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A dive into shallow water

Abstract. In this tutorial, we attempt to furnish a basic introduction on shallow water modeling with specific attention to Saint-Venant equations. We propose a selection of results, including derivation of the model, well-posedness of the Cauchy problem, existence and stability of roll-waves, kinetic formulation and the corresponding hydrodynamical limit, presented, whenever possible, in a simplified way and designed mainly for readers that are not expert in the field.

Keywords. Shallow water equations, hyperbolic-parabolic systems, roll-waves, kinetic equations.

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Contents

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1 - Introduction

The interest of the mathematical community in Shallow Water equations have been constantly increasing in the last fifty years. A parameter to roughly estimate such growing attention is the ratio of the articles containing the words “shallow water” in the title to the total publications, as registered in MathSciNet database: this index has been approximately equal 6×10^{-5} in the 50s and in the 60s and the increased to 8.6×10^{-5} in the 70s, to 2.8×10^{-4} in the 80s, to 4.4×10^{-4} in the 90s, and to 8.8×10^{-4} in the last ten years (the last two years exhibit an evident upward trend, as the same parameter amounts to 1.2×10^{-3} and 1.0×10^{-3}, in 2008 and 2009, respectively)[1]. In this respect, the Shallow Water models analysis has recently trespassed honorable “competitors” as Boltzmann and Hamilton–Jacobi research areas (see Fig. 1). Such great attrac-

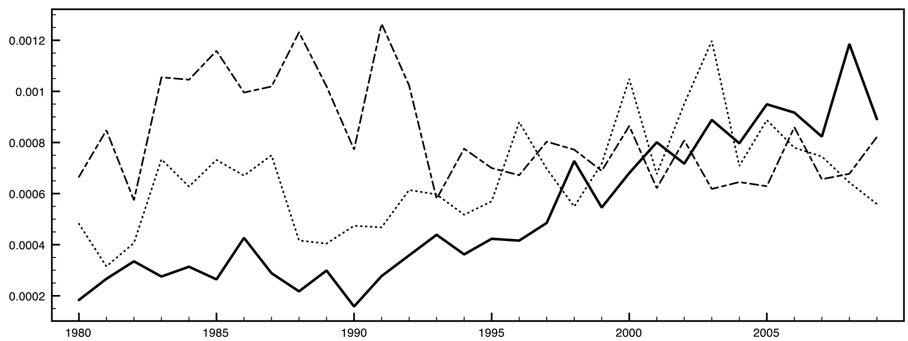

Fig. 1. The ratio of the publications on the subject to the total number of publications as registered in MathSciNet database year per year relative to the period 1980–2009: Shallow Water (continuous line), Boltzmann (dashed line), Hamilton–Jacobi (dotted line).

[1] For comparison, both the Navier–Stokes and Schrödinger equations has been stably over the level 2×10^{-3} in the last twenty years, and at about 4×10^{-3} in the last three.

tion has different roots, somewhat summarised in the fact that shallow water models settle into a lively cross-road where real phenomena and affordable theoretical studies meet and positively interact.

The term shallow refers to the fact that the water/fluid dynamics under investigation satisfy the assumption that the horizontal length scale L is much greater than the height length scale H, so that the ratio $\varepsilon = H/L$ can be considered small and, in the limit $\varepsilon \to 0$, simplified equations are formally derived. The first and simpler mathematical model has been proposed by Adhémar Jean Claude Barré de Saint-Venant (1797–1886) and published in [**46**]. The model consists in a hyperbolic system of two partial differential equations with a structure that is the same of the system for isentropic gas-dynamics in Eulerian coordinates in the case of pressure with a power-law form with exponent equal to 2. Natural modifications of the model emerge when additional physical effects are taken into account: viscosity, friction, Coriolis force... As a consequence, the shallow water equations appear to be widely applicable to describe different situations in fluid dynamics (river and channels flows, atmospheric and oceanic motions...), partially explaining the large amount of publications in the field.

This tutorial aim to trace an introductory path intended for not-expert and, thus, the content is self-contained and simplified, whenever possible. Four different areas have been selected: derivation of the equations (Sec. 2), well-posedness of the Cauchy problem (Sec. 3), existence and stability of roll-waves (Sec. 4), kinetic formulation and hydrodynamical limit (Sec. 5). The results presented here are slightly modified versions of published ones; the reader interested in a deeper dive into the subject is recommended to refer to the original articles (in particular, [**28, 36, 2**]). For recently published Lecture Notes on the same subject, we flag ref.[**5**].

2 - Derivation of the model

In the analysis of flow of incompressible fluids, two different situations are usually distinguished: the pipe flow and the open channel flow. In the former case, the geometry of the region occupied by the fluid is fixed and regarded as a datum of the problem; in the latter, the one that we will consider here, the liquid flows with a free surface and the geometry of the domain, where equations should be solved, is an unknown.

Analyzing a model over a domain with variable shape is an additional difficulty with respect to the case of a fixed form, and any reasonable reduction able to circumvent such obstacle is tempting. The shallow water assumption gives one of such reductions: when the open channel flow has a vertical scale that is small relatively to the horizontal one, it is possible to derive simplified equations by averaging the vertical variable and disregarding appropriate terms as consequence of the smallness assumption. The first derivation of such system has been performed by Saint-Venant (who also gave fundamental contribution to stress analysis), at the age of seventy-four years [**46**]; since then, the simpler model for shallow water, consisting in a system of hyperbolic conservation laws, is called Saint-Venant system. Depending on assumptions and approximations, shallow water models may also contain other terms and give raise to different type of partial differential equations. Let us also stress that the smallness assumption on the ratio between vertical and horizontal scale is closely related to the one used to derive modeling equations for thin films (see [**40**] for a detailed account on such kind of phenomena from a physical perspective).

Here, we limit ourselves to the simpler case proposing two different formal derivations. Firstly, we deduce the shallow water system starting from first principles: conservation of mass and momentum. Since such derivation is based on a number of crude simplifications, we consider a different approach stemming from a formal (but sound) reduction of the Navier-Stokes equations for incompressible fluids. In this second approach, we will also consider how the equations are modified in the presence of viscosity; on the contrary, for the sake of simplicity, we will not examine the terms relative to the presence of friction and Coriolis force, that are also widely considered in the literature. For completeness, let us also stress that shallow water model may also be obtained starting from the weak formulation of Navier-Stokes equations with free surface, as recently showed in [**8**] in the case of Bingham fluids.

By all accounts, the rigorous mathematical justification of shallow water equations starting from full incompressible Navier Stokes equations with free surface has been obtained very recently in [**10, 9**], the main differences between the two articles being related on boundary conditions at the bottom and at the free surface; specifically, [**10**] refers to no-slip conditions at the bottom and surface tension forces at the free surface, while [**9**] considers Navier conditions at the bottom and zero surface tension. We refer to the original articles for details on how such convergence result can be proved.

2.1 - *A shortcut: departing from first principles*

The simpler model for shallow water consists in a single layer of flowing water with vertical height small with respect to the horizontal dimensions. We assume the bottom, located at $z = 0$, is covered with a stationary solid layer of height $Z = Z(x, y)$ over which the water, with constant density ρ, flows. The height of the water column is described by the variable $h = h(x, y, t)$, so that the height of water surface is given by the sum $\xi := h + Z$. The unknown variables are the height h and the velocity $\boldsymbol{v}$ of the water and they are coupled by the relations arising from conservation of mass and balance of momentum.

Conservation of mass. As consequence of the reduced size of the vertical dimension, we assume that the velocity $\boldsymbol{v}$ of the water depends only on the horizontal variable (and time, of course), $\boldsymbol{v} = \boldsymbol{v}(x, y, t)$, and have identically zero vertical component. Consider a column of water situated over the square Q centered at the point (x, y) with sides ℓ_i of length $L > 0$. The mass contained in the column is (approximately) given by $M = \rho L^2 h(x, y, t)$; the time rate change of mass is due to the flux through the boundary of the column

$$\sum_{\ell_i\in\partial Q} \rho\, h\, L\, \boldsymbol{v}\cdot\boldsymbol{n}\Big|_{\ell_i} = -\rho L\Big\{h\boldsymbol{v}\cdot\boldsymbol{i}\left(x+\frac{L}{2},y,t\right)-h\boldsymbol{v}\cdot\boldsymbol{i}\left(x-\frac{L}{2},y,t\right) + h\boldsymbol{v}\cdot\boldsymbol{j}\left(x,y+\frac{L}{2},t\right)-h\boldsymbol{v}\cdot\boldsymbol{j}\left(x,y-\frac{L}{2},t\right)\Big\}$$

where $\boldsymbol{n}$ indicate the outward normal to the side ℓ_i and $\boldsymbol{i}$,$\boldsymbol{j}$ indicate the normal vectors $(1, 0, 0), (0, 1, 0)$, respectively. Thus, we obtain the balance equation

$$\frac{\partial h}{\partial t} + \frac{1}{L}\Big\{h\boldsymbol{v}\cdot\boldsymbol{i}\left(x+\frac{L}{2},y,t\right)-h\boldsymbol{v}\cdot\boldsymbol{i}\left(x-\frac{L}{2},y,t\right) + h\boldsymbol{v}\cdot\boldsymbol{j}\left(x,y+\frac{L}{2},t\right)-h\boldsymbol{v}\cdot\boldsymbol{j}\left(x,y-\frac{L}{2},t\right)\Big\} = 0.$$

Passing to the limit $L \to 0$, we obtain the usual linear transport equation

$$\frac{\partial h}{\partial t} + \operatorname{div}(h\,\boldsymbol{v}) = 0,$$

that describes the conservation of mass.

Momentum balance. Ignoring effects due to other forces different from pressure, the horizontal momentum equation given by the Newton's second law of motion reads as

$$\rho\,\frac{d\boldsymbol{V}}{dt} = -\operatorname{grad} p \tag{1}$$

where $\boldsymbol{V} = \boldsymbol{v}(x(t), y(t), t)$ is the velocity of a particle moving with the fluid, p is the pressure, grad is the horizontal component of the gradient. Since

$$\frac{d\boldsymbol{V}}{dt} = \frac{\partial\boldsymbol{v}}{\partial t} + \frac{\partial\boldsymbol{v}}{\partial x}\frac{dx}{dt} + \frac{\partial\boldsymbol{v}}{\partial y}\frac{dy}{dt} = \frac{\partial\boldsymbol{v}}{\partial t} + \boldsymbol{v}\cdot\nabla\boldsymbol{v},$$

the balance equation (1) can be rewritten as

$$\frac{\partial \boldsymbol{v}}{\partial t} + \boldsymbol{v} \cdot \nabla \boldsymbol{v} + \frac{1}{\rho}\,\mathrm{grad}\, p = 0. \tag{2}$$

To complete the derivation we need to determine an expression for the pressure p in term of the unknowns h and $\boldsymbol{v}$, and, eventually, the variables (x, y, t). Such expression is based on two assumptions relative to the pressure: one at the free surface $\xi = Z + h$ and one inside the layer. The pressure $p = p(x, y, z, t)$ at the upper surface is assumed to be constant:

$$p(x, y, \xi(x, y, t), t) = p_0 \in \mathbb{R}.$$

Additionally, the horizontal character of the fluid trajectories makes the vertical variations so small that the Archimedian principle for static fluid is applicable guaranteeing the istantaneous hydrostatic balance between vertical pressure's gradient balances gravity (generating a buoyancy effect): in formulas, we assume

$$\frac{\partial p}{\partial z} = -\,g\,\rho. \tag{3}$$

Later on, we will see how this relation can be formally deduced from the Navier-Stokes equation for incompressible fluids, when the vertical characteristic length is assumed to be much smaller than the horizontal one.

Integrating with respect to z in $[z, \xi]$ we obtain

$$p(x, y, z, t) = p(x, y, \xi, t) - \int_z^\xi \frac{\partial p}{\partial z}(x, y, \zeta, t)\, d\zeta = p_0 + g\,\rho\,(\xi - z),$$

so that $\mathrm{grad}\, p = \rho\, g\, \mathrm{grad}\,(h + Z)$. Inserting in the momentum equation (2), we deduce the system satisfied by h and $\boldsymbol{v}$, that is

$$\begin{cases} \dfrac{\partial h}{\partial t} + \mathrm{div}\,(h\,\boldsymbol{v}) = 0, \\ \dfrac{\partial \boldsymbol{v}}{\partial t} + \boldsymbol{v} \cdot \nabla \boldsymbol{v} + g\,\mathrm{grad}\,(h + Z) = 0, \end{cases}$$

or, in conservative form,

$$\begin{cases} \dfrac{\partial h}{\partial t} + \mathrm{div}\,(h\,\boldsymbol{v}) = 0, \\ \dfrac{\partial (h\,\boldsymbol{v})}{\partial t} + \mathrm{div}\left(h\,\boldsymbol{v} \otimes \boldsymbol{v} + \dfrac{1}{2}\,g\,h^2\,\mathbb{I}\right) = -g\,\mathrm{grad}\,Z\,h, \end{cases}$$

where $\boldsymbol{w} \otimes \boldsymbol{v} = (w_i\, v_j)_{ij}$ and div indicates the divergence taken row by row.

2.2 - *Derivation from incompressible Navier-Stokes equations*

Next, we propose a derivation of the shallow-water system departing from the open channel flow dynamics described by means of the Navier-Stokes equations with appropriate conditions on the free surface. Such approach permits to obtain sound derivations of the terms arising from the presence of viscosity effects, friction with bottom, ... Different regimes give raise to different limiting equations and the choice is dictated by the specific phenomenon one is modeling (see [**4, 11, 18**] and discussions therein, for other significant situations). Here, we closely follow [**28**] simplifying the presentation by disregarding both Coriolis and friction terms (see [**20**] for the one-dimensional case).

We consider the Navier-Stokes equations

$$\begin{cases} \mathrm{Div}\, \boldsymbol{U} = 0, \\ \dfrac{\partial \boldsymbol{U}}{\partial t} + \mathrm{Div}\,(\boldsymbol{U} \otimes \boldsymbol{U}) = \mathrm{Div}\, \sigma(\boldsymbol{U}) + \boldsymbol{G} \end{cases}$$

where Div is the three dimensional divergence operator, $\boldsymbol{U} \in \mathbb{R}^3$ is the velocity, $\sigma(\boldsymbol{U}) \in \mathbb{R}^3 \times \mathbb{R}^3$ is the total stress tensor, $\mathrm{Div}\, \sigma(\boldsymbol{U})$ are viscosity forces and $\boldsymbol{G} \in \mathbb{R}^3$ describes external vector fields (in this case, gravity). The system is considered in a domain of the form

$$\{(x, y, z) \, : \, Z(x, y) \leq z \leq \xi(x, y, t)\}$$

where Z describes the bottom topography and ξ the free surface, so that the function $h(t, x, y) := \xi(x, y, t) - Z(x, y)$ gives the height of the fluid column above the point of coordinates (x, y) at time t. The system has to be complemented with boundary condition at the top and at the bottom of the domain.

At the free surface, we assume viscosity of air negligible and we ask for the continuity of the normal component of the stress

$$\sigma(\boldsymbol{U})\, n_s = -\, p_0\, n_s, \qquad \text{at} \quad z = \xi(x, y, t)$$

where n_s is the outward free surface normal, and p_0 the atmospheric pressure.

At the bottom, disregarding friction terms, we assume both a wall–law and an impermeable condition

$$\begin{cases} \sigma(\boldsymbol{U})\, n_b\, \tau = \boldsymbol{0} \\ \boldsymbol{U} \cdot n_b = 0 \end{cases} \qquad \text{at} \quad z = Z(x, y)$$

where n_b is the outward normal of the domain at the bottom and τ is any vector tangent to the surface Z.

Remark 2.1. A capillarity term could be incorporated in the analysis by replacing, in the boundary condition at the free surface, the term p_0 with $p_0 - \beta\,\kappa$ where β is a capillarity coefficient, and $\kappa = \kappa(t,x,y)$ denotes the mean curvature of the surface ξ at (x,y). Also friction effect could be taken into account by adding appropriate terms in the bottom boundary conditions. Both capillarity and friction are considered in [**28**].

Neglecting turbulence effects, we assume the total stress tensor to be

$$\sigma(\boldsymbol{U}) = -p\,\mathbb{I} + 2\mu(\mathrm{D}\boldsymbol{U})^*$$

where p is the local pressure of the fluid, μ the dynamical viscosity and $(\mathrm{D}\boldsymbol{U})^*$ is the viscosity tensor, that is the symmetric part of $\mathrm{D}\boldsymbol{U}$. The proportionality of the viscosity tensor with respect to $(\mathrm{D}\boldsymbol{U})^*$ is sometimes referred to as Newton's viscosity law and therefore the fluid is said to be newtonian.

Taking in account the specific form for the total stress tensor, the Navier-Stokes equations for an incompressible fluid reads as

$$\begin{cases} \mathrm{Div}\,\boldsymbol{U} = 0, \\ \dfrac{\partial \boldsymbol{U}}{\partial t} + \mathrm{Div}(\boldsymbol{U}\otimes\boldsymbol{U} + p\,\mathbb{I}) = 2\mu\,\mathrm{Div}\,(\mathrm{D}\boldsymbol{U})^* + \boldsymbol{G} \end{cases}$$

with boundary conditions

$$\begin{aligned} &(p-p_0)\,n_s - 2\mu\,(\mathrm{D}\boldsymbol{U})^*\,n_s = 0 && \text{at } z = \xi(x,y,t), \\ &\big(pI - 2\mu\,(\mathrm{D}\boldsymbol{U})^*\big)n_b\cdot\tau = 0 && \text{at } z = Z(x,y), \\ &\boldsymbol{U}\cdot n_b = 0 && \text{at } z = Z(x,y), \end{aligned} \tag{4}$$

where τ is any tangent vector to the bottom surface Z. The conditions at the bottom correspond to the so-called Navier conditions in the absence of friction.

Next, we assume that the dynamics is *gravity driven*, meaning that the unique external force is gravity, and we disassemble horizontal and vertical directions to reveal the structure emerging from the shallow water assumption. The final system of equations will couple the height of the water column h and the vertical averaged velocity

$$\bar{\boldsymbol{v}}(x,y,t) := \frac{1}{h(x,y,t)}\int_Z^\xi \boldsymbol{v}(x,y,\zeta,t)\,d\zeta$$

where $\boldsymbol{U} = (\boldsymbol{v}, w)$. Letting $\boldsymbol{G} = (0,0,-g)$ and denoting by div the divergence operator with respect to the horizontal variables (x, y), the system becomes

$$\begin{cases} \operatorname{div} \boldsymbol{v} + \dfrac{\partial w}{\partial z} = 0, \\ \dfrac{\partial \boldsymbol{v}}{\partial t} + \operatorname{div}(\boldsymbol{v} \otimes \boldsymbol{v}) + \dfrac{\partial (w\boldsymbol{v})}{\partial z} + \operatorname{grad} p = 2\mu \operatorname{div}(\mathrm{d}\boldsymbol{v})^* + \mu \dfrac{\partial^2 \boldsymbol{v}}{\partial z^2} + \mu \operatorname{grad}\left(\dfrac{\partial w}{\partial z}\right), \\ \dfrac{\partial w}{\partial t} + \operatorname{div}(w\boldsymbol{v}) + \dfrac{\partial (w^2)}{\partial z} + \dfrac{\partial p}{\partial z} = \mu \dfrac{\partial}{\partial z} \operatorname{div} \boldsymbol{v} + \mu \Delta w + 2\mu \dfrac{\partial^2 w}{\partial z^2} - g. \end{cases}$$

The vectors n_s, n_b are given by the formulas

$$n_s = \frac{(-\operatorname{grad} \xi, 1)^t}{\sqrt{1 + |\operatorname{grad} \xi|^2}}, \qquad n_b = \frac{(\operatorname{grad} Z, -1)^t}{\sqrt{1 + |\operatorname{grad} Z|^2}}.$$

The boundary conditions at the free surface $z = \xi$ turn into

$$(p - p_0) \operatorname{grad} \xi + \mu \left\{ \frac{\partial \boldsymbol{v}}{\partial z} + \operatorname{grad} w - 2(\mathrm{d}\boldsymbol{v})^* \operatorname{grad} \xi \right\} = 0,$$
$$p - p_0 + \mu \left\{ \operatorname{grad} \xi \cdot \left(\frac{\partial \boldsymbol{v}}{\partial z} + \operatorname{grad} w \right) - 2 \frac{\partial w}{\partial z} \right\} = 0,$$

and, at the bottom $z = Z$,

$$\Big\{ \big(pI - 2\mu(\mathrm{d}\boldsymbol{v})^*\big) \operatorname{grad} Z + \mu \left(\frac{\partial \boldsymbol{v}}{\partial z} + \operatorname{grad} w \right)$$
$$- \mu \left(\frac{\partial \boldsymbol{v}}{\partial z} + \operatorname{grad} w \right) \operatorname{grad} Z - p + 2\mu \frac{\partial w}{\partial z} \Big\} \cdot \tau = 0,$$
$$\boldsymbol{v} \cdot \operatorname{grad} Z - w = 0$$

where τ is any vector normal to the surface Z. By choosing as base of this tangent plane the vectors

$$\frac{(1, 0, \partial_x Z)^t}{\sqrt{1 + |\partial_x Z|^2}}, \qquad \frac{(0, 1, \partial_y Z)^t}{\sqrt{1 + |\partial_y Z|^2}}$$

the first condition at the bottom changes into

$$\left(\big(pI - 2\mu(\mathrm{d}\boldsymbol{v})^*\big) \operatorname{grad} Z + \mu \left\{ \frac{\partial \boldsymbol{v}}{\partial z} + \operatorname{grad} w \right\} \right) \boldsymbol{i}$$
$$+ \left(-\mu \left\{ \frac{\partial \boldsymbol{v}}{\partial z} + \operatorname{grad} w \right\} \operatorname{grad} Z - p + 2\mu \frac{\partial w}{\partial z} \right) \partial_x Z = 0,$$
$$\left(\big(pI - 2\mu(\mathrm{d}\boldsymbol{v})^*\big) \operatorname{grad} Z + \mu \left\{ \frac{\partial \boldsymbol{v}}{\partial z} + \operatorname{grad} w \right\} \right) \boldsymbol{j}$$
$$+ \left(-\mu \left\{ \frac{\partial \boldsymbol{v}}{\partial z} + \operatorname{grad} w \right\} \operatorname{grad} Z - p + 2\mu \frac{\partial w}{\partial z} \right) \partial_y Z = 0,$$

that is, in vectorial form,

$$\frac{\partial \boldsymbol{v}}{\partial z} + \operatorname{grad} w - 2\,(\mathrm{d}\boldsymbol{v})^{*}\operatorname{grad} Z = \left\{\left(\frac{\partial \boldsymbol{v}}{\partial z} + \operatorname{grad} w\right)\cdot \operatorname{grad} Z - 2\,\frac{\partial w}{\partial z}\right\}\operatorname{grad} Z$$

holding at the bottom $z = Z$.

Conservation of mass. Following [**20**], we introduce the indicator function

$$\phi(t,x,y,z) = \chi_{\{z\,:\,z\in[Z(x,y),\xi(x,y,t)]\}}(z).$$

Since the particles move with speed $\boldsymbol{U}$ and the fluid is incompressible (i.e. Div $\boldsymbol{U} = 0$), there holds

$$\frac{\partial \phi}{\partial t} + \operatorname{Div}(\phi\,\boldsymbol{U}) = \frac{\partial \phi}{\partial t} + \operatorname{div}(\phi\,\boldsymbol{v}) + \frac{\partial w}{\partial z} = 0.$$

Integrating with respect to z in $[Z, +\infty)$, we infer

$$\begin{aligned} 0 &= \frac{\partial h}{\partial t} + \int_{Z}^{+\infty}\left\{\operatorname{div}(\phi\,\boldsymbol{v}) + \frac{\partial w}{\partial z}\right\} d\zeta \\ &= \frac{\partial h}{\partial t} + \operatorname{div}\left(\int_{Z}^{+\infty}\phi\,\boldsymbol{v}\,d\zeta\right) + \phi\big(\boldsymbol{v}\cdot\operatorname{grad} Z - w\big)\big|_{z=Z} = 0. \end{aligned}$$

Hence, since $\boldsymbol{v}\cdot\operatorname{grad} Z$ and w coincide at the bottom, we obtain

$$\frac{\partial h}{\partial t} + \operatorname{div}\left(\int_{Z}^{\xi}\boldsymbol{v}\,d\zeta\right) = 0,$$

that is

$$\frac{\partial h}{\partial t} + \operatorname{div}(h\,\bar{\boldsymbol{v}}) = 0.$$

For later reference, by integrating in $[Z, \xi]$ the transport equation for ϕ, we get

$$\frac{\partial h}{\partial t} + \operatorname{div}\left(\int_{Z}^{\xi}\boldsymbol{v}\,d\zeta\right) - \left(\frac{\partial \xi}{\partial t} + \boldsymbol{v}\cdot\operatorname{grad}\xi - w\right)\bigg|_{z=\xi} + (\boldsymbol{v}\cdot\operatorname{grad} d - w)\bigg|_{z=Z} = 0$$

and thus we have

$$\frac{\partial \xi}{\partial t} + \boldsymbol{v}\cdot\operatorname{grad}\xi - w = 0 \qquad \text{at} \quad z = \xi \tag{5}$$

that is a relation between the horizontal and the vertical component of the speed $\boldsymbol{U}$ at the free surface ξ.

Scaling. Next, we a-dimensionalize the system so that parameters describing the characteristic scales appear into play and relative sizes of characteristic lengths, horizontal L and vertical H become apparent. Set

$$W := \frac{HV}{L}, \qquad T := \frac{L}{V}, \qquad P := V^2$$

were W, V, T, P are characteristic scales for velocities (vertical and horizontal), time and pressure and let us introduce the quantities

$$\tilde{\boldsymbol{v}} = \frac{\boldsymbol{v}}{V}, \quad \tilde{w} = \frac{w}{W}, \quad (\tilde{x}, \tilde{y}) = \frac{(x, y)}{L}, \quad \tilde{z} = \frac{z}{H}, \quad \tilde{t} = \frac{t}{T}, \quad \tilde{p} = \frac{p}{P}.$$

Expunging for shortness the tildas and defining

$$\varepsilon := \frac{H}{L}, \qquad \mathrm{Re} := \frac{VL}{\mu}, \qquad \mathrm{Fr} := \frac{V}{\sqrt{gH}},$$

we obtain the system

$$(6) \qquad \begin{cases} \operatorname{div} \boldsymbol{v} + \dfrac{\partial w}{\partial z} = 0, \\ \dfrac{\partial \boldsymbol{v}}{\partial t} + \operatorname{div}(\boldsymbol{v} \otimes \boldsymbol{v}) + \operatorname{grad} p + \dfrac{\partial (w \boldsymbol{v})}{\partial z} \\ \qquad = \dfrac{1}{\mathrm{Re}} \left\{ 2 \operatorname{div} (\mathrm{d}\boldsymbol{v})^* + \dfrac{1}{\varepsilon^2} \dfrac{\partial^2 \boldsymbol{v}}{\partial z^2} + \operatorname{grad} \left(\dfrac{\partial w}{\partial z} \right) \right\}, \\ \varepsilon^2 \left(\dfrac{\partial w}{\partial t} + \operatorname{div}(w \boldsymbol{v}) + \dfrac{\partial (w^2)}{\partial z} \right) + \dfrac{\partial p}{\partial z} \\ \qquad = \dfrac{1}{\mathrm{Re}} \left\{ \dfrac{\partial}{\partial z} \operatorname{div} \boldsymbol{v} + \varepsilon^2 \Delta w + 2 \dfrac{\partial^2 w}{\partial z^2} \right\} - \dfrac{1}{\mathrm{Fr}^2}. \end{cases}$$

From the second equation, we infer that $\dfrac{\partial^2 \boldsymbol{v}}{\partial z^2}$ is of order $O(\varepsilon^2)$ as soon as Re is bounded away from zero.

