Riv. Mat. Univ. Parma (7) **7** (2007), 135-150

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A degenerate rainfall infiltration model with periodic data (**)

1 - Introduction

The paper deals with a degenerate nonlinear boundary value problem modelling incompressible periodic rainfall infiltration into a homogeneous, isotropic, unsaturated porous medium. The mathematical treatment of incompressible fluids through unsaturated medium, began with the work of [**25**]. A porous medium consists of a solid matrix and void pores. The pores are filled with water provided by rainfall, irrigations, leaking from the surfaces waters or underground sources. To study water infiltration supplied by rainfall is of great importance when we want to forecast the history of contamination.The flow is said to be unsaturated as long as void pores are still present. Water infiltration in unsaturated soils is formulated by the Richards equation

$$\theta_t - div(D(\theta)\nabla\theta) + \frac{\partial}{\partial x_3}K(\theta) = f$$

(see e.g. [**5**]) where $D(\theta)$ represents the water diffusivity and $K(\theta)$ the hydraulic conductivity. These functions $D(\theta)$ and $K(\theta)$ both depending nonlinearly on θ, were introduced in the soil sciences by empirical expressions and defined in a subset of R. This fact is a feature of the diffusion that develops in a porous medium which may reach the saturation θ_s when the fluid fills all free pores. For a weakly nonlinear isotropic medium, the water diffusion D and the hydraulic conductivity K are real

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(**) Received 27th February 2007 and in revised form 24th September 2007. AMS classification 35B10, 35D05, 35K65, 47J35, 76S05.

functions defined on $[0, \theta_s]$ (see [7], [24]). In the specific framework of the paper, the degeneracy occurs at only one point where the solution becomes 0 and the water diffusion $D(0)$ vanishes. The degeneracy of Richards' equation in [4], [24] appears since the water diffusivity blows up at the saturation value. In our case, this singular behavior is avoided by considering a finite value of diffusivity.

Let Ω be a bounded and regular open set of R^n ($n = 1, 2, 3$), we consider the Richards equation with nonhomogeneous flux conditions, described by the following mathematical problem

$$\theta_t - \triangle\overline{D}(\theta) + \frac{\partial}{\partial x_3}K(\theta) = f \text{ in } Q := \Omega \times P, \tag{1.1}$$

$$(K(\theta)\mathbf{i}_3 - \nabla\overline{D}(\theta)).\upsilon = \varphi(x,t) \text{ on } \Sigma_\varphi := \Gamma_\varphi \times P\ , \tag{1.2}$$

$$(K(\theta)\mathbf{i}_3 - \nabla\overline{D}(\theta)).\upsilon = a(x)\overline{D}(\theta) + f_0(x,t) \text{ on } \Sigma_a := \ \Gamma_a \times P, \tag{1.3}$$

$$\theta(x, t+\omega) = \theta(x,t) \text{ in } Q,\ \omega > 0, \tag{1.4}$$

where $P := R/\omega Z$ denotes the period interval $[0, \omega]$, so the functions defined in Q are automatically ω-time periodic. We assume that

H_D) D is a positive continuous, monotonically increasing function defined in $(0, \theta_s]$ such that $\lim\limits_{\theta\to 0} D(\theta) = 0$.

The function

$$\overline{D}(\theta) := \int_0^\theta D(s)ds,\ \theta \in [0,\ \theta_s]$$

is the Kirchhoff transform.

The boundary $\partial\Omega$ is composed of the disjoint boundaries Γ_φ, the inflow boundary and Γ_a, the outflow boundary. On Γ_φ, we consider a flux due to the rainfall and on Γ_a a flux proportional to the water diffusivity. In this model, υ is the outward normal to $\partial\Omega$ and $\mathbf{i}_3$ is the unit vector along Ox_3 downwards directed. The term $\frac{\partial}{\partial x_3}K(\theta)$ represents the contribution given by the effect of the gravitational field upon the infiltration process while the function $f > 0$ stands for a periodic water source within the domain.

