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Analytical integrations in 3D BEM (**)

1 - Introduction

Modeling elliptic problems by means of boundary integral equations (BIEs) and approximating their solution through boundary element methods (BEM) is firmly established in the academic community as well as in industry.

Several well known yet stimulating as well as modern applications and on going research topics can be effectively described via BIEs: to cite but a few, size and location of tumors from temperature measurements [**4**], mechanics of highly non linear material behaviors eventually with large strains and rotations [**5**] as well as strain gradient constitutive laws [**6**], mechanics of carbon nanotubes composites [**7**] and dislocations [**8**].

The present note aims at providing a closed form for analytical integrations [**9**] involved in 3D BIEs for elliptic problems, what seems to be of interest for computational and theoretical purposes, for isotropic homogeneous materials. Educational advantages of analytical integrations can also be envisaged, as in [**10**]. In this note: reference will be made to linear elasticity as a prototype of an elliptic problem; bulk forces in domain Ω are denoted with $\bar{\boldsymbol{f}}(\boldsymbol{y})$; displacements $\bar{\boldsymbol{u}}(\boldsymbol{y})$ are given on boundary $\Gamma_u \subset \partial\Omega$ whereas tractions $\bar{\boldsymbol{p}}(\boldsymbol{y})$ are given on boundary $\Gamma_p \subset \partial\Omega$; boundary is taken such that $\Gamma_u \cup \Gamma_p = \partial\Omega$ and $\Gamma_u \cap \Gamma_p = \emptyset$. The boundary integral formulation of

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Lamé equations stems from Somigliana's identity [**11**]:

$$
\boldsymbol{u}(\boldsymbol{x}) + \int_{\partial\Omega} \boldsymbol{G}_{up}(\boldsymbol{x}-\boldsymbol{y};\boldsymbol{l}(\boldsymbol{y}))\boldsymbol{u}(\boldsymbol{y})\,\mathrm{d}\boldsymbol{y} = \int_{\partial\Omega} \boldsymbol{G}_{uu}(\boldsymbol{x}-\boldsymbol{y})\boldsymbol{p}(\boldsymbol{y})\,\mathrm{d}\boldsymbol{y} + \int_{\Omega} \boldsymbol{G}_{uu}(\boldsymbol{x}-\boldsymbol{y})\bar{\boldsymbol{f}}(\boldsymbol{y})\,\mathrm{d}\boldsymbol{y}\,, \qquad \boldsymbol{x}\in\Omega \tag{1}
$$

which is the boundary integral representation (BIR) of displacements in the interior of Ω. Somigliana's identity is based on Green's functions (also called kernels) which represent components u_i of the displacement vector $\boldsymbol{u}$ in a point $\boldsymbol{x}$ due to: i) a unit force concentrated in space (point $\boldsymbol{y}$) and acting on the unbounded elastic space Ω_∞ in direction j (such functions are gathered in matrix $\boldsymbol{G}_{uu}(\boldsymbol{x}-\boldsymbol{y})$); ii) a unit relative displacement concentrated in space (at a point $\boldsymbol{y}$), crossing a surface with normal $\boldsymbol{l}(\boldsymbol{y})$ and acting on the unbounded elastic space Ω_∞ (in direction j) (gathered in matrix $\boldsymbol{G}_{up}(\boldsymbol{x}-\boldsymbol{y})$).

To obtain an additional integral equation, the traction operator can be applied to Somigliana's identity [1], thus obtaining the BIR of tractions on a surface of normal $\boldsymbol{n}(\boldsymbol{x})$ in the interior of the domain [**12, 13**]:

$$
\begin{aligned}
&\boldsymbol{p}(\boldsymbol{x},\boldsymbol{n}(\boldsymbol{x})) + \int_{\Gamma_p} \boldsymbol{G}_{pp}(\boldsymbol{r};\boldsymbol{n}(\boldsymbol{x});\boldsymbol{l}(\boldsymbol{y}))\boldsymbol{u}(\boldsymbol{y})\,\mathrm{d}\boldsymbol{y} + \int_{\Gamma_u} \boldsymbol{G}_{pp}(\boldsymbol{r};\boldsymbol{n}(\boldsymbol{x});\boldsymbol{l}(\boldsymbol{y}))\bar{\boldsymbol{u}}(\boldsymbol{y})\,\mathrm{d}\boldsymbol{y} \\
&= \int_{\Gamma_u} \boldsymbol{G}_{pu}(\boldsymbol{r};\boldsymbol{n}(\boldsymbol{x}))\boldsymbol{p}(\boldsymbol{y})\,\mathrm{d}\boldsymbol{y} + \int_{\Gamma_p} \boldsymbol{G}_{pu}(\boldsymbol{r};\boldsymbol{n}(\boldsymbol{x}))\bar{\boldsymbol{p}}(\boldsymbol{y})\,\mathrm{d}\boldsymbol{y} + \int_{\Omega} \boldsymbol{G}_{pu}(\boldsymbol{r};\boldsymbol{n}(\boldsymbol{x}))\bar{\boldsymbol{f}}(\boldsymbol{y})\,\mathrm{d}\boldsymbol{y}, \quad \boldsymbol{x}\in\Omega
\end{aligned} \tag{2}
$$

having denoted with $\boldsymbol{r} = \boldsymbol{x} - \boldsymbol{y}$. Such a BIR involves Green's functions (collected in matrices $\boldsymbol{G}_{pu}$ and $\boldsymbol{G}_{pp}$) which describe components (p_i) of the traction vector $\boldsymbol{p}$ on a surface of normal $\boldsymbol{n}(\boldsymbol{x})$ due to: i) a unit force concentrated in space (point $\boldsymbol{y}$) and acting on the unbounded elastic space Ω_∞ in direction j; ii) a unit relative displacement concentrated in space (at a point $\boldsymbol{y}$), crossing a surface with normal $\boldsymbol{l}(\boldsymbol{y})$ and acting on the unbounded elastic space Ω_∞ (in direction j).

BIEs for the linear elastic problem can be derived from BIRs (1) and (2) by performing the boundary limit [2] $\Omega \ni \boldsymbol{x} \to \boldsymbol{x} \in \Gamma$. In the limit process, extensively

[1] The above introduced kernels are infinitely smooth in their domain, which is the whole space $\mathbb{R}^3$ with exception of the origin (that is $\boldsymbol{x} \neq \boldsymbol{y}$).

[2] In the traction equation (5) the boundary limit must be taken at a smooth point $\boldsymbol{x} \in \partial\Omega \overset{def}{=} \Gamma$ with a well defined normal vector $\boldsymbol{n}(\boldsymbol{x})$. Strong and hypersingular kernels generate free terms - with the notation of [**14**] they will be termed $\chi^u_\Gamma(\boldsymbol{x})$ and $\chi^p_\Gamma(\boldsymbol{x})$ - in the limit process such that $\chi^u_\Gamma(\boldsymbol{x}) = \chi^p_\Gamma(\boldsymbol{x}) = \frac{1}{2}\mathbb{1}$ for smooth boundaries [**14, 15, 16, 17, 18, 19, 20**], whereas special cares are required for the discrete problem (see again [**14**]).

investigated [3] singularities of Green's functions are triggered off: their singularity-orders are collected in Table 1. Kernel $\boldsymbol{G}_{uu}$, that appears in the Single Layer Potential operator $V : H^{-\frac{1}{2}}(\Gamma) \to H^{\frac{1}{2}}(\Gamma)$, shows an integrable singularity (named "weak"); kernels $\boldsymbol{G}_{up}$, within the Double Layer Potential operator $K : H^{\frac{1}{2}}(\Gamma) \to \to H^{\frac{1}{2}}(\Gamma)$, and $\boldsymbol{G}_{pu}$, within the Adjoint Double Layer Potential operator $K' : H^{-\frac{1}{2}}(\Gamma) \to H^{-\frac{1}{2}}(\Gamma)$, present a strong singularity $O(r^{-2})$; kernel $\boldsymbol{G}_{pp}$, into the Hypersingular Integral Operator $D : H^{\frac{1}{2}}(\Gamma) \to H^{-\frac{1}{2}}(\Gamma)$, is usually named hypersingular, because it shows a singularity of $O(r^{-3})$ greater than the dimension of the integral.

Table 1. *Kernels and their singularities. Here* $\boldsymbol{r} \stackrel{def}{=} \boldsymbol{x} - \boldsymbol{y}$ *and* $r = \|\boldsymbol{r}\|$.

kernel	Asymptotical behavior when $\boldsymbol{r} \to \boldsymbol{0}$		Denomination of singularity	Relevant "integrals" when $\boldsymbol{x} \in \partial\Omega$	
	2D	3D		Nature	Symbol
$\boldsymbol{G}_{uu}$	$O(\log(r))$	$O(r^{-1})$	Weak(integrable)	Lebesgue	$\int$
$\boldsymbol{G}_{up}$,$\boldsymbol{G}_{pu}$	$O(r^{-1})$	$O(r^{-2})$	Strong	CPV	$⨍$
$\boldsymbol{G}_{pp}$	$O(r^{-2})$	$O(r^{-3})$	Hyper	HFP	$⨎$

According with their singular behavior, Green's functions may contain terms of the following kind (see for instance the expressions of kernels for linear elasticity in Appendix 1):

$$\frac{d_1^{\alpha}\, d_2^{\beta}\, d_3^{\gamma}}{r^{\alpha+\beta+\gamma+s}} \qquad \alpha \geq 0,\ \ \beta \geq 0,\ \ \gamma \geq 0,\ \ s = 1, 2, 3 \tag{3}$$

where $\boldsymbol{d} = -\boldsymbol{r} = \boldsymbol{y} - \boldsymbol{x}$ and $d_j = y_j - x_j$.

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[3] By the approach of [**21**], all singular terms cancel out in the limit process (and without recourse to any a-priori interpretation in the finite part sense). However, there exists an intimate relationship between hypersingular BIEs and finite part integrals (HFP) in the sense of Hadamard [**2**]. It has been proved that a hypersingular integral can be interpreted as a HFP in the limit as an internal point source approaches the boundary. In [**22**], the same conclusion has been obtained by an alternate definition of HFP, without the need for a limiting process.

Making recourse to the distribution theory, in [**12**] BIEs are obtained by the application of a trace operator to the representation formulae. In such an approach, the strongly singular and hypersingular integrals can be expressed by means of discontinuity jumps (also named "free terms") of these integrals on the boundary summed with the values of the integrals on the boundary existing only in the sense of Cauchy Principal Value (CPV) or in the sense of the HFP. By exploiting Green's functions properties, the commutativity of the two operations of traction and trace has also been proved, showing the consistency of all different approaches of derivations of the BIEs.

Assuming smooth boundaries, the following BIEs come out:

$$\frac{1}{2}\boldsymbol{u}(\boldsymbol{x}) + \fint_{\Gamma_p} \boldsymbol{G}_{up}(\boldsymbol{r};\boldsymbol{l}(\boldsymbol{y}))\boldsymbol{u}(\boldsymbol{y})\mathrm{d}\,\mathrm{d}\boldsymbol{y} + \fint_{\Gamma_u} \boldsymbol{G}_{up}(\boldsymbol{r};\boldsymbol{l}(\boldsymbol{y}))\bar{\boldsymbol{u}}(\boldsymbol{y})\,\mathrm{d}\boldsymbol{y} \tag{4}$$

$$= \int_{\Gamma_u} \boldsymbol{G}_{uu}(\boldsymbol{r})\boldsymbol{p}(\boldsymbol{y})\,\mathrm{d}\boldsymbol{y} + \int_{\Gamma_p} \boldsymbol{G}_{uu}(\boldsymbol{r})\bar{\boldsymbol{p}}(\boldsymbol{y})\,\mathrm{d}\boldsymbol{y} + \int_{\Omega} \boldsymbol{G}_{uu}(\boldsymbol{r})\bar{\boldsymbol{f}}(\boldsymbol{y})\,\mathrm{d}\boldsymbol{y}\ , \qquad \boldsymbol{x} \in \partial\Omega$$

$$\frac{1}{2}\,\boldsymbol{p}(\boldsymbol{x}) + \fint_{\Gamma_p} \boldsymbol{G}_{pp}(\boldsymbol{r};\boldsymbol{n}(\boldsymbol{x});\boldsymbol{l}(\boldsymbol{y}))\boldsymbol{u}(\boldsymbol{y})\,\mathrm{d}\boldsymbol{y} + \fint_{\Gamma_u} \boldsymbol{G}_{pp}(\boldsymbol{r};\boldsymbol{n}(\boldsymbol{x});\boldsymbol{l}(\boldsymbol{y}))\bar{\boldsymbol{u}}(\boldsymbol{y})\,\mathrm{d}\boldsymbol{y}$$

$$= \fint_{\Gamma_u} \boldsymbol{G}_{pu}(\boldsymbol{r};\boldsymbol{n}(\boldsymbol{x}))\boldsymbol{p}(\boldsymbol{y})\,\mathrm{d}\boldsymbol{y} + \fint_{\Gamma_p} \boldsymbol{G}_{pu}(\boldsymbol{r};\boldsymbol{n}(\boldsymbol{x}))\bar{\boldsymbol{p}}(\boldsymbol{y})\,\mathrm{d}\boldsymbol{y} \tag{5}$$

$$+ \fint_{\Omega} \boldsymbol{G}_{pu}(\boldsymbol{r};\boldsymbol{n}(\boldsymbol{x}))\bar{\boldsymbol{f}}(\boldsymbol{y})\,\mathrm{d}\boldsymbol{y}\ , \qquad \boldsymbol{x} \in \partial\Omega\,.$$

Equation (4) is referred to as "displacement equation", whereas equation (5) is named "traction equation": they permit to derive the Calderon Projector for the elastostatic operator. After imposing the fulfillment of equation (4) on the Dirichlet boundary Γ_u and of equation (5) on the Neumann boundary Γ_p, the following linear boundary integral problem comes out:

$$\begin{bmatrix} V[.] & -K[.] \\ -K'[.] & -D[.] \end{bmatrix} \begin{bmatrix} \boldsymbol{p} \\ \boldsymbol{u} \end{bmatrix} = \begin{bmatrix} \boldsymbol{f}^u \\ \boldsymbol{f}^p \end{bmatrix} \quad \begin{matrix} \boldsymbol{x} \in \Gamma_u \\ \boldsymbol{x} \in \Gamma_p \end{matrix} \tag{6}$$

with integral operators V, K, K', D defined by comparison and with notation according to [**10**]. Vectors $\boldsymbol{f}^i$, $i = u, p$, that gather all data (i.e. $\bar{\boldsymbol{p}}$, $\bar{\boldsymbol{u}}$, $\bar{\boldsymbol{f}}$) follows:

$$\begin{aligned} \boldsymbol{f}^u(\boldsymbol{x}) &:= \frac{1}{2}\bar{\boldsymbol{u}} - \int_{\Gamma_p} \boldsymbol{G}_{uu}\,\bar{\boldsymbol{p}}\,\mathrm{d}\boldsymbol{y} + \fint_{\Gamma_u} \boldsymbol{G}_{up}\,\bar{\boldsymbol{u}}\,\mathrm{d}\boldsymbol{y} - \int_{\Omega} \boldsymbol{G}_{uu}\,\bar{\boldsymbol{f}}\,\mathrm{d}\boldsymbol{y} \\ \boldsymbol{f}^p(\boldsymbol{x}) &:= -\frac{1}{2}\bar{\boldsymbol{p}} + \fint_{\Gamma_p} \boldsymbol{G}_{pu}\,\bar{\boldsymbol{p}}\,\mathrm{d}\boldsymbol{y} - \fint_{\Gamma_u} \boldsymbol{G}_{pp}\,\bar{\boldsymbol{u}}\,\mathrm{d}\boldsymbol{y} + \fint_{\Omega} \boldsymbol{G}_{pu}\,\bar{\boldsymbol{f}}\,\mathrm{d}\boldsymbol{y}\,. \end{aligned}$$

Integral problem (6) can be written in the compact form:

$$\mathcal{L}\,[\,y\,] = f \tag{7}$$

with all terms defined by comparison. Unknown vector y is made of tractions (Neumann data) $\boldsymbol{p}$ on the Dirichlet boundary Γ_u and displacements (Dirichlet data) $\boldsymbol{u}$ on the Neumann boundary Γ_p. Denote with $Y_{\mathcal{L}}$ the domain of $\mathcal{L}$ and with $F_{\mathcal{L}}$ its range. Let bilinear form $\mathcal{A}_\mathcal{L} : Y_\mathcal{L} \times Y_\mathcal{L} \to \mathbb{R}$:

$$\mathcal{A}_\mathcal{L}(a,b) \stackrel{def}{=} \int_{\partial\Omega} \mathcal{L}[\,a(\boldsymbol{y})](\boldsymbol{x})\; b(\boldsymbol{x})\,\mathrm{d}\Gamma(\boldsymbol{x})\,. \tag{8}$$

It can be proved - starting from the property of reciprocity [**13**] - that bilinear form $\mathcal{A}_\mathcal{L}$ is symmetric:

$$\mathcal{A}_\mathcal{L}(a,b) = \mathcal{A}_\mathcal{L}(b,a) \qquad\qquad \forall a,b \in Y_\mathcal{L}\,.$$

As a consequence of the mapping properties of operators V and D, problem (6) is uniquely solvable provided that some conditions are fulfilled [**23**] and the solution is a critical point of functional

$$\Psi[y] = \frac{1}{2}\mathcal{A}_\mathcal{L}(y,y) - \int_{\partial\Omega} y(\boldsymbol{x})\,f(\boldsymbol{x})\,\mathrm{d}\Gamma(\boldsymbol{x})\,.$$

Let $h > 0$ be a parameter and let $[\boldsymbol{p}_h(\boldsymbol{y}), \boldsymbol{u}_h(\boldsymbol{y})]^T \stackrel{def}{=} y_h \in Y_{\mathcal{L}h}$ be an approximation of the unknown vector field y, denoting with $Y_{\mathcal{L}h}$ a family of finite dimensional subspaces of $Y_\mathcal{L}$ such that

$$\forall y \in Y_\mathcal{L}\;,\; \inf_{y_h \in Y_{\mathcal{L}h}} \|y - y_h\| \to 0 \quad as \quad h \to 0\,. \tag{9}$$

Discretization (9) allows to transform integral problem (7) into a set of algebraic equations. Two main techniques have been successfully developed to this aim: the collocation boundary element method (CBEM) [**24**] and the symmetric Galerkin [**25**] method (SGBEM).