Boundary conditions in adimensional form are, at the free boundary $z = \xi$,

$$(7) \qquad \begin{cases} (p - p_0) \operatorname{grad} \xi + \dfrac{1}{\mathrm{Re}} \left\{ \dfrac{1}{\varepsilon^2} \dfrac{\partial \boldsymbol{v}}{\partial z} + \operatorname{grad} w - 2 (\mathrm{d}\boldsymbol{v})^* \operatorname{grad} \xi \right\} = 0, \\ p - p_0 + \dfrac{1}{\mathrm{Re}} \left\{ \operatorname{grad} \xi \cdot \left(\dfrac{\partial \boldsymbol{v}}{\partial z} + \varepsilon^2 \operatorname{grad} w \right) - 2 \dfrac{\partial w}{\partial z} \right\} = 0, \end{cases}$$

and, at the bottom $z = Z$,

$$(8) \qquad \begin{cases} \dfrac{1}{\varepsilon^2} \dfrac{\partial \boldsymbol{v}}{\partial z} + \operatorname{grad} w - 2 (\mathrm{d}\boldsymbol{v})^* \operatorname{grad} Z \\ \qquad = \left\{ \left(\dfrac{\partial \boldsymbol{v}}{\partial z} + \varepsilon^2 \operatorname{grad} w \right) \cdot \operatorname{grad} Z - 2 \dfrac{\partial w}{\partial z} \right\} \operatorname{grad} Z, \\ \boldsymbol{v} \cdot \operatorname{grad} Z - w = 0. \end{cases}$$

Note that the latter relations do not depend on the Reynolds number Re and that, at $z = Z$, there holds $\dfrac{\partial \boldsymbol{v}}{\partial z} = O(\varepsilon^2)$.

Hydrostatic approximation. Disregarding the terms of order ε^2 in the second equation of the system (6), we obtain

$$\frac{\partial p}{\partial z} = -\frac{1}{\mathrm{Fr}^2} + \frac{1}{\mathrm{Re}}\frac{\partial}{\partial z}\left(\operatorname{div}\boldsymbol{v} + 2\,\frac{\partial w}{\partial z}\right) \tag{9}$$

corresponding to relation (3) previously considered (the two equalities coincide when disregarding the viscosity term, i.e. $\mu = 0$). This equality has to be coupled with the information relative to the pressure at the free surface that is, disregarding the $O(\varepsilon^2)$–term,

$$p - p_0 = \frac{2}{\mathrm{Re}}\frac{\partial w}{\partial z}, \qquad \text{at} \quad z = \xi.$$

Integrating equation (9), we obtain

$$\begin{aligned} p(t,x,y,z) - p_0 &= \frac{1}{\mathrm{Fr}^2}(\xi - z) + \frac{1}{\mathrm{Re}}\int_{\xi}^{z}\frac{\partial}{\partial z}\operatorname{div}\boldsymbol{v}\,d\zeta + \frac{2}{\mathrm{Re}}\frac{\partial w}{\partial z}(x,y,z,t) \\ &= \frac{1}{\mathrm{Fr}^2}(\xi - z) + \frac{1}{\mathrm{Re}}\Big\{\operatorname{div}\boldsymbol{v}(x,y,z,t) - \operatorname{div}\boldsymbol{v}(x,y,\xi,t)\Big\} + \frac{2}{\mathrm{Re}}\frac{\partial w}{\partial z}(x,y,z,t). \end{aligned}$$

Using the incompressibility relation, we deduce

$$p(t,x,y,z) = p_0 + \frac{1}{\mathrm{Fr}^2}(\xi - z) - \frac{1}{\mathrm{Re}}\Big\{\operatorname{div}\boldsymbol{v}(x,y,z,t) + \operatorname{div}\boldsymbol{v}(x,y,\xi,t)\Big\}. \tag{10}$$

Again, ignoring viscosity, we obtain the usual linear dependence of the pressure with respect to the height of the water column.

Momentum equation. Since we miss the equation for $\bar{\boldsymbol{v}}$, we integrate the equation for $\boldsymbol{v}$ with respect to z in $[Z, \xi]$

$$\begin{aligned} \frac{\partial}{\partial t}\int_Z^{\xi}\boldsymbol{v}\,d\zeta + \operatorname{div}\left(\int_Z^{\xi}(\boldsymbol{v}\otimes\boldsymbol{v})\,d\zeta\right) + \int_Z^{\xi}\operatorname{grad} p\,d\zeta - \frac{2}{\mathrm{Re}}\operatorname{div}\int_Z^{\xi}(\mathrm{d}\boldsymbol{v})^*\,d\zeta& \\ = \boldsymbol{F}_{\xi} - \boldsymbol{F}_Z& \end{aligned}$$

where

$$\begin{aligned} \boldsymbol{F}_{\xi} &:= \frac{\partial \xi}{\partial t}\,\boldsymbol{v} + (\boldsymbol{v}\otimes\boldsymbol{v})\operatorname{grad}\xi - w\,\boldsymbol{v} + \frac{1}{\mathrm{Re}}\left\{\frac{1}{\varepsilon^2}\frac{\partial \boldsymbol{v}}{\partial z} + \operatorname{grad} w - 2\,(\mathrm{d}\boldsymbol{v})^*\operatorname{grad}\xi\right\} \\ \boldsymbol{F}_Z &:= (\boldsymbol{v}\otimes\boldsymbol{v})\operatorname{grad} Z - w\,\boldsymbol{v} + \frac{1}{\mathrm{Re}}\left\{\frac{1}{\varepsilon^2}\frac{\partial \boldsymbol{v}}{\partial z} + \operatorname{grad} w - 2\,(\mathrm{d}\boldsymbol{v})^*\operatorname{grad} Z\right\}. \end{aligned}$$

Using (5) and the first relation in (7), we deduce

$$F_\xi = -(\boldsymbol{v}\cdot \operatorname{grad}\xi)\,\boldsymbol{v} + (\boldsymbol{v}\otimes\boldsymbol{v})\operatorname{grad}\xi - (p-p_0)\Big|_\xi \operatorname{grad}\xi = -(p-p_0)\Big|_\xi \operatorname{grad}\xi.$$

Similarly, from (8), we infer

$$F_Z = \frac{1}{\mathrm{Re}}\left\{\left(\frac{\partial \boldsymbol{v}}{\partial z} + \varepsilon^2 \operatorname{grad} w\right)\cdot \operatorname{grad} Z - 2\frac{\partial w}{\partial z}\right\}\operatorname{grad} Z.$$

Hence

$$\begin{aligned}\frac{\partial}{\partial t}\int_Z^\xi \boldsymbol{v}\,d\zeta + \operatorname{div}\left(\int_Z^\xi (\boldsymbol{v}\otimes\boldsymbol{v})\,d\zeta\right) + \int_Z^\xi \operatorname{grad} p\,d\zeta - \frac{2}{\mathrm{Re}}\operatorname{div}\int_Z^\xi (\mathrm{d}\boldsymbol{v})^*\,d\zeta \\ = -(p-p_0)\Big|_\xi \operatorname{grad}\xi - \frac{1}{\mathrm{Re}}\left\{\left(\frac{\partial \boldsymbol{v}}{\partial z} + \varepsilon^2 \operatorname{grad} w\right)\cdot \operatorname{grad} Z - 2\frac{\partial w}{\partial z}\right\}\operatorname{grad} Z.\end{aligned}$$

Next, we consider appropriate reductions of the latter equation obtained by ignoring small terms in the limit $\varepsilon \to 0$, giving raise to an equation for the quantity $h\,\bar{\boldsymbol{v}}$. In what follows, we consider the asymptotic regime determined by the position $\mathrm{Re} \sim O(\varepsilon^{-1})$.

First order approximation. To start with, we consider the reduced system obtained by disregarding all the terms of order $O(\varepsilon)$. From rescaled system and boundary conditions, we deduce

$$\frac{\partial^2 \boldsymbol{v}}{\partial z^2} = O(\varepsilon), \qquad \frac{\partial \boldsymbol{v}}{\partial z}\Big|_\xi = O(\varepsilon),$$

hence

$$\frac{\partial \boldsymbol{v}}{\partial z}(x,y,z,t) = \frac{\partial \boldsymbol{v}}{\partial z}(x,y,\xi,t) + \int_Z^\xi \frac{\partial^2 \boldsymbol{v}}{\partial z^2}(x,y,\zeta,t)\,d\zeta = O(\varepsilon).$$

As a consequence, we have

$$\boldsymbol{v}(x,y,z,t) = \boldsymbol{v}(x,y,Z,t) + \int_Z^z \frac{\partial \boldsymbol{v}}{\partial z}(x,y,\zeta,t)\,d\zeta = \boldsymbol{v}(x,y,Z,t) + O(\varepsilon),$$

and thus

$$\boldsymbol{v}(x,y,z,t) = \bar{\boldsymbol{v}}(x,y,t) + O(\varepsilon), \qquad \boldsymbol{v}\otimes\boldsymbol{v} = \bar{\boldsymbol{v}}\otimes\bar{\boldsymbol{v}} + O(\varepsilon).$$

Hence, the momentum equation can be rewritten in the form

$$\frac{\partial}{\partial t}\big(h\,\bar{\boldsymbol{v}}\big) + \operatorname{div}\big(h\,\bar{\boldsymbol{v}}\otimes\bar{\boldsymbol{v}}\big) + \int_Z^\xi \operatorname{grad} p\,d\zeta = -(p-p_0)\big|_\xi \operatorname{grad}\xi + O(\varepsilon).$$

At first order, the pressure is given by

$$p(t,x,y,z) = p_0 + \frac{1}{\mathrm{Fr}^2}\,(\xi - z) + O(\varepsilon)$$

so that

$$p\big|_{\xi} = p_0 + O(\varepsilon) \qquad \text{and} \qquad \operatorname{grad} p = \frac{1}{\mathrm{Fr}^2}\operatorname{grad}(Z+h) + O(\varepsilon).$$

Thus, we readily obtain the first order approximation for the momentum equation

$$\frac{\partial}{\partial t}(h\,\bar{\boldsymbol{v}}) + \operatorname{div}\Big(h\bar{\boldsymbol{v}}\otimes\bar{\boldsymbol{v}} + \frac{h^2}{2\mathrm{Fr}^2}\,I\Big) = -\frac{1}{\mathrm{Fr}^2}\operatorname{grad} Z\,h + O(\varepsilon).$$

Ignoring the $O(\varepsilon)$-term and collecting together with the equation describing conservation of mass, we obtain the unviscous Saint-Venant system for shallow water. Using the conservation of mass, we can also write an equation for the average horizontal velocity $\boldsymbol{v}$

(11) $$\frac{\partial\bar{\boldsymbol{v}}}{\partial t} + \bar{\boldsymbol{v}}\cdot\nabla\bar{\boldsymbol{v}} + \frac{1}{\mathrm{Fr}^2}\operatorname{grad}(h+Z) = O(\varepsilon)$$

where $\boldsymbol{v}\cdot\nabla\boldsymbol{v}$ indicates the action of the differential $d\boldsymbol{v}$ over the vector $\boldsymbol{v}$.

Second order approximation. Next, we want to ignore term of order $O(\varepsilon^2)$ still in the regime $\mathrm{Re} \sim O(\varepsilon^{-1})$. To begin with, we obtain a better estimate of $\boldsymbol{v}$ in term of ε by taking advantage of the above approximated equation for the horizontal velocity. Rearranging the original equation for $\boldsymbol{v}$ and using the incompressibility condition, we obtain

$$\begin{aligned}\frac{1}{\varepsilon^2\mathrm{Re}}\frac{\partial^2\boldsymbol{v}}{\partial z^2} &= \frac{\partial\boldsymbol{v}}{\partial t} + \operatorname{div}(\boldsymbol{v}\otimes\boldsymbol{v}) + \operatorname{grad} p + \frac{\partial(w\,\boldsymbol{v})}{\partial z} + O(\varepsilon)\\ &= \frac{\partial\boldsymbol{v}}{\partial t} + \boldsymbol{v}\cdot\nabla\boldsymbol{v} + \operatorname{grad} p + w\frac{\partial\boldsymbol{v}}{\partial z} + O(\varepsilon).\end{aligned}$$

Since $\dfrac{\partial\boldsymbol{v}}{\partial z} = O(\varepsilon)$ and (11) holds, we also have

$$\frac{1}{\varepsilon^2\mathrm{Re}}\frac{\partial^2\boldsymbol{v}}{\partial z^2} = \operatorname{grad}\Big(p - \frac{1}{\mathrm{Fr}^2}\,\xi\Big) + O(\varepsilon) = O(\varepsilon).$$

By integration and recalling the bottom boundary conditions (8), we get

$$\frac{\partial\boldsymbol{v}}{\partial z} = \frac{\partial\boldsymbol{v}}{\partial z}\Big|_Z + O(\varepsilon^2) = O(\varepsilon^2) \qquad \text{and} \qquad \boldsymbol{v} = \boldsymbol{v}\big|_Z + O(\varepsilon^2).$$

Hence the previously found relation between the horizontal speed and its vertical average hold also at order $O(\varepsilon^2)$, that is

$$\boldsymbol{v}(x,y,z,t) = \bar{\boldsymbol{v}}(x,y,t) + O(\varepsilon^2), \qquad \boldsymbol{v}\otimes\boldsymbol{v} = \bar{\boldsymbol{v}}\otimes\bar{\boldsymbol{v}} + O(\varepsilon^2).$$

Moreover, by integration with respect to the vertical variable,

$$\int_Z^\xi \frac{\partial \boldsymbol{v}}{\partial x}\, d\zeta = \frac{\partial}{\partial x}\int_Z^\xi \boldsymbol{v}\, d\zeta - \boldsymbol{v}\big|_\xi \frac{\partial \xi}{\partial x} + \boldsymbol{v}\big|_Z \frac{\partial Z}{\partial x}$$
$$= \frac{\partial}{\partial x}(h\,\bar{\boldsymbol{v}}) - \bar{\boldsymbol{v}}\,\frac{\partial h}{\partial x} + O(\varepsilon^2) = h\,\frac{\partial \bar{\boldsymbol{v}}}{\partial x} + O(\varepsilon^2)$$

so that the following relation holds true

$$\int_Z^\xi (\mathrm{d}\boldsymbol{v})^*\, d\zeta = h(\mathrm{d}\bar{\boldsymbol{v}})^* + O(\varepsilon^2).$$

Therefore, the momentum equation can be provisionally written as

$$\frac{\partial}{\partial t}(h\,\bar{\boldsymbol{v}}) + \operatorname{div}\big(h\,\bar{\boldsymbol{v}}\otimes\bar{\boldsymbol{v}}\big) + \int_Z^\xi \operatorname{grad} p\, d\zeta - \frac{2}{\mathrm{Re}}\operatorname{div}\big(h(\mathrm{d}\bar{\boldsymbol{v}})^*\big)$$
$$= -(p-p_0)\big|_\xi \operatorname{grad}\xi - \frac{2}{\mathrm{Re}}\operatorname{div}\bar{\boldsymbol{v}}\big|_Z \operatorname{grad} Z + O(\varepsilon^2)$$

having used also the incompressibility condition.

Next, let us consider the expression for the pressure. Recalling that (10) is obtained by ignoring $O(\varepsilon^2)$ terms, we have

$$p(t,x,y,z) = p_0 + \frac{1}{\mathrm{Fr}^2}(\xi - z) - \frac{2}{\mathrm{Re}}\operatorname{div}\bar{\boldsymbol{v}} + O(\varepsilon^3), \tag{12}$$

since $\boldsymbol{v} = \bar{\boldsymbol{v}} + O(\varepsilon^2)$. Differentiating with respect to (x,y) and integrating with respect to z, we infer

$$\int_Z^\xi \operatorname{grad} p\, d\zeta = \frac{1}{\mathrm{Fr}^2}\, h\operatorname{grad}\xi - \frac{2}{\mathrm{Re}}\, h\operatorname{grad}\operatorname{div}\bar{\boldsymbol{v}} + O(\varepsilon^3),$$

substituting, we obtain

$$\frac{\partial}{\partial t}(h\,\bar{\boldsymbol{v}}) + \operatorname{div}\left(h\,\bar{\boldsymbol{v}}\otimes\bar{\boldsymbol{v}} + \frac{1}{2\mathrm{Fr}^2}\, h^2 I\right) + \frac{1}{\mathrm{Fr}^2}\, h\operatorname{grad} Z - \frac{2}{\mathrm{Re}}\operatorname{div}\big(h(\mathrm{d}\bar{\boldsymbol{v}})^*\big)$$
$$= \frac{2}{\mathrm{Re}}\, h\operatorname{grad}\operatorname{div}\bar{\boldsymbol{v}} + \frac{2}{\mathrm{Re}}\operatorname{div}\bar{\boldsymbol{v}}\operatorname{grad}(Z+h) - \frac{2}{\mathrm{Re}}\operatorname{div}\bar{\boldsymbol{v}}\operatorname{grad} Z + O(\varepsilon^2)$$
$$= \frac{2}{\mathrm{Re}}\operatorname{grad}\big(h\operatorname{div}\bar{\boldsymbol{v}}\big) + O(\varepsilon^2).$$

Hence, we get the final system

$$\begin{cases} \dfrac{\partial h}{\partial t} + \operatorname{div}(h\,\bar{\boldsymbol{v}}) = 0, \\ \dfrac{\partial}{\partial t}(h\,\bar{\boldsymbol{v}}) + \operatorname{div}\Big(h\,\bar{\boldsymbol{v}} \otimes \bar{\boldsymbol{v}} + \dfrac{1}{2\mathrm{Fr}^2}\,h^2\,I\Big) = -\dfrac{1}{\mathrm{Fr}^2}\,h\operatorname{grad} Z \\ \qquad\qquad + \dfrac{2}{\mathrm{Re}}\operatorname{div}\big(h(\mathrm{d}\bar{\boldsymbol{v}})^*\big) + \dfrac{2}{\mathrm{Re}}\operatorname{grad}\big(h\operatorname{div}\bar{\boldsymbol{v}}\big) + O(\varepsilon^2) \end{cases}$$

which gives the viscous shallow water system when ignoring the term $O(\varepsilon^2)$.

Remark 2.2. Taking into account the capillarity effects amounts in replacing the atmospheric pressure term p_0 with the difference $p_0 - \beta\,\kappa$ where β is a capillarity coefficients and κ is the mean curvature of the free surface ξ. For the function κ, there holds

$$\kappa = \varepsilon\,\Delta\xi + O(\varepsilon^3).$$

In the analysis, this term modifies the structure of the equation in the second order approximation. Precisely, in equation (12) the term p_0 is replaced by $p_0 - \beta\,\varepsilon\,\Delta\xi$; accordingly, the final momentum equation is modified by adding at the right-hand side the third-order term $\beta\,\varepsilon\,h\operatorname{grad}\Delta(Z+h)$.

Remark 2.3. When friction of the fluid with the bottom is relevant, the second condition in (4) has to be reshaped. Specifically, one can assume the following modified boundary condition

$$\big(pI - 2\mu\,(\mathrm{D}\boldsymbol{U})^*\big)n_b \cdot \tau = \big(\kappa_\ell + \kappa_t\,h|\boldsymbol{U}|\big)\,\boldsymbol{U}\cdot\tau \qquad \text{at } z = Z(x,y),$$

for any vector τ tangent to the surface $z = Z$, where κ_ℓ and κ_t are laminar and turbulent friction coefficients (the latter having the classical form proposed by the French hydraulics engineer Antoine de Chézy).

Coherently, assuming κ_ℓ and κ_t to be $O(\varepsilon)$, the right hand-side of the momentum equation is changed by the appearance of a friction term of the form

$$-a_\ell(h)\,\boldsymbol{v} - a_t(h)\,h\,|\boldsymbol{v}|\boldsymbol{v},$$

where

$$a_\ell(h) = \frac{r_0}{1+\varepsilon A\,h}, \qquad a_t(h) = \frac{\varepsilon\,r_1}{(1+\varepsilon A\,h)^2}$$

for appropriate constants r_0, r_1, A (see [28]). In Section 4, we will consider the hyperbolic shallow water model in the presence of friction. Consistently with the derivation, we will consider a linear friction term of the form $-r_0\,\boldsymbol{v}$, $r_0 > 0$, obtained by formally setting $\varepsilon = 0$.

2.3 - *Energy dissipation*

In the case of flat bottom, the viscous shallow water equations have the form

$$\begin{cases} \dfrac{\partial h}{\partial t} + \operatorname{div}(h\,\boldsymbol{v}) = 0, \\[2mm] \dfrac{\partial}{\partial t}(h\,\boldsymbol{v}) + \operatorname{div}\left(h\,\boldsymbol{v}\otimes\boldsymbol{v} + \dfrac{1}{2\mathrm{Fr}^2}\,h^2\,I\right) = \dfrac{2}{\mathrm{Re}}\,\mathcal{V}[h,\boldsymbol{v}] \end{cases}$$

where

$$\mathcal{V}[h,\boldsymbol{v}] := \operatorname{div}\big(h(\mathrm{d}\boldsymbol{v})^*\big) + \operatorname{grad}\big(h\operatorname{div}\boldsymbol{v}\big) \tag{13}$$

describe the viscosity term. As observed in [**19**], not all of the possible viscosity term are energetically consistent: depending on the form of the operator $\mathcal{V}$, the corresponding system of partial differential equations may or may not possesses a Lyapunov functional describing conservation/dissipation of the total energy (kinetic + potential energy).

The form expressed in (13) is energetically consistent. Indeed, let us consider the energy functional

$$\mathcal{E}[h,\boldsymbol{v}] := \int\left\{\frac{1}{2}\,h\,|\boldsymbol{v}|^2 + \frac{1}{2\,\mathrm{Fr}^2}\,h^2\right\}dx\,dy.$$

Assume that the couple $(h,\boldsymbol{v})$ is a classical solution of the shallow water system and that appropriate boundary conditions are satisfied (so that all boundary term in integration by parts vanish). Then

$$\frac{d}{dt}\mathcal{E}[h,\boldsymbol{v}] = \int\left\{\frac{1}{2}\,\boldsymbol{v}\cdot\frac{\partial}{\partial t}(h\,\boldsymbol{v}) + \frac{1}{2}\,\boldsymbol{v}\cdot\left(h\,\frac{\partial\boldsymbol{v}}{\partial t}\right) + \frac{1}{\mathrm{Fr}^2}\,h\,\frac{\partial h}{\partial t}\right\}dx\,dy.$$

Using the equation satisfied by h and $h\,\boldsymbol{v}$, we infer

$$\begin{aligned} \frac{d}{dt}\mathcal{E}[h,\boldsymbol{v}] &= -\int\left\{\boldsymbol{v}\cdot\operatorname{div}\left(h\,\boldsymbol{v}\otimes\boldsymbol{v} + \frac{1}{2\mathrm{Fr}^2}\,h^2\,I\right) + \left(\frac{1}{\mathrm{Fr}^2}\,h - \frac{1}{2}\,|\boldsymbol{v}|^2\right)\operatorname{div}(h\,\boldsymbol{v})\right\} \\ &\qquad + \frac{2}{\mathrm{Re}}\int\boldsymbol{v}\cdot\mathcal{V}[h,\boldsymbol{v}] \\ &= -\int\operatorname{div}\left\{\frac{1}{2}\,h\,|\boldsymbol{v}|^2\,\boldsymbol{v} + \frac{1}{\mathrm{Fr}^2}\,h^2\,\boldsymbol{v}\right\} + \frac{2}{\mathrm{Re}}\int\boldsymbol{v}\cdot\mathcal{V}[h,\boldsymbol{v}] \end{aligned}$$

so that

$$\frac{d}{dt}\mathcal{E}[h,\boldsymbol{v}] = \frac{2}{\mathrm{Re}}\int\big(\boldsymbol{v}\cdot\mathcal{V}[h,\boldsymbol{v}]\big)\,dx\,dy.$$

From the definition of $\mathcal{V}$, we reckon

$$\begin{aligned}\frac{d}{dt}\mathcal{E}[h,\boldsymbol{v}] &= \frac{2}{\mathrm{Re}}\int\big\{\boldsymbol{v}\cdot\mathrm{div}\big(h(\mathrm{d}\boldsymbol{v})^*\big)\big\}\,dx\,dy + \frac{2}{\mathrm{Re}}\int\big\{\boldsymbol{v}\cdot\mathrm{grad}\big(h\,\mathrm{div}\,\boldsymbol{v}\big)\big\}\,dx\,dy\\ &= \frac{2}{\mathrm{Re}}\int\big\{\boldsymbol{v}\cdot\mathrm{div}\big(h(\mathrm{d}\boldsymbol{v})^*\big)\big\}\,dx\,dy - \frac{2}{\mathrm{Re}}\int h\,|\mathrm{div}\,\boldsymbol{v}|^2\,dx\,dy.\end{aligned}$$

Setting $\boldsymbol{v} = (u, v)$, we have

$$\int\big(\boldsymbol{v}\cdot\mathrm{div}\big(h(\mathrm{d}\boldsymbol{v})^*\big)\big)\,dx\,dy = -\int h\Big\{(\partial_x u)^2 + \frac{1}{2}\,(\partial_y u + \partial_x v)^2 + (\partial_y v)^2\Big\}\,dx\,dy.$$

Substituting, we obtain the energy relation

$$\frac{d}{dt}\mathcal{E}[h,\boldsymbol{v}] + \frac{1}{\mathrm{Re}}\int h\Big\{2\,|d\boldsymbol{v}^*|^2 + 3\,|\mathrm{div}\,\boldsymbol{v}|^2\Big\}\,dx\,dy = 0.$$

Different choices for the viscosity term may give unconsistent energy properties (as an example, see [**7**] for a discussion relative to the case $\mathcal{V}[h,\boldsymbol{v}] = h\,\Delta\boldsymbol{v}$ considered in [**39**]).

3 - Well-posedness of the Cauchy problem

In this Section, we discuss the Cauchy problem for the shallow-water system, previously derived. To simplify the matter, we will always restrict our attention to the case of flat bottom. The unviscous system fits into the class of hyperbolic systems of conservation laws, whose existence theory is well–established (see [**27**]). In the presence of viscosity, the system exhibits a hyperbolic–parabolic structure that is known, but less popular. For this reason, we will examine the latter in details when the spatial variable is one-dimensional, and we will only sketch how to deal with the multi-dimensional case.

