defined by those distributions that can be extended to $S'(\mathbb{R}; H)$ and whose extension is in $H^\gamma((0,T); H)$. The norm in $H^\gamma((0,T); H)$ is the quotient norm. Moreover, we shall denote by Q and P respectively the Leray's projectors Q on the space of gradients vector fields and P on the space of divergence - free vector fields, namely

$$Q = \nabla \Delta^{-1} div \qquad P = I - Q.$$

Finally, The notations $L^p(L^q)$, $L^p(W^{k,q})$, $L^p(H^s)$ and $H^r(H^s)$ will abbreviate respectively the spaces $L^p([0,T]; L^q(\mathbb{R}^d))$, $L^p([0,T]; W^{k,q}(\mathbb{R}^d))$, $L^p([0,T]; H^s(\mathbb{R}^d))$ and $H^r([0,T]; H^s(\mathbb{R}^d))$.

2.2 - *Artificial compressibility approximation*

In this section we recall some previous result about the approximating system that we rewrite for convenience of the reader.

$$\begin{cases} \partial_t u^\varepsilon + \nabla p^\varepsilon = \Delta u^\varepsilon - (u^\varepsilon \cdot \nabla) u^\varepsilon - \dfrac{1}{2}(\operatorname{div} u^\varepsilon) u^\varepsilon \\ \varepsilon \partial_t p^\varepsilon + \operatorname{div} u^\varepsilon = 0. \end{cases} \tag{9}$$

As said in the Introduction we consider two initial condition, namely

$$u^\varepsilon(x,0) = u_0^\varepsilon(x) \qquad p^\varepsilon(x,0) = p_0^\varepsilon(x), \tag{10}$$

such that

$$u_0^\varepsilon \to u_0 \text{ in } L^2(\mathbb{R}^3) \qquad \text{as } \varepsilon \to 0 \tag{11}$$

$$\sqrt{\varepsilon} p_0^\varepsilon \to 0 \quad \text{in } L^2(\mathbb{R}) \qquad \text{as } \varepsilon \to 0. \tag{12}$$

In [**6**] it was showed the following result.

Theorem 2.1. *Let $(u^\varepsilon, p^\varepsilon)$ be a sequence of weak solutions in $\mathbb{R}^3$ of the system* (9), *with the initial data* (10). *Let us suppose that* (11) *and* (12) *are satisfied. Then there exists $u \in L^\infty(L^2) \cap L^2(\dot{H}^1)$ such that $u^\varepsilon \rightharpoonup u$ weakly in $L^2(\dot{H}^1)$. The gradient component Qu^ε of the vector field u^ε satisfies $Qu^\varepsilon \to 0$ strongly in $L^2(L^p)$, for any $p \in [4,6)$, while the divergence free component Pu^ε of the vector field u^ε satisfies $Pu^\varepsilon \to Pu$ strongly in $L^2(L^2_{loc})$. Moreover, $u = Pu$ is a Leray weak solution of the incompressible Navier-Stokes equations.*

3 - A priori estimates

In this section we deal with the *a priori* bounds for our approximating system. Let be $p, q, \bar{r}$ and s real numbers such that the following relations hold:

$$\frac{2}{p} + \frac{3}{q} = 4, \quad p \in [1,2], \quad q \in \left[1, \frac{3}{2}\right], \quad \frac{s}{3} := \frac{1}{q} - \frac{1}{2}, \quad \bar{r} := \frac{1}{p} - \frac{1}{2}. \tag{13}$$

These relations follow from the Sobolev embedding theorems, in particular if p, q satisfy (13) then the following embedding holds

$$L^p((0,T); L^q(\mathbb{R}^3)) \subset H^{-r}((0,T); H^{-s}(\mathbb{R}^3)). \tag{14}$$

The first *a priori* estimate regards the nonlinear terms

Lemma 3.1. *Let u^ε be a weak solutions of the system* (9), *then:*

- *there exists a constant $c_1 > 0$, independent on ε, such that*

$$\sup_{x\in(0,T)} \|u^\varepsilon(t)\|_{L^2} + \|u^\varepsilon\|_{L^2(\dot{H}^1)} \leq c_1 \tag{15}$$

- *there exists a constant $c_2 > 0$, independent on ε, such that*

$$\left\|(u^\varepsilon \cdot \nabla)u^\varepsilon + \frac{1}{2}u^\varepsilon \operatorname{div} u^\varepsilon\right\|_{H^{-r}(\dot{H}^{-s})} \leq c_2. \tag{16}$$

The proof of (15) follows by standard energy method and the proof of (16) follows by an interpolation argument and the embedding (14).