Displacement equation (4) is the starting point for the numerical approximation via the CBEM [4]. Starting from problem (7) CBEM requires the fulfillment of integral equations

$$\mathcal{L}\,[\,y_h\,] = f$$

onto a selected set of collocation points $\boldsymbol{x}_i^* \in \partial\Omega$. In this technique "integrals" of the form:

$$\int_{\Gamma_s} \boldsymbol{G}_{rs}(\boldsymbol{x}_i^* - \boldsymbol{y})\,\psi_h(\boldsymbol{y})\mathrm{d}\Gamma(\boldsymbol{y}) \qquad \mathrm{r = u,\; s = u, p} \tag{10}$$

[4] In the modeling of fracture mechanics problems an insurmountable mathematical difficulty arises in applying the CBEM making use of the displacement equation only (see e.g. [**26**], [**27**]). Several special techniques have been devised to overcome this mathematical degeneracy: among others, the special Green's functions methods [**28**], the zone method [**29**] and the Dual BEM [**30**].

must be tackled, denoting with $\psi_h(\boldsymbol{y})$ *scalar* shape functions for modeling the components of approximation y_h of the unknown vector fields along $\partial\Omega$.

The SGBEM approximation of (7) consists in finding $y_h \in Y_{\mathcal{L}h}$ critical point of the functional:

$$\Psi[y_h] = \frac{1}{2}\mathcal{A}_{\mathcal{L}}(y_h, y_h) - \int_{\partial\Omega} y_h(\boldsymbol{x})\, f(\boldsymbol{x})\, \mathrm{d}\Gamma(\boldsymbol{x})\,.$$

By imposing the stationarity of $\Psi[y_h]$ with respect to the set of nodal values, one deals with integrals of the following form:

$$\int_{\Gamma_r} \psi_k(\boldsymbol{x}) \int_{\Gamma_s} \boldsymbol{G}_{rs}(\boldsymbol{x}, \boldsymbol{y})\, \psi_h(\boldsymbol{y}) \mathrm{d}\Gamma(\boldsymbol{y}) \mathrm{d}\Gamma(\boldsymbol{x}) \qquad \mathrm{r, s = u, p} \tag{11}$$

where $\psi_k(\boldsymbol{x})$, $\psi_h(\boldsymbol{y})$ are *scalar* test and shape functions that model the components of the unknown vector fields along the boundary.

The evaluation of (10)-(11) is never a trivial task, because of the involved singularities, especially for the hypersingular kernel. Several techniques, collectable in three principal groups, have been proposed for their evaluation: (i) regularization techniques, (ii) numerical integrations, (iii) analytical integrations. By a *regularization procedure*, the strongly singular and hypersingular integrals are analytically manipulated to convert them into, at most, weakly singular integrals, which can then be computed throughout different quadrature schemes. Regularization procedures have been obtained by means of simple solutions [**31, 32**]; by applying the Stokes theorem [**33, 34**]; via integration by parts [**35**]. *Numerical methods* for the evaluation of the CPV were proposed first in [**36**]. There is nowadays an extensive literature on this subject (see, among others, [**37**]). A huge amount of literature concerns the numerical evaluation of hypersingular integrals: among others, see [**38, 39**]. *Analytical integrations* have been basically performed towards three schemes. In the first scheme (see e.g. [**15, 21**]), the source point is fixed, while the boundary around the source point is temporarily deformed to allow an analytical evaluation of contributions from singular kernels, and then the limit is taken as the deformed boundary shrinks back to the actual boundary. In a second approach, see among others [**40, 41, 42**], the source point $\boldsymbol{x}$ is first moved away from the boundary; integrals are evaluated analytically and a limit process is then performed to bring the source point back to the boundary. In all the aforementioned papers, analytical integrations are provided for all singular integrals, while standard quadrature formulae are used for non-singular integrals. In the third scheme [**43, 44, 45**], the complete analytical integration has been provided, directly evaluating HFP and CPV as well as by means of a limit to the boundary process.

The present note will provide a contribution to the analytical evaluation of integral (10) and of the inner integral in (11) in three dimensions, after having performed a tassellation of the boundary by means of flat triangles. Denoting with Γ_h a triangulation of boundary Γ and with $T_j \subset \Gamma_h$ its generic triangle, the paper is devoted to analytical integration of

$$\int_{T_j} \boldsymbol{G}_{rs}(\boldsymbol{x}_i^* - \boldsymbol{y})\,\psi_h(\boldsymbol{y})\mathrm{d}\Gamma(\boldsymbol{y}) \qquad \mathrm{r} = \mathrm{u},\ \mathrm{s} = \mathrm{u}, \mathrm{p}\,. \tag{12}$$

Shape functions $\psi_h(\boldsymbol{y})$ are taken as polynomial of arbitrary degree - thus allowing for a p-refinement technique. Integrations are performed in a local coordinate system, which is detailed in Section 2.

Differently from several papers in the literature, analytical integrations are performed for both the singular and the regular part, so that the closed form of equation (12) is obtained - see Section 4 - as a function of the collocation point x_i^*. The proposed outcome is exhaustive for the collocation approach as well as for the post-process reconstruction of primal and dual fields (temperature and flux, displacement and stress). It seems to be of interest for the Galerkin technique as well, because it firmly distinguishes the weak singularity relevant to the outer integral and the singular terms that will cancel out in the outer integration process [5]. Besides accuracy and computational efficiency, the availability of the closed form for the approximated primal and dual fields entails the possibility of analytical manipulations - see e.g. [**46**] - which are hardly possible with alternative approaches.

In the closed form of integral (12), the Lebesgue integral of $\dfrac{1}{r^3}$ over a triangle, named $I_{\triangle}^{r^{-3}}(\boldsymbol{x})$, is exploited. Such a function has been discussed in details in [**44**], and will be shortly summarized in Section 4.1.

Because integral (12) - and $I_{\triangle}^{r^{-3}}(\boldsymbol{x})$ as well - depend on the position of source point x_i^* with respect to T_j, all significant instances of the position of the source point will be analyzed. In particular, when point x_i^* belongs to triangle T_j, the integral does not exist in a classical sense. The HFP of a divergent integral has a perfect meaning though and the continuity (with respect to the source point) between the HFP and the Lebesgue integral is shown. To this aim, the HFP has been directly evaluated as first; further, the limit process to the boundary $\Omega \ni \boldsymbol{x} \to \boldsymbol{x} \in \Gamma$ has been performed.

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[5] In this regard, much work still needs to be performed, accordingly the extension of [**14**] to the 3D case will be the subject of a further publication.

2 - Shape functions

2.1 - *Definition*

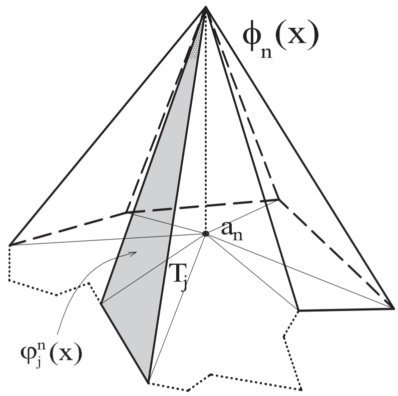

Fig. 1. – Local $\varphi_j^n(\boldsymbol{x})$ and global $\phi_n(\boldsymbol{x})$ shape functions.

Let Γ_h be a triangulation of boundary Γ, $T_j \subset \Gamma_h$ its generic triangle (considered as an open set) and $\boldsymbol{a}_n$ a generic node of Γ_h. Collect in set $\mathcal{T}_n := \{T_j \; s.t. \; \boldsymbol{a}_n \in \overline{T_j}\}$ all triangles of Γ_h sharing node $\boldsymbol{a}_n$ (see figure 1). Choose over T_j a local (lagrangian) basis $\varphi_j := \{\varphi_j^1, \varphi_j^2, ..., \varphi_j^{M(j)}\}$ and denote with $\varphi_j^{n(j)}$ the unique element of φ_j such that $\varphi_j^{n(j)}(\boldsymbol{a}_n) = 1$. Define shape function $\phi_n(\boldsymbol{x})$ (see figure 1) as a piecewise continuous function over Γ_h whose value is zero outside $\mathcal{T}_n$, as follows:

$$\phi_n \in C^0(\Gamma_h) \qquad supp(\phi_n) = \mathcal{T}_n \qquad \phi_n|_{T_j} = \varphi_j^{n(j)} \,. \tag{13}$$

2.2 - *Representation*

A suitable choice of an orthogonal cartesian coordinate system [6] allows an effective representation for $\varphi_j^{n(j)}(\boldsymbol{y})$.

Let $\mathcal{L} \equiv \{y_1, y_2, y_3\}$ define a local coordinate system such that: i) a vertex of T_j is the origin; ii) the plane $y_1 = 0$ contains T_j; iii) the plane $y_3 = 0$ is orthogonal to the side of T_j opposite to the origin. In $\mathcal{L}$, T_j is defined by:

$$T_j := \{\boldsymbol{y} \in \mathbb{R}^3 \quad \text{s.t. } y_1 = 0 \,;\, 0 < y_2 < \bar{y}_2 \,;\, a y_2 - y_3 < 0 \,;\, b y_2 - y_3 > 0\}$$

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[6] The choice of an orthogonal coordinate system is arbitrary because the jacobian is unit and no distortions are introduced with regard to the Hadamard's finite part [**47**].

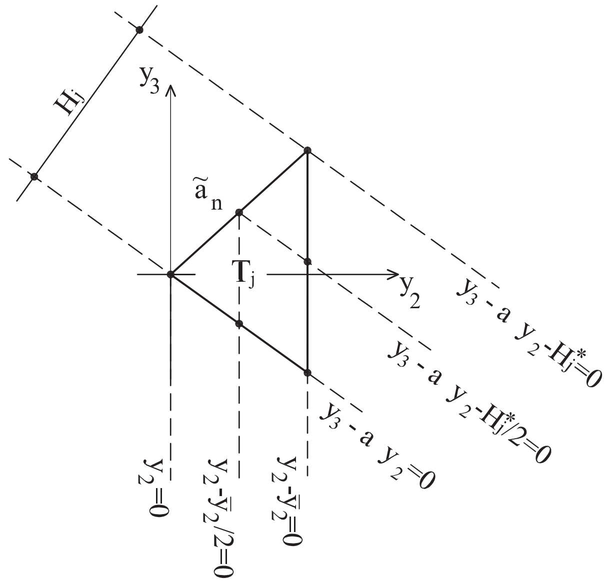

Fig. 2. – Local coordinate system $\mathcal{L}$; here $H_j^* = \sqrt{1+a^2}H_j$.

where a and b denote the slopes of the two sides of T_j that cross the origin (see figure 2). Selecting arbitrarily one of these two sides, say $y_3 - a\ y_2 = 0$, denote with H_j the height of T_j, namely the segment orthogonal to a side emanating from the vertex opposite to it - see figure 2. Shape functions can be readily expressed in terms of H_j in the form:

(14) $$\varphi_j^n(\boldsymbol{y}) = \boldsymbol{y}_3^T\, \Lambda_j^n\, \boldsymbol{y}_2$$

where vectors $\boldsymbol{y}_3$ and $\boldsymbol{y}_2$ are defined by:

$$\boldsymbol{y}_3^T = \{1, y_3, y_3^2, ...\} \qquad \boldsymbol{y}_2^T = \{1, y_2, y_2^2, ...\}$$

and matrix Λ_j^n depends on node $\boldsymbol{a}_n$. For the six-node element of figure 2 with reference to node $\tilde{\boldsymbol{a}}_n$, matrix Λ_j^n reads:

$$\Lambda_j^n = \begin{bmatrix} 0 & \dfrac{4a}{(a-b)\bar{y}_2} & -\dfrac{4a}{(a-b)\bar{y}_2{}^2} \\ -\dfrac{4}{(a-b)\bar{y}_2} & \dfrac{4}{(a-b)\bar{y}_2{}^2} & 0 \end{bmatrix}.$$

For linear shape functions and with reference to the node at the origin, it reads:

(15) $$\Lambda_j^n = \begin{bmatrix} 1 & -\dfrac{1}{\bar{y}_2} \end{bmatrix}, \qquad \varphi_j^n(\boldsymbol{y}) = \Lambda_j^n\, \boldsymbol{y}_2\,.$$

Because of the simplicity of form (15), it may be computationally worth for linear shape functions to consider three different local reference systems - one at each vertex - instead of making use of expression (14), what would lead to a more involved matrix Λ_j^n.

2.3 - *Discrete approximation of unknown fields*

Collect in vector $\boldsymbol{\phi}_i^u$ all shape functions defined by equation (13) for the discrete approximation $\boldsymbol{u}_h(\boldsymbol{y})$ of the Dirichlet field $\boldsymbol{u}(\boldsymbol{y})$ relevant to direction i

$$\boldsymbol{\phi}_i^u(\boldsymbol{y}) = \{\phi_1(\boldsymbol{y}), ..., \phi_{N_i}(\boldsymbol{y})\} \,. \tag{16}$$

For scalar problems $i = 1$ will be generally omitted, for vector problems $i = 1, 2, 3$. N_i is the number of unknowns for direction i. Vector $\boldsymbol{\phi}_i^p$ for the discrete approximation $\boldsymbol{p}_h(\boldsymbol{y})$ of Neumann field $\boldsymbol{p}(\boldsymbol{y})$ relevant to direction i is defined analogously. Accordingly,

$$\boldsymbol{u}_h(\boldsymbol{y}) = \sum_j \boldsymbol{e}_j \otimes \boldsymbol{\phi}_j^u(\boldsymbol{y})\, \hat{\boldsymbol{u}}_j \,, \qquad \boldsymbol{p}_h(\boldsymbol{y}) = \sum_j \boldsymbol{e}_j \otimes \boldsymbol{\phi}_j^p(\boldsymbol{y})\, \hat{\boldsymbol{p}}_j \,. \tag{17}$$

In the former equation: i) tensor product $\otimes : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^{n \times m}$ is defined as: $(\boldsymbol{a} \otimes \boldsymbol{b})\boldsymbol{c} = (\boldsymbol{b} \cdot \boldsymbol{c})\boldsymbol{a}$; ii) for vector problems $\boldsymbol{e}_j$ is the unit vector in direction j, for scalar problems it is merely the number 1; iii) $\hat{\boldsymbol{u}}, \hat{\boldsymbol{p}}$ are the (discrete) unknowns.