The problem of showing existence of classical solutions for the viscous shallow water has been addressed in [**12**] (Dirichlet problem, Hölder regularity), [**24**] (periodic case, Sobolev regularity), [**48**] (Dirichlet problem, Sobolev regularity) [**49**] (Cauchy problem, Sobolev regularity). More recently, low-regularity existence results has been proved by means of Littlewood–Paley decomposition, [**51, 13**]. A different path concerns the analysis of weak solutions. We will not discuss anything in this direction; for reader convenience, we limit ourselves to flag the articles [**25**] (entropy solutions for the hyperbolic system, 1-d) and [**6**] (weak solutions for viscous shallow water equations, 2-d).

We will also restrict the attention to the case of a *strictly positive height* h. In fact, if the unknown h vanishes at some zones (as in the case of the dam-break problem, a classical numerical experiment in this context), the flow admits some dry region, and the shallow water system suffers of the same kind of degenerations appearing in gas-dynamics in the case of the presence of vacuum.

3.1 - *The shallow water system without viscosity*

The shallow water system in the absence of viscosity is

$$\begin{cases} \dfrac{\partial h}{\partial t} + \operatorname{div}(h\,\boldsymbol{v}) = 0, \\ \dfrac{\partial}{\partial t}(h\,\boldsymbol{v}) + \operatorname{div}\Big(h\,\boldsymbol{v}\otimes\boldsymbol{v} + \dfrac{1}{2\mathrm{Fr}^2}\,h^2\,I\Big) = -\dfrac{1}{\mathrm{Fr}^2}\,h\,\mathrm{grad}\,Z. \end{cases} \tag{14}$$

Under an appropriate change of variable, the system is symmetrizable and therefore it fits in a general framework found by Friedrichs and widely explored in [**27**]. Indeed, assuming $Z =$ constant and setting $\boldsymbol{m} = h\,\boldsymbol{v} = (p, q)$, the unviscous shallow water system reads

$$\begin{cases} \dfrac{\partial h}{\partial t} + \dfrac{\partial p}{\partial x} + \dfrac{\partial q}{\partial y} = 0, \\ \dfrac{\partial p}{\partial t} + \dfrac{\partial}{\partial x}\left(\dfrac{p^2}{h} + \dfrac{h^2}{2\,\mathrm{Fr}^2}\right) + \dfrac{\partial}{\partial y}\left(\dfrac{p\,q}{h}\right) = 0 \\ \dfrac{\partial q}{\partial t} + \dfrac{\partial}{\partial x}\left(\dfrac{p\,q}{h}\right) + \dfrac{\partial}{\partial y}\left(\dfrac{q^2}{h} + \dfrac{h^2}{2\,\mathrm{Fr}^2}\right) = 0 \end{cases}$$

that is, in vectorial form,

$$\frac{\partial w}{\partial t} + \mathbb{A}_x(w)\,\frac{\partial w}{\partial x} + \mathbb{A}_y(w)\,\frac{\partial w}{\partial y} = 0$$

where $w := (h, p, q)$ and

$$\mathbb{A}_x(w) := \begin{pmatrix} 0 & 1 & 0 \\ -p^2/h^2 + h/\mathrm{Fr}^2 & 2p/h & 0 \\ -p\,q/h^2 & q/h & p/h \end{pmatrix}$$

and

$$\mathbb{A}_y(w) := \begin{pmatrix} 0 & 0 & 1 \\ -p\,q/h^2 & q/h & p/h \\ -q^2/h^2 + h/\mathrm{Fr}^2 & 0 & 2q/h \end{pmatrix}.$$

We look for a symmetrizer of the system, i.e. a symmetric matrix $\mathbb{A}_0 > \mathbb{0}$

$$\mathbb{A}_0 = \mathbb{A}_0(w) = \begin{pmatrix} a & b & c \\ b & d & e \\ c & e & f \end{pmatrix}$$

such that $\mathbb{A}_0\,\mathbb{A}_x$ and $\mathbb{A}_0\,\mathbb{A}_y$ are symmetric. In order to satisfy such constraint, two sets of conditions have to be satisfied, namely

$$\begin{cases} a + \dfrac{2\,b\,p}{h} + \dfrac{c\,q}{h} = (\mathbb{A}_0\mathbb{A}_x)_{12} = (\mathbb{A}_0\mathbb{A}_x)_{21} = -\dfrac{d\,p^2}{h^2} + \dfrac{d\,h}{\mathrm{Fr}^2} - \dfrac{e\,p\,q}{h^2} \\ \dfrac{c\,p}{h} = (\mathbb{A}_0\mathbb{A}_x)_{13} = (\mathbb{A}_0\mathbb{A}_x)_{31} = -\dfrac{e\,p^2}{h^2} + \dfrac{e\,h}{\mathrm{Fr}^2} - \dfrac{f\,p\,q}{h^2} \\ c + \dfrac{2\,e\,p}{h} + \dfrac{f\,q}{h} = (\mathbb{A}_0\mathbb{A}_y)_{23} = (\mathbb{A}_0\mathbb{A}_x)_{32} = \dfrac{e\,p}{h} \end{cases}$$

and

$$\begin{cases} a + \dfrac{b\,p}{h} + \dfrac{2\,c\,q}{h} = (\mathbb{A}_0\mathbb{A}_y)_{13} = (\mathbb{A}_0\mathbb{A}_y)_{31} = -\dfrac{e\,p\,q}{h^2} - \dfrac{f\,q^2}{h^2} + \dfrac{f\,h}{\mathrm{Fr}^2} \\ \dfrac{b\,q}{h} = (\mathbb{A}_0\mathbb{A}_y)_{12} = (\mathbb{A}_0\mathbb{A}_y)_{21} = -\dfrac{d\,p\,q}{h^2} - \dfrac{e\,q^2}{h^2} + \dfrac{e\,h}{\mathrm{Fr}^2} \\ b + \dfrac{d\,p}{h} + \dfrac{2\,e\,q}{h} = (\mathbb{A}_0\mathbb{A}_x)_{23} = (\mathbb{A}_0\mathbb{A}_y)_{32} = \dfrac{e\,q}{h}. \end{cases}$$

From the third equations of the two systems, we get

$$c = -\frac{e\,p}{h} - \frac{f\,q}{h}, \qquad\qquad b = -\frac{d\,p}{h} - \frac{e\,q}{h}$$

and introducing the latter relations in the second lines of the two systems, we infer $e\,h/\mathrm{Fr}^2 = 0$, that is $e = 0$. Substituting, we get

$$\begin{cases} a = \dfrac{d\,p^2}{h^2} + \dfrac{f\,q^2}{h^2} + \dfrac{d\,h}{\mathrm{Fr}^2} = \dfrac{d\,p^2}{h^2} + \dfrac{f\,q^2}{h^2} + \dfrac{f\,h}{\mathrm{Fr}^2} \\ b = -\dfrac{d\,p}{h}, \qquad c = -\dfrac{f\,q}{h}, \qquad e = 0 \end{cases}$$

so that d coincides with f. The final expression for the symmetrizer $\mathbb{A}_0$ is

$$\mathbb{A}_0(\mathrm{w}) = d(w)\begin{pmatrix} |\boldsymbol{v}|^2 + \dfrac{1}{\mathrm{Fr}^2}\,h & -u & -v \\ -u & 1 & 0 \\ -v & 0 & 1 \end{pmatrix}$$

where $d = d(w)$ is an arbitrary positive function and $\boldsymbol{v} = (u, v)$. Choosing $d = d(w) = 1/h$, the symmetrizer $\mathbb{A}_0$ turns to be the hermitian matrix of the energy, already determined in the previous section,

$$\mathbb{A}_0(w) = \mathrm{d}^2 E \qquad \text{where} \quad E(h, \boldsymbol{v}) := \frac{1}{2}\,h\,|\boldsymbol{v}|^2 + \frac{1}{2\,\mathrm{Fr}^2}\,h^2.$$

The general theory for hyperbolic systems of conservation laws applies and the following statement holds concerning existence of solutions to the Cauchy problem for (14) with initial conditions

$$h(x,y,0) = h_0(x,y), \qquad h\boldsymbol{v}(x,y,0) = (h\boldsymbol{v})_0(x,y). \tag{15}$$

Theorem 3.1. *Let $h_0, (h\boldsymbol{v})_0 \in H^s$ for some $s > 2$ and $h_0 \geq c_0 > 0$ for some c_0. Then there exists $T > 0$ such that the Cauchy problem* (14)-(15) *has a unique classical solution belonging to $C^1(\mathbb{R}^2 \times [0,T])$ with* $\inf h > 0$.

Here we want to pay attention to the system for shallow water taking in account the viscosity term and we will not survey details on the proof and properties of solutions to the hyperbolic system (14); we suggest to the interested reader the classical reference book [**27**].

3.2 - *Adding viscosity: hyperbolic–parabolic systems*

The shallow water system with viscosity fits into the class of hyperbolic–parabolic system (for a short review on the local existence theory for a class of such system see [**45**]).

Here, we restrict to the one-dimensional case and flat bottom:

$$\begin{cases} \dfrac{\partial h}{\partial t} + \dfrac{\partial}{\partial x}(h\,u) = 0, \\ \dfrac{\partial}{\partial t}(h\,u) + \dfrac{\partial}{\partial x}\left(h\,u^2 + \dfrac{h^2}{2\,\mathrm{Fr}^2}\right) = \dfrac{4}{\mathrm{Re}}\,\dfrac{\partial}{\partial x}\left(h\,\dfrac{\partial u}{\partial x}\right). \end{cases} \tag{16}$$

Fixed a reference state $(\bar{h}, \bar{u})$, we consider a solution of the form $(\bar{h}, \bar{u}) + (h, u)$. Without restriction, by invariance with respect to galileian transformation, we can assume $\bar{u} = 0$. Then the perturbation (h, u) satisfies

$$\begin{cases} \dfrac{\partial h}{\partial t} + \bar{h}\,\dfrac{\partial u}{\partial x} + \dfrac{\partial}{\partial x}(h\,u) = 0, \\ \dfrac{\partial u}{\partial t} + \dfrac{\partial}{\partial x}\left(\dfrac{1}{2}\,u^2 + \dfrac{h}{\mathrm{Fr}^2}\right) = \dfrac{4}{\mathrm{Re}}\,\dfrac{1}{\bar{h}+h}\,\dfrac{\partial}{\partial x}\left((\bar{h}+h)\,\dfrac{\partial u}{\partial x}\right). \end{cases}$$

Dividing by $\bar{h}$ and denoting still by h the ratio $h/\bar{h}$, we get the system

$$\begin{cases} \dfrac{\partial h}{\partial t} + u\,\dfrac{\partial h}{\partial x} + (1+h)\dfrac{\partial u}{\partial x} = 0, \\ \dfrac{\partial u}{\partial t} + \dfrac{\partial}{\partial x}\left(\dfrac{1}{2}\,u^2 + g\,h\right) = \dfrac{\nu}{1+h}\,\dfrac{\partial h}{\partial x}\,\dfrac{\partial u}{\partial x} + \nu\,\dfrac{\partial^2 u}{\partial x^2} \end{cases} \tag{17}$$

where $g = \bar{h}/\mathrm{Fr}^2$ and $\nu := 4/\mathrm{Re}$, to be considered for $(h, u) \in (-1, \infty) \times \mathbb{R}$ where h now denotes the ratio between the perturbation of the height of the water with respect to the reference height $\bar{h}$ to $\bar{h}$ itself, i.e. $(h - \bar{h})/\bar{h}$, in the original variable.

To prove a local existence of solutions of the Cauchy problem for (17), we apply an iterating procedure based on the resolution of the (decoupled) hyperbolic–parabolic system

$$\frac{\partial h}{\partial t} + U\,\frac{\partial h}{\partial x} = F_1, \qquad \frac{\partial u}{\partial t} - \nu\,\frac{\partial^2 u}{\partial x^2} = F_2 \tag{18}$$

for given functions U, F_1, F_2. Iterative procedures of this kind have been used in the fluid-dynamics context since [**35**], where a local existence result for viscous, compressible, heat-conducting fluids is proved. Differently with respect to [**35**], where the author reduces the analysis to an iteration problem for a single parabolic equation for the speed variable, here we deal with a coupled hyperbolic-parabolic iterative system. The two procedures are conceptually equivalent, since reduction to a single parabolic equation amounts in incorporating into the coefficients of the reduced equation itself the resolution formulas of the hyperbolic part of the problem.

To start with, we determine some estimates on the solutions h and u to the previous equation needed to close the iterative argument, based on the following result on ordinary differential inequalities.

Lemma 3.1. *Given $b = b(t)$ such that $b(t) \geq 0$ for any t, let $y = y(t)$ be such that $y(t) > 0$ for any t and*

$$y'(t) + b(t) \leq 2\,a(t)\,\sqrt{y(t)} + M\,y(t)$$

then there holds

$$y(t) + \int_0^t b(\tau)\,e^{M(t-\tau)}\,d\tau \leq 2\left\{ y(0) + t\int_0^t a^2(\tau)\,e^{-M\tau}\,d\tau \right\} e^{Mt}. \tag{19}$$

Proof. First, we consider the case $M = 0$. Set

$$Y(t) := y(t) + B(t) \qquad \text{where} \quad B(t) := \int_0^t b(\tau)\,d\tau.$$

Since $b \geq 0$, $B \geq 0$ and $0 \leq y \leq Y$. Hence

$$Y'(t) \leq 2\,a(t)\,\sqrt{y(t)} \leq 2\,a(t)\,\sqrt{Y(t)}.$$

Then, integrating the relation $\big(\sqrt{Y}\big)' \leq a(t)$, we obtain

$$\sqrt{y(t) + B(t)} \leq \sqrt{y(0)} + \int_0^t a(\tau)\,d\tau.$$

Squaring and applying the Jensen inequality, we obtain

$$y(t) + B(t) \le 2\left\{ y(0) + t \int_0^t a^2(\tau)\, d\tau \right\}.$$

If M is any real number the function $z := y\, e^{-M\,t}$ satisfies the inequality

$$z'(t) + b(t)\, e^{-Mt} \le 2\, a(t)\, e^{-Mt/2} \sqrt{z(t)}.$$

Then, by Step 1, we infer

$$z(t) + \int_0^t b(\tau)\, e^{-M\tau}\, d\tau \le 2\left\{ y(0) + t \int_0^t a^2(\tau)\, e^{-M\,\tau}\, d\tau \right\}$$

and multiplying by e^{Mt} we obtain the conclusion. □

Next, we derive estimates for the solution of the non-homogeneous transport equation for the unknown h in (18). Multiplying by h and integrating, we get

$$\frac{d}{dt}|h|_{L^2}^2 \le 2\,|F_1|_{L^2}|h|_{L^2} + \int_{\mathbb{R}} (\partial_x U)\, h^2\, dx \le 2\,|F_1|_{L^2}|h|_{L^2} + |\partial_x U|_{L^\infty}\, |h|_{L^2}^2.$$

Applying the inequality (19) with $y = |h|_{L^2}^2$, $a = |F_1|_{L^2}$, $b \equiv 0$ and $M := \sup_{\tau\in[0,t]} |\partial_x U|_{L^\infty}(\tau)$, we deduce

$$|h|_{L^2}^2(t) \le 2\left\{ |h_0|_{L^2}^2 + t \int_0^t |F_1|_{L^2}^2(\tau)\, e^{-M\tau}\, d\tau \right\} e^{Mt}. \tag{20}$$

Differentiating with respect to x the equation of h in (18), we get

$$\frac{\partial}{\partial t}(\partial_x h) + \frac{\partial}{\partial x}(U\, \partial_x h) = \partial_x F_1.$$

Multiplying by $\partial_x h$, we obtain

$$\begin{aligned}
\frac{d}{dt}|\partial_x h|_{L^2}^2 &\le 2\,|\partial_x F_1|_{L^2}|\partial_x h|_{L^2} - 2\int \partial_x(U\, \partial_x h)\, \partial_x h\, dx \\
&\le 2\,|\partial_x F_1|_{L^2}|\partial_x h|_{L^2} + 2\int U\, \partial_x h\, \partial_x^2 h\, dx \\
&\le 2\,|\partial_x F_1|_{L^2}|\partial_x h|_{L^2} - \int \partial_x U\, (\partial_x h)^2\, dx \\
&\le 2|\partial_x F_1|_{L^2}|\partial_x h|_{L^2} + |\partial_x U|_{L^\infty}\, |\partial_x h|_{L^2}^2,
\end{aligned}$$

so that, still with $M := \sup_{\tau\in[0,t]} |\partial_x U|_{L^\infty}(\tau)$,

$$|\partial_x h|^2_{L^2}(t) \leq 2\left\{ |\partial_x h_0|^2_{L^2} + t \int_0^t |\partial_x F|^2_{L^2}(\tau)\, e^{-M\tau}\, d\tau \right\} e^{Mt}.$$

For $k \geq 2$, differentiating k-times with respect to x the equation of h, we get

$$(\partial_x^k h)_t + \partial_x^k (U\, \partial_x h) = \partial_x^k F_1$$

and, multiplying by $\partial_x^k h$ in L^2, we obtain

$$\frac{d}{dt} |\partial_x^k h|^2_{L^2} \leq 2|\partial_x^k F_1|_{L^2} |\partial_x^k h|_{L^2} - 2 \int \partial_x^k (U\, \partial_x h)\, \partial_x^k h\, dx.$$

Let us consider the integral term at the right-hand side. Since

$$\partial_x^k (U\, \partial_x h) = \sum_{j=0}^{k} \binom{k}{j} \partial_x^{k-j} U\, \partial_x^{j+1} h,$$

we need to estimate terms of the form

$$\int \partial_x^{k-j} U\, \partial_x^{j+1} h\, \partial_x^k h\, dx, \qquad j = 0, \dots, k.$$

There holds: for $j = 0$,

$$\begin{aligned}\left| \int \partial_x^k U\, \partial_x h\, \partial_x^k h\, dx \right| &\leq |\partial_x^k U|_{L^2}\, |\partial_x h|_{L^\infty}\, |\partial_x^k h|_{L^2} \\ &\leq |\partial_x U|_{H^{k-1}}\, |\partial_x h|^2_{H^1}\end{aligned}$$

for $j = 1, \dots, k-1$,

$$\begin{aligned}\left| \int \partial_x^{k-j} U\, \partial_x^{j+1} h\, \partial_x^k h\, dx \right| &\leq |\partial_x U|_{W^{k-2,\infty}}\, |\partial_x^{j+1} h|_{L^2}\, |\partial_x^k h|_{L^2} \\ &\leq |\partial_x U|_{H^{k-1}}\, |\partial_x h|^2_{H^1}\end{aligned}$$

for $j = k$,

$$\begin{aligned}\left| \int U\, \partial_x^{k+1} h\, \partial_x^k h\, dx \right| \leq \left| \int \partial_x U\, (\partial_x^k h)^2\, dx \right| &\leq |\partial_x U|_{L^\infty}\, |\partial_x^k h|^2_{L^2} \\ &\leq |\partial_x U|_{H^{k-1}}\, |\partial_x h|^2_{H^1}.\end{aligned}$$

Recalling that $\displaystyle\sum_{j=0}^{k} \binom{k}{j} = 2^k$ we obtain, for $k \geq 2$, the estimate

$$\frac{d}{dt} |\partial_x^k h|^2_{L^2} \leq 2|\partial_x^k F_1|_{L^2} |\partial_x^k h|_{L^2} + 2^{k+1}\, |\partial_x U|_{H^{k-1}}\, |\partial_x h|^2_{H^{k-1}}.$$

Therefore, summing with respect to k,

$$\frac{d}{dt}|\partial_x h|^2_{H^{k-1}} \leq 2|\partial_x F_1|_{H^{k-1}}|\partial_x h|_{H^{k-1}} + 2^{k+1}\,|\partial_x U|_{H^{k-1}}\,|\partial_x h|^2_{H^{k-1}}.$$

By setting $M_k[U] := \sup_{\tau\in[0,t]} |\partial_x U|_{H^{k-1}}(\tau)$, we get

$$|\partial_x h|^2_{H^{k-1}}(t) \leq 2\left\{ |\partial_x h_0|^2_{H^{k-1}} + t\int_0^t |\partial_x F_1|^2_{H^{k-1}}(\tau)\, e^{-C_k M_k[U]\tau}\, d\tau \right\} e^{C_k M_k[U]\, t}$$

for some constant C depending only on k.

Summing with (20), we obtain the following result.

Proposition 3.1. *Let $U, F_1 : \mathbb{R}\times[0,T]\to\mathbb{R}$ be functions such that, for some $s\geq 2$, there hold*

$$\partial_x U \in L^\infty([0,T];\, H^{s-1}), \qquad F_1 \in L^2([0,T];\, H^s),$$

and, for any integer $k\in[2,s]$, set $M_k[U] := \sup_{t\in[0,T]} |\partial_x U|_{H^{k-1}}$.

Then, if h solves the transport equation $\partial_t h + U\,\partial_x h = F_1$, there holds

$$|h|^2_{H^k}(t) \leq 2\left\{ |h_0|^2_{H^k} + t\int_0^t |F_1|^2_{H^k}(\tau)\, d\tau \right\} e^{C_k M_k[U]\, t}, \tag{21}$$

for some constant C_k depending only on k.

Now, we turn our attention to the solution of the non-homogeneous heat equation for the unknown u in (18). Since

$$\frac{\partial}{\partial t}(\partial_x^k u) - \nu\,\frac{\partial^2}{\partial x^2}(\partial_x^k u) = \partial_x^k F_2,$$

there holds

$$\frac{d}{dt}|\partial_x^k u|^2_{L^2} + 2\nu|\partial_x^{k+1} u|^2_{L^2} \leq 2|\partial_x^k F_2|_{L^2}|\partial_x^k u|_{L^2}.$$

Applying (19) with $b = 2\,\nu|\partial_x^{k+1}u|^2_{L^2}$, $a = |\partial_x^k F_2|_{L^2}$ and $M = 0$, we deduce

$$|\partial_x^k u|^2_{L^2}(t) + 2\,\nu\int_0^t |\partial_x^{k+1} u|^2_{L^2}(\tau)\, d\tau \leq 2\left\{ |\partial_x^k u_0|^2_{L^2} + t\int_0^t |\partial_x^k F_2|^2_{L^2}(\tau)\, d\tau \right\}. \tag{22}$$

Thanks to the presence of the operator ∂_x^2, we can get better estimates with respect to the regularity of F_2 by integrating by parts and moving a space derivative from the forcing term $\partial_x^k F_2$ to the derivative of the solution $\partial_x^k u$. Precisely, for $k \geq 1$, there holds

$$\frac{d}{dt}|\partial_x^k u|_{L^2}^2 + 2\nu|\partial_x^{k+1}u|_{L^2}^2 = 2\int \partial_x^k F_2\, \partial_x^k u\, dx = -2\int \partial_x^{k-1} F_2\, \partial_x^{k+1} u\, dx,$$

so that, by Cauchy–Schwarz inequality and Young inequalities, we get

$$\frac{d}{dt}|\partial_x^k u|_{L^2}^2 + 2\nu|\partial_x^{k+1}u|_{L^2}^2 \leq 2\left|\int \partial_x^{k-1} F_2\, \partial_x^{k+1} u\, dx\right| \leq \frac{1}{\nu}|\partial_x^{k-1}F_2|_{L^2}^2 + \nu|\partial_x^{k+1}u|_{L^2}^2$$

and thus we infer

$$\frac{d}{dt}|\partial_x^k u|_{L^2}^2 + \nu|\partial_x^{k+1}u|_{L^2}^2 \leq \frac{1}{\nu}|\partial_x^{k-1}F_2|_{L^2}^2.$$

By integrating in $[0,t]$, we readily obtain

$$|\partial_x^k u|_{L^2}^2(t) + \nu\int_0^t |\partial_x^{k+1}u|_{L^2}^2(\tau)\,d\tau \leq |\partial_x^k u_0|_{L^2}^2 + \frac{1}{\nu}\int_0^t |\partial_x^{k-1}F_2|_{L^2}^2(\tau)\,d\tau. \tag{23}$$

Collecting (22) and (23), we deduce the following result.

Proposition 3.2. *Let $F_2 : \mathbb{R}\times[0,T]\to\mathbb{R}$ be a function such that, for some integer $s \geq 1$,*

$$F_2 \in L^2([0,T]\,;\, H^{s-1}).$$

Then, if u is a solution to the non-homogeneous diffusion equation $\partial_t u - \nu\,\partial_x^2 u = F_2$, there holds for any $k \in [1,s]$,

$$|u|_{H^k}^2(t) + \nu\int_0^t |\partial_x u|_{H^k}^2(\tau)\,d\tau \leq 2\,|u_0|_{H^k}^2 + C\int_0^t |F_2|_{H^{k-1}}^2(\tau)\,d\tau \tag{24}$$

with $C = C(T,\nu) = 2T + \nu^{-1}$.

Estimates (21) and (24) can be applied to the linear non-homogeneous equations (18) and used to prove a local existence result by applying a fixed point argument.

Given $\sigma, T > 0$, consider the spaces

$$X_T^\sigma := C^0([0,T]; H^\sigma),$$
$$Y_T^\sigma := \{u \in C^0([0,T]; H^\sigma)\,:\, \partial_x u \in L^2([0,T]; H^\sigma)\}$$

with the norms

$$\|h\|_{X_T^\sigma} := \left\{ \sup_{t\in[0,T]} |h|^2_{H^\sigma}(t) \right\}^{1/2},$$

$$\|u\|_{Y_T^\sigma} := \left\{ \sup_{t\in[0,T]} |u|^2_{H^\sigma}(t) + \nu \int_0^T |\partial_x u|^2_{H^\sigma}(\tau)\, d\tau \right\}^{1/2}$$

and the product normed space

$$\mathcal{X}_T^\sigma = X_T^\sigma \times Y_T^\sigma, \qquad \|(h,u)\|_{\mathcal{X}_T^\sigma} := \left\{ \|h\|^2_{X_T^\sigma} + \|u\|^2_{Y_T^\sigma} \right\}^{1/2}.$$

Let $F_1 = F_1(w, \partial_x u)$ and $F_2 = F_2(w, \partial_x w)$ be such that $F_1(w,0) = F_2(w,0) = 0$ for any w. Given $W = (H, U)$, let $w = (h, u) = \mathcal{T} W$ be the solution to

$$\begin{cases} \partial_t h + U\, \partial_x h = F_1(W, \partial_x U), \\ \partial_t u - \nu\, \partial_x^2 u = F_2(W, \partial_x W) \end{cases}$$

with initial data $(h_0, u_0) \in H^{\sigma+1}$. In the case we are treating, functions F_1 and F_2 have the explicit form

$$F_1(w, \partial_x u) = -(1+h)\, \partial_x u,$$
$$F_2(w, \partial_x w) = -g\, \partial_x h - u\, \partial_x u + \frac{\nu}{1+h}\, \partial_x h\, \partial_x u.$$

Fixed $T > 0$, estimates (21) and (24), given for $k \geq 2$, give

$$\|h\|^2_{X_T^k} \leq 2 \left\{ |h_0|^2_{H^k} + T \int_0^T |F_1|^2_{H^k}(\tau)\, d\tau \right\} \exp\{C\, \|U\|_{Y_T^k}\, T\},$$
$$\|u\|^2_{Y_T^k} \leq 2\, |u_0|^2_{H^k} + C\, T \sup_{t\in[0,T]} |F_2|^2_{H^{k-1}},$$

that gives, if $\|U\|_{Y_T^k} \leq R$, then

$$\|(h,u)\|^2_{\mathcal{X}_T^k} \leq 2\, e^{C\,R\,T}\, |(h_0, u_0)|^2_{H^k} + 2\, e^{C\,R\,T}\, T \int_0^T |F_1|^2_{H^k}(\tau)\, d\tau + C\, T \sup_{t\in[0,T]} |F_2|^2_{H^{k-1}}$$

with for some constant $C > 0$ (independent on R).