To study our problem, we shall do the following structural assumptions on the data

H_k) $\left\{ \begin{array}{c} K \text{ is a positive, bounded and continuous function with} \\ 0 < k_m \leqslant K(s) \leqslant k_M, \forall s \in [0,\ \theta_s]; \end{array} \right\}$

H_f) $f \in L^2(Q)$;

$\mathrm{H}\varphi$) $\varphi \in L^2(\ \Sigma_\varphi)$;

H_0) $f_0 \in L^2(\Sigma_a)$.

A fairly general situation including the present type of operator is considered in [**1**]. The hydraulic process describes the evolution of the volumetric water content θ present per unit volume of soil. Physical arguments lead to consider $\theta \geqslant 0$. There is an extensive mathematical literature on the infiltration in porous medium problems closely related to the one considered here, [**4**], [**9**], [**11**], [**12**], [**15**], [**16**], [**23**], [**26**], [**27**], [**28**]. Degenerate equations are considered in [**13**], [**14**]. In most cases, the authors are not looking for periodic solutions. However, papers like [**22**] and [**17**] are focussed particularly on this issue. Generally, the Richards model behaves hysteretically if infiltration is followed by evaporation. We assume that only infiltration takes place, so we can neglect the hysteretic aspect. A consequence of the degeneracy of the equation is that we do not expect to have classical solutions. Therefore, we need to introduce the concept of weak periodic solutions.

The paper is organized as follows. In Section 2, we introduce the space of ω-periodic functions where solutions are sought and give the definition of weak solution. Section 3 is devoted to prove the existence of weak periodic solutions θ_n for the regularized problem. Uniform estimates are established to pass to the limit on θ_n. In Section 4, we use Schauder's fixed point theorem to get the existence of weak periodic solutions. Finally, in Section 5 the Hölder continuity assumption on the inverse of the Kirchhoff transform (see below), is used to show the existence of weak periodic solutions to (1.1)-(1.4).

2 - Preliminaries

H$_a$) a is a positive continuous and bounded function such that $0 < a_m \leqslant a(x) \leqslant a_M$, $\forall x \in \Gamma_a$.

Next, let us introduce the functional framework for the periodic solutions of problem.

We consider the Hilbert space

$$V := L^2(P; W^{1,2}(\Omega))$$

endowed with the norm

$$\|v\|_V := \left(\int_Q \mid \nabla v(x,t) \mid^2 dxdt + \int_{\Sigma_a} a(x) \mid v(x,t) \mid^2 dSdt \right)^{1/2}$$

equivalent with the usual norm in V, and its topological dual space

$$V^* = L^2(P; (W^{1,2}(\Omega))^*)$$

with $\|.\|_*$ norm. The duality pairing between V and V^* shall be written as $\langle ., . \rangle$.

To prove the existence of solutions, we extend D and K by continuity to the left of zero and to the right of the saturation value θ_s, preserving the properties of the original functions.

$$D(\theta)\left\{\begin{array}{c} = 0, \text{ if } \theta < 0 \\ = D(\theta), \text{ if } 0 \leqslant \theta \leqslant \theta_s \\ = D(\theta_s), \text{ if } \theta > \theta_s \end{array}\right\}$$

$$K(\theta)\left\{\begin{array}{c} = k_m, \text{ if } \theta < 0 \\ = K(\theta), \text{ if } 0 \leqslant \theta \leqslant \theta_s \\ = k_M, \text{ if } \theta > \theta_s \end{array}\right\}$$

(see [**24**]).

The mathematical approach to periodicity shall be of static type, that is we will transform problem (1.1)-(1.4) in an abstract problem to which we apply some techniques of the maximal monotone mappings theory. Because of the degeneracy of equation (1.1), we consider a nondegenerate regularized problem obtained replacing the term $\overline{D}(\theta)$ with $D_n(s) := \overline{D}(s) + s/n$, for any $s \in R$ and $n \in N$, (see [**18**]).