Since we are going to use Fourier transform with respect to time we need to extend all the function from $[0, T]$ to $\mathbb{R}$. We define by $\overline{u^\varepsilon}$ the following extension of u^ε

$$\overline{u^\varepsilon} = \begin{cases} (t+1)u_0^\varepsilon & on\ [-1, 0] \\ u^\varepsilon & (0, T+1) \\ 0 & on\ [T+1, \infty]. \end{cases}$$

Let $\varphi \in C^\infty(\mathbb{R})$ be such that $supp(\varphi) \subset (-1, T+1)$ and $\varphi \equiv 1$ on $[0, T]$, we denote with abuse of notation

$$u^\varepsilon = \varphi\overline{u}^\varepsilon.$$

Next we define the following function

$$f^\varepsilon = \begin{cases} (1+t)\varphi' u_0^\varepsilon + \varphi u_0^\varepsilon - (1+t)\varphi\left((u^\varepsilon \cdot \nabla)u^\varepsilon + \frac{1}{2}u^\varepsilon \operatorname{div} u^\varepsilon\right) & t \in (-1, 0) \\ -\varphi\left((u^\varepsilon \cdot \nabla)u^\varepsilon + \frac{1}{2}u^\varepsilon \operatorname{div} u^\varepsilon\right) + \varphi' u^\varepsilon & t \notin (-1, 0). \end{cases}$$

It follows that u^ε e f^ε are well defined on $(-\infty, +\infty)$. By using (8) it follows that there exist $c > 0$, independent on ε, such that

$$\|f^\varepsilon\|_{H^{-r}(\dot{H}^{-s})} \leq c. \tag{17}$$

Lemma 3.2. *Let $(u^\varepsilon, p^\varepsilon)$ be a solution of the system* (9), *there exists $c > 0$, independent on ε, such that*

- *for all $\alpha \in \left[\frac{1}{4}, \frac{1}{2}\right)$ and $\tau < \overline{\tau} = \frac{2}{5}(1+\alpha)$, one has*

$$\|u^\varepsilon\|_{H^{-\tau}(\dot{H}^{-\alpha})} \leq c \tag{18}$$

- *for all $s \in \left[\frac{1}{2}, \frac{3}{2}\right)$ and $r > \overline{r}$, one has*

$$\|\partial_t u^\varepsilon\|_{H^{-r}(\dot{H}^{-s})} \leq c, \tag{19}$$

$$\|\Delta u^\varepsilon\|_{H^{-r}(\dot{H}^{-s})} \leq c. \tag{20}$$

Proof. We start by proving (18). By taking the Fourier transform with respect to the time of the system (9) we obtain

$$\begin{cases} 2i\pi k\tilde{u}^\varepsilon - \Delta\tilde{u}^\varepsilon + \nabla\tilde{p}^\varepsilon = \tilde{f}^\varepsilon \\ \varepsilon 2i\pi k\tilde{p}^\varepsilon + \operatorname{div}\tilde{u}^\varepsilon = 0. \end{cases} \tag{21}$$

Let $\alpha > 0$, we multiply the first equation of (21) by the complex conjugated of $-\Delta^{-\alpha}\tilde{u}^\varepsilon$ and the second by the complex conjugate of $-\Delta^{-\alpha}\tilde{p}^\varepsilon$. By summing up and by taking the imaginary part we obtain the following inequality

$$|k||(\tilde{u}^\varepsilon, \Delta^{-\alpha}\tilde{u}^\varepsilon)| \leq |(\tilde{f}^\varepsilon, \Delta^{-\alpha}\tilde{u}^\varepsilon)|^2. \tag{22}$$