Both functions F_1 and F_2 are such that $F(w, 0) = 0$ for any w. Hence, for $|w|_{H^1} \leq R$, we have

$$|F_1(w, \partial_x u)|^2_{L^2} \leq C_R\, |\partial_x u|^2_{L^2}, \qquad |F_2(w, \partial_x w)|^2_{L^2} \leq C_R\, |\partial_x w|^2_{L^2}$$

where C_R is a constant depending on R. Moreover, for any smooth function Φ, for $|f|_{H^{s+1}} \le R$, there holds

$$|\partial_x \Phi(f)|_{H^s} \le C_R |\partial_x f|_{H^s}.$$

Hence, for $|w|_{H^k} \le R$, we deduce

$$|F_1|^2_{H^{k-1}} \le C_R \left(|\partial_x w|^2_{H^{k-1}} + |\partial_x u|^2_{H^k}\right), \qquad |F_2|^2_{H^{k-1}} \le C_R\, |\partial_x w|^2_{H^{k-1}}.$$

Finally, we need to control the L^2 norm of $\partial_x^k F_1$. The function F_1 has the form $\Phi(w)\, u$ where $\Phi(h,u) = -(1+h)$. Since there holds, for $k \ge 2$,

$$\partial_x^k(\Phi(w)\,\partial_x u) = \Phi'(w)\,\partial_x^k w\,\partial_x u + \sum_{j=2}^{k} \Psi_j(w,\dots,\partial_x^{j-1} w)\,\partial_x^j u$$

then, for $|w|_{H^k} \le R$, we have

$$|\partial_x^k(\Phi(w)\,\partial_x u)|^2_{L^2} \le C_R\left(|\partial_x^k w|^2_{L^2} + |\partial_x^2 u|^2_{H^{k-1}}\right) \le C_R\left(|\partial_x w|^2_{H^{k-1}} + |\partial_x u|^2_{H^k}\right).$$

Therefore, for $|(h,u)|_{H^k} \le R$, we infer

$$|F_1|^2_{H^k} \le C_R\left(|\partial_x w|^2_{H^{k-1}} + |\partial_x u|^2_{H^k}\right).$$

As a consequence, for $\|W\|_{X^k_T} \le R$ we have the estimate

$$\begin{aligned}\|(h,u)\|^2_{\mathcal{X}^k_T} &\le 2\,e^{C_R T}|(h_0,u_0)|^2_{H^k} + C_R\,T\left(\sup_{t\in[0,T]} |\partial_x W|^2_{H^{k-1}} + \int_0^T |\partial_x U|^2_{H^k}(\tau)\,d\tau\right)\\ &\le 2\,e^{C_R T}\,|(h_0,u_0)|^2_{H^k} + C_R\,T|W|^2_{X^k_T}\\ &\le 2\,e^{C_R T}\,|(h_0,u_0)|^2_{H^k} + C_R\,R^2\,T.\end{aligned}$$

Therefore the function $w=(h,u)$ satisfies the same bound of $W=(H,U)$ for $t\in[0,T]$ if the following condition is satisfied

$$|(h_0,u_0)|^2_{H^k} \le \frac{1}{2}\,R^2(1 - C_R\,T)\,e^{-C_R T}.$$

The value at the right-hand side tends to $\frac{1}{2}\,R^2$ as $T \to 0$. Therefore, given the initial datum (h_0,u_0), by choosing $R \ge 2|(h_0,u_0)|_{H^k}$ we determine an invariant region for the transformation $\mathcal{T}$ whenever $T \le 1/C_R$.

As soon as the H^2 norm of u is bounded, also the L^∞ norm of $\partial_x u$ is. Hence, integrating along the characteristics, we infer

$$1 + h = (1+h_0)\exp\left(-\int_0^t \partial_x u\,dt\right) \ge \frac{\inf\,(1+h_0)(x)}{\exp(T\,\sup|\partial_x u|)}.$$

In particular, the function $1+h$ remains uniformly bounded away from 0 as soon as h_0 is and as soon as the value of R (bounding also the H^2 norm of u) is fixed.

Now that we have determined an invariant region for the transformation $\mathcal{T}$, we want to show that $\mathcal{T}$ is a contraction (choosing a smaller T, if needed). Given the unknown $W=(H,U)$ and $W+Z=(H+K,U+V)$, set

$$z=(k,v):=\mathcal{T}(H+K,U+V)-\mathcal{T}(H,U).$$

The function z solves the Cauchy problem for the system

$$\begin{cases} \partial_t k+(U+V)\,\partial_x k=\mathcal{F}_1:=-V\,\partial_x h+F_1(W+Z,\partial_x(U+V))-F_1(W,\partial_x U),\\ \partial_t v-\nu\,\partial_x^2 v=\mathcal{F}_2:=F_2(W+Z,\partial_x(W+Z))-F_2(W,\partial_x W)\end{cases}$$

with zero initial data. Our aim is to estimate $\|(k,v)\|_{\mathcal{X}_T^2}$ in terms of $\|(K,V)\|_{\mathcal{X}_T^2}$.

Fixed $\bar{T}>0$, by (21) and (24), we extrapolate

$$|(k,v)|^2_{H^2}(t)+\nu\int_0^t|\partial_x v|^2_{H^2}(\tau)\,d\tau\le C\left(t\int_0^t|\mathcal{F}_1|^2_{H^2}(\tau)\,d\tau+\int_0^t|\mathcal{F}_2|^2_{H^1}(\tau)\,d\tau\right).$$

We need to estimate the terms $\mathcal{F}_i$ for $i=1,2$.

Lemma 3.2. *Given $\Phi\in C^k$, $k\ge 1$, set $\Delta\Phi(f;g):=\Phi(f+g)-\Phi(f)$. Then for any $R>0$ there exists $C_R>0$ such that*

(25) $$|\Delta\Phi(f;g)|_{H^k}\le C_R\,|g|_{H^k}$$

for any f,g such that $|f|_{H^k},|f+g|_{H^k}\le R$.

Proof. We prove the statement by induction over k.

Consider the case $k=1$. For any $R>0$, there exists C_R such that

$$|\Delta\Phi(f;g)|_{L^2}\le C_R\,|g|_{L^2}$$

for any f,g such that $|f|_{H^1},|f+g|_{H^1}\le R$. Moreover, since

$$\partial_x\Delta\Phi=\Delta(\partial_x\Phi)=\Delta(\mathrm{d}\Phi f_x)=\Delta(\mathrm{d}\Phi)f_x+\mathrm{d}\Phi(f+g)\,g_x$$

we obtain

$$\begin{aligned}|\partial_x\Delta\Phi|_{L^2}&\le|\Delta(\mathrm{d}\Phi)f_x|_{L^2}+|\mathrm{d}\Phi(f+g)\,g_x|_{L^2}\\&\le|\Delta(\mathrm{d}\Phi)|_{L^\infty}\,|f_x|_{L^2}+|\mathrm{d}\Phi(f+g)|_{L^\infty}\,|g_x|_{L^2}\\&\le C_R\,|g|_{H^1}\,|f_x|_{L^2}+C_R\,|g_x|_{L^2}\le C_R\,|g|_{H^1}.\end{aligned}$$

Next, let the assertion to be true for $k-1$. We need to estimate

$$|\partial_x^k\Delta\Phi|_{L^2}=|\partial_x^{k-1}\Delta(\partial_x\Phi)|_{L^2}=|\partial_x^{k-1}\Delta(\,\mathrm{d}\Phi\,\partial_x f)|_{L^2}.$$

Since $\varDelta(\,\mathrm{d}\varPhi\,\partial_x f) = \varDelta(\mathrm{d}\varPhi)\,\partial_x f + \varPhi(f+g)\,\partial_x g$ and $|\partial_x^k(uv)|_{L^2} \leq C|u|_{H^k}|v|_{H^k}$, we obtain

$$\begin{aligned}|\partial_x^{k-1}\varDelta(\,\mathrm{d}\varPhi\,\partial_x f)|_{L^2} &\leq C|\varDelta(\mathrm{d}\varPhi)|_{H^{k-1}}|\partial_x f|_{H^{k-1}} + C|\varPhi(f+g)|_{H^{k-1}}|\partial_x g|_{H^{k-1}}\\ &\leq C_R\,|g|_{H^{k-1}}|\partial_x f|_{H^{k-1}} + C_R\,|g|_{H^k} \leq C_R\,|g|_{H^k}\end{aligned}$$

that gives the conclusion. □

Applying (25), we deduce that, for $\|W\|_{\mathcal{X}^3_T}, \|W+Z\|_{\mathcal{X}^3_T} \leq R$ bounded in H^3,

$$\begin{aligned}\|(k,v)\|^2_{H^2}(t) + \nu\int_0^t |\partial_x v|^2_{H^2}(\tau)\,d\tau &\leq C\,t\,\|(K,V)\|^2_{\mathcal{X}^2_T} + C\,t\int_0^t |V\,\partial_x h|^2_{H^2}(\tau)\,d\tau\\ &\leq C\,t\,\|(K,V)\|^2_{\mathcal{X}^2_T}\end{aligned}$$

and, taking the supremum over t, we obtain

$$\|(k,v)\|^2_{\mathcal{X}^2_T} \leq C\,t\,\|(K,V)\|^2_{\mathcal{X}^2_T}.$$

Therefore, for t sufficiently small, the transformation $\mathcal{T}$ is a contraction and it has a unique fixed point. Altogether, the above procedure prove the following result.

Theorem 3.2. *Given any reference state* $(\bar h, \bar u)$, *let* $(h_0 - \bar h, v_0 - \bar v) \in H^3(\mathbb{R})$ *with* $\inf h_0 > 0$. *Then there exists* $T > 0$ *such that the Cauchy problem for the system* (16) *has a unique* (*classical*) *solution* (h, v).

Let us briefly analyze how the situation changes for the two-dimensional model (without going in details). The system reads as

$$\text{(26)}\qquad \begin{cases}\dfrac{\partial h}{\partial t} + \mathrm{div}(h\,\boldsymbol{v}) = 0,\\[2mm] \dfrac{\partial}{\partial t}(h\,\boldsymbol{v}) + \mathrm{div}\left(h\,\boldsymbol{v}\otimes\boldsymbol{v} + \dfrac{1}{2\mathrm{Fr}^2}\,h^2\,I\right) = \dfrac{2}{\mathrm{Re}}\,\mathcal{V}[h,\boldsymbol{v}]\end{cases}$$

where

$$\mathcal{V}[h,\boldsymbol{v}] := \mathrm{div}\big(h(\mathrm{d}\boldsymbol{v})^*\big) + \mathrm{grad}\big(h\,\mathrm{div}\,\boldsymbol{v}\big).$$

Setting $\boldsymbol{v} = (u, v)$, the system can be rewritten as

$$\begin{cases}\dfrac{\partial h}{\partial t} + \dfrac{\partial}{\partial x}(h\,u) + \dfrac{\partial}{\partial y}(h\,v) = 0,\\[2mm] \dfrac{\partial u}{\partial t} + u\,\dfrac{\partial u}{\partial x} + v\,\dfrac{\partial u}{\partial y} + g\,\dfrac{\partial h}{\partial x} = \nu\,\mathcal{V}_1[h,u,v],\\[2mm] \dfrac{\partial v}{\partial t} + u\,\dfrac{\partial v}{\partial x} + v\,\dfrac{\partial v}{\partial y} + g\,\dfrac{\partial h}{\partial y} = \nu\,\mathcal{V}_2[h,u,v],\end{cases}$$

where $g = 1/\mathrm{Fr}^2$, $\nu = 2/\mathrm{Re}$ and

$$\mathcal{V}_1 := \frac{1}{h}\left\{2\frac{\partial}{\partial x}\left(h\,\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial x}\left(h\,\frac{\partial v}{\partial y}\right) + \frac{1}{2}\frac{\partial}{\partial y}\left(h\,\frac{\partial v}{\partial x}\right) + \frac{1}{2}\frac{\partial}{\partial y}\left(h\,\frac{\partial u}{\partial y}\right)\right\}$$

$$\mathcal{V}_2 := \frac{1}{h}\left\{\frac{1}{2}\frac{\partial}{\partial x}\left(h\,\frac{\partial v}{\partial x}\right) + \frac{1}{2}\frac{\partial}{\partial x}\left(h\,\frac{\partial u}{\partial y}\right) + \frac{\partial}{\partial y}\left(h\,\frac{\partial u}{\partial x}\right) + 2\frac{\partial}{\partial y}\left(h\,\frac{\partial v}{\partial y}\right)\right\}.$$

Therefore, the system can be recast as

$$\begin{cases} \dfrac{\partial h}{\partial t} + \boldsymbol{v}\cdot \operatorname{grad} h = f(h, \mathrm{d}\boldsymbol{v}), \\ \dfrac{\partial \boldsymbol{v}}{\partial t} - \nu\left(B^{11}\dfrac{\partial^2 \boldsymbol{v}}{\partial x^2} + 2B^{12}\dfrac{\partial^2 \boldsymbol{v}}{\partial x \partial y} + B^{22}\dfrac{\partial^2 \boldsymbol{v}}{\partial y^2}\right) = g(h, \boldsymbol{v}, \mathrm{d}h, \mathrm{d}\boldsymbol{v}), \end{cases}$$

where

$$B^{11} := \begin{pmatrix} 2 & 0 \\ 0 & \frac{1}{2} \end{pmatrix}, \qquad B^{12} := \begin{pmatrix} 0 & \frac{3}{4} \\ \frac{3}{4} & 0 \end{pmatrix}, \qquad B^{22} := \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 2 \end{pmatrix},$$

and $g := (g_1, g_2)$,

$$f(h, \mathrm{d}\boldsymbol{v}) := -h\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right),$$

$$g_1(h, \boldsymbol{v}, \mathrm{d}h, \mathrm{d}\boldsymbol{v}) := -u\,\frac{\partial u}{\partial x} - v\,\frac{\partial u}{\partial y} - g\,\frac{\partial h}{\partial x} + \frac{\nu}{h}\left(2\frac{\partial h}{\partial x}\frac{\partial u}{\partial x} + \frac{\partial h}{\partial x}\frac{\partial v}{\partial y} + \frac{1}{2}\frac{\partial h}{\partial y}\frac{\partial v}{\partial x} + \frac{1}{2}\frac{\partial h}{\partial y}\frac{\partial u}{\partial y}\right),$$

$$g_2(h, \boldsymbol{v}, \mathrm{d}h, \mathrm{d}\boldsymbol{v}) := -u\,\frac{\partial v}{\partial x} - v\,\frac{\partial v}{\partial y} - g\,\frac{\partial h}{\partial y} + \frac{\nu}{h}\left(\frac{1}{2}\frac{\partial h}{\partial x}\frac{\partial v}{\partial x} + \frac{1}{2}\frac{\partial h}{\partial x}\frac{\partial u}{\partial y} + \frac{\partial h}{\partial y}\frac{\partial u}{\partial x} + 2\frac{\partial h}{\partial y}\frac{\partial v}{\partial y}\right).$$

Such representation permits to prove a local existence theorem, by applying the approach presented in [**45**] and taken from the PhD thesis by S. Kawashima. We stress here that such strategy has been recently improved in [**47**] by significantly enlarging the class of initial data.

The strategy, following the same line of the one-dimensional case, requires the following key assumptions:

i. the second order operator in the equations for $\boldsymbol{v}$ is symmetric and the corresponding symbol is positive definite;

ii. the source terms f, g are zero at any constant state.

Assumption **ii.** is immediate consequence of the specific form of the functions f, g_1, g_2. Regarding the former request, we need to consider the matrix

$$\mathcal{B}(\xi,\eta) := B^{11}\xi^2 + B^{12}\xi\,\eta + B^{22}\eta^2 = \begin{pmatrix} 2\xi^2 + \dfrac{1}{2}\,\eta^2 & \dfrac{3}{2}\xi\eta \\ \dfrac{3}{2}\xi\eta & \dfrac{1}{2}\xi^2 + 2\,\eta^2 \end{pmatrix}$$

obtained by formally replacing derivatives with respect to x with multiplication by ξ and derivatives with respect to y with multiplication by η. Since $\det\mathcal{B}(\xi,\eta) = \xi^4 + \eta^4$, the corresponding bilinear form is positive definite (for $(\xi,\eta) \neq (0,0)$) and condition **ii.** is satisfied.

Hence, the following result holds.

Theorem 3.3. *Given any reference state* $(\bar{h}, \bar{\boldsymbol{v}})$, *let* $s \geq 4$ *and* $(h_0 - \bar{h}, v_0 - \bar{v}) \in H^s(\mathbb{R})$ *with* $\inf h_0 > 0$. *Then there exists* $T > 0$ *such that the Cauchy problem for the system* (26) *has a unique* (*classical*) *solution* $(h, \boldsymbol{v})$ *satisfying the regularity properties*

$$\begin{aligned} &h - \bar{h} \in C^0([0,T], H^s) \cap C^1([0,T], H^{s-1}), \\ &\boldsymbol{v} - \bar{\boldsymbol{v}} \in C^0([0,T], H^{s,2}) \cap C^1([0,T], H^{s-2}) \cap L^2([0,T], H^{s+1}) \end{aligned}$$

and

$$\begin{aligned} \sup_{t\in[0,T]} |(h - \bar{h}, \boldsymbol{v} - \bar{\boldsymbol{v}})|^2_{H^s} + \int_0^T \Big(|h - \bar{h}|^2_{H^s} + |\boldsymbol{v} - \bar{\boldsymbol{v}}|^2_{H^{s+1}} \Big)\, d\tau \\ \leq C\,|(h_0 - \bar{h}, \boldsymbol{v}_0 - \bar{\boldsymbol{v}})|^2_{H^s} \end{aligned}$$

for some constant $C > 0$ (*depending on* T).

See [**45**] for a sketch of the proof.

For any reasonable viscosity term, the same theory for hyperbolic-parabolic systems applies. As an example, in [**49**], the viscosity considered has the form

$$\mathcal{V}[h, \boldsymbol{v}] = \mathrm{div}\big(h\,\mathrm{d}\boldsymbol{v}\big) = \begin{pmatrix} \partial_x(h\,\partial_x u) + \partial_y(h\,\partial_y u) \\ \partial_x(h\,\partial_x v) + \partial_y(h\,\partial_y v) \end{pmatrix}$$

corresponding to $B^{11} = B^{22} = I$ and $B^{12} = 0$. Also in this case, the bilinear form $B^{11}\xi^2 + B^{12}\xi\,\eta + B^{22}\eta^2$ is symmetric positive definite and therefore the local existence theorem applies. In the case of the (energetically inconsistent) choice $\mathcal{V}[h, \boldsymbol{v}] = h\,\Delta\boldsymbol{v}$, the form of matrices B^{ij} is still $B^{11} = B^{22} = I$ and $B^{12} = 0$ and, even if energy does not decrease, local existence is guaranteed.

3.3 - *A glimpse to large-time behavior*

Exploring the large-time behaviour is, of course, a much more complicate problem. In fact, it encompasses two different matters: global existence and stability. Frequently, such aspects are treated together, since solutions possessing a special symmetry are often globally defined and asymptotically stable. As a consequence, initial data close to such distinct configurations likely generate solutions that are defined for any positive time and that converge to the special solution itself.

Here, we concentrate on the analysis of linear stability of a very particular class of solutions: the constant states. Linearizing a systems of partial differential equations at a reference solution consists in writing the equations satisfied by perturbations of the fixed state and then disregarding all of the non-linear terms. In the case of perturbations of a constant state, one is faced to treat a constant coefficient linear system of PDEs that can be converted in a system of linear ODEs by means of Fourier transform. Applying Inverse Transform procedure, it is possible to obtain precise information on the Green function of the original linearized problem. Such classical strategy has been explored in [**26**] in the case of a general class of hyperbolic–parabolic systems in one space dimension, containing also, as a special case, the shallow water system with flat topography and viscosity.

Here, we concentrate on the two-dimensional viscous system for shallow water with flat topography

$$\begin{cases} \dfrac{\partial h}{\partial t} + \mathrm{div}(h\,\boldsymbol{v}) = 0, \\ \dfrac{\partial}{\partial t}(h\,\boldsymbol{v}) + \mathrm{div}\left(h\,\boldsymbol{v}\otimes\boldsymbol{v} + \dfrac{1}{2}\,g\,h^2\,I\right) = \nu\,\mathcal{V}[h,\boldsymbol{v}] \end{cases} \tag{27}$$

where the viscosity $\mathcal{V}$ is given by (13) and $g = 1/\mathrm{Fr}^2$, $\nu = 2/\mathrm{Re}$. Let us fix a reference state $(\bar{h},\bar{\boldsymbol{v}})$ and consider initial data that are small perturbation of such state. Because of galileian invariance of the system, we can assume without loss of generality $\bar{\boldsymbol{v}}$ to be zero.

The perturbation $(h,\boldsymbol{v})$ satisfies the nonlinear system

$$\begin{cases} \dfrac{\partial h}{\partial t} + \mathrm{div}\big((\bar{h}+h)\,\boldsymbol{v}\big) = 0, \\ \dfrac{\partial}{\partial t}((\bar{h}+h)\,\boldsymbol{v}) + \mathrm{div}\left((\bar{h}+h)\,\boldsymbol{v}\otimes\boldsymbol{v} + \dfrac{1}{2}\,g\,(\bar{h}+h)^2\,I\right) = \nu\,\mathcal{V}[\bar{h}+h,\boldsymbol{v}]. \end{cases}$$

Assuming all nonlinear terms to be negligible, we obtain the linearized equations at $(\bar{h},\bar{\boldsymbol{v}})$ for the perturbation $(h,\boldsymbol{v})$

$$\begin{cases} \dfrac{\partial h}{\partial t} + \bar{h}\,\mathrm{div}\,\boldsymbol{v} = 0, \\ \dfrac{\partial \boldsymbol{v}}{\partial t} + g\,\mathrm{grad}\,h = \nu\,\mathcal{V}[1,\boldsymbol{v}]. \end{cases} \tag{28}$$

In the one-dimensional case, the speed $\boldsymbol{v}$ can be easily eliminated and the height perturbation h turns to solve the scalar equation

$$\frac{\partial^2 h}{\partial t^2} = g\,\bar{h}\,\frac{\partial^2 h}{\partial x^2} + \nu\,\frac{\partial^3 h}{\partial t\,\partial^2 x}.$$

In the absence of viscosity, the equation satisfied at the linearized level by the perturbation h is the wave equation; when $\nu > 0$, presence of the viscosity terms translates into the presence of a third order dissipating term (of Sobolev type).

Going back to the bi-dimensional case, when considering the Cauchy problem for (28), it is natural to follow an approach based on the Fourier–Laplace transform and looking for (complex-valued) solutions in the form $(h, \boldsymbol{v}) = (H, \boldsymbol{V}) \cdot \exp(\lambda\,t + i(\xi\,x + \eta\,y))$. The linearized system (28) transforms into the linear system

$$\mathbb{A}(\lambda, \xi, \eta)\begin{pmatrix} H \\ \boldsymbol{V} \end{pmatrix} = 0$$

where

$$\mathbb{A}(\lambda, \xi, \eta) = \begin{pmatrix} \lambda & i\,\bar{h}\,\xi & i\,\bar{h}\,\eta \\ i\,\gamma\,\xi & \lambda + 2\,\nu\,\xi^2 + \dfrac{1}{2}\,\nu\,\eta^2 & \dfrac{3}{2}\,\nu\,\xi\,\eta \\ i\,\gamma\,\eta & \dfrac{3}{2}\,\nu\,\xi\,\eta & \lambda + \dfrac{1}{2}\,\nu\,\xi^2 + 2\,\nu\,\eta^2 \end{pmatrix}.$$

Therefore, the dispersion relation of the linear system (28) is

$$\det\mathbb{A}(\lambda, \xi, \eta) = \lambda\left(\lambda^2 + \frac{5}{2}\,\nu\,\rho^2\,\lambda + \nu^2\,\rho^4\right) + \gamma\,\bar{h}\,\rho^2\left(\lambda + \frac{1}{2}\,\nu\,\rho^2\right) = 0,$$

where $\rho^2 := \xi^2 + \eta^2$.

Given (ξ, η), the sign of the real part of λ determines if the corresponding solution is dissipated or not. Poorly speaking, if $\mathrm{Re}\lambda \leq -c_0 < 0$, for some $c_0 > 0$, for any choice of (ξ, η), then the solution is stable, since any possible periodic perturbation decrease (exponentially fast) in time. Due to presence of a conservative structure, a simple modification of the reference height $\bar{h}$ produces a perturbation, arbitrary small in the uniform norm, that is not dissipated in time. Hence, the request of the real part of λ to be uniformly negative is never satisfied, since the choice $(\xi, \eta) = 0$ gives $\lambda = 0$. The typical prototype of PDE exhibiting such behavior is the classical linear heat equation (when considered in the whole space).

Inspired by the standard case of linear diffusion, we expect to find a slower rate decay (algebraic, instead of exponential) if a relaxed version of the request on the real part of $\lambda = \lambda(\xi, \eta)$ is satisfied: namely, $\mathrm{Re}\lambda \leq 0$ with equality holding if and only if $\lambda = 0$ and $\mathrm{Re}\,\lambda \leq -\,c_0 < 0$ for any $|\lambda| \geq \varepsilon > 0$ for some $c_0, \varepsilon > 0$. The rate of decay is then determined by the local behavior of λ at $(\xi, \eta) = (0, 0)$.

First of all, let us note that there are no values $(\xi,\eta)\neq(0,0)$ such that $\lambda\in i\mathbb{R}$. Indeed, assume by contradiction that for some $\lambda=i\theta$, $\theta\in\mathbb{R}$, there holds

$$\det\mathbb{A}(\xi,\eta,i\theta)=\frac{\nu}{2}\Big(\gamma\,\bar h\,\rho^2-5\,\theta^2\Big)\rho^2+i\,\theta\Big(\nu^2\rho^4+\gamma\,\bar h\,\rho^2-\theta^2\Big)=0,$$

then, we should also have, for some $\rho\neq 0$,

$$\gamma\,\bar h\,\rho^2=5\,\theta^2,\qquad \nu^2\rho^4+\gamma\,\bar h\,\rho^2=\theta^2$$

implying $\nu=\gamma\,\bar h=0$.