3 - Problem formulation

Boundary element methods (BEMs) come out substituting the unknown Dirichlet and Neumann fields with the discrete approximations (17) into BIEs (4)-(5). The collocation BEM (see e.g. [24]) requires the fulfillment of the discrete primal equation onto a selected set of collocation points, $\boldsymbol{x}_i \in \Gamma$. In this technique, one has to deal with "integrals" of the following form (no Einstein summation rule):

$$\int_{\Gamma_s} \boldsymbol{G}_{rs}(\boldsymbol{x}_i, \boldsymbol{y})\, \boldsymbol{e}_j \otimes \boldsymbol{\phi}_j(\boldsymbol{y}) \mathrm{d}\Gamma_y \qquad r = u;\; s = u, p;\; j = 1, 2, 3;\; i = 1, 2, ..., N_i$$

having denoted with N_i the number of collocation points. The symmetric Galerkin BEM requires the evaluation of integrals of the form:

$$\int_{\Gamma_r} \boldsymbol{\phi}_i(\boldsymbol{x}) \otimes \boldsymbol{e}_i \int_{\Gamma_s} \boldsymbol{G}_{rs}(\boldsymbol{x}, \boldsymbol{y})\, \boldsymbol{e}_j \otimes \boldsymbol{\phi}_j(\boldsymbol{y}) \mathrm{d}\Gamma_y \mathrm{d}\Gamma_{\boldsymbol{x}} \qquad r, s = u, p;\;\; i, j = 1, 2, 3;$$

that arise performing the first variation of functional $\boldsymbol{\Psi}(\boldsymbol{u}_h, \boldsymbol{p}_h)$ with respect to the (discrete) unknowns $\hat{\boldsymbol{u}}, \hat{\boldsymbol{p}}$. In the present work reference will be made to the generic "integral"

$$\int_{\Gamma_s} \boldsymbol{G}_{rs}(\boldsymbol{x},\boldsymbol{y})\,\boldsymbol{e}_j \otimes \boldsymbol{\phi}_j(\boldsymbol{y})\mathrm{d}\Gamma_y \qquad r,s=u,p;\quad i,j=1,2,3; \tag{18}$$

that pertains to the collocation BEM when $r=u$ and to the SGBEM as the inner "integral". By definition (16) of vector $\boldsymbol{\phi}_j(\boldsymbol{y})$, "integral" (18) can be reduced to:

$$\int_{supp(\phi_n)} \boldsymbol{G}_{rs}(\boldsymbol{x},\boldsymbol{y})\,\phi_n(\boldsymbol{y})\mathrm{d}\Gamma_y = \sum_j \int_{T_j} \boldsymbol{G}_{rs}(\boldsymbol{x},\boldsymbol{y})\,\varphi_j^{n(j)}(\boldsymbol{y})\mathrm{d}\Gamma_y \overset{def}{=} \sum_j \boldsymbol{F}_{rs}{}_j^n(\boldsymbol{x}) \tag{19}$$

and further simplified by the following variable change. Denoting with $\boldsymbol{d} = \boldsymbol{y} - \boldsymbol{x}$, $r = \|\boldsymbol{d}\|$, the binomial expansion for y_a^i reads:

$$y_a^i = (x_a + d_a)^i = \sum_{k=0}^{i} \binom{i}{k} x_a^{(i-k)} d_a^k \qquad a=2,3\,.$$

It is straightforward to rewrite equation (14) as follows:

$$\varphi_j^n(\boldsymbol{y}) = \boldsymbol{d}_3^T\,{\boldsymbol{X}^{(3)}}^T\,\Lambda_j^n\,\boldsymbol{X}^{(2)}\,\boldsymbol{d}_2 \tag{20}$$

where:

$$\boldsymbol{d}_a^T = \{1, d_a, d_a^2, ..., d_a^{N_j}\}\,, \qquad \boldsymbol{X}_{ij}^{(a)} = \binom{i-1}{j-1} x_a^{(i-j)}$$

$$i,j = 1,2,....,N_j+1; \qquad a=2,3\,.$$

For linear shape functions one has for instance:

$$\varphi_j^n(\boldsymbol{y}) = 1\begin{bmatrix}1 & -\dfrac{1}{\bar{y}_2}\end{bmatrix}\begin{bmatrix}1\\ y_2\end{bmatrix} = 1\,{\boldsymbol{X}^{(3)}}^T\begin{bmatrix}1 & -\dfrac{1}{\bar{y}_2}\end{bmatrix}\boldsymbol{X}^{(2)}\begin{bmatrix}1\\ d_2\end{bmatrix} \tag{21}$$

with:

$$\boldsymbol{X}^{(2)} = \begin{pmatrix}1 & x_2\\ 0 & 1\end{pmatrix} \qquad \boldsymbol{X}^{(3)} = (\,1\,)\,.$$

Denoting with $k_a = a\,x_2 - x_3 = 0$ and $k_b = b\,x_2 - x_3 = 0$ the expressions defining the equations of the two sides of T_j that cross the origin (see figure 2), integral (19) becomes:

$$\boldsymbol{F}_{rs}{}_j^n(\boldsymbol{x}) = \int_{-x_2}^{\bar{y}_2 - x_2} \int_{ad_2+k_a}^{bd_2+k_b} \boldsymbol{G}_{rs}(\boldsymbol{d})\,\boldsymbol{d}_3^T\;\mathrm{d}d_3\,{\boldsymbol{X}^{(3)}}^T\,\Lambda_j^n\,\boldsymbol{X}^{(2)}\,\boldsymbol{d}_2\,\mathrm{d}d_2 \tag{22}$$

which, in the easy case of linear shape functions, reduces to:

$$(23) \qquad \boldsymbol{F}_{rs_j}^{\,n}(\boldsymbol{x}) = \int\limits_{-x_2}^{\bar{y}_2 - x_2} \int\limits_{ad_2+k_a}^{bd_2+k_b} \boldsymbol{G}_{rs}(\boldsymbol{d})\, \mathrm{d}d_3\; \boldsymbol{d}_2^T\; \mathrm{d}d_2 \begin{bmatrix} 1 & -\dfrac{x_2}{\bar{y}_2} \\ & \dfrac{1}{\bar{y}_2} \end{bmatrix} = \mathbb{K}_{rs}(\boldsymbol{x}) \begin{bmatrix} 1 & -\dfrac{x_2}{\bar{y}_2} \\ & \dfrac{1}{\bar{y}_2} \end{bmatrix}.$$

For scalar problems $\boldsymbol{F}_{rs_j}^{\,n}(\boldsymbol{x})$ is a scalar function, while $\mathbb{K}_{rs}$ is a vector of dimension 2. For vector problems, $\boldsymbol{F}_{rs_j}^{\,n}(\boldsymbol{x})$ is a matrix of the same order of kernel $\boldsymbol{G}_{rs}$, whereas $\mathbb{K}_{rs}$ is a third order matrix, whose third dimension is equal to 2.

In what follows, analytical integrations will be carried out with reference to integral (22) but tables will be presented only for $\mathbb{K}_{rs}(\boldsymbol{x})$ for paucity of space. Generalizations are quite easy, and a technical report will be devoted to the publication of outcomes for shape functions up to order 3.

$\mathbb{K}_{rs}(\boldsymbol{x})$ depends on the kernel $\boldsymbol{G}_{rs}$, on the selected element T_j and on the position of the point $\boldsymbol{x}$. The weakly-singular kernel $\boldsymbol{G}_{uu}(\boldsymbol{d})$, the strongly singular kernels $\boldsymbol{G}_{up}(\boldsymbol{d}, \boldsymbol{l}(\boldsymbol{y}))$ and $\boldsymbol{G}_{pu}(\boldsymbol{d}, \boldsymbol{n}(\boldsymbol{x}))$ and the hyper-singular kernel $\boldsymbol{G}_{pp}(\boldsymbol{d}, \boldsymbol{n}(\boldsymbol{x}), \boldsymbol{l}(\boldsymbol{y}))$ are *singular with respect to* $\boldsymbol{y}$ depending on the position of $\boldsymbol{x}$ with respect to T_j. The item $\boldsymbol{x} \notin \overline{T_j}$ (that for all kernels leads to a Lebesgue inner integral) and the item $\boldsymbol{x} \in T_j$ (that leads to an improper integral for $\boldsymbol{G}_{uu}$, to a Cauchy principal value (CPV) for $\boldsymbol{G}_{up}$ and $\boldsymbol{G}_{pu}$, and to a finite part of Hadamard (HFP) for $\boldsymbol{G}_{pp}$) will be therefore separately discussed. An interesting property of continuity (with respect to the source point $\boldsymbol{x}$) between the CPV, the HFP and the Lebesgue integral is shown. To this aim, the CPV and the HFP has been directly evaluated as first; further, the limit process to the boundary has been performed.

4 - Analytical integrations

In view of Green's functions contributions (3) and of equation (22), integrals of the following kind must be dealt with:

$$\int\limits_{ad_2+k_a}^{bd_2+k_b} \frac{d_3^k}{r^{2m+1}}\, \mathrm{d}d_3 \qquad k, m \in \mathbb{N}_0\,.$$

The identity:

$$(24) \qquad \frac{x^{2k}}{a^2+x^2} = (-1)^k \frac{a^{2k}}{a^2+x^2} + \sum_{j=1}^{k} \binom{k}{j} (-1)^{k-j} (a^2+x^2)^{j-1} (a^2)^{k-j}$$

which comes out from the binomial expansion rule, permits to obtain the following recursive relationship, that seems to be useful for analytical integrations:

$$
\frac{d_3^k}{r^{2m+1}} = (-a^2)^{\widehat{k}} \frac{d_3^{k_{[2]}}}{r^{2m+1}} + \sum_{j=1}^{\widehat{k}} \binom{\widehat{k}}{j} (-1)^{\widehat{k}-j} \sum_{h=0}^{j-1} \binom{j-1}{h} a^{2(\widehat{k}-1-h)} \frac{d_3^{2h+k_{[2]}}}{r^{2m-1}} \tag{25}
$$

$$k, m \in \mathbb{N}_0$$

where $a^2 = d_1^2 + d_2^2$ is the squared projection of the distance on the plane $d_3 = 0$. Here and in the rest of the paper the following notation will be considered:

$\widehat{k} = k \div 2$ integer division $k \div 2$.

$k_{[2]} = k - 2\widehat{k}$ remainder of the (integer) division $k \div 2$.

4.1 - *Preliminaries*

A preliminary work [**44**] concerned the analytical integration [7] of the hypersingular function $\frac{1}{r^3}$ over triangle T_j, as the sum of two factors $I_{\triangle}^{r^{-3}}(\boldsymbol{x}, d_2)$ (in the same spirit of [**10**], appendix A):

$$
I_{\triangle}^{r^{-3}}(\boldsymbol{x}) = \int_{-x_2}^{\bar{y}_2 - x_2} \int_{ad_2+k_a}^{bd_2+k_b} \frac{1}{r^3} \, \mathrm{d}d_3 \, \mathrm{d}d_2 = I_{\triangle}^{r^{-3}}(\boldsymbol{x}, d_2) \Big|_{-x_2}^{\bar{y}_2 - x_2} \tag{26}
$$

where $I_{\triangle}^{r^{-3}}(.,.) : \{\mathbb{R}^3 \backslash T_j \times [-x_2, \bar{y}_2 - x_2]\} \to \mathbb{R}$ was defined by:

$$
I_{\triangle}^{r^{-3}}(\boldsymbol{x}, d_2) = \int \mathrm{d}d_2 \int_{ad_2+k_a}^{bd_2+k_b} \frac{1}{r^3} \, \mathrm{d}d_3 = I_{\triangle}^{r^{-3}}(\boldsymbol{x}, d_2, d_3) \Big|_{d_3 = ad_2+k_a}^{d_3 = bd_2+k_b} \tag{27}
$$

Focusing on the upper extremum $d_3 = bd_2 + k_b$, there are two candidate functions for $I_{\triangle}^{r^{-3}}(\boldsymbol{x}, d_2, bd_2 + k_b)$, namely:

$$
f_1^b = \frac{1}{2d_1} \arctan \frac{2d_1(bd_1^2 - k_b d_2)\sqrt{d_1^2 + d_2^2 + (bd_2 + k_b)^2}}{(b^2-1)d_1^4 + (k_b d_2)^2 - d_1^2((1+b^2)d_2^2 + 4bd_2k_b + k_b^2)}
$$

$$
f_2^b = -\frac{1}{2d_1} \arctan \frac{(b^2-1)d_1^4 + (k_b d_2)^2 - d_1^2((1+b^2)d_2^2 + 4bd_2k_b + k_b^2)}{2d_1(bd_1^2 - k_b d_2)\sqrt{d_1^2 + d_2^2 + (bd_2 + k_b)^2}}
$$

[7] It has been defined in the previous section: $r = \|\mathbf{d}\|$.

which are linked by:

$$f_1^b - f_2^b = \frac{\pi}{4d_1}\,\mathrm{sgn}\,\frac{(b^2-1)d_1^4 + (k_b d_2)^2 - d_1^2((1+b^2)d_2^2 + 4bd_2k_b + k_b^2)}{2d_1(bd_1^2 - k_b d_2)\sqrt{d_1^2 + d_2^2 + (bd_2 + k_b)^2}}. \tag{28}$$

The (unique) function $I_{\triangle}^{r^{-3}}(\boldsymbol{x}, d_2, bd_2 + k_b)$ can be caught in studying the domain in which f_1^b and f_2^b are defined. Within a domain where both f_1^b and f_2^b are defined, they have the same derivative, for they differ by a constant. Within a domain in which only f_1^b (or f_2^b) is everywhere defined, f_1^b (or f_2^b) is the unique primitive. The analysis is quite involved: a flow chart summarizes it in figure 3.

if $x_1 \neq 0$ (*that is, the field point* $\mathbf{x}$ *does not lie in the plane of the triangle* T_j; $z_1, z_2, z_{13}, z_{23}, \eta_{13}, \eta_{23}, f_1^b, f_2^b$ *as in* [44] *paragraphs* 3.1.1, 3.1.2)

if $x_1^2(1+b^2) < k_b^2$ **then** $f^b = \begin{cases} -\infty < d_2 \le z_{13} & f_2^b \\ z_{13} \le d_2 \le z_{23} & f_1^b + \frac{\pi}{4x_1}\,\mathrm{sgn}\,(\eta_{13}) \\ z_{23} \le d_2 < +\infty & f_2^b + \frac{\pi}{4x_1}(\,\mathrm{sgn}\,(\eta_{13}) - \mathrm{sgn}\,(\eta_{23})) \end{cases}$

else if $x_1^2(1-b^2) \ge k_b^2$ **then** $f^b = f_1^b$

else if $z_3 < z_1 < z_2$ **then** $f^b = \begin{cases} -\infty < d_2 \le z_{13} & f_1^b \\ z_{13} \le d_2 < +\infty & f_2^b - \frac{\pi}{4x_1}\,\mathrm{sgn}\,(\eta_{13}) \end{cases}$

else $f^b = \begin{cases} -\infty < d_2 \le z_{23} & f_2^b \\ z_{23} \le d_2 < +\infty & f_1^b + \frac{\pi}{4x_1}\,\mathrm{sgn}\,(\eta_{23}) \end{cases}$

else (*that is, the field point* $\mathbf{x}$ *lies in the plane of the triangle* T_j, *see* [44] *section* 3.2)

if $k_b \neq 0$ **then** $f^b = -\frac{\sqrt{d_2^2 + (bd_2 + k_b)^2}}{d_2\, k_b}$

else $f^b = -\frac{b}{\sqrt{d_2^2 + b^2\, d_2^2}}$

Fig. 3. – A flow chart of $f^b(\mathbf{x}, d_2)$.

$I_{\triangle}^{r^{-3}}(\boldsymbol{x})$ is well defined only when $\boldsymbol{x} \notin \overline{T_j}$; nevertheless, for its interest in the context of BEM, it has been evaluated also in the sense of finite part of Hadamard for $\boldsymbol{x} \in T_j$. A relationship between the two instances is set performing the "limit to the boundary" $\overline{T_j} \not\ni \boldsymbol{x} \to \boldsymbol{x} \in T_j$ and, as expected, the limit to the boundary does not coincide with the finite part of Hadamard [8]. In local reference $\mathcal{L}$ of figure 2 it holds:

$$I_{\triangle}^{r^{-3}}(\boldsymbol{x} \in T_j) = \oint_{T_j} \frac{1}{r^3}\, \mathrm{d}\Gamma_y - \frac{2\pi}{x_1} + o(x_1^{-1})\,. \tag{29}$$

The reader is referred to [44] for details.