Now we choose α such that $\alpha \leq s \leq 1 + 2\alpha$. Since $s \in \left[\frac{1}{2}, \frac{3}{2}\right)$ we have that $\alpha \in \left[\frac{1}{4}, \frac{1}{2}\right)$. By interpolation we get

$$\|\tilde{u}^\varepsilon\|_{\dot{H}^s} \leq \|\tilde{u}^\varepsilon\|^{\gamma}_{\dot{H}^\alpha}\|\tilde{u}^\varepsilon\|^{1-\gamma}_{\dot{H}^{1+2\alpha}}, \tag{23}$$

where $\gamma = \dfrac{2\alpha + 1 - s}{1 + \alpha}$. Inserting (23) in (22) we have

$$|k|\|\tilde{u}^\varepsilon\|^{2-\gamma}_{H^{-\alpha}} \leq c\|\tilde{f}_\varepsilon\|_{H^{-s}}\|\tilde{u}^\varepsilon\|^{1-\gamma}_{H^1}. \tag{24}$$

We set $\nu = \dfrac{2\gamma}{2-\gamma}$, then we have

$$|k|^{\frac{2}{2-\gamma}-\nu}\|\tilde{u}^\varepsilon\|^2_{H^{-\alpha}} \leq c(1 + |k|)^{-\nu}\|\tilde{f}^\varepsilon\|^{\frac{2}{2-\gamma}}_{H^{-s}}\|\tilde{u}^\varepsilon\|^{\frac{2(1-\gamma)}{2-\gamma}}_{H^1}. \tag{25}$$

By integrating (25) with respect to the time and by using Hölder inequality we get

$$\int_{\mathbb{R}} dk|k|^{\frac{2}{2-\gamma}}\|\tilde{u}^\varepsilon\|^2_{H^{-\alpha}} \leq c\|f^\varepsilon\|^{\frac{2}{2-\gamma}}_{H^{-r}(H^{-s})}\|\tilde{u}^\varepsilon\|^{\frac{2(1-\gamma)}{2-\gamma}}_{L^2(H^1)}. \tag{26}$$

If we set $\bar{\tau} := \dfrac{1+\alpha}{1+s}(1 - \bar{r})$ then (18) follows.

The estimates (19) and (20) can be proved by using the same argument. See [8] for further details. □

The following lemma is the core of the paper.

Lemma 3.3. *Let* $(u^\varepsilon, p^\varepsilon)$ *be a weak solution of* (9), *there exist a constant* $c > 0$, *independent on* ε, *such that for* $s \in \left[\frac{1}{2}, \frac{3}{2}\right)$, $r > \bar{r} = \frac{3}{4} - \frac{s}{2}$, $\delta \in \left(0, \frac{1}{12}\right)$ *and* $\beta \in \left(0, \frac{1-12\delta}{10}\right)$ *we have*

$$\|p^\varepsilon\|_{H^{-r}(\dot{H}^{1-s})} \leq c \tag{27}$$

and

$$\|\sqrt{\varepsilon}p^{\varepsilon}\|_{\dot{H}^{\frac{1}{2}+\frac{\beta}{4}}(H^{-\frac{1}{2}+\delta})} \leq c. \tag{28}$$

Proof. The estimate (27) can be proved by observing that from the first equation of the system (9) we have

$$\nabla p^{\varepsilon} = f^{\varepsilon} + \Delta u^{\varepsilon} - \partial_t u^{\varepsilon} \tag{29}$$

and by using (19), (20) and (17). Now we are going to prove (28). Let us consider the second equation of (21)

$$2\pi i k \varepsilon \tilde{p^{\varepsilon}} + \operatorname{div} u^{\varepsilon} = 0. \tag{30}$$

By multiplying (30) by $\Delta^{-\frac{1}{2}+\delta}\tilde{p^{\varepsilon}}$ we obtain

$$|k|\,\|\sqrt{\varepsilon}\tilde{p^{\varepsilon}}\|^2_{H^{-\frac{1}{2}+\delta}} \leq c|(\Delta^{-\frac{1}{2}+\delta}u^{\varepsilon}, \nabla p^{\varepsilon})|. \tag{31}$$