Formally, as $(\xi,\eta)\to(0,0)$, we deduce the following formal asymptotics expression for λ

$$\lambda=\pm i\sqrt{\gamma\,\bar h\,\rho^2}-\nu\,\rho^2+o(\rho^2),\qquad \lambda=-\frac{1}{2}\,\nu\,\rho^2+o(\rho^2)$$

as $\rho\to 0$, corresponding the three different advective-diffusive modes with characteristic speeds $\sqrt{\gamma\bar h}$ and 0. Hence, all the elementary modes with $(\xi,\eta)\neq(0,0)$ decays in time. We leave to the reader to check the behavior of λ as $\rho=\sqrt{\xi^2+\eta^2}\to\infty$.

These properties give an heuristic argument in favor of stability of the constant states (at least at linearized level) for the viscous equation with $\mathcal{V}$ given by (13). In [**49**], the asymptotic stability of constant states has been rigorously proved in the case of a viscosity of the form

$$\mathcal{V}[h,\boldsymbol{v}]=\operatorname{grad}(h\operatorname{div}\boldsymbol{v}).$$

In this case, it is possible to eliminate $\operatorname{div}\boldsymbol{v}$ from the linearized equation and show that the (linearized) perturbation h satisfies the third order partial differential equations

$$\frac{\partial^2 h}{\partial t^2}=\Delta\Big(g\,\bar h\,h+\nu\frac{\partial h}{\partial t}\Big),$$

that is analogous to the one already found in the case of one space dimension.

With a more detailed analysis, it is possible to determine the shape of asymptotic profile of perturbation of constant states, amounting in a superposition of nonlinear diffusion waves (see [**23**] for the one-dimensional case).

System (27) possesses also other special interesting solutions, in particular, the so-called viscous shock waves, i.e. planar travelling wave solutions connecting proper asymptotic states. Such kind of solutions has been widely explored in the context of gas-dynamics, particularly in one space dimension. A pionereeing result, based on energy estimates, is contained in [**33**]. The approach of stability by means of pointwise estimates of the resolvent kernel of the linearized equation has been performed in [**31, 32**] for general hyperbolic–parabolic systems, giving complete results relative to the nonlinear stability of small shocks for isentropic gas dynamics. More contributions relative to the case of large shocks can be found in [**1**].

4 - Roll–waves

A distinctive feature of the shallow water system is the presence of a term describing the topography of the region where the flow is occuring. The interaction between the dynamics of the fluid and the geometry of the physical domain generates a rich class of different behaviors, due to the fact that the topography term appear in the equation as a zero order term, possibly generating a sort of reactive response. The aim of the Section is to present a framework where transport mechanism and geometry structure cooperate in the formation of remarkable structures: the roll-waves. Such structures, frequently observed in reality and experimentally reproducible in laboratory, come out as surface signals propagated by the flow of water along an open channel over an incline. They consist in almost periodic patterns with wave-form and possessing smooth parts separated by breaking jumps. The regime of the flow is sub-critical at the left of the jump and super-critical at the right, meaning that the velocity of the fluid, relative to speed of the wave, rises from lower to higher value of the corresponding characteristic velocity.

As described by Dressler in [**17**], roll-waves are consistent with the shallow water description given by the hyperbolic Saint-Venant model, relative to topography with constant slope $\boldsymbol{m}$, *without viscosity* and *in presence of friction*. Mathematically, this means that the system

$$
\begin{cases} \dfrac{\partial h}{\partial t} + \mathrm{div}\,(h\,\boldsymbol{v}) = 0, \\ \dfrac{\partial}{\partial t}(h\,u) + \mathrm{div}\left(h\,\boldsymbol{v}\otimes\boldsymbol{v} + \dfrac{1}{2}\,g\,h^2\,\mathbb{I}\right) = g\,\boldsymbol{m}\,h - r(h,\boldsymbol{v})\boldsymbol{v}, \end{cases} \tag{29}
$$

supports planar traveling wave solutions, i.e. solutions of the form $(h,\boldsymbol{v})(\boldsymbol{x},t) = (H,\boldsymbol{V})(\boldsymbol{k}\cdot\boldsymbol{x} - ct)$, for some unit vector $\boldsymbol{k}$ and speed c, with properties similar to the physically observed phenomenon. The couple $(H,\boldsymbol{V})$ is the profile of the wave, and the constant $c \in \mathbb{R}$ is its speed. Coherently with the real observed phenomenon, the presence of the friction term $r(h,\boldsymbol{v})\boldsymbol{v}$, where $r = r(h,\boldsymbol{v})$ is a real-valued function to be specified later on, turns to be indispensable.

In this Section, we portray the existence theory of such solutions and we discuss the problem of stability, following the approach considered in [**36**]. Since we deal with the shallow water system without viscosity, the approach has to take into account the possibility of jumps at both level of existence and stability.

The analysis of roll-waves in the presence of viscosity is a widely open direction of research. In [**37**], existence and linear stability of periodic viscous roll waves has been analyzed. Recently, in [**22**], it has been proved that spectral stability, namely, the assumption that all of the point spectrum of the linearized operator at the periodic wave is contatined in the stable half-plane, implies both linear and nonlinear stability. Nowadays, more detailed results on stability properties are not available.

4.1 - *Traveling waves for a scalar balance law*

A simplified model for the description of roll–waves, considered in [**38**], is given by the following one-dimensional viscous scalar balance law

$$\frac{\partial u}{\partial t}+\frac{\partial}{\partial x}\left(\frac{1}{2}\,u^2\right)=a\,(u-c)+\nu\,\frac{\partial^2 u}{\partial x^2}.$$

Rather than regarding at this equation as a realistic alternative model, we consider it a starting benchmark case to understand the basic mathematical features of the problem.

Disregarding the viscosity term, $\nu=0$, and setting $u\mapsto u+c$, $x\mapsto x-ct$, we end up with the scalar hyperbolic balance law

$$\frac{\partial u}{\partial t}+\frac{\partial}{\partial x}\left(\frac{1}{2}\,u^2\right)=a\,u \qquad a>0. \tag{30}$$

A general study of traveling wave solutions for scalar balance laws can be found in [**29**]. Here, we concentrate on determining traveling wave solutions to (30): for $c\in\mathbb{R}$, inserting the ansatz $u(x,t)=U(x-ct)$ into the equation, we obtain

$$-cU'+\left(\frac{1}{2}\,U^2\right)'=a\,U \qquad\Longleftrightarrow\qquad U'=\frac{a\,U}{U-c}. \tag{31}$$

Solutions to this equation are implicitly given by the expression

$$a(\xi-\xi_0)=\int_{U_0}^{U(\xi)}\frac{s-c}{s}\,ds=U(\xi)-U_0-c\ln\left|\frac{U(\xi)}{U_0}\right|.$$

For $c\neq 0$, it is readily seen that the equation has no global solutions except the trivial one $U\equiv 0$. For $c=0$, the equation reduces to $U'=a$ and there exist a one-parameter family of global solution: $U(\xi)=a(\xi-\xi_0)$ for arbitrary $\xi_0\in\mathbb{R}$.

Motivated by the quest for bounded solution, we consider traveling waves with at least a discontinuity point. By translation invariance, we can assume without restriction that the jump occurs at $\xi=0$. Denoting by $u_\pm := u(0\pm)$, the values $u_\pm$ has to satisfy the Rankine–Hugoniot and the entropy condition, that is

$$c = \frac{u_+ + u_-}{2} \qquad u_+ < u_- .$$

As a consequence, $u_+ < c < u_-$.

For $c \neq 0$, there is no global solution even in the class of discontinuous traveling waves. Indeed, let us consider the case $c > 0$: assuming $\xi = 0$ to be a discontinuity point, for $\xi < 0$, the solution U stays above the level c and it reaches for some finite $\xi_1 < 0$ the level c itself. For $\xi < \xi_1$, it is not possible to prolunge the solution.

In the case $c = 0$, the situation is different. Both numerator and denominator of the right hand side of (31) vanish at $U = c = 0$ and the solutions can trespass the singularity of the equation. Thanks to this property, the set of bounded travelling waves with zero speed becomes very crowded...

Proposition 4.1. *The (steady) traveling waves ϕ of the equation* (30) *are in one-to-one correspondence with the open subsets A of the real line different from* $\mathbb{R}$, *the correspondence being determined by the condition*

$$\mathbb{R} \setminus A = \{x \in \mathbb{R} : \phi(x) = 0\}.$$

Such solutions are bounded if and only if the corresponding open set A does not contain any unbounded interval of the form $(-\infty, b)$ *or* $(a, +\infty)$. *Finally, the traveling wave is periodic if and only if the set A is periodic.*

Proof. Let A be an open subset of $\mathbb{R}$. Then there exists a countable union of pairwise disjoint open intervals (a_k, b_k), $k \in \mathbb{N}$, such that

$$A = \bigcup_{k \in \mathbb{N}} (a_k, b_k).$$

If (a_k, b_k) is bounded, we define the solution U in (a_k, b_k) by setting

$$U(\xi) = a(\xi - a_k) \quad \xi \in \left[a_k, \frac{a_k + b_k}{2}\right], \qquad U(\xi) = a(\xi - b_k) \quad \xi \in \left[\frac{a_k + b_k}{2}, b_k\right].$$

For $(a_k, +\infty)$, we set $U(\xi) = a(\xi - a_k)$ and, similarly, for $(-\infty, b_k)$, $U(\xi) = a(\xi - b_k)$. We task the reader to verify that the solution defined is an entropy stationary solution.

Viceversa, let U be a traveling wave solution of the balance equation. Let $C := \{\xi \in \mathbb{R} : U(\xi) = 0\}$. Then it is possible to prove that the set C is closed and that the solution U is given by the above construction relative to the open set $A = \mathbb{R} \setminus C$. $\square$

All of the traveling waves just built are (highly) unstable. Indeed, solutions of (30) with initial data in L^1 satisfy the relation

$$\int_{\mathbb{R}} u(x,t)\,dx = e^{a\,t}\int_{\mathbb{R}} u_0(x)\,dx.$$

The traveling waves previously described are stationary solutions and they necessarily have zero mass. Any small perturbation corresponding to an initial datum with non-zero initial mass gives raise to a solution with exponential rapidly diverging mass. A finer description of the dynamics, based on the theory of generalized characteristics, can be found in [**30**], showing that the family of traveling wave solutions describe completely the asymptotic behavior of solutions of the Cauchy problem for equation (30) (also, if the zero-mass constraint is forced, stability of the waves family is completely recovered).

4.2 - *Roll-waves for the Saint-Venant system*

Next, we turn our attention to the system (29), where, to be uniform with Section 2, we choose $r_0(h,\boldsymbol{v}) = r_0 > 0$. This choice is different with respect to the one selected by Dressler [**17**] (and considered in [**36**]). Assuming a topography of the form

$$Z(x,y) = -m\,x$$

and looking for solution with vanishing second component of the velocity $\boldsymbol{v}$, we reduce to the one-dimensional hyperbolic system

$$\begin{cases} \dfrac{\partial h}{\partial t} + \dfrac{\partial}{\partial x}(h\,u) = 0, \\ \dfrac{\partial}{\partial t}(h\,u) + \dfrac{\partial}{\partial x}\left(h\,u^2 + \dfrac{1}{2}\,g\,h^2\right) = g\,m\,h - r_0\,u. \end{cases} \tag{32}$$

A different frame could be considered where the height of the water is calculated perpendicularly to the slope itself, the velocity parallel to it and the horizontal coordinate is measured in the direction of the slope. The two description are not completely equivalent when considering solutions with jumps because of the different meaning of the height variable.

As in the toy scalar model, also in the case of the hyperbolic shallow water system, roll-waves profiles turn to be necessarily discontinuous; thus, we primarily concentrate on the problem of determining the entropy jump conditions for the hyperbolic system under consideration.

The admissibility conditions for jumps of the non-homogenous system (32) are the same of the homogeneous hyperbolic system

$$(33)\qquad \begin{cases} \dfrac{\partial h}{\partial t} + \dfrac{\partial}{\partial x}(h\,u) = 0, \\ \dfrac{\partial}{\partial t}(h\,u) + \dfrac{\partial}{\partial x}\left(h\,u^2 + \dfrac{1}{2}\,g\,h^2\right) = 0. \end{cases}$$

Such conditions are determined by the choice of a couple entropy/entropy flux, that, in the present setting, are given by

$$\mathcal{E}(h,u) := \frac{1}{2}h\,u^2 + \frac{1}{2}g\,h^2, \qquad \mathcal{Q}(h,u) := \frac{1}{2}\,h\,u^3 + g\,h^2\,u,$$

corresponding to the physical energy/energy flux of the system.

Given $h_\pm > 0$, $u_\pm, c \in \mathbb{R}$, let (h_-, u_-) and (h_+, u_+) be an entropic discontinuity of (33) with speed c, that is we assume that the function

$$(H,U)(x,t) := \begin{cases} (h_-, u_-) & \text{for } x < ct, \\ (h_+, u_+) & \text{for } x > ct, \end{cases}$$

is a weak solution satisfying, in the sense of distributions, the entropy inequality

$$(34)\qquad \frac{\partial \mathcal{E}}{\partial t} + \frac{\partial \mathcal{Q}}{\partial x} \leq 0.$$

The request of weak solution translates in the Rankine–Hugoniot conditions

$$[h(u-c)] = 0, \qquad \left[h\,u\,(u-c) + \frac{1}{2}\,g\,h^2\right] = 0,$$

where $[f] := f_+ - f_-$ denotes the jump of the function f. Setting $v := u - c$, we obtain

$$(35)\qquad [h\,v] = 0, \qquad \left[h\,v^2 + \frac{1}{2}\,g\,h^2\right] = 0.$$

The entropy condition (34) reads as $[\mathcal{Q} - c\,\mathcal{E}] \leq 0$. Still denoting $u - c$ by v, there holds

$$\mathcal{Q} - c\,\mathcal{E} = \frac{1}{2}\,h\,v^3 + g\,h^2 v + c\left(h\,v^2 + \frac{1}{2}\,g\,h^2\right) + \frac{1}{2}\,c^2\,h\,v.$$

Hence, by using (35), the entropy condition (34) translate into

$$(36)\qquad \left[\frac{1}{2}\,h\,v^3 + g\,h^2 v\right] \leq 0.$$

By squaring the first equation in (35), we obtain a system for the quantities $v_\pm^2$:

$$h_+^2 v_+^2 - h_-^2 v_-^2 = 0, \qquad h_+ v_+^2 - h_- v_-^2 = \frac{1}{2}\,g(h_-^2 - h_+^2)$$

whose solutions are

$$v_+^2 = \frac{g\,h_-}{2\,h_+}\,(h_- + h_+), \qquad v_-^2 = \frac{g\,h_+}{2\,h_-}\,(h_- + h_+).$$

Inserting these relations in (36), we obtain

$$\begin{aligned} 0 \geq \left[\frac{1}{2}\,h\,v^3 + g\,h^2 v\right] &= \frac{1}{2}\,h_+\,v_+\{v_+^2 - v_-^2 + 2g\,(h_+ - h_-)\} \\ &= -\frac{g\,v_+}{4\,h_-}(h_+ - h_-)^3. \end{aligned}$$

Hence, the entropy condition is satisfied (if and) only if

(37) $$(u_+ - c)(h_+ - h_-) \geq 0.$$

In particular, if $u_\pm > c$, then $h_- < h_+$ so that the jump condition describe the realistic phenomenon of the hydraulic jump consisting in an abrupt rise of the fluid surface and a corresponding decrease of the velocity. Conversely, if $u_\pm < c$, then $h_- > h_+$. As we will see later on, the latter situation enters into play in the case of roll-waves.

In the case $u_\pm > c$, since $h_- < h_+$, we also have

$$\frac{v_+^2}{g\,h_+} = \frac{h_-}{h_+}\cdot\frac{h_- + h_+}{2\,h_+} < 1 < \frac{h_+}{h_-}\cdot\frac{h_- + h_+}{2\,h_-} = \frac{v_-^2}{g\,h_-},$$

and, similarly, if $u_\pm < c$,

$$\frac{v_+^2}{g\,h_+} = \frac{h_-}{h_+}\cdot\frac{h_- + h_+}{2\,h_+} > 1 > \frac{h_+}{h_-}\cdot\frac{h_- + h_+}{2\,h_-} = \frac{v_-^2}{g\,h_-}.$$

These formulas means that when the velocity is greater than the speed of propagation of the jump, across the shock, the fluid jumps from super-critical ($|u_- - c| > \sqrt{gh_-}$) to sub-critical regime ($|u_+ - c| < \sqrt{gh_+}$); in the opposite case, the fluid jumps from sub-critical regime to super-critical regime.

Remark 4.1. An alternative procedure to obtain the entropy condition on the jump is to determine the traveling wave solutions for the viscous Saint-Venant system and then taking the vanishing viscosity limit. Precisely, let us look for solutions $(H(x - ct), U(x - ct))$ to the hyperbolic–parabolic system

$$\begin{cases} \dfrac{\partial h}{\partial t} + \dfrac{\partial}{\partial x}(h\,u) = 0, \\ \dfrac{\partial}{\partial t}(h\,u) + \dfrac{\partial}{\partial x}\left(h\,u^2 + \dfrac{1}{2}\,g\,h^2\right) = \nu\dfrac{\partial}{\partial x}\left(h\,\dfrac{\partial u}{\partial x}\right) \end{cases}$$

the couple (H,U) satisfies

$$\begin{cases} \big(H(U-c)\big)' = 0, \\ \Big(H\,U(U-c) + \dfrac{1}{2}\,g\,H^2\Big)' = \nu(H\,U')'. \end{cases}$$

Setting $V = U - c$, assuming $(H,V)(-\infty) = (h_-, v_-)$, and integrating in $(-\infty,\xi)$, we obtain

$$\begin{cases} H\,V = h_-\,v_-, \\ H\,V^2 + \dfrac{1}{2}\,g\,H^2 - h_-\,v_-^2 - \dfrac{1}{2}\,g\,h_-^2 = \nu\,H\,V' = -\nu\,H'\,V, \end{cases}$$

that is, by eliminating the variable V,

$$H' = -\frac{g}{2\,\nu\,h_-\,v_-}\left\{H^3 - \frac{2}{g}\Big(h_-\,v_-^2 + \frac{1}{2}\,g\,h_-^2\Big)H + \frac{2\,h_-^2\,v_-^2}{g}\right\}.$$

Factorising, we obtain the autonomous equation

$$H' = -\frac{g}{2\,\nu\,h_-\,v_-}(H-h_-)(H-h_+)(H+h_-+h_+),$$

from which we deduce that an heteroclinic connection from h_- to h_+ exists if and only if condition (37) is satisfied.

Now, we are ready to run into the existence problem of roll-waves for (32). Inserting the *ansatz* $(h,u)(x,t) = (H,U)(x-ct)$ with $c > 0$ to be determined, we obtain the system of ordinary differential equation

$$\begin{pmatrix} U-c & H \\ g & U-c \end{pmatrix}\begin{pmatrix} H' \\ U' \end{pmatrix} = \frac{1}{H}\begin{pmatrix} 0 \\ g\,m\,H - r_0\,U \end{pmatrix}$$

or, in normal form,

$$\begin{cases} H' = \dfrac{r_0\,U - g\,m\,H}{(U-c)^2 - gH}, \\ U' = \dfrac{(c-U)(r_0\,U - g\,m\,H)}{H[(U-c)^2 - gH]}. \end{cases}$$

By the conservation of mass, $H(c-U) = \kappa \in \mathbb{R}$; hence all trajectories lives on some hyperbola with asymptotics lines $H = 0$ and $U = c$. Consistently with real roll-waves structure, from now on, we concentrate on the case $\kappa > 0$. By using the relation $U(H) = (c\,H - \kappa)/H$, we can write a scalar differential equation for the variable H

(38) $$H' = \frac{H(g\,m\,H^2 - c\,r_0\,H + \kappa\,r_0)}{gH^3 - \kappa^2},$$

which possesses only monotone solutions for any value of the parameter κ. Therefore, oscillating behavior may appear only admitting the possibility of discontinuous solutions.

Let ξ_0 be a jump point of the wave profile. Then, setting $H_\pm := H(\xi_0 \pm)$ and $U_\pm := U(\xi_0 \pm)$, there holds

$$\frac{(U_- - c)^2}{g H_-} < 1 < \frac{(U_+ - c)^2}{g H_+}.$$

In order to have a second jump point, it is necessary that the ratio $\frac{(U_+ - c)^2}{g H_+}$ cross (decreasingly) the threshold 1. Because of translation invariance, we can assume without restriction that the value 1 is reached at $\xi = 0$. In order for such crossing point to exist, the couple (H_0, U_0) of the value of (H, U) at 0 has to satisfy the relations

$$r_0 U - g m H = 0, \qquad (U - c)^2 - gH = 0;$$

thus, setting $m_0 := m/r_0$,

$$H_0 = \frac{\sqrt{1 + 4 m_0 c} - 1}{\sqrt{1 + 4 m_0 c} + 1} \frac{c}{g m_0}, \quad U_0 = \frac{\sqrt{1 + 4 m_0 c} - 1}{\sqrt{1 + 4 m_0 c} + 1} c. \tag{39}$$

The corresponding value for κ is $\kappa_0 := (c - U_0) H_0$. Note that the couple (H_0, U_0) is a constant solution of the system (32).

For such choice, equation (38) reduces to

$$H' = P(H) := \frac{mH(H - H_1)}{H^2 + H_0 H + H_0^2} \qquad \text{where } H_1 := \frac{c}{g m_0} - H_0. \tag{40}$$

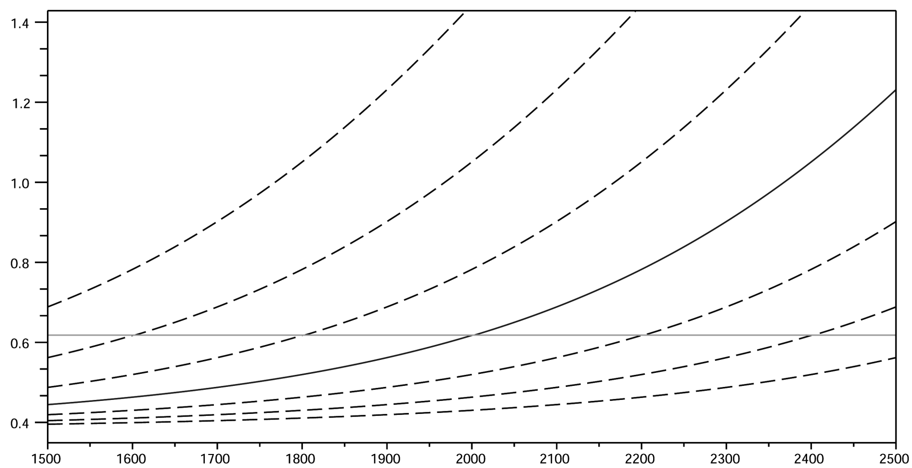

Fig. 2. The reference profile $\hat{H} = \hat{H}(\xi)$ (continuous line) together with some of its translations $\hat{H}_\delta = \hat{H}(\xi + \delta)$ (dashed lines) and the critical height H_0.

We denote by $\hat{H}$, the solution of the Cauchy problem for (40) determined by the initial condition $H(0) = H_0$. The function $\hat{H}$ is globally defined and it is monotone increasing if and only if $H_1 < H_0$, that is if and only if

$$\frac{c}{g\,m_0} < 2\,H_0.$$

By substitution, this condition reduces to

$$\frac{\sqrt{1+4\,m_0\,c}-1}{\sqrt{1+4\,m_0\,c}+1} > \frac{1}{2}$$

that gives the condition $m_0\,c > 2$. In this case, the asymptotics of $\hat{H}$ are described by the limits $\hat{H}(-\infty) = H_1$ and $\hat{H}'(+\infty) = m > 0$.

In order to construct a roll-wave, we start by following the reference solution $\hat{H}$ in an interval of the form $[0, \xi_*]$ where ξ_* is the first jump point at the right hand side of 0. Because of the Rankine–Hugoniot conditions, the left/right limits

$$(H_\pm, U_\pm) := \lim_{\varepsilon\to 0^+} (H, U)(\xi_* \pm \varepsilon)$$

enjoy the relations

$$H_\pm(c - U_\pm) = \kappa_0, \qquad \frac{1}{2}\,g\,H_-^2 - \kappa_0\,U_- = \frac{1}{2}\,g\,H_+^2 - \kappa_0\,U_+\,.$$

Hence, given H_-, the value H_+ is the unique value such that

$$H_+ < H_0, \qquad \text{and} \qquad F(H_+) = F(H_-),$$

where

$$F(\sigma) = \frac{1}{2}\,g\,\sigma^2 + \frac{\kappa_0^2}{\sigma} - \kappa_0 c\,. \tag{41}$$

The solution H at the right of ξ_* is determined by the solution of (40) with initial condition $H(\xi_*) = H_+$ and such solution is given by the same profile $\hat{H}$ shifted by some amount if and only if $H_+ > H_1$. Hence, denoting by H_2 the unique value in $(0, +\infty)$ such that $F(H_1) = F(H_2)$, a jump at ξ_* lead to a shifted profile $\hat{H}_\delta(\xi) := \hat{H}(\xi + \delta)$ for some δ if and only if $H_- < H_2$. By joining many pieces of translated profiles $\hat{H}$, it is possible to build many different roll-wave profiles corresponding to the same speed c.

We are ready to estabilish the following result.

Proposition 4.2. *Assume $m, r_0, g > 0$. Then, for any c such that*

$$c > c_* := \frac{2\,r_0}{m}, \tag{42}$$

the bounded traveling waves with speed c of the system (32) *are in one-to-one correspondence with the open subsets A of the real line that do not contain any unbounded interval of the form* $(a, +\infty)$. *The correspondence is determined by the condition*

$$\mathbb{R} \setminus A = \{\xi \in \mathbb{R} : H(\xi) = H_0\}.$$

Finally, the traveling wave is periodic if and only if the set A is periodic.