4.2 - *Lebesgue integrals*

Consider $\boldsymbol{x} \notin \overline{T_j}$, which implies $r \neq 0$ and the Lebesgue nature of integral (22). Exploiting recursively identity (25), the first contribution of $\boldsymbol{F}_{rs_j}^{\,n}(\boldsymbol{x})$, namely:

$$\int_{ad_2+k_a}^{bd_2+k_b} \boldsymbol{G}_{rs}(\boldsymbol{d})\, \boldsymbol{d}_3^T \, \mathrm{d}d_3$$

is reduced to the sum of a set of basic integrals; the following identities, that can be easily proved by induction, are required for most kernels:

$$\begin{aligned}\int \frac{x^j}{\sqrt{a^2+x^2}}\mathrm{d}x \\ = \mu(j)\, a^j \, \log(x+\sqrt{a^2+x^2}) + \sum_{k=0}^{\widetilde{j-1}} \eta(k,j)\; a^{2(\widetilde{j}-k-1)}\; x^{2k+1-j_{[2]}}\; \sqrt{a^2+x^2}\end{aligned} \tag{30}$$

$$\int \frac{x^{n_{[2]}}}{(a^2+x^2)\sqrt{a^2+x^2}}\mathrm{d}x = \frac{x - \left(a^2+x\right)(n_{[2]})}{a^2\sqrt{a^2+x^2}} \tag{31}$$

$$\int \frac{x^{n_{[2]}}}{(a^2+x^2)^2\sqrt{a^2+x^2}}\mathrm{d}x = \frac{3\,a^2\,x + 2\,x^3 - \left(a^4+3\,a^2\,x+2\,x^3\right)n_{[2]}}{3\,a^4\,(a^2+x^2)^{\frac{3}{2}}} \tag{32}$$

$$\int \frac{x^{n_{[2]}}}{(a^2+x^2)^3\sqrt{a^2+x^2}}\mathrm{d}x = \frac{\left(15\,a^4\,x+20\,a^2\,x^3+8\,x^5\right)(1-n_{[2]}) - 3\,a^6 n_{[2]}}{15\,a^6\,(a^2+x^2)^{\frac{5}{2}}} \tag{33}$$

where:

$\widehat{j} = j \div 2$ division in $\mathbb{N}$.

$j_{[2]} = j - 2\widehat{j}$ remainder of $j \div 2$ in $\mathbb{N}$.

---

[8] It will be shown on the contrary that, as in two dimension, the limit to the boundary does coincide with the finite part of Hadamard in the analytical integration of the hypersingular kernel.

$\widetilde{j} = j - \widehat{j}$ complementary part of $j \div 2$ in $\mathbb{N}$.

$$\mu(j) = (1 - j_{[2]})(-1)^{\widehat{j}} \frac{j!}{(\widehat{j}!\, 2^{\widehat{j}})^2}$$

$$\eta(k,j) = (-1)^{k-1-\widetilde{j}} \left((1 - j_{[2]}) \frac{j!}{(2k+1)!} \left(\frac{k!}{\widehat{j}!\, 2^{\widehat{j-k}}} \right)^2 + j_{[2]} \frac{(2k)!}{j!} \left(\frac{\widehat{j-1}!}{k!\, 2^{k-\widehat{j-1}}} \right)^2 \right)$$

In view of (30-33), it comes out:

$$(34) \quad \int_{ad_2+k_a}^{bd_2+k_b} \boldsymbol{G}_{rs}(\boldsymbol{d})\, d_3^k\, \mathrm{d}d_3$$
$$= \sum_{l,m=0}^{3} \sum_{j=0}^{N(k,l,m)} \frac{\boldsymbol{A}_{lm}^j(\boldsymbol{x})}{(d_1^2+d_2^2)^l} \frac{d_2^j}{r^{2m-1}} + \sum_{j=0}^{k} \boldsymbol{B}^j(\boldsymbol{x})\, d_2^j \log(d_3+r) \Bigg|_{d_3=ad_2+k_a}^{d_3=bd_2+k_b}$$

For scalar problems $\boldsymbol{A}_{lm}^j$ and $\boldsymbol{B}^j$ are scalar functions, while for vector problems they are matrices functions of the same order (here termed N_{rs}: for elasticity, $N_{rs}=3$) of kernel $\boldsymbol{G}_{rs}$. Of course, $\boldsymbol{A}_{lm}^j$ and $\boldsymbol{B}^j$ depend upon the considered kernel. Integral (34) can be recast in the vector formalism of equation (22), namely:

$$(35) \quad \int_{ad_2+k_a}^{bd_2+k_b} \boldsymbol{G}_{rs}(\boldsymbol{d})\, \boldsymbol{d}_3^T\, \mathrm{d}d_3 = \sum_{l,m=0}^{3} \frac{\widetilde{\boldsymbol{d}}_2^{\,T}\, \mathbb{A}_{lm}(\boldsymbol{x})}{(d_1^2+d_2^2)^l\, r^{2m-1}} + \widetilde{\widetilde{\boldsymbol{d}}}_2^{\,T}\, \mathbb{B}(\boldsymbol{x}) \log(d_3+r) \Bigg|_{d_3=ad_2+k_a}^{d_3=bd_2+k_b}$$

Vectors $\widetilde{\boldsymbol{d}}_2$ and $\widetilde{\widetilde{\boldsymbol{d}}}_2$ are special instances of $\boldsymbol{d}_2$, with different length, here termed $\widetilde{N}_2$ and $\widetilde{\widetilde{N}}_2$ respectively; the latter is equal to the length of $\boldsymbol{d}_3$, here termed N_3; $\widetilde{N}_2$ depends also upon l and m. For scalar problems $\mathbb{A}_{lm}$ is a matrix $\widetilde{N}_2 \times N_3$ of functions of the source point $\boldsymbol{x}$, while for vector problems $\mathbb{A}_{lm}$ is a fourth order matrix $\widetilde{N}_2 \times (N_{rs} \times N_{rs}) \times N_3$; analogously, for scalar problems $\mathbb{B}$ is a matrix $\widetilde{\widetilde{N}}_2 \times N_3$ of functions of the source point $\boldsymbol{x}$, while for vector problems $\mathbb{B}$ is a fourth order matrix $\widetilde{\widetilde{N}}_2 \times (N_{rs} \times N_{rs}) \times N_3$.

Substituting expression (35) into equation (22), the latter will be rewritten as:

$$(36) \quad \boldsymbol{F}_{rs}{}_j^n(\boldsymbol{x}) = \sum_{l,m=0}^{3} \int_{-x_2}^{\bar{y}_2-x_2} \frac{\widetilde{\boldsymbol{d}}_2^{\,T}\, \mathbb{A}_{lm}\, \boldsymbol{X}^{(3)^T}\, \Lambda_j^n\, \boldsymbol{X}^{(2)}\, \boldsymbol{d}_2}{(d_1^2+d_2^2)^l\, r^{2m-1}} \Bigg|_{d_3=ad_2+k_a}^{d_3=bd_2+k_b} \mathrm{d}d_2$$
$$+ \int_{-x_2}^{\bar{y}_2-x_2} \widetilde{\widetilde{\boldsymbol{d}}}_2^{\,T}\, \mathbb{B}\, \boldsymbol{X}^{(3)^T}\, \Lambda_j^n\, \boldsymbol{X}^{(2)}\, \boldsymbol{d}_2 \log(d_3+r) \Bigg|_{d_3=ad_2+k_a}^{d_3=bd_2+k_b} \mathrm{d}d_2$$

$$= \sum_{l,m=0}^{3} \int_{-x_2}^{\bar{y}_2 - x_2} \frac{\widetilde{\boldsymbol{d}}_2^{\,T} \, \mathbb{C}_{lmj}^{\,n} \, \boldsymbol{d}_2}{(d_1^2 + d_2^2)^l \, r^{2m-1}} \Bigg|_{d_3 = a d_2 + k_a}^{d_3 = b d_2 + k_b} \mathrm{d} d_2 + \int_{-x_2}^{\bar{y}_2 - x_2} \widetilde{\widetilde{\boldsymbol{d}}}_2^{\,T} \, \mathbb{D}_j^n \, \boldsymbol{d}_2 \log(d_3 + r) \Bigg|_{d_3 = a d_2 + k_a}^{d_3 = b d_2 + k_b} \mathrm{d} d_2$$

with $\mathbb{C}_{lmj}^{\,n}$ and $\mathbb{D}_j^n$ defined by comparison in the previous equation. To completely solve analytical integrations for $\boldsymbol{F}_{rsj}^{\,n}(\boldsymbol{x})$ the following integrals are required:

$$\int_{-x_2}^{\bar{y}_2 - x_2} \frac{1}{(d_1^2 + d_2^2)^l} \, \frac{d_2^h}{r^{2m-1}} \Bigg|_{d_3 = a d_2 + k_a}^{d_3 = b d_2 + k_b} \mathrm{d} d_2 \qquad l, m = 0, 1, 2, 3; \quad h \in \mathbb{N}_0 \tag{37}$$

$$\int_{-x_2}^{\bar{y}_2 - x_2} d_2^h \log(d_3 + r) \Bigg|_{d_3 = a d_2 + k_a}^{d_3 = b d_2 + k_b} \mathrm{d} d_2 \qquad h \in \mathbb{N}_0 \, . \tag{38}$$

4.2.1 - Weakly singular kernel

In dealing with the weakly singular kernel $\boldsymbol{G}_{uu}$, matrices $\mathbb{C}_{lm}$ vanish for $l, m = 2, 3$. Accordingly, only the following integrals are of interest in this case:

$$\int_{-x_2}^{\bar{y}_2 - x_2} \frac{d_2^k}{r} \Bigg|_{d_3 = a d_2 + k_a}^{d_3 = b d_2 + k_b} \mathrm{d} d_2 \tag{39}$$

$$\int_{-x_2}^{\bar{y}_2 - x_2} \frac{d_2^k}{x_1^2 + d_2^2} \, \frac{1}{r} \Bigg|_{d_3 = a d_2 + k_a}^{d_3 = b d_2 + k_b} \mathrm{d} d_2 \tag{40}$$

$$\int_{-x_2}^{\bar{y}_2 - x_2} d_2^k \log(d_3 + r) \Bigg|_{d_3 = a d_2 + k_a}^{d_3 = b d_2 + k_b} \mathrm{d} d_2 \, . \tag{41}$$

With reference to $d_3 = b d_2 + k_b$ as a prototype, making use of the affine transformation

$$\zeta_2 = \frac{d_2 + b \, d_3}{\sqrt{1 + b^2}} \Bigg|^{d_3 = b d_2 + k_b} \tag{42}$$

integral (39) becomes:

$$(43)\qquad \sum_{j=0}^{k} \binom{k}{j} \frac{(-\,b\,k_b)^{k-j}}{(\sqrt{1+b^2})^{2k+1-j}} \int\limits_{\frac{b\,k_b}{\sqrt{1+b^2}}-\sqrt{1+b^2}x_2}^{\frac{b\,k_b}{\sqrt{1+b^2}}+\sqrt{1+b^2}(\bar{y}_2-x_2)} \frac{\zeta_2^j}{\sqrt{d_1^2+\frac{k_b^2}{1+b^2}+\zeta_2^2}}\,\mathrm{d}\zeta_2$$

which has a closed form owing to outcome (30).

By defining with $\gamma : \mathbb{N}\times\mathbb{N}\times\mathbb{N}_0\times\mathbb{R}\to\mathbb{R}$ the function:

$$(44)\qquad \gamma(k,j,h,d_1) \overset{def}{=} \binom{\widehat{k}}{j}\binom{j-1}{h}(-\,1)^{\widehat{k-j}}\,(d_1^2)^{\widehat{k-1-h}}$$

and by means of identity (24), equation (40) will be rewritten as:

$$(45)\qquad (-\,1)^{\widehat{k}}d_1^{\widehat{2k}} \int\limits_{-x_2}^{\bar{y}_2-x_2} \frac{d_2^{k_{[2]}}}{(x_1^2+d_2^2)}\,\frac{1}{r}\Bigg|_{d_3=ad_2+k_a}^{d_3=bd_2+k_b} \mathrm{d}d_2$$

$$+\sum_{j=1}^{\widehat{k}}\sum_{h=0}^{j-1}\gamma(k,j,h,d_1)\int\limits_{-x_2}^{\bar{y}_2-x_2}\frac{d_2^{\widehat{k+2h}}}{r}\Bigg|_{d_3=ad_2+k_a}^{d_3=bd_2+k_b}\mathrm{d}d_2\,.$$

Integral (45) has a closed form in view of (43) and (76) of Appendix 2. Finally, the closed form of integral (41) is given in Appendix 2 by equation (80).

Algebraic manipulations lead from (43), (45) and (80) to the following tabular expression for $\boldsymbol{F}_{rs_j^n}(\boldsymbol{x})$ in the case of linear shape functions (see eq. (23)) with regard to the weakly singular kernel $\boldsymbol{G}_{uu}$ in the local coordinate system $\mathcal{L}$:

$$(46)\qquad \mathbb{K}_{uu}(\boldsymbol{x}) = \kappa\,\widehat{\mathbb{K}}^{uu}(\boldsymbol{x},d_2,d_3)\,\Big|_{d_3=ad_2+k_a}^{d_3=bd_2+k_b}\Big|_{d_2=-x_2}^{d_2=\bar{y}_2-x_2}$$

with:

$$(47)\qquad \widehat{\mathbb{K}}^{uu}(\boldsymbol{x},d_2,d_3)=\mathbb{L}^{uu}\log(\zeta_2+r)\,+\mathbb{A}^{uu}\,\operatorname{arctanh}\frac{d_3}{r}\,+\mathbb{I}^{uu}\,I_{\triangle}^{r^{-3}}(\boldsymbol{x},d_2,d_3)\,+\mathbb{R}^{uu}\,r\,.$$

In identities (46, 47):

κ is a constant for the problem under consideration. For potential problems, $\kappa=\dfrac{1}{4a\pi}$ while for linear elasticity, $\kappa=\dfrac{1}{16\pi}\dfrac{1}{G(1-\nu)}$

$I_{\triangle}^{r^{-3}}(\boldsymbol{x},d_2,d_3)$ has been defined in Section 4.1

$\mathbb{L}^{uu}$, $\mathbb{A}^{uu}$, $\mathbb{I}^{uu}$, $\mathbb{R}^{uu}$ are matrices of the same order of $\mathbb{K}^{uu}$, whose expressions, for potential problems, are collected in Appendix 3.

Identity (46), which holds for linear shape functions in view of eq. (23), can straightforwardly be extended to polynomial shape functions of arbitrarily degree over flat triangles accordingly to (36).

4.2.2 - Strongly singular kernel

In order to evaluate (36) for kernels $\boldsymbol{G}_{up}$ and $\boldsymbol{G}_{pu}$, integrals (39-41) are required. Moreover, one has to deal with the following integrals:

$$\int_{-x_2}^{\bar{y}_2-x_2} \frac{d_2^k}{\left(x_1^2+d_2^2\right)^2} \frac{1}{r}\Bigg|_{d_3=ad_2+k_a}^{d_3=bd_2+k_b} \mathrm{d}d_2 \tag{48}$$

$$\int_{-x_2}^{\bar{y}_2-x_2} \frac{d_2^k}{r^{\frac{3}{2}}}\Bigg|_{d_3=ad_2+k_a}^{d_3=bd_2+k_b} \mathrm{d}d_2 \tag{49}$$

$$\int_{-x_2}^{\bar{y}_2-x_2} \frac{d_2^k}{\left(x_1^2+d_2^2\right)} \frac{1}{r^{\frac{3}{2}}}\Bigg|_{d_3=ad_2+k_a}^{d_3=bd_2+k_b} \mathrm{d}d_2\,. \tag{50}$$

By means of identities (24) and (44) integral (48) will be rewritten as:

$$\begin{aligned}(-1)^{\widehat{k}} d_1^{\widehat{2k}} \int_{-x_2}^{\bar{y}_2-x_2} \frac{d_2^{k_{[2]}}}{\left(x_1^2+d_2^2\right)^2} \frac{1}{r}\Bigg|_{d_3=ad_2+k_a}^{d_3=bd_2+k_b} \mathrm{d}d_2 \\ +\sum_{j=1}^{\widehat{k}} \sum_{h=0}^{j-1} \gamma(k,j,h,d_1) \int_{-x_2}^{\bar{y}_2-x_2} \frac{d_2^{\widehat{k}+2h}}{\left(x_1^2+d_2^2\right)} \frac{1}{r}\Bigg|_{d_3=ad_2+k_a}^{d_3=bd_2+k_b} \mathrm{d}d_2\,.\end{aligned} \tag{51}$$

Integral (51) has a closed form in view of (77) of Appendix 2. Making use of affine transformation (42), integral (49) becomes:

$$\sum_{j=0}^{k} \binom{k}{j} \frac{(-b\,k_b)^{k-j}}{(\sqrt{1+b^2})^{2k+1-j}} \int_{\frac{b\,k_b}{\sqrt{1+b^2}}-\sqrt{1+b^2}x_2}^{\frac{b\,k_b}{\sqrt{1+b^2}}+\sqrt{1+b^2}(\bar{y}_2-x_2)} \frac{\zeta_2^j}{\left(d_1^2+\frac{k_b^2}{1+b^2}+\zeta_2^2\right)^{\frac{3}{2}}} \mathrm{d}\zeta_2 \tag{52}$$

which has a closed form in view of (24, 30, 31). Finally, following the same path of reasoning used for integral (40), one obtains the closed form of (50) in view of (52, 82) of Appendix 2.