From (31) we have

$$|k|^{1+\frac{\beta}{2}}\|\sqrt{\varepsilon}p^{\varepsilon}\|_{H^{-\frac{1}{2}+\delta}} \leq (1+|k|)^{\frac{1}{2}+\beta}\|\tilde{u^{\varepsilon}}\|_{H^{-\frac{1}{2}+3\delta}} \frac{\|\tilde{p^{\varepsilon}}\|_{\dot{H}^{\frac{1}{2}-\frac{\delta}{2}}}}{(1+|k|)^{\frac{1}{2}+\frac{\beta}{2}}}, \tag{32}$$

where $\beta \in \left(0, \dfrac{1-12\delta}{10}\right)$. By integrating with respect to the time (32) and by using the Hölder inequality we have

$$\|\sqrt{\varepsilon}p^{\varepsilon}\|_{\dot{H}^{\frac{1}{2}+\frac{\beta}{4}}(H^{-\frac{1}{2}+\delta})} \leq \|u^{\varepsilon}\|_{\dot{H}^{\frac{1}{2}+\beta}(H^{-\frac{1}{2}+3\delta})}\|p^{\varepsilon}\|_{H^{-\frac{1}{2}-\frac{\beta}{2}}(\dot{H}^{\frac{1}{2}-\delta})}. \tag{33}$$

The right hand-side of (33) is bounded by using (18) and (27), provided $\delta \in \left(0, \dfrac{1}{12}\right)$ and $\beta \in \left(0, \dfrac{1-12\delta}{10}\right)$.

4 - Convergence to a suitable weak solution

In this section we give the proof of the Theorem 1.1.

Let us multiply the first equation of (9) by $u^{\varepsilon}\phi$ with $\phi \in C_0^{\infty}(\mathbb{R}^3 \times \mathbb{R})$, $\phi > 0$, then we have,

$$\begin{aligned} &\int_0^T dt(\partial_t u^{\varepsilon}, u^{\varepsilon}\phi) - \int_0^T dt(\Delta u^{\varepsilon}, u^{\varepsilon}\phi) = -\int_0^T dt(u^{\varepsilon}\cdot\nabla u^{\varepsilon}, u^{\varepsilon}\phi) \\ &-\int_0^T dt\frac{1}{2}(u^{\varepsilon}\operatorname{div} u^{\varepsilon}, u^{\varepsilon}\phi) - \int_0^T dt(\nabla p^{\varepsilon}, u^{\varepsilon}\phi). \end{aligned} \tag{34}$$

By integrating by parts we obtain

$$\begin{aligned}(35)\qquad &\int_0^T dt(|\nabla u^\varepsilon|^2,\phi) \le \int_0^T dt\frac{|u^\varepsilon|^2}{2}(\phi_t + \Delta\phi)\\ &\int_0^T dt(u^\varepsilon\frac{|u^\varepsilon|^2}{2},\nabla\phi) + \int_0^T dt(u^\varepsilon p^\varepsilon,\nabla\phi) + \int_0^T dt(p^\varepsilon \operatorname{div} u^\varepsilon,\phi).\end{aligned}$$

We estimate each term of (35) separately.

By weak lower semicontinuity and the fact that $u^\varepsilon \rightharpoonup u$ weakly in $L^2(\dot{H}^1)$ we have that

$$(36)\qquad \int_0^T dt(|\nabla u|^2,\phi) \le \liminf_{\varepsilon\to 0}\int_0^T dt(|\nabla u^\varepsilon|^2,\phi).$$

Since $u^\varepsilon \to u$ strongly in $L^2(L^2_{loc})$, we get

$$(37)\qquad \int_0^T dt\left(\frac{|u^\varepsilon|^2}{2},\phi_t + \Delta\phi\right) \to \int_0^T dt\left(\frac{|u|^2}{2},\phi_t + \Delta\phi\right) \qquad \text{as } \varepsilon \to 0.$$

Next, by interpolation we have that $u^\varepsilon \to u$ strongly in $L^2(L^3)$ and that u^ε is bounded in $L^4(L^3)$, so it follows

$$(38)\qquad \int_0^T dt(u^\varepsilon\frac{|u^\varepsilon|^2}{2},\nabla\phi) \to \int_0^T dt(u\frac{|u|^2}{2},\phi) \qquad \text{as } \varepsilon \to 0.$$