Proof. We can proceed exactly in the same spirit of Proposition 4.1. Given an open subset A of $\mathbb{R}$, we can decompose it as a countable union of pairwise disjoint open intervals (a_k, b_k). For (a_k, b_k) bounded, we define the solution H in (a_k, b_k) by setting

$$H(\xi) = \hat{H}(\xi - a_k) \quad \xi \in [a_k, \xi_k], \qquad H(\xi) = \hat{H}(\xi - b_k) \quad \xi \in [\xi_k, b_k],$$

where the jump point ξ_k is dictated by the condition $F(H(\xi_k -)) = F(H(\xi_k +))$, where the function F is defined in (41). Such condition dictate a single value of ξ, since the map $F(\hat{H}(\xi - a_k)) - F(\hat{H}(\xi - b_k))$ is monotone decreasing and

$$F(\hat{H}(\xi - a_k)) - F(\hat{H}(\xi - b_k))\Big|_{\xi = a_k} < 0 < F(\hat{H}(\xi - a_k)) - F(\hat{H}(\xi - b_k))\Big|_{\xi = b_k}.$$

If (a_k, b_k) is unbounded, then $a_k = -\infty$ and $b_k = b \in \mathbb{R}$ and we simply set $H(\xi) = \hat{H}(\xi - b)$ for any $\xi < b$. □

The condition on the speed (42) can be interpreted in term of stability/instability of the value (H_0, U_0). Indeed, the dispersion relation for the linearization of (32) at $(\bar{H}, \bar{U})$, with $g\, m\, \bar{H} - r_0 \bar{U} = 0$, is

$$\lambda^2 + \left(2\, \bar{U}\, \mu + \frac{r_0}{\bar{H}}\right)\lambda + \left(g\, m + \frac{r_0\, \bar{U}}{\bar{H}}\right)\mu + (\bar{U}^2 - g\, \bar{H})\, \mu^2 = 0$$

where λ and μ denote time and space derivative, respectively. For $\mu = i\, \xi$ and $\lambda = i\, \theta$, the dispersion relation becomes

$$g\, \bar{H}\, \xi^2 - (\theta + \bar{U}\, \xi)^2 + \frac{r_0}{\bar{H}}(\theta + 2\, \bar{U}\, \xi)\, i = 0.$$

Hence, purely imaginary values for λ corresponding to $\xi \neq 0$ appears if and only if $\theta = -2\, \bar{U}\, \xi$ and $g\, \bar{H} - \bar{U}^2 = 0$, translating into

$$\bar{H} = H_* := \frac{r_0^2}{g\, m^2}.$$

Moreover, the expansion for the branch $\lambda = \lambda(\mu)$ such that $\lambda(0) = 0$ is given by

$$\lambda = -2\, \bar{U}\, \mu + \frac{g\, m^2}{r_0^3\, \bar{H}}\left(H_* - \bar{H}\right) \mu^2 + o(\mu^2) \qquad \mu \to 0,$$

showing that, for small μ, $\mu = i\,\xi$, the real part of λ is negative. Therefore, the equilibrium state $(\bar{H}, \bar{U})$ is linearly stable (linearly unstable, respectively) if and only if $\bar{H} < H_*$ ($\bar{H} > H_*$, resp.).

From (39) we infer that H_0 is increasing with respect to c and $H_0 \to H_*$ as $c \to c_*$, with c_* defined in (42). Hence, the admissible value for the speed c are all the ones that correspond to unstable values for the equilibrium state (H_0, U_0).

In case of different choices for the friction term, one can still prove statements in the same spirit. Of course, the basic condition on the parameters strongly depends on the specific considered case.

4.3 - *Stability for discontinuous waves*

Analyzing the stability of roll-waves for the unviscous shallow water system is a delicate issue because of the presence of discontinuities. The case of scalar equations with convex flux is simplified by the possibility of using the theory of generalized characteristics, introduced in [**14, 15**]. Such approach is well-defined also in the system case (as soon as one deals with genuinely nonlinear hyperbolic systems) but is much more complicate to employ.

A different approach consists in assuming from the beginning the structure of the set of discontinuities of the solution and to deal with a system consisting of the differential system (to be satisfied in the region where the solution is smooth) and a number of algebraic constraints (given by both Rankine–Hugoniot and entropy conditions, to be satisfied along the jump set).

First of all, let us consider the case of a scalar hyperbolic balance law

$$\frac{\partial u}{\partial t} + \frac{\partial f(u)}{\partial x} = g(u), \tag{43}$$

where f and g are smooth functions with f strictly convex. Given a steady state $U = U(x)$ with a single jump point at $x = 0$, let $U^{\pm}$ be the values $U(0\pm)$. By the entropy condition, we know that the following inequalities hold

$$f'(U^+) < 0 < f'(U^-).$$

Let us consider a solution $\tilde{u}$ to the scalar equation (43) having the form

$$\tilde{u}(x,t) = \begin{cases} U(x - \xi(t)) + u^-(x - \xi(t), t) & x < \xi(t), \\ U(x - \xi(t)) + u^+(x - \xi(t), t) & x > \xi(t), \end{cases}$$

for some (small) $\xi = \xi(t)$. Then the functions $u^{\pm}$ solve for $\pm y > 0$,

$$\frac{\partial u^{\pm}}{\partial t} + \big(f'(U + u^{\pm}) - \xi'\big)\frac{\partial u^{\pm}}{\partial y} = g(U + u^{\pm}) - f'(U + u^{\pm})\,U' + \xi'\,U', \tag{44}$$

where U, U' are calculated at $y = x - \xi(t)$. For $u^\pm$ and ξ' small, there hold $f'(U + u^\pm) - \xi' \sim f'(U^\pm)$. Hence, the characteristic speed is positive for $y < 0$ and negative for $y > 0$; therefore there is no need of boundary conditions at $y = 0$ in both equations for $u^\pm$.

The shock location ξ is determined by the usual Rankine–Hugoniot condition

$$\xi' \big[U\big]_0 + \xi' \big(u^+(0,t) - u^-(0,t)\big) = f\big(U^+ + u^+(0,t)\big) - f\big(U^- - u^-(0,t)\big) \tag{45}$$

to be interpreted as a nonlinear transmission condition for $u^\pm$ at ξ.

Since $f'(U)\,U' = g(U)$, there holds

$$g(U(y) + u) - f'(U(y) + u)\,U'(y) = \big(g'(U) - f''(U)\,U'\big)\,u + \text{h.o.t.}$$

and the linearization of the system (44) and (45) is

$$\begin{cases} \dfrac{\partial u^\pm}{\partial t} + f'(U)\,\dfrac{\partial u^\pm}{\partial y} = \big(g'(U) - f''(U)\,U'\big)\,u^\pm + \xi'\,U' & \pm y > 0, \\ \xi' \big[U\big]_0 = f'(U^+)\,u^+(0,t) - f'(U^-)\,u^-(0,t) \end{cases}$$

where U, U' are calculated at y.

Following the classical Laplace–Fourier point of view, given $\lambda \in \mathbb{C}$, we look for solution with the form

$$u^\pm(y,t) = e^{\lambda t}\,v^\pm(y), \qquad \xi(t) = e^{\lambda t}\,\zeta \tag{46}$$

with $v^\pm, \zeta$ to be determined and we readily obtain the eigenvalue problem

$$\begin{cases} f'(U)\,\dfrac{dv^\pm}{dy} = \Big(g'(U) - f''(U)\,U' - \lambda\Big)\,v^\pm + \lambda\,U'\,\zeta & \pm y > 0, \\ \lambda\big[U\big]_0\,\zeta = f'(U^+)\,v^+(0) - f'(U^-)\,v^-(0). \end{cases} \tag{47}$$

As a basic example, let us consider equation (30), i.e. let us choose

$$f(s) = \frac{1}{2}\,s^2, \qquad g(s) = a\,s$$

and a corresponding stationary solution $U = U(x)$ such that $U'(x) = a$ for any $x \neq 0$. Then the system (47) becomes

$$\begin{cases} U\,\dfrac{dv^\pm}{dy} + \lambda\,v^\pm = a\,\lambda\,\zeta & \pm y > 0, \\ \lambda\big[U\big]_0\,\zeta = U^+\,v^+(0) - U^-\,v^-(0). \end{cases}$$

Multiplying the equation for $v^\pm$ by $U^{\lambda/a-1}$ and recalling that $U' = a$, we get

$$\frac{d}{dy}(U^{\lambda/a}\,v^\pm) = U^{\lambda/a}\,\frac{dv^\pm}{dy} + \frac{\lambda}{a}\,U^{\lambda-1}\,U'\,v^\pm = \lambda\,\zeta\,U^{\lambda/a-1}\,U' = \frac{d}{dy}(a\,\zeta\,U^{\lambda/a}).$$

Hence

$$v^{\pm}(y) = \zeta\, U'(y) + \frac{C}{U^{\lambda/a}(y)} = a\,\zeta + \frac{C}{U^{\lambda/a}(y)} \qquad C \in \mathbb{R}. \tag{48}$$

Since U becomes zero at finite y and we look for bounded solution, for $\mathrm{Re}\lambda > 0$, we are forced to choose $v^{\pm}(y) = a\,\zeta$. Therefore, the above system reduces to the single equation

$$U^{+}\, a\,\zeta - U^{-}\, a\,\zeta - \lambda\big[U\big]_0\,\zeta = \big[U\big]_0(a-\lambda)\zeta = 0.$$

For $\lambda \neq a$, the system has no non-trivial solutions; for $\lambda = a$, there is a one-parameter family of solutions given by $(v^{-}, v^{+}, \zeta) = \kappa(a, a, 1)$, $\kappa \in \mathbb{R}$ and, thus, the wave is spectrally unstable.

Such instability can also be recognized at the nonlinear level. Indeed, by assumption, the stationary solution U is given by

$$U(x) = \begin{cases} a\,x + U_{-} & x < 0,\\ a\,x + U_{+} & x > 0,\end{cases}$$

in a neighborhood $(-\varepsilon, \varepsilon)$ of $x = 0$. Hence, an initial data u_0 coinciding with U outside $(-\varepsilon, \varepsilon)$ and given in $(-\varepsilon, \varepsilon)$ by

$$u_0(x) = \begin{cases} a\,x + U_{-} & x < \zeta,\\ a\,x + U_{+} & x > \zeta,\end{cases}$$

for some ζ such that $|\zeta| < \varepsilon$, determine (locally in time) a solution coinciding with U for $|x| > \varepsilon$ and such that in $(-\varepsilon, \varepsilon)$

$$u(x,t) = \begin{cases} a\,x + U_{-} & x < \zeta\, e^{a\,t},\\ a\,x + U_{+} & x > \zeta\, e^{a\,t}.\end{cases}$$

The location of the discontinuity is strongly unstable and can be easily destroyed by a local perturbation.

Let us apply the same procedure to analyze the stability of roll-waves. This program has been completely accomplished in [**36**] in the case of a friction term of quadratic type and in the case of periodic roll-wave with large period. The analysis is based on a precise description of the solutions to a degenerate linear system of o.d.e. in an appropriate regime for the parameters of the system. Here, we restrict ourself to sketch the approach, without proposing any complete stability result.

For a general system of balance laws of the form (30) where $\boldsymbol{u} \in \mathbb{R}^N$ and f and g are smooth functions from $\mathbb{R}^n$ into itself, we can apply the same procedure used in the scalar case. Let $\boldsymbol{U} = \boldsymbol{U}(x - ct)$ be a traveling wave solution with a discontinuity at 0, then assuming that the perturbed solution has a single jump, we get the linearized system

$$\begin{cases} \dfrac{\partial \boldsymbol{u}^\pm}{\partial t} + \dfrac{\partial}{\partial y}\big\{ \big(\mathrm{d}f(\boldsymbol{U}) - c\,\mathbb{I}\big)\,\boldsymbol{u}^\pm \big\} = \mathrm{d}g(\boldsymbol{U})\,\boldsymbol{u}^\pm + \xi'\,\boldsymbol{U}' & \pm y > 0, \\ \xi'\,\big[\boldsymbol{U}\big]_0 = \big[\big(\mathrm{d}f(\boldsymbol{U}) - c\,\mathbb{I}\big)\,\boldsymbol{u}^\pm\big]_0. \end{cases}$$

Looking for solution in the form (46), we obtain

$$\begin{cases} \dfrac{d}{dy}\big\{ \big(\mathrm{d}f(\boldsymbol{U}) - c\,I\big)\,\boldsymbol{v}^\pm \big\} + \lambda\,\boldsymbol{v}^\pm = \mathrm{d}g(\boldsymbol{U})\,\boldsymbol{v}^\pm + \lambda\,\zeta\,\boldsymbol{U}' & \pm y > 0, \\ \big[\big(\mathrm{d}f(\boldsymbol{U}) - c\,\mathbb{I}\big)\,\boldsymbol{v}^\pm\big]_0 = \lambda\,\zeta\big[\boldsymbol{U}\big]_0. \end{cases}$$

In the case of shallow water system, the conservative variables are h and $p = h\,u$, and the jacobian matrices of the functions f and g are explicitly given by

$$\mathrm{d}f(h,p) = \begin{pmatrix} 0 & 1 \\ -\dfrac{p^2}{h^2} + g\,h & \dfrac{2p}{h} \end{pmatrix} \quad \text{and} \quad \mathrm{d}g(h,p) = \begin{pmatrix} 0 & 0 \\ g\,m + \dfrac{r_0\,p}{h^2} & -\dfrac{r_0}{h} \end{pmatrix}.$$

The unique case of roll-wave with a single jump point at $y = 0$ correspond to the solution (H, U) with H such that

$$H(-\infty) = H_1, \quad H(y_0) = H_0, \quad H(0-) = H_2, \quad H(y) = H_1 \text{ in } \{y<0\},$$

for some $y_0 < 0$. Thanks to the relations $(H\,U)' = c\,H'$ and $[HU]_0 = c[H]_0$, the variable $\boldsymbol{v}^\pm := (k^\pm, q^\pm)$ solves the resolvent system

$$\begin{cases} \dfrac{d}{dy}((\mathbb{A}(H) - c\,\mathbb{I})\boldsymbol{v}) + \big(\mathbb{B}(H) + \lambda\,\mathbb{I}\big)\boldsymbol{v} = \lambda\,\zeta\,H'\,(1,c)^t, \\ \big[\big(\mathbb{A}(H) - c\,\mathbb{I}\big)\boldsymbol{v}\big]_0 = \lambda\,\zeta[H]_0(1,c)^t \end{cases} \tag{49}$$

where, setting $V_\pm = V_\pm(H) := U(H) \pm \sqrt{g\,H}$, the matrices $\mathbb{A} = \mathbb{A}(H)$ and $\mathbb{B} = \mathbb{B}(H)$ are defined by

$$\mathbb{A} := \begin{pmatrix} 0 & 1 \\ -V_-\,V_+ & V_- + V_+ \end{pmatrix} \quad \text{and} \quad \mathbb{B} := \frac{1}{H}\begin{pmatrix} 0 & 0 \\ -g\,m\,H - r_0\,U & r_0 \end{pmatrix}.$$

Asking for bounded solutions gives an additional constraint in the region $y < 0$ since the matrix $\mathbb{A}(H) - c\,\mathbb{I}$ degenerates at $y = y_0$ and solutions may exhibit blow-up behavior as in the scalar case (see (48)). Such degeneration can be used to prove existence/non-existence of eigenvalues under appropriate regimes.

To go a little further and to peep at the kind of problems one is faced, we restrict the analysis to the case of H_0 sufficiently close to the critical value $H_* := r_0^2/gm^2$ (so that the profile (H, U) is a small perturbation of the constant state H_0) and we content ourself to show that the resolvent system has no non-trivial solution for λ real and positive. This is not particularly surprising since the linearized equation at H_* has no positive real eigenvalues.

For $y > 0$, H is constant, hence the linear system of ordinary differential equations for $\boldsymbol{v}$ is homogeneous with constant coefficients

$$(\mathbb{A}(H_1) - c\,\mathbb{I})\,\frac{d\boldsymbol{v}}{dy} + \big(\mathbb{B}(H_1) + \lambda\,\mathbb{I}\big)\boldsymbol{v} = 0. \tag{50}$$

Setting $U_1 := (cH_1 - \kappa_0)/H_1$, there holds

$$\begin{aligned}&\det\Big((\mathbb{A}(H_1) - c\,\mathbb{I})\mu + \mathbb{B}(H_1) + \lambda\,\mathbb{I}\Big)\\ &\qquad = \big[(c - U_1)^2 - g\,H_1\big]\mu^2 - \left[\left(2\lambda + \frac{r_0}{H_1}\right)(c - U_1) - g\,m\right]\mu + \lambda\left(\frac{r_0}{H_1} + \lambda\right).\end{aligned}$$

For λ real, there holds

$$2\,\mathrm{Re}\,\mu = \left(2\lambda + \frac{r_0}{H_1}\right)(c - U_1) - g\,m,$$

hence the real part of the eigenvalue μ is positive for any choice of $\lambda > 0$ if the following condition is satisfied

$$r_0\,(c - U_1) > g\,m\,H_1.$$

By using the relation $(c - U_1)H_1 = (c - U_0)H_0$ and the explicit expression for U_0, H_0 and H_1 (see (39)-(40)), it is possible to check that the above condition is equivalent to (42). Hence, for λ real and strictly positive, equation (50) has no non-zero bounded solution in $(0, \infty)$, so that $\boldsymbol{v} = 0$ in this region. The resolvent system (49) reduces to the Cauchy problem in $(-\infty, 0)$

$$\begin{cases} \dfrac{d}{dy}((\mathbb{A}(H) - c\,\mathbb{I})\boldsymbol{v}) + \big(\mathbb{B}(H) + \lambda\,\mathbb{I}\big)\boldsymbol{v} = \lambda\,\zeta\,H'\,(1, c)^t, \\ \boldsymbol{v}(0) = -\lambda\,\zeta[H]_0\big(\mathbb{A}(H_2) - c\,\mathbb{I}\big)^{-1}(1, c)^t \end{cases}$$

where ζ is a parameter. In the case $\zeta = 0$, the system is homogeneous, so that the problem has no nontrivial solutions; hence we can assume, without restriction, $\zeta \neq 0$.

Moreover, for $\lambda \neq 0$, by changing the independent variable and rescaling the variable $\boldsymbol{v}$ by the factor $\lambda\,\zeta$, we can rewrite the previous system as

$$\begin{cases} \dfrac{d}{dH}((\mathbb{A}(H) - c\,\mathbb{I})\boldsymbol{v}) + \dfrac{1}{P(H)}\big(\mathbb{B}(H) + \lambda\,\mathbb{I}\big)\boldsymbol{v} = (1, c)^t, \\ \boldsymbol{v}(H_2) = -[H]_0(\mathbb{A}(H_2) - c\,\mathbb{I}\big)^{-1}(1, c)^t \end{cases}$$

where $P = P(H)$ is defined in (40).

Since the matrix $(\mathbb{A}(H) - c\,\mathbb{I})$ degenerate at H_0, we can choose a row vector w_0 such that $w_0(\mathbb{A}(H_0) - c\,\mathbb{I}) = 0$. Multiplying by w and calculating at H_0, we obtain the (algebraic) conditions

$$\tag{51} \begin{cases} w_0\left\{\dfrac{d\mathbb{A}}{dH}(H_0) + \dfrac{1}{P(H_0)}\big(\mathbb{B}(H_0) + \lambda\,\mathbb{I}\big)\right\}\boldsymbol{v}(H_0) = w_0\,(1, c)^t, \\ \boldsymbol{v}(H_2) = -[H]_0\big(\mathbb{A}(H_2) - c\,\mathbb{I}\big)^{-1}(1, c)^t. \end{cases}$$

Next we can patiently perform expansion in the limiting regime $c \to c_* = 2\,r_0/m$, corresponding to $H \to H_* := r_0^2/g\,m^2$, $U \to U_* := r_0/m$, obtaining

$$\begin{aligned} w_0 &= (0, 1) + o(1), & P(H_0) &= \frac{r_0\,m^2}{3}(c - c_*) + o(c - c_*), \\ \frac{d\mathbb{A}}{dH}(H_0) &= O(1), & \mathbb{B}(H_0) &= g\,m\begin{pmatrix} 0 & 0 \\ -2 & m/r_0 \end{pmatrix} + o(1) \end{aligned}$$

$$[H]_0\big(\mathbb{A}(H_2) - c\,\mathbb{I}\big)^{-1} = -\frac{2}{3\,g}\begin{pmatrix} 0 & -1 \\ 0 & -2r_0/m \end{pmatrix} + o(1).$$

Multiplying the first equation in (51) by $c - c_*$ and then passing to the limit, we obtain the relations

$$(0\ 1)\begin{pmatrix} \lambda & 0 \\ -2 & m/r_0 + \lambda \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 0 & -2r_0/m \end{pmatrix}\begin{pmatrix} 1 \\ 2\,r_0/m \end{pmatrix} = 0$$

that reduces to the equation $r_0^2\,\lambda/m^2 = 0$, hence $\lambda = 0$.

5 - Kinetic formulation

A milestone in the foundation of fluid-dynamics is to understand the link between the system of partial differential equations describing the evolution at macroscopic and mesoscopic scale. At the latter level, the description is usually given by the Boltzmann equations

$$\tag{52} \partial_t f^\varepsilon + \xi \cdot \mathrm{grad}_x f^\varepsilon = \frac{1}{\varepsilon}\,\mathcal{Q}[f^\varepsilon],$$

where ε is the mean free path of particles, $f^\varepsilon(x, t; \xi) \in \mathbb{R}$ density of particles at $(x, t) \in \mathbb{R}^2 \times [0, \infty)$ with velocity $\xi \in \mathbb{R}^2$, $\mathcal{Q}[f^\varepsilon]$ is a (collisional) operator. One of the basic goal is to rigorously prove the so-called hydrodynamic limit, meaning that appropriate integrated quantities of the solution f^ε converge as $\varepsilon \to 0$ to the solution of a corresponding system of PDEs (a striking result relative to the incompressible Navier-Stokes equation is proved in [**21**]). For a review on Boltzmann kinetic equation we refer to [**50**].

Generalizing, the idea of providing a mesoscopic description for a macroscopic system of hyperbolic/parabolic partial differential equations turned out to be useful from many point of view, somewhat independently on the physical interpretation of the equation (52). Analysis of optimal regularity of solutions, efficient numerical schemes has been proposed departing from such a point of view (see [**44**] and descendants), the main advantage being the semilinear structure of the kinetic formulation and the possibility of the use of linear techniques (a recent review on the subject can be found in [**42**]).

Here, we consider the Saint-Venant system in dimension $n = 2$ without viscosity and with bottom topography with the aim of shortly deriving a kinetic formulation in BGK form for the macroscopic system and presenting a result relative to the hydrodynamical limit for smooth solutions proved in [**2**].

5.1 - *A kinetic version for the Saint-Venant model*

Let the real-valued function $f = f(x, t; \xi)$ be a solution to the equation

$$\partial_t f + \xi \cdot \mathrm{grad}_x f + \boldsymbol{F}(x) \cdot \mathrm{grad}_\xi f = \frac{1}{\varepsilon}\, \mathcal{Q}[f] \tag{53}$$

where the collision operator $\mathcal{Q}$ maps the set of functions $\{f : \mathbb{R}^2 \to \mathbb{R}\}$ into itself and $\boldsymbol{F}$, whose rôle is to take into account the presence of a macroscopic force acting on particles, is a function from $\mathbb{R}^2$ into itself. Next, we want to describe the specific choice of the operator $\mathcal{Q}$ and of the function $\boldsymbol{F}$ in order to obtain in the limit $\varepsilon \to 0^+$ the unviscous shallow water equations, at least at a formal level.

Assuming that $f \to 0$ as $|\xi| \to \infty$ and the constraint

$$\int_{\mathbb{R}^2} \mathcal{Q}[f]\, d\xi = 0 \qquad \forall f : \mathbb{R}^2 \to \mathbb{R},$$

the variable h is recovered by integration with respect to ξ in $\mathbb{R}^2$: indeed, one can easily obtain

$$\partial_t h + \mathrm{div}_x \boldsymbol{J} = 0 \qquad \text{where} \qquad \begin{cases} h(x,t) := \displaystyle\int_{\mathbb{R}^2} f(x,t;\xi)\, d\xi \\ \boldsymbol{J}(x,t) := \displaystyle\int_{\mathbb{R}^2} \xi f(x,t;\xi)\, d\xi. \end{cases} \tag{54}$$

In general, let $\boldsymbol{K} = \boldsymbol{K}(\xi)$ be a function from $\mathbb{R}^2$ to $\mathbb{R}^p$ such that

$$\int_{\mathbb{R}^2} \boldsymbol{K}(\xi)\, \mathcal{Q}[f]\, d\xi = \boldsymbol{0} \qquad \forall f : \mathbb{R}^2 \to \mathbb{R}.$$

Then, setting

$$\boldsymbol{U}(x,t) := \int_{\mathbb{R}^2} \boldsymbol{K}(\xi) f(x,t;\xi)\, d\xi,$$

multiplying by $\boldsymbol{K}(\xi)$ and integrating with respect to ξ, we obtain a system satisfied by the macroscopic variable $\boldsymbol{U}$

$$\partial_t \boldsymbol{U} + \mathrm{div}_x \left(\int_{\mathbb{R}^n} (\boldsymbol{K}(\xi) \otimes \xi) f\, d\xi \right) - \boldsymbol{F}(x) \int_{\mathbb{R}^n} \mathrm{grad}_\xi \boldsymbol{K}(\xi) f\, d\xi = 0.$$

Looking at the form of the flux term $\boldsymbol{J}$ in (54), it is natural to choose $\boldsymbol{K}(\xi) = (1, \xi)$. Then, setting

$$h(x,t) := \int_{\mathbb{R}^2} f(x,t;\xi)\, d\xi, \qquad (h\,\boldsymbol{v})(x,t) := \int_{\mathbb{R}^2} \xi f(x,t;\xi)\, d\xi,$$

we obtain the following system for the macroscopic variable $\boldsymbol{U} = (h, h\boldsymbol{v})$

$$(55) \qquad \begin{cases} \partial_t h + \mathrm{div}_x (h\,\boldsymbol{v}) = 0, \\ \partial_t (h\,\boldsymbol{v}) + \mathrm{div}_x\, \mathbb{J} = \boldsymbol{F}(x)\, h, \end{cases} \quad \text{where} \quad \mathbb{J}(x,t) := \int_{\mathbb{R}^n} (\xi \otimes \xi)\, \mathcal{M}[f]\, d\xi.$$

In order to formally obtain the shallow water system, the potential term $\boldsymbol{F}$, introducing a bias in the evolution of the kinetic variable f, has to be chosen equal to $-\mathrm{grad}_x Z$, where Z denotes the bottom topography, consistently with the fact that a slope in the bottom will generate a transition of particles speed in the direction of the slope itself.

Everything is now in the hands of the collisional term $\mathcal{Q}$. Its rôle is to drive, in a time-scale of order ε, the solutions f toward a corresponding equilibrium configuration $\mathcal{M}[f]$, usually called maxwellian or Maxwell–Boltzmann configuration, that in the original gas-dynamics context has the form of a gaussian with shape determined by the macroscopic variables.