Algebraic manipulations lead to the following tabular expression for ${\boldsymbol F_{rs}}_j^n(\boldsymbol x)$ in the case of linear shape functions (see eq. (23)) with regard to the strongly singular kernels $\boldsymbol G_{up}$, $\boldsymbol G_{pu}$ in the local coordinate system $\mathcal{L}$:

$$\mathbb{K}^{up}(\boldsymbol x) = \kappa\, \widehat{\mathbb{K}}^{up}(\boldsymbol x, d_2, d_3)\, \Big|_{d_3=ad_2+k_a}^{d_3=bd_2+k_b} \Big|_{d_2=-x_2}^{d_2=\bar y_2-x_2}$$

(53)

$$\mathbb{K}^{pu}(\boldsymbol x) = \kappa\, \widehat{\mathbb{K}}^{pu}(\boldsymbol x, d_2, d_3)\, \Big|_{d_3=ad_2+k_a}^{d_3=bd_2+k_b} \Big|_{d_2=-x_2}^{d_2=\bar y_2-x_2}$$

with:

(54) $\widehat{\mathbb{K}}_{up}(\boldsymbol x, d_2, d_3)$

$$= \mathbb{L}^{up} \log(\zeta_2 + r) + \mathbb{A}^{up} \operatorname{arctanh}\frac{d_3}{r} + \mathbb{I}^{up}\, I_{\triangle}^{r^{-3}}(\boldsymbol x, d_2, d_3) + \mathbb{R}^{up}\, r + \mathbb{S}^{up}\, \frac{1}{r}$$

(55) $\widehat{\mathbb{K}}_{pu}(\boldsymbol x, d_2, d_3)$

$$= \mathbb{L}^{pu} \log(\zeta_2 + r) + \mathbb{A}^{pu} \operatorname{arctanh}\frac{d_3}{r} + \mathbb{I}^{pu}\, I_{\triangle}^{r^{-3}}(\boldsymbol x, d_2, d_3) + \mathbb{R}^{pu}\, r + \mathbb{S}^{pu}\, \frac{1}{r}\ .$$

In identities (53 - 55):

κ is a constant for the problem under consideration. For potential problems, $\kappa = \dfrac{1}{4\pi}$ while for linear elasticity, $\kappa = \dfrac{1}{8\pi(1-\nu)}$

$I_{\triangle}^{r^{-3}}(\boldsymbol x, d_2, d_3)$ has been defined in Section 4.1

$\mathbb{L}^{up}$, $\mathbb{A}^{up}$, $\mathbb{I}^{up}$, $\mathbb{R}^{up}$, $\mathbb{S}^{up}$ are matrices of the same order of $\mathbb{K}^{up}$, whose expressions, for potential problems, are collected in Appendix 3.

$\mathbb{L}^{pu}$, $\mathbb{A}^{pu}$, $\mathbb{I}^{pu}$, $\mathbb{R}^{pu}$, $\mathbb{S}^{pu}$ are matrices of the same order of $\mathbb{K}^{pu}$, whose expressions, for potential problems, are collected in Appendix 3.

Identities (53), which hold for linear shape functions in view of eq. (23), can straightforwardly be extended to polynomial shape functions of arbitrarily degree over flat triangles accordingly to (36).

4.2.3 - Hyper singular kernel

The evaluation of integrals (39-41) and (48-50) is required in order to evaluate (36) for the hyper singular kernel $\boldsymbol G_{pp}$. Moreover, the following integrals are involved by the hypersingular integral operator D:

$$\int_{-x_2}^{\bar{y}_2-x_2} \frac{d_2^k}{\left(x_1^2+d_2^2\right)^3} \frac{1}{r}\Bigg|_{d_3=ad_2+k_a}^{d_3=bd_2+k_b} \mathrm{d}d_2 \tag{56}$$

$$\int_{-x_2}^{\bar{y}_2-x_2} \frac{d_2^k}{\left(x_1^2+d_2^2\right)^2} \frac{1}{r^{\frac{3}{2}}}\Bigg|_{d_3=ad_2+k_a}^{d_3=bd_2+k_b} \mathrm{d}d_2 \tag{57}$$

$$\int_{-x_2}^{\bar{y}_2-x_2} \frac{d_2^k}{r^{\frac{5}{2}}}\Bigg|_{d_3=ad_2+k_a}^{d_3=bd_2+k_b} \mathrm{d}d_2 \tag{58}$$

$$\int_{-x_2}^{\bar{y}_2-x_2} \frac{d_2^k}{\left(x_1^2+d_2^2\right)} \frac{1}{r^{\frac{5}{2}}}\Bigg|_{d_3=ad_2+k_a}^{d_3=bd_2+k_b} \mathrm{d}d_2\,. \tag{59}$$

The path of reasoning used for solving integral (48) will be adopted in handing (56) and (57). By making recourse to function (44) they will be rewritten as:

$$(-1)^{\widehat{k}} d_1^{2\widehat{k}} \int_{-x_2}^{\bar{y}_2-x_2} \frac{d_2^{k_{[2]}}}{\left(x_1^2+d_2^2\right)^3} \frac{1}{r}\Bigg|_{d_3=ad_2+k_a}^{d_3=bd_2+k_b} \mathrm{d}d_2 \tag{60}$$

$$+\sum_{j=1}^{\widehat{k}} \sum_{h=0}^{j-1} \gamma(k,j,h,d_1) \int_{-x_2}^{\bar{y}_2-x_2} \frac{d_2^{\widehat{k}+2h}}{\left(x_1^2+d_2^2\right)^2} \frac{1}{r}\Bigg|_{d_3=ad_2+k_a}^{d_3=bd_2+k_b} \mathrm{d}d_2$$

and

$$(-1)^{\widehat{k}} d_1^{2\widehat{k}} \int_{-x_2}^{\bar{y}_2-x_2} \frac{d_2^{k_{[2]}}}{\left(x_1^2+d_2^2\right)^2} \frac{1}{r^{\frac{3}{2}}}\Bigg|_{d_3=ad_2+k_a}^{d_3=bd_2+k_b} \mathrm{d}d_2 \tag{61}$$

$$+\sum_{j=1}^{\widehat{k}} \sum_{h=0}^{j-1} \gamma(k,j,h,d_1) \int_{-x_2}^{\bar{y}_2-x_2} \frac{d_2^{\widehat{k}+2h}}{\left(x_1^2+d_2^2\right)} \frac{1}{r^{\frac{3}{2}}}\Bigg|_{d_3=ad_2+k_a}^{d_3=bd_2+k_b} \mathrm{d}d_2$$

respectively. Integrals (60-61) have closed forms in view of (78, 83) of Appendix 2. By means of affine transformation (42), integral (58) becomes:

$$\text{(62)} \qquad \sum_{j=0}^{k} \binom{k}{j} \frac{(-\,b\,k_b)^{k-j}}{(\sqrt{1+b^2})^{2k+1-j}} \int_{\frac{b\,k_b}{\sqrt{1+b^2}}-\sqrt{1+b^2}x_2}^{\frac{b\,k_b}{\sqrt{1+b^2}}+\sqrt{1+b^2}(\bar{y}_2-x_2)} \frac{\zeta_2^j(d_1^2+\frac{k_b^2}{1+b^2}+\zeta_2^2)^{-2}}{\sqrt{d_1^2+\frac{k_b^2}{1+b^2}+\zeta_2^2}}\,\mathrm{d}\zeta_2$$

which has a closed form in view of (24, 30-32). Finally, following the same path of reasoning used for equation (40), the closed form of (59) comes out in view of (61) and of (84) of Appendix 2.

Algebraic manipulations lead to the following tabular expression for $\boldsymbol{F}_{rs_j}^{\ n}(\boldsymbol{x})$ in the case of linear shape functions (see eq. (23)) with regard to the hyper singular kernel $\boldsymbol{G}_{pp}$ in the local coordinate system $\mathcal{L}$:

$$\text{(63)} \qquad \mathbb{K}_{pp}(\boldsymbol{x}) = \kappa\;\widehat{\mathbb{K}}^{pp}(\boldsymbol{x},d_2,d_3)\,\Big|_{d_3=ad_2+k_a}^{d_3=bd_2+k_b}\Big|_{d_2=-x_2}^{d_2=\bar{y}_2-x_2}$$

with:

$$\text{(64)} \qquad \widehat{\mathbb{K}}^{pp}(\boldsymbol{x},d_2,d_3) = \mathbb{L}^{pp}\log(\zeta_2+r) + \mathbb{A}^{pp}\,\operatorname{arctanh}\frac{d_3}{r}$$
$$+\,\mathbb{I}^{pp}\,I_{\triangle}^{r^{-3}}(\boldsymbol{x},d_2,d_3) + \mathbb{R}^{pp}\,r + \mathbb{S}^{pp}\,\frac{1}{r} + \mathbb{H}^{pp}\,\frac{1}{r^3}\;.$$

In identities (63, 64):

κ is a constant for the problem under consideration. For potential problems $\kappa = \dfrac{a}{4\pi}$ whereas for linear elasticity, $\kappa = \dfrac{G}{8\pi(1-\nu)}$

$I_{\triangle}^{r^{-3}}(\boldsymbol{x},d_2,d_3)$ has been defined in Section 4.1

$\mathbb{L}^{pp}$, $\mathbb{A}^{pp}$, $\mathbb{I}^{pp}$, $\mathbb{R}^{pp}$, $\mathbb{S}^{pp}$, $\mathbb{H}^{pp}$ are matrices of the same order of $\mathbb{K}^{pp}$, whose expressions, for potential problems, are collected in Appendix 3.

Identity (63), which holds for linear shape functions in view of eq. (23), can straightforwardly be extended to polynomial shape functions of arbitrarily degree over flat triangles accordingly to (36).

4.3 - *Singular integrals*

4.3.1 - Hadamard's finite part

In the limit process $\Omega \ni \boldsymbol{x} \to \boldsymbol{x} \in \Gamma$, the singularity [9] of Green's function $\boldsymbol{G}_{pp}$ is triggered off. Provided that regularity requirements are satisfied [**48**], peculiarities of Green's functions [**12**] allow to interpret hypersingular integral (22) as a Hadamard's [**2, 3**] Finite Part (HFP) [**14**].

Consider first analytical integrations for point $\boldsymbol{x} \in T_j \subset \Gamma$; elastostatic kernel $\boldsymbol{G}_{pp}$ and potential kernel G_{qq} in the local coordinate system $\mathcal{L}$ simplify as:

$$\boldsymbol{G}_{pp}(\boldsymbol{d};\boldsymbol{e}_1;\boldsymbol{e}_1) = -\frac{G\nu}{(1-\nu)}\left\{2\,(\boldsymbol{e}_1 \otimes \boldsymbol{e}_1) + \frac{(1-2\nu)}{\nu}\boldsymbol{I} \quad + 3\frac{\boldsymbol{d} \otimes \boldsymbol{d}}{r^2}\right\}\mathrm{G}_{qq};$$

$$\mathrm{G}_{qq}(\boldsymbol{d};\boldsymbol{e}_1;\boldsymbol{e}_1) = -\frac{1}{4\pi r^3}$$

The definition of the finite part can be given as follows:

Definition 1. Let $\varepsilon \to I(\varepsilon)$ denote a complex-valued function which is continuous in $(0, \varepsilon_0)$ and assume that

$$I(\varepsilon) = I_0 + I_1\ log(\varepsilon) + \sum_{j=2}^{m} I_j\ \varepsilon^{1-j} + o(1); \quad \varepsilon \to 0$$

where $I_j \in \mathbb{C}$. Then I_0 is called the finite part of $I(\varepsilon)$. In dealing with integrals, the finite part I_0 of a (usually) divergent integral $\int_{-\infty}^{+\infty} \phi(t)\mathrm{d}t$ is denoted by the symbol $\displaystyle ⨎_{-\infty}^{+\infty} \phi(t)\mathrm{d}t$.

Definition 2. Define with:

$$T_j^{\varepsilon} = \{\boldsymbol{y} \in T_j : |y_2 - x_2| < \varepsilon \text{ and } |y_3 - x_3| < \varepsilon\}$$

the domain in figure 4. In agreement with equation (23), define with:

$$I_{\square}\,(\boldsymbol{x},\varepsilon) \stackrel{def}{=} \int\limits_{T_j \setminus T_j^{\varepsilon}} \boldsymbol{G}_{pp}(\boldsymbol{d};\boldsymbol{e}_1;\boldsymbol{e}_1)\ \mathrm{d}d_3\,\mathrm{d}d_2$$

---

[9] Kernel $\boldsymbol{G}_{pp}$ shows a singularity of $O(r^{-3})$ which is greater than the dimension of the integral, whence its name "hypersingular".

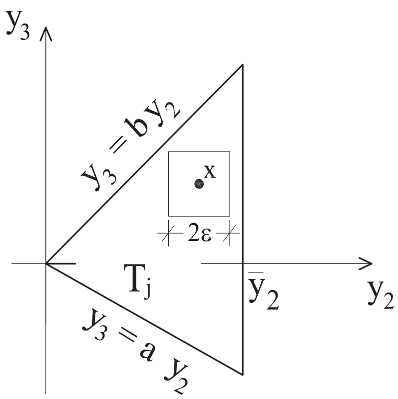

Fig. 4. – Geometrical description of a square neighborhood.

By direct integration:

(65)
$$I_{\Box}(\boldsymbol{x},\varepsilon) = \frac{2\kappa}{k_{\vartheta}}\left\{ \frac{\nu}{r}\begin{pmatrix} 0 & 0 & 0 \\ 0 & d_2 & d_3 \\ 0 & d_3 & -d_2 \end{pmatrix} - \frac{r}{d_2}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1-\nu \end{pmatrix} \right\}\Bigg|_{d_3=ad_2+k_a,\vartheta=a}^{d_3=bd_2+k_b,\vartheta=b}\Bigg|_{d_2=-x_2}^{d_2=\bar{y}_2-x_2} + I_2\frac{1}{\varepsilon} + o(1); \quad \varepsilon \to 0$$

where:

(66)
$$\kappa = \frac{G}{8\pi(1-\nu)}; \qquad I_2 = 4\kappa\sqrt{2}\begin{pmatrix} 2 & 0 & 0 \\ 0 & 2-\nu & 0 \\ 0 & 0 & 2-\nu \end{pmatrix}$$

and the finite part immediately follows from its definition. The same result comes out from a limit process, by taking $d_1 \to 0^+$ in equations (63, 64). Considering only the term pertaining to a constant shape function, it holds in fact ($\vartheta = a, b$):

$$\lim_{d_1\to 0^+} \mathbb{L}^{pp} = \lim_{d_1\to 0^+} \mathbb{A}^{pp} = \lim_{d_1\to 0^+} \mathbb{I}^{pp} = \lim_{d_1\to 0^+} \mathbb{H}_{pp} = 0$$

$$\lim_{d_1\to 0^+} \mathbb{R}_{pp} = -\frac{2}{d_2\, k_{\vartheta}}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1-\nu \end{pmatrix}$$

$$\lim_{d_1\to 0^+} \mathbb{S}_{pp} = \frac{2\,\nu}{k_{\vartheta}}\begin{pmatrix} 0 & 0 & 0 \\ 0 & d_2 & d_3 \\ 0 & d_3 & -d_2 \end{pmatrix}.$$

4.3.2 - Cauchy's principal value

Considerations about the nature of the singularity in the boundary limit apply to the CPV as well. Consider first the point $\boldsymbol{x} \in \Gamma$, so that kernel $\boldsymbol{G}_{pu}$ in the local coordinate system $\mathcal{L}$ simplifies as:

$$\boldsymbol{G}_{pu}(\boldsymbol{d};\boldsymbol{e}_1) = \frac{1}{4\pi}\frac{(1-2\nu)}{(1-\nu)}\frac{1}{r^3}\boldsymbol{SKW}(\boldsymbol{d}\otimes\boldsymbol{e}_1)\,.$$