In order to estimate the last two terms in (35) we have to use carefully the estimates of the Lemma 3.2 and Lemma 3.3. We start by estimating

$$\int_0^T dt(u^\varepsilon p^\varepsilon,\nabla\phi).$$

Let $\eta > 0$, we set $r = \frac{2}{5} + \eta$, $s = \frac{3}{10}$. This choice implies that $\|p^\varepsilon\|_{H^{-r}(H^s)}$ is uniformly bounded. So $p^\varepsilon \rightharpoonup p$ weakly in $H^{-r}((0,T),H^s_{loc}(\mathbb{R}^3))$. Now let $\eta' \in [0,\frac{1}{20}]$ and set $\alpha = \frac{3}{10} - \eta' \le \frac{1}{4}$, $\tau = \frac{2}{5}(1+\alpha-\varepsilon')$. This choice implies that $\|u^\varepsilon\|_{H^\tau(H^{-\alpha})}$ is uniformly bounded and as a consequence $u^\varepsilon \rightharpoonup u$ weakly in $H^\tau((0,T);H^{-\alpha})$. By standard compactness lemma we have that the immersion of $H^\tau(H^{-\alpha})$ in $H^r((0,T);H^{-s}(\mathbb{R}^3))$ is compact provided $\tau > r$ and $s > \alpha$. Since $\eta' \in \left(0,\frac{1}{20}\right)$ we have $s > \alpha$, and if we choose $\eta = \frac{1}{50}(3 - 20\eta')$ we have $\tau > r$ and so we obtain that $u^\varepsilon \to u$ strongly in $H^r((0,T); H^{-s}_{loc}(\mathbb{R}^3))$. Now it follows easily that

$$\int_0^T dt(p^\varepsilon u^\varepsilon, \nabla\phi) \to \int_0^T dt(pu, \nabla\phi) \qquad \text{as } \varepsilon \to 0. \tag{39}$$

By using the second equation of (9) we have that

$$\begin{aligned} \left| \int_0^T dt(p^\varepsilon, \operatorname{div}\phi) \right| &\leq \sqrt{\varepsilon} \left| \int_0^T dt(\sqrt{\varepsilon}p^\varepsilon, p^\varepsilon \phi_t) \right| \\ &\leq \sqrt{\varepsilon} \|\sqrt{\varepsilon}p^\varepsilon\|_{\dot{H}^{\frac{1}{2}+\frac{\beta}{2}}(H^{-\frac{1}{2}+\delta})} \|p^\varepsilon\|_{H^{-\frac{1}{2}-\beta}(\dot{H}^{\frac{1}{2}-\delta})} . \end{aligned}$$

So by choosing δ and β as in the Lemma 3.3 we have that

$$\int_0^T dt(p^\varepsilon, \operatorname{div}\phi) \to 0 \qquad \text{as } \varepsilon \to 0.$$

References

[1] H. Beirão da Veiga, *On the suitable weak solutions to the Navier-Stokes equations in the whole space*, J. Math. Pures Appl. **64** (1985), 77-86.

[2] L. Caffarelli, R. Kohn and L. Nirenberg, *Partial regularity of suitable weak solutions of the Navier-Stokes equations*, Comm. Pure Appl. Math. **35** (1982), 771-831.

[3] A. J. Chorin, *Numerical solution of the Navier-Stokes equations*, Math. Comp. **22** (1968), 745-762.

[4] A. J. Chorin, *On the convergence of discrete approximations to the Navier-Stokes equations*, Math. Comp. **23** (1969), 341-353.

[5] B. Desjardins, E. Grenier, P.-L. Lions and N. Masmoudi, *Incompressible limit for solutions of the isentropic Navier-Stokes equations with Dirichlet boundary conditions*, J. Math. Pures Appl. **78** (1999), 461-471.

[6] D. Donatelli and P. Marcati, *A dispersive approach to the artificial compressibility approximations of the Navier-Stokes equations in* 3*D*, J. Hyperbolic Differ. Equ. **3** (2006), 575-588.

[7] D. Donatelli and P. Marcati, *Leray weak solutions of the Incompressible Navier Stokes system on exterior domains via the artificial compressibility method*, Preprint 2009, arXiv:0902.0476v1 [math AP].

[8] D. Donatelli and S. Spirito, *Weak solutions of Navier-Stokes equations*

constructed by artificial compressibility method are suitable, Preprint 2010, arXiv:1001.4484v1 [math.AP].