As $\varepsilon \to 0^+$, one may imagine that the solution f is exactly maxwellian, $f \sim \mathcal{M}[f]$; as a consequence, we formally obtain the hyperbolic shallow water system by setting

$$\int_{\mathbb{R}^2} (\xi \otimes \xi)\, \mathcal{M}[f]\, d\xi \sim h\,\boldsymbol{v} \otimes \boldsymbol{v} + \frac{1}{2}\, g\, h^2\, \mathbb{I}.$$

In 1954, Bhatnagar, Gross and Krook proposed a form for the collision term $\mathcal{Q}$ giving raise to a simplified version of the original Boltzmann-like model, still preserving a number of significant properties of the complete model. Such choice, dictated by the idea of generating a simple dynamics leading the solution f toward the corresponding maxwellian state $\mathcal{M}[f]$, is based on asking $\mathcal{Q}$ to be written as

$$\mathcal{Q}[f] = \mathcal{M}[f] - f \tag{56}$$

where the operator $\mathcal{M}[f]$ has the form

$$\mathcal{M}[f] := M(\boldsymbol{U};\xi) \qquad \text{where} \quad \boldsymbol{U}(x,t) = \int_{\mathbb{R}^2} \boldsymbol{K}(\xi) f(x,t;\xi)\,d\xi,$$

for some appropriate real-valued function $M = M(\boldsymbol{U};\xi)$. A kinetic model (53) with the choice (56) is referred to as a BGK kinetic model (for a general framework of such systems, see the fundamental article [**3**]) and any function f that is a fixed point for the operator $\mathcal{M}$ is called a maxwellian of the system. To get a system for the macroscopic variable $\boldsymbol{U}$ of the form (55), the function M has to be chosen so that

$$\int_{\mathbb{R}^2} \mathcal{M}[f]\,d\xi = \int_{\mathbb{R}^2} f\,d\xi \qquad \text{and} \qquad \int_{\mathbb{R}^2} \xi\,\mathcal{M}[f]\,d\xi = \int_{\mathbb{R}^2} \xi f\,d\xi$$

that is, in term of the variables h and $h\,\boldsymbol{v}$,

$$h = \int_{\mathbb{R}^2} M(h,\boldsymbol{v};\xi)\,d\xi \qquad \text{and} \qquad h\,\boldsymbol{v} = \int_{\mathbb{R}^2} \xi\,M(h,\boldsymbol{v};\xi)\,d\xi.$$

An innocent choice could be $M(h,\boldsymbol{v};\xi) := h\,\delta_{\xi=\boldsymbol{v}}$, where δ denote the usual Dirac delta, describing the idea that, at mesoscopic level, all of the particles in location (x,t) move with the macroscopic speed $\boldsymbol{v}$. Regardless with the problem emerging from the distributional context, such choice would not fulfill the request of determining an approximation of shallow water system, since

$$\int_{\mathbb{R}^2} (\xi\otimes\xi)\,M(h,\boldsymbol{v};\xi)\,d\xi = \int_{\mathbb{R}^2} (\xi\otimes\xi)\,h\,\delta_{\xi=\boldsymbol{v}}\,d\xi = h\,\boldsymbol{v}\otimes\boldsymbol{v},$$

and no pressure term appears in the corresponding term $\mathbb{J}$. In fact, pressure effect is determined by the fact that, at equilibrium, the density function f is spread around the (average) speed $\boldsymbol{v}$ that is macroscopically observed.

Assuming the transport process to be isotropic, a realistic class for the function M is

$$M(h,\boldsymbol{v};\xi) = m(|\xi - \boldsymbol{v}|, h) \tag{57}$$

for some function $m = m(\rho,h)$. The following (straightforward) Lemma shows the structure of the macroscopic equations corresponding to different choices of function m.

Lemma 5.1. *Let the function $m = m(\rho, h)$ be such that*

$$h := 2\pi \int_0^{+\infty} \rho\, m(\rho; h)\, d\rho < +\infty. \tag{58}$$

Then, for the function $M = M(h, \boldsymbol{v}; \xi)$ defined as in (57), *there hold*

$$\int_{\mathbb{R}^2} M(h, \boldsymbol{v}; \xi)\, d\xi = h,$$

$$\int_{\mathbb{R}^2} \xi\, M(h, \boldsymbol{v}; \xi)\, d\xi = h\,\boldsymbol{v},$$

$$\int_{\mathbb{R}^2} (\xi \otimes \xi) M(h, \boldsymbol{v}; \xi)\, d\xi = h\boldsymbol{v} \otimes \boldsymbol{v} + P(h)\,\mathbb{I},$$

where

$$P(h) := \pi \int_0^{+\infty} \rho^3\, m(\rho; h)\, d\rho$$

for any $h \geq 0$, $\boldsymbol{v} \in \mathbb{R}^2$.

Proof. It is just a matter of standard integrals computations: setting $\zeta = \xi - \boldsymbol{v}$, we get

$$\int_{\mathbb{R}^2} m(|\xi - \boldsymbol{v}|; h)\, d\xi = \int_{\mathbb{R}^2} m(|\zeta|; h)\, d\zeta = 2\pi \int_0^{+\infty} \rho\, m(\rho; h)\, d\rho = h;$$

$$\begin{aligned} \int_{\mathbb{R}^2} \xi\, m(|\xi - \boldsymbol{v}|; h)\, d\xi &= \int_{\mathbb{R}^2} \zeta\, m(|\zeta|; h)\, d\zeta + \boldsymbol{v} \int_{\mathbb{R}^2} m(|\zeta|; h)\, d\zeta \\ &= 2\pi\boldsymbol{v} \int_0^{+\infty} \rho\, m(\rho; h)\, d\rho = h\,\boldsymbol{v}; \end{aligned}$$

$$\begin{aligned} \int_{\mathbb{R}^2} (\xi \otimes \xi) m(|\xi - \boldsymbol{v}|; h)\, d\xi &= \int_{\mathbb{R}^2} (\zeta + \boldsymbol{v}) \otimes (\zeta + \boldsymbol{v})\, m(|\xi - \boldsymbol{v}|; h)\, d\zeta \\ &= (\boldsymbol{v} \otimes \boldsymbol{v}) \int_{\mathbb{R}^2} m(|\zeta|; h)\, d\zeta + \int_{\mathbb{R}^2} (\zeta \otimes \zeta)\, m(|\zeta|; h)\, d\zeta. \end{aligned}$$

Since

$$\int_{\mathbb{R}^2} \zeta_i\,\zeta_j\, m(|\zeta|;h)\,d\zeta = 0 \qquad\qquad \text{if } i \neq j,$$

$$\int_{\mathbb{R}^2} \zeta_i\,\zeta_j\, m(|\zeta|;h)\,d\zeta = \pi \int_0^{+\infty} \rho^3\, m(\rho;h)\,d\rho \qquad \text{if } i = j,$$

the conclusion holds. □

The easiest example for the m is the characteristic function of an interval:

$$m(\rho;h) = C(h)\,\chi_{[0,R(h)]}(\rho)$$

with $C = C(h)$ and $R = R(h)$ functions to be determined. Condition (58) translate into the relation $C\,R^2 = h/\pi$. Thus, we get

$$P(h) = \pi \int_0^{+\infty} \rho^3\, m(\rho;h)\,d\rho = C\,\pi \int_0^{R} \rho^3\,d\rho = \frac{1}{4}\,C\,R^4\,\pi = \frac{1}{4}\,R^2\,h,$$

and we realize that shallow water model corresponds to the choice

$$R(h) = \sqrt{2\,g\,h}\,, \qquad C(h) = \frac{1}{2\,\pi\,g}.$$

A suggestive choice for m is the gaussian form

$$m(\rho;h) = A(h)\,\exp\bigl(-\rho^2/B(h)\bigr).$$

In this case, request (58) becomes the constraint $AB = h/\pi$ and

$$P(h) = \pi \int_0^{+\infty} \rho^3\, m(\rho;h)\,d\rho = A\,\pi \int_0^{+\infty} \rho^3\, e^{-B\,\rho^2}\,d\rho = \frac{1}{2}\,A\,B^2\,\pi = \frac{1}{2}\,B\,h,$$

so that the Saint-Venant model accords with $A(h) = 1/g\,\pi$ and $B(h) = g\,h$.

From now on, we consider the first type of function M with a simple choice for the functions C and R, namely $C(h) = 1$ and $R(h) = \sqrt{h/\pi}$, so that

$$M(h,\boldsymbol{v};\xi) := \chi_{\{\xi\in\mathbb{R}^2\,:\,|\xi-\boldsymbol{v}|^2\leq h/\pi\}} = \begin{cases} 1 & \text{if } |\xi-\boldsymbol{v}|^2 \leq h/\pi, \\ 0 & \text{if } |\xi-\boldsymbol{v}|^2 > h/\pi, \end{cases} \tag{59}$$

and the formal hydrodynamical limit is

$$\begin{cases} \partial_t h + \mathrm{div}_x(h\,\boldsymbol{v}) = 0, \\ \partial_t(h\,\boldsymbol{v}) + \mathrm{div}_x\,\mathbb{J} = \boldsymbol{F}(x)\,h, \end{cases} \qquad \text{where} \quad \mathbb{J}(x,t) := h\,\boldsymbol{v}\otimes\boldsymbol{v} + \frac{1}{4\,\pi}\,h^2\,\mathbb{I}. \tag{60}$$

The macroscopic variable $\boldsymbol{U}$, determined by the kinetic solution f, solves the system (60) up to an error term arising in the flux term. In vectorial notation, the approximate solution $\boldsymbol{U}$ satisfies the relation

$$\partial_t \boldsymbol{U} + \mathrm{div}_x A(\boldsymbol{U}) - B(x, \boldsymbol{U}) = \mathrm{div}_x \left(A(\boldsymbol{U}) - \int_{\mathbb{R}^n} \big(\boldsymbol{K}(\xi) \otimes \xi\big) f \, d\xi \right) \tag{61}$$

with the manifest meaning for the functions A and B.

If we consider the system (60) together with the entropy η, defined by

$$\eta(h, h\,\boldsymbol{v}) := \frac{1}{2}\, h\, |\boldsymbol{v}|^2 + \Phi(h) \qquad \text{where} \quad \Phi(h) := \frac{1}{4\pi}\, h^2,$$

correspondingly, the kinetic system (53) can be provided of a (corresponding) kinetic entropy, defined as

$$\mathcal{H}[f] := \int_{\mathbb{R}^2} H(f, \xi)\, d\xi \qquad \text{where} \ \ H(f, \xi) := \frac{1}{2}\, |\xi|^2 f,$$

so that, thanks to properties described in Lemma 5.1, we deduce the entropy compatibility condition

$$\mathcal{H}\big[\mathcal{M}[f]\big] = \frac{1}{2} \int_{\mathbb{R}^2} |\xi|^2\, \mathcal{M}[f]\, d\xi = \frac{1}{2}\, h\, |\boldsymbol{v}|^2 + \frac{1}{4\pi}\, h^2 = \eta(h, h\,\boldsymbol{v}), \tag{62}$$

meaning that *the kinetic entropy at a maxwellian state coincides with the entropy of the corresponding macroscopic variables*. The relation is even more remarkable, since maxwellian states minimize the kinetic entropy.

Proposition 5.1. *For any f such that $(1 + |\xi|^2) f \in L^1(\mathbb{R}^2)$ and $f(\xi) \in [0, 1]$ for any ξ, there holds the* minimization principle

$$\mathcal{H}\big[\mathcal{M}[f]\big] \le \mathcal{H}[f] \tag{63}$$

with the equality holding if and only if $\mathcal{M}[f] = f$.

Proof. For any $\boldsymbol{v} \in \mathbb{R}^2$, there holds

$$\int_{\mathbb{R}^2} |\xi|^2 \big(f - \mathcal{M}[f]\big)\, d\xi = \int_{\mathbb{R}^2} \big(|\xi - \boldsymbol{v}|^2 + 2\,\boldsymbol{v} \cdot \xi + |\boldsymbol{v}|^2\big) \big(f - \mathcal{M}[f]\big)\, d\xi.$$

Since f and $\mathcal{M}[f]$ have same mass and same first moment, we have

$$\int_{\mathbb{R}^2} |\xi|^2 \big(f - \mathcal{M}[f]\big)\, d\xi = \int_{\mathbb{R}^2} |\xi - \boldsymbol{v}|^2 \big(f - \mathcal{M}[f]\big)\, d\xi.$$

By applying the change of variable $\xi \mapsto \xi + \boldsymbol{v}$, we reduce to the case $\boldsymbol{v} = 0$. Such choice bears

$$\begin{aligned}\int_{\mathbb{R}^2} |\xi|^2 (f - \mathcal{M}[f])\, d\xi &= - \int_{|\xi|^2 \in [0,h/\pi]} |\xi|^2 (1-f)\, d\xi + \int_{|\xi|^2 \notin [0,h/\pi]} |\xi|^2 f\, d\xi \\ &\geq -\frac{h}{\pi} \int_{|\xi|^2 \in [0,h/\pi]} (1-f)\, d\xi + \frac{h}{\pi} \int_{|\xi|^2 \notin [0,h/\pi]} f\, d\xi \\ &= \frac{h}{\pi} \left(\int_{\mathbb{R}^2} f\, d\xi - h \right) = 0,\end{aligned}$$

with the equality holding if and only if f is the characteristic function of the set $\{\xi \in \mathbb{R}^2 \,:\, |\xi|^2 \leq h/\pi\}$. □

The above result suggests to restrict attention to solution f taking values in the interval $[0, 1]$ (that is the range of the function M). The BGK-structure (56) guarantees that if initial datum for (53) satisfy this condition, then the same property holds for any positive time; in other words, the set $\{f \,:\, f(\xi) \in [0, 1]\}$ is an invariant region for the kinetic equation we are dealing with. Indeed, given $\delta > 0$, assume that there is some $\bar{t} > 0$ such that there exist $\bar{x}$ and $\bar{\xi}$ for which $f(\bar{x}, \bar{t}; \bar{\xi}) = 1 + \delta$ with $\bar{t}$ minimum with such property. Then

$$\partial_t f(\bar{x}, \bar{t}; \bar{\xi}) \geq 0, \qquad \mathrm{grad}_x f(\bar{x}, \bar{t}; \bar{\xi}) = \mathrm{grad}_\xi f(\bar{x}, \bar{t}; \bar{\xi}) = 0,$$

and thus

$$0 \leq \{\partial_t f + \xi \cdot \mathrm{grad}_x f + \boldsymbol{F} \cdot \mathrm{grad}_\xi f\}(\bar{x}, \bar{t}; \bar{\xi}) = \frac{1}{\varepsilon}(\mathcal{M}[f] - f)(\bar{x}, \bar{t}; \bar{\xi}) \leq -\frac{\delta}{\varepsilon} < 0,$$

that is a contradiction. Hence $f \leq 1 + \delta$ for any $\delta > 0$ and, therefore, $f \leq 1$. Similarly we can prove that $f \geq 0$. From now on, for any x, t we assume that $f(x, t; \cdot)$ belong to the set

$$L^1_w(\mathbb{R}^2; [0,1]) := \{f \in L^1(\mathbb{R}^2; [0,1]) \,:\, (1 + |\xi|^2) f(\xi) \in L^1(\mathbb{R}^2; [0,1])\}. \tag{64}$$

The minimization principle (63) indicates that, in the absence of the source term, i.e. if $\boldsymbol{F} \equiv 0$, the integral with respect to ξ and to x of the kinetic entropy is a Lyapunov functional for (53): indeed

$$\begin{aligned}\frac{d}{dt}\mathcal{H}[f] &= -\frac{1}{2} \int_{\mathbb{R}^2} |\xi|^2 \left\{ \xi \cdot \mathrm{grad}_x f + \frac{1}{\varepsilon}(f - \mathcal{M}[f]) \right\} d\xi \\ &= -\frac{1}{2} \mathrm{grad}_x \left(\int_{\mathbb{R}^2} |\xi|^2 \, \xi f\, d\xi \right) - \frac{1}{\varepsilon} \int_{\mathbb{R}^2} \{H(f, \xi) - H(\mathcal{M}[f], \xi)\}\, d\xi,\end{aligned}$$

and thus, integrating with respect to x,

$$\frac{d}{dt}\int_{\mathbb{R}^2} \mathcal{H}[f]\,dx \le -\frac{1}{\varepsilon}\int_{\mathbb{R}^2}\Big\{\mathcal{H}[f]-\mathcal{H}\big[\mathcal{M}[f]\big]\Big\}dx \le 0.$$

When the source term is present, there holds

$$\frac{d}{dt}\int_{\mathbb{R}^2} \mathcal{H}[f]\,dx = \iint_{\mathbb{R}^2\times\mathbb{R}^2} \Big\{\boldsymbol{F}(x)\cdot\xi f - \frac{1}{\varepsilon}\{H(f,\xi)-H(\mathcal{M}[f],\xi)\}\Big\}\,d\xi\,dx$$

hence

$$\begin{aligned}\frac{d}{dt}\int_{\mathbb{R}^2} \mathcal{H}[f]\,dx &\le |\boldsymbol{F}|_{L^\infty}\iint_{\mathbb{R}^2\times\mathbb{R}^2} |\xi| f\,d\xi\,dx \le \frac{1}{2}|\boldsymbol{F}|_{L^\infty}\iint_{\mathbb{R}^2\times\mathbb{R}^2}(1+|\xi|^2)f\,d\xi\,dx\\ &\le |\boldsymbol{F}|_{L^\infty}\left(\frac{1}{2}\iint_{\mathbb{R}^2\times\mathbb{R}^2} f_0\,d\xi\,dx + \int_{\mathbb{R}^2}\mathcal{H}[f]\,dx\right)\end{aligned}$$

where $f_0(x;\xi) := f(x,0;\xi)$. Therefore, the following estimate holds true

$$\int_{\mathbb{R}^2}\mathcal{H}[f]\,dx \le e^{a\,t}\int_{\mathbb{R}^2}\mathcal{H}[f_0]\,dx + \frac{1}{2}\left(e^{a\,t}-1\right)\iint_{\mathbb{R}^2\times\mathbb{R}^2} f_0\,d\xi\,dx,$$

where a denotes the $L^\infty-$norm of the function $\boldsymbol{F}$.

5.2 - *A few words on the Cauchy problem*

Before dealing with the singular limit $\varepsilon \to 0^+$, let us consider the evolution defined by the kinetic equation for ε fixed (for definiteness equal to 1) and let us consider the Cauchy problem

$$\begin{cases}\partial_t f + \xi\cdot\mathrm{grad}_x f = \mathcal{M}[f]-f,\\ f(x,0;\xi) := f_0(x;\xi),\end{cases}$$

where $\mathcal{M}[f] := M(h,\boldsymbol{v};\xi)$ with M given in (59) and f_0 is a given initial datum.

Global existence results for BGK models have been considered by many authors in literature with particular attention to the significant case of kinetic description of rarified gas dynamics (see [**41, 43**] and descendants). Here, we consider the specific model presented in the previous Section to show the basic difficulties arising in estabilishing existence of a global solution to the problem.

The natural strategy to prove well-posedness is to use a standard iteration procedure of the form

$$\begin{cases} \partial_t f^{n+1} + \xi \cdot \mathrm{grad}_x f^{n+1} = \mathcal{M}[f^n] - f^{n+1}, \\ f^{n+1}(x,0;\xi) := f_0(x;\xi), \end{cases} \tag{65}$$

where the initial datum f_0 is assumed to take values in $[0,1]$. First of all, it is easy to estabilish that, under the assumption

$$\int_{\mathbb{R}^2} f_0(x - \xi\, t; \xi)\, d\xi \geq c_0 > 0,$$

all the functions of the sequence $\{f^n\}$ verify

$$h^n(x,t) := \int_{\mathbb{R}^2} f^n(x,t;\xi)\, d\xi \geq c_0\, e^{-t} \quad \text{for any} \quad t > 0.$$

Indeed, the positivity of M guarantees that, for any n,

$$\partial_t f^n + \xi \cdot \mathrm{grad}_x f^n + f^n \geq 0,$$

so that the function f^n is greater than or equal to the function g, solution to the linear Cauchy problem

$$\begin{cases} \partial_t g + \xi \cdot \mathrm{grad}_x g + g = 0, \\ g(x,0;\xi) := f_0(x;\xi), \end{cases}$$

that is explicitly given by

$$g(x,t;\xi) = f_0(x - \xi\, t; \xi)\, e^{-t} \qquad x \in \mathbb{R}^2,\ t > 0.$$

To prove convergence of the iteration algorithm (65), one is faced with the problem of proving Lipschitz continuity of the operator $\mathcal{M}$, in some appropriate functional space. Here, we concentrate on the dependence with respect to ξ and consider the space (64) endowed with the norm

$$|f|_{L^1_w} := \int_{\mathbb{R}^2} (1 + |\xi|^2)|f(\xi)|\, d\xi.$$

Given $f_1, f_2 \in L^1_w$, we have

$$\begin{aligned} |\mathcal{M}[f_1] - \mathcal{M}[f_2]|_{L^1_w} \leq & \int_{\mathbb{R}^2} (1 + |\xi|^2)\big|M(h_1, \boldsymbol{v}_1; \xi) - M(h_2, \boldsymbol{v}_1; \xi)\big|\, d\xi \\ & + \int_{\mathbb{R}^2} (1 + |\xi|^2)\big|M(h_2, \boldsymbol{v}_1; \xi) - M(h_2, \boldsymbol{v}_2; \xi)\big|\, d\xi. \end{aligned}$$

The first integral at the righthand side is easily estimated; indeed, for $h_1 \le h_2$, there holds

$$\int_{\mathbb{R}^2} (1+|\xi|^2)\big|M(h_1,\boldsymbol{v}_1;\xi) - M(h_2,\boldsymbol{v}_1;\xi)\big|\,d\xi = \int_{h_1 \le \pi|\xi|^2 \le h_2} (1+|\xi+\boldsymbol{v}|^2)\,d\xi$$
$$= \int_{h_1 \le \pi|\xi|^2 \le h_2} (1+|\boldsymbol{v}|^2+|\xi|^2)\,d\xi = \left(1+|\boldsymbol{v}|^2+\frac{1}{2\pi}(h_1+h_2)\right)(h_2-h_1).$$

Concerning the second term, for $|\boldsymbol{v}_1-\boldsymbol{v}_2|^2 \ge 4\,h/\pi$, we have

$$\int_{\mathbb{R}^2} (1+|\xi|^2)\big|M(h,\boldsymbol{v}_1;\xi) - M(h,\boldsymbol{v}_2;\xi)\big|\,d\xi = h(1+|\boldsymbol{v}_1|^2+|\boldsymbol{v}_2|^2)+\frac{1}{\pi}\,h^2,$$

and, for $|\boldsymbol{v}_1-\boldsymbol{v}_2|^2 < 4\,h/\pi$, setting $\Sigma := \{\xi\,:\,\sqrt{h/\pi} \le |\xi| \le |\boldsymbol{v}_2-\boldsymbol{v}_1| + \sqrt{h/\pi}\}$,

$$\int_{\mathbb{R}^2} (1+|\xi|^2)\big|M(h,\boldsymbol{v}_1;\xi) - M(h,\boldsymbol{v}_2;\xi)\big|\,d\xi \le \int_{\Sigma} (1+|\xi|^2+|\boldsymbol{v}|^2)\,d\xi$$
$$= \Phi(h,\boldsymbol{v}_1,\boldsymbol{v}_2)\,|\boldsymbol{v}_1-\boldsymbol{v}_2|$$

where Φ is a smooth function. Thus, *for bounded* $h_1, h_2, \boldsymbol{v}_1, \boldsymbol{v}_2$, we have

$$|\mathcal{M}[f_1]-\mathcal{M}[f_2]|_{L^1_w} \le C(|h_1-h_2|+|\boldsymbol{v}_1-\boldsymbol{v}_2|) \le C\,|f_1-f_2|_{L^1_w},$$

where the constant C depends on the $L^\infty-$norm of $h_1, h_2, \boldsymbol{v}_1, \boldsymbol{v}_2$. The basic question is *how to get L^∞ bounds on the macroscopic variables?*

Following [**43**], we deduce such a priori estimates by means of the L^∞-norm of high order moments of the function f. Precisely, let us set

$$N_q(f) := \sup_{\xi\in\mathbb{R}^2} |\xi|^q f(\xi), \quad \text{and} \quad m_1 := \int_{\mathbb{R}^2} |\xi| f\,d\xi.$$

Then, since $0 \le f \le 1$, there holds

$$h := \int_{\mathbb{R}^2} f\,d\xi = \int_{|\xi|\le R} f\,d\xi + \int_{|\xi|>R} f\,d\xi \le \pi R^2 + \frac{m_1}{R},$$

for any $R > 0$. Minimizing with respect to R, we obtain

$$h \le C\,m_1^{2/3} \tag{66}$$

for some constant $C > 0$. Moreover, we have

$$m_1 = \int_{|\xi|\le R} |\xi| f\,d\xi + \int_{|\xi|>R} |\xi| f\,d\xi \le R\,h + \frac{1}{R}\int_{|\xi|>R} \frac{|\xi|^q f}{|\xi|^{q-2}}\,d\xi,$$

so that, for $q > 4$,

$$m_1 \le R\,h + \frac{N_q}{R^{q-3}} \qquad \forall R > 0.$$

Minimizing again with respect to R, we deduce

$$m_1 \le C\,N_q^{1/(q-2)}\,h^{(q-3)/(q-2)} \qquad (q > 4) \tag{67}$$

for some constant $C > 0$. Combining (66) and (67), we infer

$$h \le C\,N_q^{2/q}, \qquad m_1 \le C\,N_q^{3/q}.$$

To close the estimate, it is necessary to bound the quantity N_q. To this aim, we note that the dynamics of the term $g_q := |\xi|^q f$ is determined by the kinetic equation

$$\partial_t g_q + \xi \cdot \mathrm{grad}_x g_q = |\xi|^q \mathcal{M}[f] - g_q.$$

Under the assumption of uniform positivity of h, the source term $|\xi|^q \mathcal{M}[f]$ is estimated by

$$\begin{aligned} |\xi|^q \mathcal{M}[f] &\le (|\xi - \boldsymbol{v}|^q + |\boldsymbol{v}|^q)\mathcal{M}[f] \le C\,h^{q/2} + |\boldsymbol{v}|^q \\ &\le C\,h^{q/2} + \frac{|m_1|^q}{|h|^q} \le C\,(N_q + N_q^3). \end{aligned}$$

By using this control, it is possible to show that the value N_q, $q > 4$, is bounded for small times. Matching all together, these arguments lead to an existence result of a solution for the Cauchy problem, locally in time.

Global existence needs a control of the term $|\xi|^q \mathcal{M}[f]$ by means of a first order power of N_q, so that, with the help of Gronwall Lemma, one can determine an exponential growth for the term N_q. Such program has been completely accomplished in the case of rarified gas dynamics with gaussian maxwellian functions in [**43**] (see also [**34**]).

5.3 - *The hydrodynamical limit*

In the remaining part of this Section, we concentrate on the problem of establishing rigorously the validity of the hydrodynamical limit $\varepsilon \to 0$, i.e. we want to show that the family f^ε of solutions to the kinetic equation (53) converges as $\varepsilon \to 0$ to the solution of the corresponding hydrodynamical limiting system (60), as soon as the solution of the hydrodynamical limit remains smooth. The result we present here is a simplified presentation of the more general result proved by Berthelin and Vasseur, [**2**].

Let U be a solution to the macroscopic system (60) and let $\boldsymbol{U}^\varepsilon$ be the macroscopic variable corresponding to the solution f^ε of the kinetic system (53), to be considered as an approximation of $\boldsymbol{U}^\varepsilon$. Following [**16**], we introduce the concept of the modulated function of a given function Ψ: given functions $\boldsymbol{U}$ and $\boldsymbol{U}^\varepsilon$, set

$$\Psi(\boldsymbol{U}^\varepsilon;\boldsymbol{U}) := \Psi(\boldsymbol{U}^\varepsilon) - \Psi(\boldsymbol{U}) - \mathrm{d}\Psi(\boldsymbol{U})(\boldsymbol{U}^\varepsilon - \boldsymbol{U}). \tag{68}$$

If Ψ is convex, $\Psi(\boldsymbol{U}^\varepsilon;\boldsymbol{U})$ is quadratic in $\boldsymbol{U}^\varepsilon - \boldsymbol{U}$ and its integral can be used to control the L^2 distance between $\boldsymbol{U}^\varepsilon$ and $\boldsymbol{U}$.