By direct integration ($\boldsymbol{x} \in T_j$):

$$\int_{T_j\setminus T_j^{\varepsilon}} \boldsymbol{G}_{pu}(\boldsymbol{d};\boldsymbol{e}_1)\,\mathrm{d}\Gamma = \tag{67}$$

$$2\kappa\left[a\,\mathrm{arctanh}\frac{d_3}{r} - \frac{\beta}{\sqrt{1+\vartheta^2}}\log(\zeta_2+r)\right]_{\substack{d_3=ad_2+k_a\\ \vartheta=a}}^{\substack{d_3=bd_2+k_b\\ \vartheta=b}}\Bigg|_{d_2=-x_2}^{d_2=\bar{y}_2-x_2} + o(1); \qquad \varepsilon\to 0$$

where:

$$\kappa = \frac{1-2\,\nu}{8\,\pi\,(1-\nu)}, \quad a = \boldsymbol{SKW}(\boldsymbol{e}_1\otimes\boldsymbol{e}_2), \quad \beta = \vartheta a - \boldsymbol{SKW}(\boldsymbol{e}_1\otimes\boldsymbol{e}_3)$$

which is the CPV by definition. Identity (67) comes out even through a limit process, by taking $d_1 \to 0^+$ in (55). Considering only the terms pertaining to a constant shape function, it holds in fact:

$$\lim_{d_1\to 0^+}\mathbb{L}^{pu} = \begin{pmatrix} 0 & -b & 1\\ b & 0 & 0\\ -1 & 0 & 0\end{pmatrix}\frac{1-2\,\nu}{\sqrt{1+b^2}};$$

$$\lim_{d_1\to 0^+}\mathbb{A}^{pu} = 2\,(1-2\,\nu)\,\boldsymbol{SKW}(\boldsymbol{e}_1\otimes\boldsymbol{e}_2);$$

$$\lim_{d_1\to 0^+}\mathbb{R}^{pu} = \lim_{d_1\to 0^+}\mathbb{S}^{pu} = 0\,.$$

Strongly singular kernels $\boldsymbol{G}_{up}$ and $\boldsymbol{G}_{pu}$ generate free terms [14] that holds $\frac{1}{2}\,\mathbb{1}$ for smooth boundaries in the limit process $\Omega \ni \boldsymbol{x} \to \boldsymbol{x} \in \Gamma$. Such free terms arise in the limit:

$$\lim_{d_1\to 0^+}\left[\mathbb{I}^{pu}\,I_{\triangle}^{r^{-3}}(\boldsymbol{x},d_2,d_3)\right]_{d_3=ad_2+k_a}^{d_3=bd_2+k_b}\Bigg|_{d_2=-x_2}^{d_2=\bar{y}_2-x_2}\,. \tag{68}$$

In fact, taking into account of equation (29) and the expansion:

$$\mathbb{I}^{pu} = 2\,d_1(1-\nu)\mathbb{1} + o(d_1); \quad d_1 \to 0$$

it can be easily shown that:

$$\kappa\lim_{d_1\to 0^+}\left[\mathbb{I}^{pu}\,I_{\triangle}^{r^{-3}}(\boldsymbol{x},d_2,d_3)\right]_{d_3=ad_2+k_a}^{d_3=bd_2+k_b}\Bigg|_{d_2=-x_2}^{d_2=\bar{y}_2-x_2} = \frac{1}{2}\,[\,1 \quad 0\,]\,. \tag{69}$$

By inserting outcome (69) into equation (23), it turns out:

$$\frac{1}{2}\,[1 \quad 0]\begin{bmatrix} 1-\dfrac{x_2}{\bar{y}_2} \\ \dfrac{1}{\bar{y}_2} \end{bmatrix} = \frac{1}{2}\,\varphi_j^n(\boldsymbol{x})$$

which is the discrete counterpart of the free-term for smooth boundaries.

5 - Discussion

5.1 - *Remarks on HFPs*

As an alternative to Definition 1, the finite part of Hadamard can be defined with reference to a circular neighborhood around the singularity point $\boldsymbol{x} \in T_j$, by introducing polar coordinates at $\boldsymbol{x}$. In 3D problems, this appears to be a standard usual way to define the finite part of Hadamard (see e.g. [**47**]).

Let $B_\varepsilon(\boldsymbol{x}) = \{\boldsymbol{y} \;/\; |\boldsymbol{y}-\boldsymbol{x}| < \varepsilon\}, \varepsilon > 0$ be the ε-ball about $\boldsymbol{x}$. For $k \geq 0$, set

$$I_\varepsilon^k[\psi] := \int_{T_i \setminus B_\varepsilon} r^{(-k-2)}\psi(\boldsymbol{y})\mathrm{d}\Gamma_y \ .$$

This integral exists in the ordinary sense. Introduce polar coordinates at $\boldsymbol{x}$ and examine I_ε as $\varepsilon \to 0$. For simplicity let ψ be "sufficiently" smooth and denote by $R(\theta)$ a parametrization of T_j with respect to the variable θ. Then:

$$I_\varepsilon^k[\psi] = \int_0^\omega \int_\varepsilon^{R(\theta)} r^{(-k-2)} \sum_{|a|\leq k} \frac{1}{a!}(D^a\psi)(\boldsymbol{x})\, r^{|a|} cos^{a_1}(\theta) sin^{a_2}(\theta)\, r \mathrm{d}r\mathrm{d}\theta + R_\varepsilon(\psi)$$

with $R_\varepsilon(\psi)$ a weakly singular, integrable kernel. Performing the inner integral, one finds:

$$v := (|a| - k)$$

$$I_\varepsilon^k[\psi] = R_\varepsilon(\psi) \tag{70}$$

$$+\sum_{|a|\leq k}\frac{1}{a!}(D^a\psi)(\boldsymbol{x})\int_0^\omega cos^{a_1}(\theta)sin^{a_2}(\theta)\left\{\begin{array}{ll} lnR(\theta) - ln\varepsilon & \text{if} |a| = k \\ v^{-1}(R^v(\theta) - \varepsilon^v) & \text{otherwise} \end{array}\right\}\mathrm{d}\theta$$

In the limit $\varepsilon \to 0$, all terms with $|a| \leq k$ may diverge:

$$I_\varepsilon^k[\psi] \sim C_0\, log(\varepsilon) + \sum_{j=1}^k C_j\, \varepsilon^{-j} + finite\ part\ (I(\varepsilon)^k)\ .$$

Hence, for $\psi \in C^{k,\mu}(T_i)$ (the functions which are k times Hoelder continuously differentiable with exponent μ), with some $\mu > 0$, an alternative definition of finite part of Hadamard reads:

Definition 1b.

$$\displaystyle = \!\!\!\!\!\!\int_{T_i} \frac{1}{r^{2+k}} \psi(\boldsymbol{y}) \mathrm{d}\Gamma_y \overset{def}{=} \sum_{|a| \le k} \frac{1}{a!} (D^a \psi)(\boldsymbol{x}) \int_0^{\omega} cos^{a_1}(\theta) sin^{a_2}(\theta) \begin{Bmatrix} lnR(\theta) & \text{if}|a| = k \\ v^{-1} R^v(\theta) & \text{otherwise} \end{Bmatrix} \mathrm{d}\theta$$

$$+ \int_0^{\omega} \int_0^{R(\theta)} r^{(-k-2)} \left[\psi(\boldsymbol{y}) - \sum_{|a| \le k} \frac{1}{a!} (D^a \psi)(\boldsymbol{x}) \, (\boldsymbol{y} - \boldsymbol{x})^a \right] \mathrm{d}\Gamma_y \ .$$

Results given in the previous section can be re-obtained with this more sophisticated definition of finite part of Hadamard, too [10]. It holds in fact for $k = 1$ (apex ^{1} will be omitted):

$$I_\varepsilon(\psi) = \int_{T_i \backslash B_\varepsilon} \frac{1}{r^3} \psi(\boldsymbol{y}) \mathrm{d}\Gamma_y = \int_0^{\omega} \psi(0, \theta) \int_{\varepsilon}^{R(\theta)} \frac{1}{r^2} \, \mathrm{d}r \mathrm{d}\theta + \int_0^{\omega} \int_{\varepsilon}^{R(\theta)} \frac{1}{r} \dot{\psi}(r, \theta) \, \mathrm{d}r \mathrm{d}\theta$$

having set:

$$\dot{\psi}(r, \theta) := \frac{1}{r} [\psi(r, \theta) - \psi(0, \theta)] \ .$$

Straightforward passages lead to:

$$\begin{aligned} I_\varepsilon(\psi) = & \frac{1}{\varepsilon} \int_0^{\omega} \psi(0, \theta) \, \mathrm{d}\theta - \int_0^{\omega} \frac{\psi(0, \theta)}{R(\theta)} \, \mathrm{d}\theta \\ & + \int_0^{\omega} \int_{\varepsilon}^{R(\theta)} \ddot{\psi}(r, \theta) \, \mathrm{d}r \mathrm{d}\theta + \int_0^{\omega} \psi_r(0, \theta) \int_{\varepsilon}^{R(\theta)} \frac{1}{r} \, \mathrm{d}r \mathrm{d}\theta \end{aligned}$$

where it has been defined:

$$\ddot{\psi}(r, \theta) := \frac{1}{r} [\dot{\psi}(r, \theta) - \psi_r(0, \theta)] = \frac{1}{r^2} [\psi(r, \theta) - \psi(0, \theta) - r \, \psi_r(0, \theta)] \ .$$

---

[10] For the sake of brevity, here φ_j^n will be substituted by ψ .

The expression that corresponds to (70) reads:

$$I_\varepsilon(\psi) = \frac{1}{\varepsilon}\int_0^\omega \psi(0,\theta)\,\mathrm{d}\theta - ln(\varepsilon)\int_0^\omega \psi_{,r}(0,\theta)\mathrm{d}\theta + \int_0^\omega \psi_{,r}(0,\theta)ln(R(\theta))\mathrm{d}\theta$$
$$- \int_0^\omega \frac{\psi(0,\theta)}{R(\theta)}\,\mathrm{d}\theta + R_\varepsilon(\psi)$$

$$R_\varepsilon(\psi) = \int_0^\omega \int_\varepsilon^{R(\theta)} \ddot{\psi}(r,\theta)\,\mathrm{d}r\mathrm{d}\theta$$

and the finite part of Hadamard reads as follows:

$$⨎_{T_i} \frac{1}{r^3}\psi(\boldsymbol{y})\mathrm{d}\Gamma_y = \int_0^\omega \int_0^{R(\theta)} \ddot{\psi}(r,\theta)\,\mathrm{d}r\mathrm{d}\theta + \int_0^\omega \psi_{,r}(0,\theta)ln(R(\theta))\mathrm{d}\theta - \int_0^\omega \frac{\psi(0,\theta)}{R(\theta)}\,\mathrm{d}\theta\,.$$

As already pointed out, $\boldsymbol{y} \in T_j$ implies in the local reference $\mathcal{L}$:

$$\psi(r,\theta) = (1 - \frac{y_2}{\bar{y}_2}) = (1 - \frac{x_2}{\bar{y}_2} + \frac{rcos(\theta)}{\bar{y}_2})\,.$$

Moreover, $\boldsymbol{x} \in T_j$, implies $\omega = 2\pi$ so that:

$$\int_0^\omega \psi_{,r}(0,\theta)\mathrm{d}\theta = \frac{1}{\bar{y}_2}\int_0^{2\pi} cos(\theta)\mathrm{d}\theta = 0$$

$$\int_0^\omega \psi(0,\theta)\,\mathrm{d}\theta = 2\pi\left(1 - \frac{x_2}{\bar{y}_2}\right)\,.$$

Collecting all terms:

$$I_\varepsilon(\psi) = \frac{2\pi}{\varepsilon}\left(1 - \frac{x_2}{\bar{y}_2}\right) + ⨎_{T_i} \frac{1}{r^3}\psi(\boldsymbol{y})\mathrm{d}\Gamma_y + o(1); \qquad \varepsilon \to 0\,. \tag{71}$$

In order to compare (71) and (65), the different behavior of the square and circular neighborhood as $\varepsilon \to 0$ must be described. Reference making to figure 5 and defining with:

$$I_{\boxed{\varepsilon}}(\psi) \stackrel{def}{=} \int_{T_j \setminus T_j^\varepsilon} \frac{1}{r^3}\psi(\boldsymbol{y})\mathrm{d}\Gamma_y$$

one recognizes that:

$$I_\varepsilon(\psi) = I_{\boxed{\varepsilon}}(\psi) + \int\limits_{T_i^\varepsilon \setminus B_\varepsilon} \frac{1}{r^3}\,\psi(\boldsymbol{y})\,\mathrm{d}\Gamma_y\ . \tag{72}$$

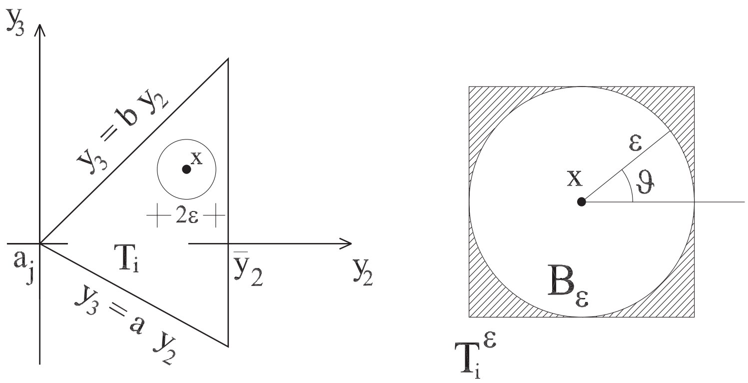

Fig. 5. – Circular and square neighborhood.

In polar coordinates around $\boldsymbol{x}$ it holds:

$$\begin{aligned}\int\limits_{T_i^\varepsilon \setminus B_\varepsilon} \frac{1}{r^3}\,\psi(\boldsymbol{y})\,\mathrm{d}\Gamma_y &= \int\limits_{-\pi/4}^{\pi/4}\int\limits_{\varepsilon}^{\varepsilon/cos(\theta)} + \int\limits_{\pi/4}^{3\pi/4}\int\limits_{\varepsilon}^{\varepsilon/sin(\theta)} + \int\limits_{3\pi/4}^{5\pi/4}\int\limits_{\varepsilon}^{-\varepsilon/cos(\theta)} \\ &+ \int\limits_{-3\pi/4}^{-\pi/4}\int\limits_{\varepsilon}^{-\varepsilon/sin(\theta)} \frac{\psi(r,\theta)}{r^2}\,\mathrm{d}r\mathrm{d}\theta = \frac{2\pi - 4\sqrt{2}}{\varepsilon}\left(1 - \frac{x_2}{\bar{y}_2}\right).\end{aligned} \tag{73}$$

Substituting (71), (73) into (72), it comes out:

$$I_{\boxed{\varepsilon}}(\psi) = \frac{4\sqrt{2}}{\varepsilon}\left(1 - \frac{x_2}{\bar{y}_2}\right) + \;⨎\limits_{T_i} \frac{1}{r^3}\psi(\boldsymbol{y})\mathrm{d}\Gamma_y + o(1); \qquad \varepsilon \to 0\ . \tag{74}$$

Term I_2 in equation (66) can be easily obtained from eq. (74) and the expression of $\boldsymbol{G}_{pp}(\boldsymbol{d};\boldsymbol{e}_1;\boldsymbol{e}_1)$ in the local reference $\mathcal{L}$.

5.2 - *Concluding Remarks*

Analytical integrations have been performed in the present note for both the singular and the regular part, so that the closed form of equation (12) is obtained as a function of the collocation point. The proposed outcomes are exhaustive for the collocation approach as well as for the post-process reconstruction of primal and dual fields (temperature and flux, displacement and stress). It seems to be of interest for the Galerkin technique as well, because it firmly distinguishes the weakly singular terms relevant to the outer integral and the singular terms in the outer integration process. In this regard, a preliminary work has been put forward that aims at showing that for "edge adjacent" elements all singular terms cancel out, whereas they just vanish for panels joint by a vertex. Logarithmic singularities require the use of special cubature schemes. All these topics will be considered in a further publication. Besides accuracy and computational efficiency, the availability of the closed form for the approximated primal and dual fields entails the possibility of analytical manipulations - see e.g. [**46**] - which are hardly possible with alternative approaches. Indeed, closed forms (47), (54), (55), and (64) allow the extension to three dimensional fracture mechanics of the important result of Gray and Paulino: in a nutshell, it has been shown that - as it happens in two dimensions [**49**] - the linear term of the expansion of crack opening and sliding about the crack tip vanishes. The argument of the proof is that a linear term in such an expansion induces a logarithmic singularity for stresses at the crack front that is not compatible with the asymptotical behavior of the stress field. Even in this regard, details will be published in a further publication. Analytical integration for static (steady state) problems are the main ingredient for the evaluation of closed forms of integrals pertaining to time dependent problems, such as elasto-dynamics and acoustics, which have been recently considered [**50**].