[9] J. L. GUERMOND, *Faedo-Galerkin weak solutions of the Navier-Stokes equations with Dirichlet boundary conditions are suitable*, J. Math. Pures Appl. 88 (2007), 87-106.

[10] J.-L. LIONS, *Quelques méthodes de résolution des problèmes aux limites non linéaires*, Dunod, Gauthier-Villars, Paris 1969.

[11] J.-L. LIONS, *On some problems connected with Navier-Stokes equations*, Nonlinear evolution equations (Proc. Sympos., Univ. Wisconsin, Madison, Wis., 1977), Publ. Math. Res. Center Univ. Wisconsin **40** (Academic Press, New York-London 1978), 59-84.

[12] J.-L. LIONS, *Sur l'existence de solutions des équations de Navier-Stokes*, C. R. Acad. Sci. Paris **248** (1959), 2847-2849.

[13] P.-L. LIONS and N. MASMOUDI, *Incompressible limit for a viscous compressible fluid*, J. Math. Pures Appl. (9) **77** (1998), 585-627.

[14] P.-L. LIONS, *Mathematical topics in fluid mechanics. Vol. 1. Incompressible models*, Oxford Science Publications, The Clarendon Press, Oxford Univ. Press, New York 1996.

[15] A. P. OSKOLKOV, *A certain quasilinear parabolic system with small parameter that approximates a system of Navier-Stokes equations*, Zap. Naučn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI) **21** (1971), 79-103.

[16] A. PROHL, *Projection and quasi-compressibility methods for solving the incompressible Navier-Stokes equations*, Advances in Numerical Mathematics, B. G. Teubner, Stuttgart 1997.

[17] R. RANNACHER, *On Chorin's projection method for the incompressible Navier-Stokes equations*, The Navier-Stokes equations II-theory and numerical methods (Proc. 2nd Conference, Oberwolfach, 1991), Lecture Notes in Math., **1530** (Springer-Verlag, Berlin 1992), 167-183.

[18] V. SCHEFFER, *Partial regularity of solutions to the Navier-Stokes equations*, Pacific J. Math. **66** (1976), 535-552.

[19] V. SCHEFFER, *Hausdorff measure and the Navier-Stokes equations*, Comm. Math. Phys. **55** (1977), 97-112.

[20] V. SCHEFFER, *The Navier-Stokes equations in space dimension four*, Comm. Math. Phys. **61** (1978), 41-68.

[21] V. SCHEFFER, *The Navier-Stokes equations on a bounded domain*, Comm. Math. Phys. **73** (1980), 1-42.

[22] J. SERRIN, *The initial value problem for the Navier-Stokes equations*, Nonlinear Problems (Proc. Sympos., Madison, Wis.), Univ. of Wisconsin Press, Madison, Wis. 1963, 69-98.

[23] J. SIMON, *Compact sets in the space* $L^p(0,T;B)$, Ann. Mat. Pura Appl. **146** (1987), 65-96.

[24] E. M. STEIN, *Harmonic analysis: real-variable methods, orthogonality, and oscillatory integrals*, Princeton University Press, Princeton, NJ 1993.

[25] R. TÉMAM, *Sur l'approximation de la solution des équations de Navier-Stokes par la méthode des pas fractionnaires. I*, Arch. Rational Mech. Anal. **32** (1969), 135-153.

[**26**] R. Témam, *Sur l'approximation de la solution des équations de Navier-Stokes par la méthode des pas fractionnaires. II*, Arch. Rational Mech. Anal. **33** (1969), 377-385.

[**27**] R. Témam, *Navier-Stokes equations. Theory and numerical analysis*, Reprint of the 1984 edition, AMS Chelsea Publishing, Providence, RI 2001.

[**28**] N. N. Yanenko, B. G. Kuznetsov and Sh. Smagulov, *On the approximation of the Navier-Stokes equations for an incompressible fluid by evolutionary-type equations*, Numerical methods in fluid dynamics, "Mir", Moscow 1984, 290-314.

Stefano Spirito
University of L'Aquila
Via Vetoio, 1
Coppito, 67100, Italy
e-mail: stefano.spirito@dm.univaq.it