In the present context, the modulated entropy $\eta(\boldsymbol{U}^\varepsilon;\boldsymbol{U})$ is given by

$$\eta(\boldsymbol{U}^\varepsilon;\boldsymbol{U}) = \frac{1}{2}\,h^\varepsilon\,|\boldsymbol{v}^\varepsilon - \boldsymbol{v}|^2 + (h^\varepsilon - h)^2$$

since

$$\frac{1}{2}\,h^\varepsilon\,|\boldsymbol{v}^\varepsilon|^2 - \frac{1}{2}\,h\,|\boldsymbol{v}|^2 - \left(-\frac{1}{2}\,|u|^2, u\right)\cdot(h^\varepsilon - h, h^\varepsilon\,\boldsymbol{v}^\varepsilon - h\,\boldsymbol{v}) = \frac{1}{2}\,h^\varepsilon\,|\boldsymbol{v}^\varepsilon - \boldsymbol{v}|^2$$

and $\Phi(h^\varepsilon;h) = \dfrac{1}{4\,\pi}(h^\varepsilon - h)^2$. The main tool to prove the hydrodynamical limit is to control the space integral of the modulated entropy $\eta(\boldsymbol{U}^\varepsilon;\boldsymbol{U})$ and, specifically, to prove an estimate of the form

$$\int_{\mathbb{R}^2} \eta(\boldsymbol{U}^\varepsilon;\boldsymbol{U})(x,t)\,dx \le C\sqrt{\varepsilon} \qquad \forall\, t\in[0,T]. \tag{69}$$

To reveal the quadratic structure of the entropy, we use the representation

$$\eta(\boldsymbol{U}_1;\boldsymbol{U}_0) = \int_0^1 \theta\,\mathrm{d}^2\eta(\boldsymbol{U}_\theta)(\boldsymbol{U}_1 - \boldsymbol{U}_0)\cdot(\boldsymbol{U}_1 - \boldsymbol{U}_0)\,d\theta$$

where $\boldsymbol{U}_\theta := (1-\theta)\boldsymbol{U}_0 + \theta\,\boldsymbol{U}_1$. Since

$$\begin{aligned}
\mathrm{d}^2\eta(\boldsymbol{U}_\theta)(\boldsymbol{U}_1 - \boldsymbol{U}_0)\cdot(\boldsymbol{U}_1 - \boldsymbol{U}_0) &= \left(\frac{|\boldsymbol{v}_\theta|^2}{h_\theta} + \frac{1}{2\pi}\right)(h_1 - h_0)^2 \\
&\quad - \frac{2}{h_\theta}\,(h_1 - h_0)\boldsymbol{v}_\theta\cdot(h_1\boldsymbol{v}_1 - h_0\boldsymbol{v}_0) + \frac{1}{h_\theta}\,|h_1\boldsymbol{v}_1 - h_0\boldsymbol{v}_0|^2 \\
&= \frac{1}{2\pi}(h_1 - h_0)^2 + \frac{1}{h_\theta}|h_1\boldsymbol{v}_1 - h_0\boldsymbol{v}_0 - (h_1 - h_0)\boldsymbol{v}_\theta|^2 \\
&= \frac{1}{2\pi}(h_1 - h_0)^2 + \frac{h_{(1-\theta)}}{h_\theta}\,|\boldsymbol{v}_1 - \boldsymbol{v}_0|^2,
\end{aligned}$$

estimate (69) reads as

$$\frac{1}{2\pi}\int_{\mathbb{R}^2} |h^\varepsilon - h|^2\,dx + \int_{\mathbb{R}^2} \frac{\theta\,h + (1-\theta)\,h^\varepsilon}{(1-\theta)h + \theta\,h^\varepsilon}\,|\boldsymbol{v}^\varepsilon - \boldsymbol{v}|^2\,dx \le C\sqrt{\varepsilon},$$

for any $t \in [0,T]$, from which L^2-convergence for the variable h^ε follows and, whenever h^ε and h are uniformly positive, also for the velocities $\boldsymbol{v}^\varepsilon$.

Now, our principal aim is to estabilish estimate (69). Given $\boldsymbol{U}$ smooth solution to the system

$$\partial_t \boldsymbol{U} + \mathrm{div}_x A(\boldsymbol{U}) - B(x,\boldsymbol{U}) = 0$$

and $\boldsymbol{U}^\varepsilon$ the macroscopic variable corresponding to the solution f^ε of the kinetic model (53), there holds

$$\begin{aligned}\partial_t \eta(\boldsymbol{U}^\varepsilon; U) = \{&\partial_t \eta(\boldsymbol{U}^\varepsilon) + \mathrm{div}_x G(\boldsymbol{U}^\varepsilon) - \mathrm{d}\eta(\boldsymbol{U}^\varepsilon)\, B(x,\boldsymbol{U}^\varepsilon)\}\\ &- \mathrm{d}\eta(\boldsymbol{U})\{\partial_t \boldsymbol{U}^\varepsilon + \mathrm{div}_x A(\boldsymbol{U}^\varepsilon) - B(x,\boldsymbol{U}^\varepsilon)\}\\ &+ R_A(\boldsymbol{U}^\varepsilon;\boldsymbol{U}) + R_B(x,\boldsymbol{U}^\varepsilon;\boldsymbol{U})\end{aligned}$$

where

$$\begin{aligned}R_A(\boldsymbol{U}^\varepsilon;\boldsymbol{U}) &:= \mathrm{div}_x(G(\boldsymbol{U}) - G(\boldsymbol{U}^\varepsilon)) + \mathrm{d}^2\eta(\boldsymbol{U})\,\mathrm{div}_x A(\boldsymbol{U})(\boldsymbol{U}^\varepsilon - \boldsymbol{U})\\ &\qquad + \mathrm{d}\eta(\boldsymbol{U})\,\mathrm{div}_x(A(\boldsymbol{U}^\varepsilon) - A(\boldsymbol{U}))\\ R_B(x,\boldsymbol{U}^\varepsilon;\boldsymbol{U}) &:= \mathrm{d}\eta(\boldsymbol{U}^\varepsilon)\, B(x,\boldsymbol{U}^\varepsilon) - \mathrm{d}\eta(\boldsymbol{U})\, B(x,\boldsymbol{U}^\varepsilon)\\ &\qquad - \mathrm{d}^2\eta(\boldsymbol{U})\, B(x,\boldsymbol{U})(\boldsymbol{U}^\varepsilon - \boldsymbol{U}).\end{aligned}$$

The term R_B is zero: indeed, using $B(x,\boldsymbol{U}) = (0, \boldsymbol{F}(x)\, h)$, we reckon

$$\begin{aligned}&\mathrm{d}\eta(\boldsymbol{U}^\varepsilon)\, B(x,\boldsymbol{U}^\varepsilon) = \boldsymbol{F}(x)\, h^\varepsilon \boldsymbol{v}^\varepsilon, \quad \mathrm{d}\eta(\boldsymbol{U})\, B(x,\boldsymbol{U}^\varepsilon) = \boldsymbol{F}(x)\, h^\varepsilon \boldsymbol{v}\\ &\mathrm{d}^2\eta(\boldsymbol{U})\, B(x,\boldsymbol{U})(\boldsymbol{U}^\varepsilon - \boldsymbol{U}) = \boldsymbol{F}(x)\, h^\varepsilon \boldsymbol{v}^\varepsilon - \boldsymbol{F}(x)\, h^\varepsilon \boldsymbol{v}.\end{aligned}$$

Integrating in space, we get

$$\begin{aligned}\frac{d}{dt}\int_{\mathbb{R}^2} \big(\eta(\boldsymbol{U}^\varepsilon;\boldsymbol{U}) - \eta(\boldsymbol{U}^\varepsilon)\big) &= -\int_{\mathbb{R}^2} \mathrm{d}\eta(\boldsymbol{U}^\varepsilon)\, B(x,\boldsymbol{U}^\varepsilon) + \int_{\mathbb{R}^2} R_A(\boldsymbol{U}^\varepsilon;\boldsymbol{U})\\ &\quad - \int_{\mathbb{R}^2} \mathrm{d}\eta(\boldsymbol{U})\{\partial_t U^\varepsilon + \mathrm{div}_x A(\boldsymbol{U}^\varepsilon) - B(x,\boldsymbol{U}^\varepsilon)\}\end{aligned} \tag{70}$$

where, after manipulations, the term R_A takes the form

$$R_A(\boldsymbol{U}^\varepsilon;\boldsymbol{U}) = \mathrm{div}_x\big(\dots\big) + \sum_{j,k} \partial_j \eta(\boldsymbol{U})\, \partial_{x_k} A(\boldsymbol{U}^\varepsilon;\boldsymbol{U}).$$

Since $A(\boldsymbol{U}) = (h\,\boldsymbol{v}, h\,\boldsymbol{v}\otimes\boldsymbol{v} + P(h)\,\mathbb{I})$, there holds

$$A(\boldsymbol{U}^\varepsilon;\boldsymbol{U}) = \big(0, h^\varepsilon(\boldsymbol{v}^\varepsilon - \boldsymbol{v})\otimes(\boldsymbol{v}^\varepsilon - \boldsymbol{v}) + P(h^\varepsilon;h)\,\mathbb{I}\big)$$

hence $|A(\boldsymbol{U}^\varepsilon;\boldsymbol{U})| \le C\,\eta(\boldsymbol{U}^\varepsilon;\boldsymbol{U})$, and the space integral of the remainder R_A can be controlled in term of the space integral of the modulated entropy.

The integral relative to the source term in (70) can be estimated by means of the kinetic entropy: indeed, since $B(x,\boldsymbol{U}) = (0, \boldsymbol{F}(x)\, h)$, we have

$$\frac{d}{dt}\iint\limits_{\mathbb{R}^2\times\mathbb{R}^2} H(f^\varepsilon,\xi)\,d\xi\,dx \le \iint\limits_{\mathbb{R}^2\times\mathbb{R}^2} \boldsymbol{F}(x)\,\xi f^\varepsilon\,d\xi\,dx$$

$$= \int\limits_{\mathbb{R}^2} \boldsymbol{F}(x)\,h^\varepsilon \boldsymbol{U}^\varepsilon\,dx = \int\limits_{\mathbb{R}^2} \mathrm{d}\eta(\boldsymbol{U}^\varepsilon)\,B(x,\boldsymbol{U}^\varepsilon)\,dx.$$

Thus, setting

$$\varDelta^\varepsilon(t) := \int\limits_{\mathbb{R}^2}\left\{\eta(\boldsymbol{U}^\varepsilon;\boldsymbol{U}) + \int\limits_{\mathbb{R}^2} H(f^\varepsilon,\xi)\,d\xi - \eta(\boldsymbol{U}^\varepsilon)\right\}dx,$$

we obtain, for some constant C depending on the L^∞ norm of $\boldsymbol{U}$,

$$\frac{d\varDelta^\varepsilon}{dt} \le -\int\limits_{\mathbb{R}^2} \mathrm{d}\eta(\boldsymbol{U})\{\partial_t\boldsymbol{U}^\varepsilon + \mathrm{div}_x A(\boldsymbol{U}^\varepsilon) - B(x,\boldsymbol{U}^\varepsilon)\}dx + C\int\limits_{\mathbb{R}^2}\eta(\boldsymbol{U}^\varepsilon;\boldsymbol{U})\,dx.$$

Note that if $\varDelta^\varepsilon \le C\sqrt{\varepsilon}$, then also (69) holds.

Because of (61), we deduce

$$\frac{d\varDelta^\varepsilon}{dt} \le -\int\limits_{\mathbb{R}^2} \mathrm{d}\eta(\boldsymbol{U})\,\mathrm{div}_x\left(A(\boldsymbol{U}^\varepsilon) - \int\limits_{\mathbb{R}^2}(\boldsymbol{K}(\xi)\otimes\xi)f^\varepsilon\,d\xi\right)dx + C\,\varDelta^\varepsilon$$

$$= \int\limits_{\mathbb{R}^2}\left(\mathrm{d}^2\eta(\boldsymbol{U})\,\mathrm{grad}_x\boldsymbol{U}\right)\left(A(\boldsymbol{U}^\varepsilon) - \int\limits_{\mathbb{R}^2}(\boldsymbol{K}(\xi)\otimes\xi)f^\varepsilon\,d\xi\right)dx + C\,\varDelta^\varepsilon.$$

Thus, denoting by C a constant depending also on the $W^{1,\infty}$ norm of $\boldsymbol{U}$,

$$\frac{d\varDelta^\varepsilon}{dt} \le C\int\limits_{\mathbb{R}^2}\left|A(\boldsymbol{U}^\varepsilon) - \int\limits_{\mathbb{R}^2}(\boldsymbol{K}(\xi)\otimes\xi)f^\varepsilon\,d\xi\right|dx + C\,\varDelta^\varepsilon$$

and, as a consequence, by Gronwall Lemma, we obtain

$$\varDelta^\varepsilon(t) \le \varDelta^\varepsilon(0)\,e^{Ct} + \int\limits_0^t e^{C(t-\tau)}\left\{\int\limits_{\mathbb{R}^n}\left|A(\boldsymbol{U}^\varepsilon) - \int\limits_{\mathbb{R}^n}(\boldsymbol{K}(\xi)\otimes\xi)f^\varepsilon\,d\xi\right|dx\right\}d\tau.$$

Since $\boldsymbol{K}(\xi) = (1, v)$ and

$$A(\boldsymbol{U}) = (A_1(\boldsymbol{U}), A_2(\boldsymbol{U})) = (h\,\boldsymbol{v}, h\,\boldsymbol{v}\otimes\boldsymbol{v} + \frac{1}{4\,\pi}\,h^2\,\mathbb{I}),$$

there holds

$$A_1(\boldsymbol{U}^\varepsilon) - \int_{\mathbb{R}^2} \xi f^\varepsilon\,d\xi = h^\varepsilon\,\boldsymbol{U}^\varepsilon - \int_{\mathbb{R}^2} \xi f^\varepsilon\,d\xi = 0.$$

Therefore, we only need to control

$$\int_{\mathbb{R}^2} \left| A_2(\boldsymbol{U}^\varepsilon) - \int_{\mathbb{R}^2} (\xi\otimes\xi) f^\varepsilon\,d\xi \right| dx = \int_{\mathbb{R}^2} \left| \int_{\mathbb{R}^2} (\xi\otimes\xi)\,\big(f^\varepsilon - \mathcal{M}[f^\varepsilon]\big)\,d\xi \right| dx.$$

First of all, we deal with the functional

$$\mathcal{D}[f] := \int_{\mathbb{R}^2} |\xi|^2 (f - \mathcal{M}[f])\,d\xi.$$

Since

$$\begin{aligned}\int_{\mathbb{R}^2} \mathcal{D}[f^\varepsilon]\,dx &= -\varepsilon \iint_{\mathbb{R}^2\times\mathbb{R}^2} |\xi|^2 (\partial_t f^\varepsilon + \boldsymbol{F}(x)\,\mathrm{grad}_\xi f^\varepsilon)\,d\xi\,dx \\ &= 2\varepsilon\left\{-\frac{d}{dt}\int_{\mathbb{R}^2} \mathcal{H}[f^\varepsilon]\,dx + \iint_{\mathbb{R}^2\times\mathbb{R}^2} \boldsymbol{F}(x)\,\xi f^\varepsilon\,d\xi\,dx\right\},\end{aligned}$$

integrating with respect to $t \in [0, T]$, we obtain

$$\begin{aligned}\int_0^T \int_{\mathbb{R}^2} \mathcal{D}[f]\,dx\,dt &\le 2\varepsilon\left\{\int_{\mathbb{R}^2} \mathcal{H}[f_0^\varepsilon]\,dx + \int_0^t \iint_{\mathbb{R}^2\times\mathbb{R}^2} |\boldsymbol{F}(x)|\,|\xi| f^\varepsilon\,d\xi\,dx\right\} \\ &\le \varepsilon\left\{2\int_{\mathbb{R}^2} \mathcal{H}[f_0^\varepsilon]\,dx + |\boldsymbol{F}|_{L^\infty} \int_0^t \iint_{\mathbb{R}^2\times\mathbb{R}^2} (1+|\xi|^2) f^\varepsilon\,d\xi\,dx\right\} \\ &\le \varepsilon\left\{2\int_{\mathbb{R}^2} \mathcal{H}[f_0^\varepsilon]\,dx + t\,|\boldsymbol{F}|_{L^\infty} \iint_{\mathbb{R}^2\times\mathbb{R}^2} f_0^\varepsilon\,d\xi\,dx \right. \\ &\qquad\qquad \left. + 2\,|\boldsymbol{F}|_{L^\infty} \int_0^t \int_{\mathbb{R}^2\times\mathbb{R}^2} \mathcal{H}[f^\varepsilon]\,dx\right\}.\end{aligned}$$

Therefore, we have the preliminary estimate

$$\int_0^T \int_{\mathbb{R}^2} \mathcal{D}[f]\,dx\,dt = \int_0^T \iint_{\mathbb{R}^2\times\mathbb{R}^2} |\xi|^2 (f - \mathcal{M}[f])\,d\xi\,dx\,dt \le C\,\varepsilon,$$

where the constant C depends also on the final time T.

Since f^ε and $\mathcal{M}[f^\varepsilon]$ have same mass and first-order moment, for any $\zeta \in \mathbb{R}^n$, there holds

$$\int_{\mathbb{R}^2} (\xi \otimes \xi)\left(f^\varepsilon - \mathcal{M}[f^\varepsilon]\right) d\xi = \int_{\mathbb{R}^2} (\xi - \zeta)\otimes(\xi - \zeta)\left(f^\varepsilon - \mathcal{M}[f^\varepsilon]\right) d\xi,$$

so that we obtain

$$\left| \int_{\mathbb{R}^2} (\xi \otimes \xi)\left(f^\varepsilon - \mathcal{M}[f^\varepsilon]\right) d\xi \right| \le \int_{\mathbb{R}^2} |\xi - \zeta|^2\, |f^\varepsilon - \mathcal{M}[f^\varepsilon]|\, d\xi$$

where we can assume, without loss of generality, the first moment $h^\varepsilon \boldsymbol{U}^\varepsilon$ of f^ε to be zero. The next (and final) lemma helps in estimating the right-hand side of the last inequality.

Lemma 5.2. *Let $\chi := \chi(\xi)$ be the characteristic function of the set $\{|\xi| \le 1\}$. Let $\mathcal{F}$, $\mathcal{D}$ be the maps from $L^1_w(\mathbb{R}^2; [0,1])$ to $\mathbb{R}$ defined by*

$$\mathcal{F}[f] := \int_{\mathbb{R}^2} |\xi|^2\, |f(\xi) - \chi(\xi/R)|\, d\xi,$$

$$\mathcal{D}[f] := \int_{\mathbb{R}^2} |\xi|^2 \left(f(\xi) - \chi(\xi/R)\right) d\xi$$

where $R := \left(|f|_{L^1}/\pi\right)^{1/2}$. Then there exists constants $C_1, C_2 > 0$ such that

$$\mathcal{D}[f] \le \mathcal{F}[f] \le C_1\, |f|_{L^1} \sqrt{\mathcal{D}[f]} + C_2\, \mathcal{D}[f] \tag{71}$$

for any $f \in L^1_w(\mathbb{R}^2; [0,1])$.

Proof. Given $R_1, R_2 \in [0,\infty]$ with $R_1 < R_2$ and a integrable function $\phi = \phi(\xi)$, set

$$\int_{R_1}^{R_2} \phi(\xi)\,d\xi := \int_{\mathbb{R}^2} \phi(\xi)\,\chi_{\{\xi\,:\,R_1 \le |\xi| \le R_2\}}(\xi)\,d\xi.$$

By definition of R, there holds

$$\pi R^2 = \int_0^R f\,d\xi + \int_R^{+\infty} f\,d\xi = \pi R^2 - \int_0^R (1-f)\,d\xi + \int_R^{+\infty} f\,d\xi.$$

Hence

$$\int_0^R (1-f)\,d\xi = \int_R^{+\infty} f\,d\xi. \tag{72}$$

We split the integrals in the definitions of $\mathcal{F}$ and $\mathcal{D}$ into the sum of two terms: one given by the integral in a annulus containing $\{|v| = R\}$ and the other given by the integral in the complement of the annulus. Precisely, given R_1, R_2 with $R_1 < R < R_2$ to be chosen, set

$$\mathcal{F}_{in}[f] := \int_{R_1}^{R_2} |\xi|^2\,|f(\xi) - \chi(\xi/R)|\,d\xi, \qquad \mathcal{F}_{out}[f] := \mathcal{F}[f] - \mathcal{F}_{in}[f]$$

$$\mathcal{D}_{in}[f] := \int_{R_1}^{R_2} |\xi|^2\,\big(f(\xi) - \chi(\xi/R)\big)\,d\xi, \quad \mathcal{D}_{out}[f] := \mathcal{D}[f] - \mathcal{D}_{in}[f].$$

We will estimate $\mathcal{F}_{out}[f]$ in term of $\mathcal{D}_{out}[f]$ and $\mathcal{F}_{in}[f]$ in term of $\mathcal{D}_{in}[f]$ for appropriate choice of R_1, R_2.

Given $R_2 > R$ such that

$$0 < \int_R^{R_2} f\,d\xi < \int_R^{+\infty} f\,d\xi,$$

we choose $R_1 < R$ so that

$$\int_{R_1}^{R} (1-f)\,d\xi = \int_R^{R_2} f\,d\xi =: M. \tag{73}$$

Using (72), we deduce from (73) the relation

$$\int_0^{R_1} (1-f)\,d\xi = \int_{R_2}^{+\infty} f\,d\xi. \tag{74}$$

Then, thanks to (73),

$$\begin{aligned}\mathcal{D}_{in}[f] &= -\int_{R_1}^{R} |\xi|^2 \left(1-f\right) d\xi + \int_{R}^{R_2} |\xi|^2 f\, d\xi \\ &\geq -R^2 \int_{R_1}^{R} \left(1-f\right) d\xi + R^2 \int_{R}^{R_2} f\, d\xi = 0.\end{aligned}$$

Similarly, from (74) it follows

$$\begin{aligned}\mathcal{D}_{out}[f] &= -\int_{0}^{R_1} |\xi|^2 \left(1-f\right) d\xi + \int_{R_2}^{+\infty} |\xi|^2 f\, d\xi \\ &\geq -R_1^2 \int_{0}^{R_1} \left(1-f\right) d\xi + R_2^2 \int_{R_2}^{+\infty} f\, d\xi = (R_2^2 - R_1^2) \int_{R_2}^{+\infty} f\, d\xi \geq 0.\end{aligned}$$

Hence, both $\mathcal{D}_{in}[f]$ and $\mathcal{D}_{out}[f]$ are non-negative.

Moreover, from the latter inequality, it follows

$$\begin{aligned}\mathcal{F}_{out}[f] &= \mathcal{D}_{out}[f] + 2\int_{0}^{R_1} |\xi|^2 (1-f)\, d\xi \\ &\leq \mathcal{D}_{out}[f] + 2R_1^2 \int_{0}^{R_1} (1-f)\, d\xi = \mathcal{D}_{out}[f] + 2R_1^2 \int_{R_2}^{+\infty} f\, d\xi \\ &\leq \mathcal{D}_{out}[f] + \frac{2R_1^2}{R_2^2 - R_1^2} \mathcal{D}_{out}[f];\end{aligned}$$

thus

$$\mathcal{F}_{out}[f] \leq \frac{R_2^2 + R_1^2}{R_2^2 - R_1^2}\, \mathcal{D}_{out}[f] \leq \frac{R_2^2 + R^2}{R_2^2 - R^2}\, \mathcal{D}_{out}[f].$$

Next, let us consider the term $\mathcal{F}_{in}$:

$$\begin{aligned}\mathcal{F}_{in}[f] &= \int_{R_1}^{R} |\xi|^2 (1-f)\, d\xi + \int_{R}^{R_2} |\xi|^2 f\, d\xi \\ &\leq R^2 \int_{R_1}^{R} (1-f)\, d\xi + R_1^2 \int_{R}^{R_2} f\, d\xi = (R^2 + R_2^2) M.\end{aligned} \tag{75}$$

At the same time, for $a<R<\beta$ to be chosen, we have

$$\begin{aligned}\mathcal{D}_{in}[f] = &- \int_{R_1}^{a} |\xi|^2(1-f)\,d\xi + \int_{a}^{R} |\xi|^2 f\,d\xi - \int_{a}^{R} |\xi|^2\,d\xi \\ &+ \int_{\beta}^{R_2} |\xi|^2 f\,d\xi - \int_{R}^{\beta} |\xi|^2\,(1-f)\,d\xi + \int_{R}^{\beta} |\xi|^2\,d\xi\end{aligned}$$

hence

$$\begin{aligned}\mathcal{D}_{in}[f] &\geq a^2\left(\int_{a}^{R} d\xi - M\right) + \beta^2\left(M - \int_{R}^{\beta} d\xi\right) + \frac{\pi}{2}\left(\beta^4 - 2R^4 + a^4\right) \\ &\geq a^2\{\pi(R^2-a^2) - M\} + \beta^2\left\{M - \pi(\beta^2 - R^2)\right\} + \frac{\pi}{2}\left(\beta^4 - 2R^4 + a^4\right).\end{aligned}$$

Thus, by choosing a, β such that $R^2 - a^2 = \beta^2 - R^2 = M/\pi$, the following inequality holds

$$\mathcal{D}_{in}[f] \geq \frac{\pi}{2}\left\{\left(R^2 + \frac{M}{\pi}\right)^2 - 2R^4 + \left(R^2 - \frac{M}{\pi}\right)^2\right\} = \frac{M^2}{\pi}.$$

Collecting the latter inequality together with (75), we obtain

$$\mathcal{F}_{in}[f] \leq (R_2^2 + R^2)\sqrt{\pi}\,\sqrt{\mathcal{D}_{in}[f]}.$$

Hence,

$$\mathcal{F}[f] = \mathcal{F}_{in}[f] + \mathcal{F}_{out}[f] \leq (R_2^2 + R^2)\sqrt{\pi}\,\sqrt{\mathcal{D}_{in}[f]} + \frac{R_2^2 + R^2}{R_2^2 - R^2}\,\mathcal{D}_{out}[f]$$

that gives the conclusion by choosing $R_2 = a\,R$ for arbitrary $a > 1$. □

To prove estimate (69) is a straightforward consequence of the inequalities

$$\int_{\mathbb{R}^2} \eta(\boldsymbol{U}^\varepsilon; \boldsymbol{U})\,dx \leq \varDelta^\varepsilon(t) \leq C\,\varDelta^\varepsilon(0) + C\int_0^t \int_{\mathbb{R}^2} \mathcal{F}[f^\varepsilon]\,dx\,d\tau \leq C\,\varDelta^\varepsilon(0) + C\sqrt{\varepsilon}$$

holding for $t<T$, with $C > 0$ depending on $T > 0$.

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