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Appendix 1 – Green's functions

The expressions of Green's functions for 3D Laplace and linear elasticity follows. Here $\boldsymbol{n}(\boldsymbol{x})$ and $\boldsymbol{l}(\boldsymbol{y})$ are the normals at the boundary at $\boldsymbol{x}$ and $\boldsymbol{y}$, respectively. Vectors $\boldsymbol{d}$ and $\boldsymbol{r}$ are defined as $\boldsymbol{d} = -\boldsymbol{r} = (\boldsymbol{y} - \boldsymbol{x})$.

1.1 - *Laplace equation*

$$G_{uu}(\boldsymbol{r}) = \frac{1}{4\alpha\pi}\frac{1}{r}$$

$$G_{pu}(\boldsymbol{r};\boldsymbol{n}(\boldsymbol{x})) = -\frac{1}{4\pi}\frac{\boldsymbol{r}\cdot\boldsymbol{n}}{r^3}$$

$$G_{up}(\boldsymbol{d};\boldsymbol{l}(\boldsymbol{y})) = \frac{1}{4\pi}\frac{\boldsymbol{r}\cdot\boldsymbol{l}}{r^3}$$

$$G_{pp}(\boldsymbol{r};\boldsymbol{n}(\boldsymbol{x});\boldsymbol{l}(\boldsymbol{y})) = \frac{\alpha}{4\pi}\frac{1}{r^3}\left(3\frac{(\boldsymbol{r}\cdot\boldsymbol{l})\,(\boldsymbol{r}\cdot\boldsymbol{n})}{r^2} - \boldsymbol{n}\cdot\boldsymbol{l}\right).$$

1.2 - *Linear elasticity*

$$\boldsymbol{G}_{uu}(\boldsymbol{d}) = \frac{1}{16\pi}\frac{1}{G(1-\nu)}\frac{1}{r}\left(\frac{\boldsymbol{d}\otimes\boldsymbol{d}}{r^2} + (3-4\nu)\,\boldsymbol{I}\right)$$

$$\boldsymbol{G}_{pu}(\boldsymbol{d};\boldsymbol{n}(\boldsymbol{x})) = -\frac{1}{8\pi}\frac{1}{(1-\nu)}\frac{1}{r^3}\Big[(1-2\nu)(2\,\boldsymbol{SKW}(\boldsymbol{d}\otimes\boldsymbol{n}) - (\boldsymbol{d}\cdot\boldsymbol{n})\,\boldsymbol{I})$$

$$-\,3\,(\boldsymbol{d}\cdot\boldsymbol{n})\frac{\boldsymbol{d}\otimes\boldsymbol{d}}{r^2}\Big]$$

$$\boldsymbol{G}_{up}(\boldsymbol{d};\boldsymbol{l}(\boldsymbol{y})) = -\frac{1}{8\pi}\frac{1}{(1-\nu)}\frac{1}{r^3}\Big[(1-2\nu)(2\,\boldsymbol{SKW}(\boldsymbol{d}\otimes\boldsymbol{l}) + (\boldsymbol{d}\cdot\boldsymbol{l})\,\boldsymbol{I})$$

$$+\,3\,(\boldsymbol{d}\cdot\boldsymbol{l})\frac{\boldsymbol{d}\otimes\boldsymbol{d}}{r^2}\Big]$$

$$\boldsymbol{G}_{pp}(\boldsymbol{d};\boldsymbol{n}(\boldsymbol{x});\boldsymbol{l}(\boldsymbol{y})) = \frac{G\nu}{4\pi(1-\nu)}\frac{1}{r^3}\Bigg\{2\,\boldsymbol{SYM}(\boldsymbol{l}\otimes\boldsymbol{n}) + 2\,\boldsymbol{SKW}(\boldsymbol{l}\otimes\boldsymbol{n})\ \frac{3\nu-1}{\nu}$$

$$+3\ \frac{(3\nu-1)}{\nu}\left[\boldsymbol{SKW}(\boldsymbol{d}\otimes\boldsymbol{l})\ \frac{\boldsymbol{d}\cdot\boldsymbol{n}}{r^2} - \boldsymbol{SKW}(\boldsymbol{d}\otimes\boldsymbol{n})\ \frac{\boldsymbol{d}\cdot\boldsymbol{l}}{r^2}\right]$$

$$+3\ \frac{(1-\nu)}{\nu}\left[\boldsymbol{SYM}(\boldsymbol{d}\otimes\boldsymbol{l})\ \frac{\boldsymbol{d}\cdot\boldsymbol{n}}{r^2} + \boldsymbol{SYM}(\boldsymbol{d}\otimes\boldsymbol{n})\ \frac{\boldsymbol{d}\cdot\boldsymbol{l}}{r^2}\right]$$

$$+3\ \frac{\boldsymbol{d}\otimes\boldsymbol{d}}{r^2}\ \left[(\boldsymbol{l}\cdot\boldsymbol{n}) - \frac{5}{\nu}\frac{(\boldsymbol{d}\cdot\boldsymbol{n})(\boldsymbol{d}\cdot\boldsymbol{l})}{r^2}\right]$$

$$+\left[3\frac{(\boldsymbol{d}\cdot\boldsymbol{n})(\boldsymbol{d}\cdot\boldsymbol{l})}{r^2} + (\boldsymbol{l}\cdot\boldsymbol{n})\ \frac{(1-2\nu)}{\nu}\right]\boldsymbol{I}\Bigg\}\,.$$

Appendix 2 – Fundamental Lebesgue integrals

The following identities, that can be proved by induction when $d_1 \neq 0$, are the keynote of the inner integration. Here, the following notation will be considered:

$\widehat{k} = k \div 2$ integer division $k \div 2$.

$k_{[2]} = k - 2\widehat{k}$ remainder of the (integer) division $k \div 2$.

and $I_{\triangle}^{r^{-3}}(\boldsymbol{x}, d_2, d_3)$ has been defined in Section 4.1.

Proposition. *By defining with:*

$$\lambda(\vartheta) = \frac{1}{\vartheta^2\, d_1^2 + k_\vartheta^2}\ ; \qquad d_3 = \vartheta d_2 + k_\vartheta \tag{75}$$

they hold:

$$\int_{-x_2}^{\bar{y}_2 - x_2} \frac{d_2^{n_{[2]}}}{d_1^2 + d_2^2}\ \frac{1}{r}\Bigg|_{\vartheta=a}^{\vartheta=b} \mathrm{d}d_2 \tag{76}$$

$$= \lambda(\vartheta)\left[\alpha \operatorname{arctanh}\frac{d_3}{r} + \beta\, I_{\triangle}^{r^{-3}}(\boldsymbol{x}, d_2, d_3)\Bigg|_{\vartheta=a}^{\vartheta=b}\right]_{d_2=-x_2}^{d_2=\bar{y}_2-x_2}$$

$$\int_{-x_2}^{\bar{y}_2-x_2} \frac{d_2^{n_{[2]}}}{\left(d_1^2+d_2^2\right)^2} \left.\frac{1}{r}\right|_{\vartheta=a}^{\vartheta=b} \mathrm{d}d_2 \tag{77}$$

$$= \lambda^3(\vartheta)\left[\gamma \operatorname{arctanh}\frac{d_3}{r} + \delta\, I_\triangle^{r^{-3}}(\mathbf{x}, d_2, d_3) + \zeta\, r\Big|_{\vartheta=a}^{\vartheta=b}\right]_{d_2=-x_2}^{d_2=\bar{y}_2-x_2}$$

$$\int_{-x_2}^{\bar{y}_2-x_2} \frac{d_2^{n_{[2]}}}{\left(d_1^2+d_2^2\right)^3} \left.\frac{1}{r}\right|_{\vartheta=a}^{\vartheta=b} \mathrm{d}d_2 \tag{78}$$

$$= \lambda^5(\vartheta)\left[\eta \operatorname{arctanh}\frac{d_3}{r} + \theta\, I_\triangle^{r^{-3}}(\mathbf{x}, d_2, d_3) + \iota\, r\Big|_{\vartheta=a}^{\vartheta=b}\right]_{d_2=-x_2}^{d_2=\bar{y}_2-x_2}$$

where:

$$\alpha = -\,k_\vartheta\, n_{[2]} + \vartheta\,(1-n_{[2]}) \tag{79}$$

$$\beta = \frac{\vartheta\, d_1}{2}\; n_{[2]} + \frac{k_\vartheta}{2\, d_1}\;(1-n_{[2]})$$

$$\gamma = \frac{k_\vartheta}{2}\left(4k_\vartheta^2 - \frac{3+2\,\vartheta^2}{\lambda(\vartheta)}\right) n_{[2]} + \frac{\vartheta}{2}\left(\frac{1+2\vartheta^2}{\lambda(\vartheta)} - 4\,k_\vartheta^2\right)(1-n_{[2]})$$

$$\delta = \vartheta\,\frac{d_1^2\left[d_1^2\,\vartheta^2(1+\vartheta^2) - 3k_\vartheta^2\right] - k_\vartheta^4}{4\,d_1}\; n_{[2]}$$

$$+\; k_\vartheta\,\frac{d_1^2\Big(3\,\vartheta^2\,d_1^2\,(1+\vartheta^2) - k_\vartheta^2(1-4\,\vartheta^2)\Big) + k_\vartheta^4}{4\,d_1^3}\;(1-n_{[2]})$$

$$\zeta = \frac{\vartheta^2\,d_1^2 + 2\,\vartheta\, d_2\, k_\vartheta - k_\vartheta^2}{2\left(d_1^2+{d_2}^2\right)\lambda(\vartheta)}\; n_{[2]} - \frac{\vartheta\, d_1^2\,(\vartheta\, d_2 - 2\,k_\vartheta) - d_2\,k_\vartheta^2}{2d_1^2\left(d_1^2+d_2^2\right)\lambda(\vartheta)}\;(1-n_{[2]})$$

$$\eta = \frac{k_\vartheta}{8}\left(\frac{-8d_1^2\vartheta^6 - 8\left(3d_1^2+k_\vartheta^2\right)\vartheta^4 - 3\left(5d_1^2 - 8k_\vartheta^2\right)\vartheta^2 + 45k_\vartheta^2}{\lambda(\vartheta)} - 48\,k_\vartheta^4\right) n_{[2]}$$

$$+\;\frac{\vartheta}{8}\left(\frac{8d_1^2\vartheta^6 + 8\left(d_1^2+k_\vartheta^2\right)\vartheta^4 + \left(3d_1^2 - 40k_\vartheta^2\right)\vartheta^2 - 33k_\vartheta^2}{\lambda(\vartheta)} + 48\,k_\vartheta^4\right)(1-n_{[2]})$$

$$\theta = \frac{\vartheta}{16}\left(3\,\mathscr{I}^8\,d_1^5 + 6\,\mathscr{I}^6\,d_1^5 + \mathscr{I}^4\,d_1\left(3\,d_1^4 - 30\,d_1^2\,k_{\mathscr{I}}^2 - 10\,k_{\mathscr{I}}^4\right)\right.$$

$$\left.+\frac{-2\,\mathscr{I}^2\,d_1^2\left(15\,d_1^4\,k_{\mathscr{I}}^2 + 15\,d_1^2\,k_{\mathscr{I}}^4 + 4\,k_{\mathscr{I}}^6\right) + k_{\mathscr{I}}^4\left(15\,d_1^4 + 6\,d_1^2\,k_{\mathscr{I}}^2 - k_{\mathscr{I}}^4\right)}{d_1^3}\right) n_{[2]}$$

$$+\frac{k_{\mathscr{I}}}{16}\left(15\,\mathscr{I}^8\,d_1^5 + 10\,\mathscr{I}^6\,d_1^3\left(3\,d_1^2 + 4\,k_{\mathscr{I}}^2\right) + \mathscr{I}^4\,d_1\left(15\,d_1^4 + 10\,d_1^2\,k_{\mathscr{I}}^2 + 38\,k_{\mathscr{I}}^4\right)\right.$$

$$\left.+\frac{\mathscr{I}^2\,d_1^2\left(-30\,d_1^4\,k_{\mathscr{I}}^2 - 22\,d_1^2\,k_{\mathscr{I}}^4 + 16\,k_{\mathscr{I}}^6\right) + k_{\mathscr{I}}^4\left(3\,d_1^4 - 2\,d_1^2\,k_{\mathscr{I}}^2 + 3\,k_{\mathscr{I}}^4\right)}{d_1^5}\right)(1 - n_{[2]})$$

$$\imath = \frac{\mathscr{I}^2\,d_1^2 + 2\,\vartheta\,d_2\,k_{\mathscr{I}} - k_{\mathscr{I}}^2}{4\left(d_1^2 + d_2^2\right)^2\lambda^3(\mathscr{I})}\;n_{[2]} + \frac{-\mathscr{I}^2\,d_1^2\,d_2 + 2\,\vartheta\,d_1^2\,k_{\mathscr{I}} + d_2\,k_{\mathscr{I}}^2}{4d_1^2\left(d_1^2 + d_2^2\right)^2\lambda^3(\mathscr{I})}\;(1 - n_{[2]})$$

$$+\frac{n_{[2]}}{8d_1^2\left(d_1^2 + d_2^2\right)\lambda(\mathscr{I})}\Big\{3\mathscr{I}^6 d_1^6 + 11\mathscr{I}^5 d_1^4 d_2 k_{\mathscr{I}} + \mathscr{I}^4(3d_1^6 - 6d_1^4 k_{\mathscr{I}}^2)$$

$$+\mathscr{I}^3(12d_1^4 d_2 k_{\mathscr{I}} + 10d_1^2 d_2 k_{\mathscr{I}}^3) + \mathscr{I}^2(-18d_1^4 k_{\mathscr{I}}^2 - 9d_1^2 k_{\mathscr{I}}^4)$$

$$+\mathscr{I}(-12d_1^2 d_2 k_{\mathscr{I}}^3 - d_2 k_{\mathscr{I}}^5) + 3d_1^2 k_{\mathscr{I}}^4\Big\}$$

$$+\frac{1 - n_{[2]}}{8d_1^4\left(d_1^2 + d_2^2\right)\lambda(\mathscr{I})}\Big\{-6\mathscr{I}^6 d_1^6 d_2 + 13\mathscr{I}^5 d_1^6 k_{\mathscr{I}} + \mathscr{I}^4(-3d_1^6 d_2 + 3d_1^4 d_2 k_{\mathscr{I}}^2)$$

$$+\mathscr{I}^3(12d_1^6 k_{\mathscr{I}} + 14d_1^4 k_{\mathscr{I}}^3) + \mathscr{I}^2(18d_1^4 d_2 k_{\mathscr{I}}^2 + 12d_1^2 d_2 k_{\mathscr{I}}^4)$$

$$+\mathscr{I}(-12d_1^4 k_{\mathscr{I}}^3 + d_1^2 k_{\mathscr{I}}^5) - 3d_1^2 d_2 k_{\mathscr{I}}^4 + 3d_2 k_{\mathscr{I}}^6\Big\}.$$

Proposition. *By defining with* $d_3 = \vartheta d_2 + k_{\mathscr{I}}$ *it holds:*

$$\int_{-x_2}^{\bar{y_2}-x_2} d_2^k \log(d_3 + r)\Big|_{\vartheta=a}^{\vartheta=b} \mathrm{d}d_2 - \sum_{n=0}^{k}\varpi(n)\int_{-x_2}^{\bar{y_2}-x_2}\frac{d_2^{k-n}}{r}\Bigg|_{\vartheta=a}^{\vartheta=b}\mathrm{d}d_2 \tag{80}$$

$$= \left[\mu\,\mathrm{arctanh}\frac{d_3}{r} + \nu\, I_{\triangle}^{r^{-3}}(\boldsymbol{x}, d_2, d_3) + \xi\log(d_3 + r)\Bigg|_{\vartheta=a}^{\vartheta=b}\right]_{d_2=-x_2}^{d_2=\bar{y_2}-x_2}$$

where:

$$\varpi(n) = (-1)^{\widehat{n}} \frac{d_1^n}{k+1} (k_b - (k_b + b\, d_1)\, n_{[2]})$$

$$\xi = \frac{d_2^{k+1}}{k+1}$$

$$\nu = (-1)^{\widehat{k}} \frac{d_1^{k+2}}{k+1} (k_{[2]} - 1)$$

$$\mu = (-1)^{\widehat{k}} \frac{d_1^{k+1}}{(k+1)} k_{[2]}\, .$$

Proposition. *With assumptions* (75) *and*

$$\Lambda(\vartheta) = \frac{1}{d_1^2(1+\vartheta^2) + k_\vartheta^2} \tag{81}$$

they hold:

$$\int_{-x_2}^{\bar{y}_2 - x_2} \frac{d_2^{n_{[2]}}}{d_1^2 + d_2^2} \frac{1}{r^3}\bigg|_{\vartheta=a}^{\vartheta=b} \mathrm{d}d_2 \tag{82}$$

$$= \lambda(\vartheta) \left[\rho \operatorname{arctanh} \frac{d_3}{r} + \varrho\, I_\triangle^{r^{-3}}(\mathbf{x}, d_2, d_3) + \sigma\, \frac{1}{r}\bigg|_{\vartheta=a}^{\vartheta=b} \right]_{d_2=-x_2}^{d_2=\bar{y}_2 - x_2}$$

$$\int_{-x_2}^{\bar{y}_2 - x_2} \frac{d_2^{n_{[2]}}}{\left(d_1^2 + d_2^2\right)^2} \frac{1}{r^3}\bigg|_{\vartheta=a}^{\vartheta=b} \mathrm{d}d_2 \tag{83}$$

$$= \lambda^2(\vartheta) \left[\varsigma \operatorname{arctanh} \frac{d_3}{r} + \tau\, I_\triangle^{r^{-3}}(\mathbf{x}, d_2, d_3) + \upsilon\, r + \phi\, \frac{1}{r}\bigg|_{\vartheta=a}^{\vartheta=b} \right]_{d_2=-x_2}^{d_2=\bar{y}_2 - x_2}$$

where:

$$\rho = \lambda^2(\vartheta)\, k_\vartheta (3\vartheta^2 d_1^2 - k_\vartheta^2)\, n_{[2]} + \lambda^2(\vartheta)\, \vartheta(-\vartheta^2 d_1^2 + 3k_\vartheta^2)\, (1 - n_{[2]})$$

$$\varrho = -\lambda^2(\vartheta)\, \frac{\vartheta d_1(\vartheta^2 d_1^2 - 3k_\vartheta^2)}{2}\, n_{[2]} + \lambda^2(\vartheta)\, \frac{k_\vartheta(-3\vartheta^2 d_1^2 + k_\vartheta^2)}{2\, d_1}\, (1 - n_{[2]})$$

$$\sigma = \Big\{ \lambda(\vartheta) \left[2\vartheta^2(\vartheta^2 - 1)d_1^2 - 2\vartheta(1+\vartheta^2)d_2 k_\vartheta \right]$$

$$- \Lambda(\vartheta) \left[\vartheta(1+\vartheta^2)(\vartheta d_1^2 - d_2 k_\vartheta) \right] + \left(1 - \vartheta^2 \right) \Big\} \; n_{[2]}$$

$$+ \Big\{ \lambda(\vartheta) \left[2\vartheta((1+\vartheta^2)\vartheta d_2 + (\vartheta^2 - 1)k_\vartheta) \right]$$

$$- \Lambda(\vartheta) \left[(1+\vartheta^2)(d_2(1+\vartheta^2) + \vartheta k_\vartheta) \right] \Big\} \; (1 - n_{[2]})$$

$$\varsigma = \lambda^3(\vartheta) \, 3 \, k_\vartheta \, \frac{5\vartheta^4 d_1^4 + 4\vartheta^6 d_1^4 + k_\vartheta^4 - 2\vartheta^2 k_\vartheta^2 (5d_1^2 + 2k_\vartheta^2)}{2} \; n_{[2]}$$

$$+ \lambda^3(\vartheta) \, \vartheta \, \frac{-\vartheta^4(3+4\vartheta^2)d_1^4 + 2\vartheta^2(15+8\vartheta^2)d_1^2 k_\vartheta^2 + 5(-3+4\vartheta^2)k_\vartheta^4}{2} \; (1 - n_{[2]})$$

$$\tau = \lambda^3(\vartheta) \, 3\vartheta \, \frac{\vartheta^4(1+\vartheta^2)d_1^6 - 5\vartheta^2(2+\vartheta^2)d_1^4 k_\vartheta^2 - 5(-1+\vartheta^2)d_1^2 k_\vartheta^4 + k_\vartheta^6}{4\, d_1} \; n_{[2]}$$

$$+ \lambda^3(\vartheta) \, k_\vartheta \, \frac{-15\vartheta^4(1+\vartheta^2)d_1^6 - 5\vartheta^2(-6+\vartheta^2)d_1^4 k_\vartheta^2}{4\, d_1^3} \; (1 - n_{[2]})$$

$$+ \lambda^3(\vartheta) \, k_\vartheta \, \frac{(-3+11\vartheta^2)d_1^2 k_\vartheta^4 + k_\vartheta^6}{4\, d_1^3} \; (1 - n_{[2]})$$

$$\upsilon = \left\{ -\lambda^2(\vartheta) \, 4\vartheta^3 d_1^2(\vartheta d_1^2 + d_2 k_\vartheta) + 2\lambda(\vartheta)\vartheta(2\vartheta d_1^2 + d_2 k_\vartheta) - \frac{1}{2} \right\} \; \frac{n_{[2]}}{d_1^2 + d_2^2}$$

$$+ \left\{ \lambda^2(\vartheta) \, 4\vartheta^3 d_1^2(\vartheta d_2 - k_\vartheta) - 2\lambda(\vartheta)\vartheta(2\vartheta d_2 - k_\vartheta) + \frac{d_2}{2d_1^2} \right\} \; \frac{1 - n_{[2]}}{d_1^2 + d_2^2}$$

$$\phi = \big\{ \lambda^2(\vartheta) \, 8\vartheta^3 d_1^2(\vartheta(-1+\vartheta^2)d_1^2 - (1+\vartheta^2)d_2 k_\vartheta) + \lambda(\vartheta) \, 4\vartheta(2\vartheta(1-2\vartheta^2)d_1^2$$

$$+ (1+\vartheta^2)d_2 k_\vartheta) - \; (1 - 6\vartheta^2 + \vartheta^4) + \; \Lambda(\vartheta) \, \vartheta(1+\vartheta^2)^2(\vartheta d_1^2 - d_2 k_\vartheta) \big\} \; n_{[2]}$$

$$+ \big\{ \lambda^2(\vartheta) \, 8\vartheta^3 d_1^2(\vartheta(1+\vartheta^2)d_2 + (-1+\vartheta^2)k_\vartheta)$$

$$+ \lambda(\vartheta) \, 4\vartheta(k_\vartheta - \vartheta(2(1+\vartheta^2)d_2 + 3\vartheta k_\vartheta))$$

$$+ \; \Lambda(\vartheta) \, (1+\vartheta^2)^2 \, (d_2(1+\vartheta^2) + \vartheta k_\vartheta) \big\} \; (1 - n_{[2]}) \, .$$

Proposition. *With assumptions* (75, 81) *it holds:*

$$\int_{-x_2}^{\bar{y}_2-x_2} \frac{d_2^{n_{[2]}}}{d_1^2+d_2^2}\,\frac{1}{r^5}\bigg|_{\vartheta=a}^{\vartheta=b} \mathrm{d}d_2 \tag{84}$$
$$= \lambda(\vartheta)\left[\varphi \operatorname{arctanh}\frac{d_3}{r} + \chi\, I_{\triangle}^{r^{-3}}(\mathbf{x}, d_2, d_3) + \psi\,\frac{1}{r} + \omega\,\frac{1}{r^3}\bigg|_{\vartheta=a}^{\vartheta=b}\right]_{d_2=-x_2}^{d_2=\bar{y}_2-x_2}$$

where:

$$\varphi = \lambda^4(\vartheta)\Big\{ -k_{\vartheta}\,(5\vartheta^4 d_1^4 - 10\vartheta^2 d_1^2 k_{\vartheta}^2 + k_{\vartheta}^4)\, n_{[2]} + \vartheta\,(\vartheta^4 d_1^4 - 10\vartheta^2 d_1^2 k_{\vartheta}^2 + 5k_{\vartheta}^4)\,(1-n_{[2]})\Big\}$$

$$\chi = \lambda^4(\vartheta)\left\{ \frac{\vartheta^5 d_1^5 - 10\vartheta^3 d_1^3 k_{\vartheta}^2 + 5\vartheta d_1 k_{\vartheta}^4}{2}\, n_{[2]} + \frac{5\vartheta^4 d_1^4 k_{\vartheta} - 10\vartheta^2 d_1^2 k_{\vartheta}^3 + k_{\vartheta}^5}{2d_1}\,(1-n_{[2]})\right\}$$

$$\begin{aligned}
\psi = \lambda(\vartheta)\Bigg\{ & -\lambda^2(\vartheta) 8\vartheta^3 d_1^2(\vartheta(-1+\vartheta^2)d_1^2 - (1+\vartheta^2)d_2 k_{\vartheta}) \\
& + \lambda(\vartheta) 4\vartheta(\vartheta(-2+3\vartheta^2+\vartheta^4)d_1^2 - (1+\vartheta^2)^2 d_2 k_{\vartheta}) \\
& - \Lambda^2(\vartheta)\,\frac{2\vartheta(1+\vartheta^2)^2(\vartheta d_1^2 - d_2 k_{\vartheta})((1+2\vartheta^2)d_1^2 + 2k_{\vartheta}^2)}{3} \\
& + \Lambda(\vartheta)\,\frac{k_{\vartheta}(3\vartheta(1+3\vartheta^2+2\vartheta^4)d_2 + (3-5\vartheta^2-2\vartheta^4)k_{\vartheta})}{3} \\
& - \Lambda(\vartheta)\,\frac{(-3+7\vartheta^2+18\vartheta^4+8\vartheta^6)d_1^2}{3}\Bigg\}\, n_{[2]} \\
+ \lambda(\vartheta)\Bigg\{ & -\lambda^2(\vartheta) 8\vartheta^3 d_1^2(\vartheta(1+\vartheta^2)d_2 + (-1+\vartheta^2)k_{\vartheta}) \\
& + \lambda(\vartheta) 4\vartheta(-k_{\vartheta} + \vartheta(2+\vartheta^2)(d_2(1+\vartheta^2) + \vartheta k_{\vartheta})) \\
& - \Lambda^2(\vartheta)\,\frac{2(1+\vartheta^2)^2(d_2(1+\vartheta^2) + \vartheta k_{\vartheta})((1+2\vartheta^2)d_1^2 + 2k_{\vartheta}^2)}{3} \\
& - \Lambda(\vartheta)\,\frac{(1+7\vartheta^2+6\vartheta^4)(d_2(1+\vartheta^2) + \vartheta k_{\vartheta})}{3}\Bigg\}\,(1-n_{[2]})
\end{aligned}$$

$$\omega = \left\{ \lambda(\vartheta)\, 2\vartheta \, \frac{\vartheta(-1+\vartheta^2)d_1^2 - (1+\vartheta^2)d_2 k_\vartheta}{3} \right.$$
$$\left. + \Lambda(\vartheta) \, \frac{k_\vartheta(k_\vartheta + \vartheta(d_2(1+\vartheta^2) - \vartheta k_\vartheta)) - (-1+\vartheta^2+2\vartheta^4)d_1^2}{3} \right\} n_{[2]}$$
$$+ \left\{ \lambda(\vartheta)\, 2\vartheta \, \frac{\vartheta(1+\vartheta^2)d_2 + (-1+\vartheta^2)k_\vartheta}{3} \right.$$
$$\left. - \Lambda(\vartheta) \, \frac{(1+\vartheta^2)(d_2(1+\vartheta^2) + \vartheta k_\vartheta)}{3} \right\} (1 - n_{[2]}) \, .$$

Appendix 3 – Matrices for the potential kernel

3.1 - *Weakly singular kernel*

Making reference to the notation of formulae (46-47) and assuming $u_b = 1 + b^2$ and $z_b = u_b \, d_1^2 + k_b^2$, they hold:

$$\mathbb{L}^{uu} = \frac{1}{\sqrt{u_b}} \left[k_b, -\frac{b z_b}{2 u_b} \right]; \qquad \mathbb{A}^{uu} = \left[d_2, \frac{1}{2} \left(d_1^2 + d_2^2 \right) \right];$$
$$\mathbb{I}^{uu} = [-d_1^2, 0]; \qquad \mathbb{R}^{uu} = \left[0, \frac{k_b}{2 u_b} \right].$$

3.2 - *Strongly singular kernels*

3.2.1 - Kernel $\boldsymbol{G}_{up}$

Assuming $u_b = 1 + b^2$, they hold:

$$\mathbb{L}^{up} = \left[0, -\frac{b d_1}{\sqrt{u_b}} \right]; \qquad \mathbb{A}^{up} = [0, d_1]; \qquad \mathbb{I}^{up} = [-d_1, 0];$$
$$\mathbb{R}^{up} = 0; \qquad \mathbb{S}^{up} = 0;$$

3.2.2 - Kernel $\boldsymbol{G}_{pu}$

Assuming $u_b = 1 + b^2$, they hold:

$$\mathbb{L}^{pu} = \left[\frac{b n_2 - n_3}{\sqrt{u_b}}, \frac{d_1 n_1 b \, u_b + k_b (b \, n_3 + n_2)}{u_b^{3/2}} \right]; \qquad \mathbb{A}^{pu} = [-n_2, -d_1 n_1];$$
$$\mathbb{I}^{pu} = [d_1 n_1, -d_1^2 n_2]; \qquad \mathbb{R}^{pu} = \left[0, \frac{b n_2 - n_3}{u_b} \right]; \qquad \mathbb{S}^{pu} = 0 \, .$$

3.3 - *Hyper singular kernel*

Assuming $u_b = 1 + b^2$ and $z_b = u_b\, d_1^2 + k_b^2$, they hold:

$$\mathbb{L}^{pp} = [0, -\frac{n_1\, b}{\sqrt{u_b}}]; \qquad \mathbb{A}^{pp} = [0, n_1]; \qquad \mathbb{I}^{pp} = [0, d_1 n_2]; \qquad \mathbb{H}^{pp} = 0;$$

$$\mathbb{S}^{pp} = -\frac{d_1}{z_b(z_b - d_1^2)}\,[\boldsymbol{n}\cdot\boldsymbol{\alpha}, \boldsymbol{n}\cdot\boldsymbol{\beta}];$$

$$\mathbb{R}^{pp} = \frac{1}{(d_1^2 + d_2^2)(z_b - d_1^2)}\,[\boldsymbol{n}\cdot\gamma, -d_1\,\boldsymbol{n}\cdot\gamma^{\perp}];$$

where:

$$\begin{aligned}
\boldsymbol{\alpha} &= \big[2bd_1k_b^2 + u_b d_1\big(bd_1^2 + d_2 k_b\big), b\,u_b\,d_1^2\,d_2 \\
&\quad - k_b\big(d_1^2 + k_b^2\big), (u_b\,d_2 + bk_b)\big(b^2 d_1^2 + k_b^2\big)\big];\\
\boldsymbol{\beta} &= \big[b\,u_b\,d_2\,d_1^3 - k_b d_1\,(d_1^2 + k_b^2), -2b^2 d_2 k_b d_1^2 \\
&\quad - \big(d_1^2 + k_b^2\big)(b\,d_1^2 + d_2 k_b), -\big(b^2 d_1^2 + k_b^2\big)\big(d_1^2 + k_b^2 + bd_2\,k_b\big)\big];\\
\gamma &= \big[bd_1^2 + d_2\,k_b, d_1(bd_2 - k_b), 0\big];\\
\gamma^{\perp} &= [-\gamma_2, \gamma_1, 0];\\
\boldsymbol{n} &= [n_1, n_2, n_3]\,.
\end{aligned}$$

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Abstract

Some results on the analytical integration of kernels in elliptic [**1**] *problems (potential, Stokes, elasticity) for 3D Boundary Element Methods are presented for isotropic homogeneous materials. Adopting polynomial shape functions of arbitrary degree on flat triangular discretizations, integrations are performed for Lebesgue integrals working in a local coordinate system. For singular integrals, both a limit to the boundary as well as the finite part of Hadamard* [**2, 3**] *approach have been pursued.*

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