Definition 2.2. A function θ is called a weak periodic solution of (1.1)-(1.4) if

$$\theta \in L^2(P; W^{1,2}(\Omega)),\ \theta_t \in L^2(P; (W^{1,2}(\Omega))^*)$$

and satisfies

$$\int_Q \theta_t \zeta dxdt + \int_Q \nabla \overline{D}(\theta) \nabla \zeta dxdt - \int_Q K(\theta) \frac{\partial}{\partial x_3} \zeta dxdt + \int_{\Sigma_a} a(x) \overline{D}(\theta) \zeta dSdt$$
$$= \int_Q f(x,t) \zeta dxdt - \int_{\Sigma_\varphi} \varphi(x,t) \zeta dSdt - \int_{\Sigma_a} f_0(x,t) \zeta dSdt,\ \forall \zeta \in V. \tag{2.1}$$

3 - The approximating problem

Fixed $w \in L^2(Q)$ and defined $D_n(s) := \overline{D}(s) + s/n$, for any $s \in R$ and $n \in N$, the nondegenerate regularized problem assumes the form

$$\theta_{nt} - div(D'_n(w) \nabla \theta_n) + \frac{\partial}{\partial x_3} K(w) = f \text{ in } Q, \tag{3.1}$$

$$(K(w)\mathbf{i}_3 - D'_n(w) \nabla \theta_n).\upsilon = \varphi(x,t) \text{ on } \Sigma_\varphi, \tag{3.2}$$

$$(K(w)\mathbf{i}_3 - D'_n(w)\nabla\theta_n).\upsilon = a(x)D_n(\theta_n) + f_0(x,t) \text{ on } \Sigma_a, \tag{3.3}$$

$$\theta_n(x, t+\omega) = \theta_n(x,t) \text{ in } Q,\ \omega > 0 \tag{3.4}$$

with

$\mathrm{H}_n)\ \dfrac{1}{n} \leqslant D'_n(s) = D(s) + \dfrac{1}{n} \leqslant 1 + D(\theta_s),\ \forall s \in R,\ n \in N.$

Its weak periodic solution is a function $\theta_n \in V$ with $\theta_{nt} \in V^*$ such that

$$\int_Q \theta_{nt}\zeta dxdt + \int_Q D'_n(w)\nabla\theta_n\nabla\zeta dxdt - \int_Q K(w)\frac{\partial}{\partial x_3}\zeta dxdt + \int_{\Sigma_a} a(x)D_n(\theta_n)\zeta dsdt = \int_Q f(x,t)\zeta dxdt - \int_{\Sigma_\varphi} \varphi(x,t)\zeta dsdt - \int_{\Sigma_a} f_0(x,t)\zeta dsdt,\ \forall\zeta \in V. \tag{3.5}$$

The approach to periodicity of solutions for problem (3.5) is based on the next result.

Theorem 3.1. ([**3**], [8], [**20**]). *Let L be a linear, closed, densely defined operator from the reflexive Banach space V to V^*, L maximal monotone and let A be a bounded, hemicontinuous monotone mapping from V into V^*. Then, $L + A$ is maximal monotone in $V \times V^*$. Moreover, if $L + A$ is coercive then $Range(L + A) = V^*$.*

In order to use Theorem 3.1, we must define the operators L and A.

The set

$$\mathcal{D} := \{v \in L^2(P; W^{1,2}(\Omega)) : v_t \in L^2(P; (W^{1,2}(\Omega))^*)\}$$

is dense in V because of the density of $C^\infty(\overline{Q}) \subset \mathcal{D}$ in V.

Let

$$L : \mathcal{D} \to V^*$$

be the linear operator with

$$\langle L\theta_n, \zeta\rangle := \int_Q \theta_{nt}\zeta dxdt, \text{ for any } \zeta \in V.$$

This operator L is closed, skew-adjoint (i.e. $L = -L^*$) and maximal monotone (see [**20**], Lemma 1.1, p. 313).

Given $w \in L^2(Q)$, we define

$$A : V \to V^*$$

by setting

$$\langle A\theta_n, \zeta\rangle := \int_Q D'_n(w)\nabla\theta_n \nabla\zeta dxdt + \int_{\Sigma_a} a(x)D_n(\theta_n)\zeta dSdt.$$

The properties of the operator A are contained in the following result.

Proposition 3.2. *If assumptions* H_D) *and* H_a), H_n) *are satisfied, then* A *is*

i) *hemicontinuous;*
ii) *monotone;*
iii) *coercive.*

Proof. i) By the Hölder inequality one has

$$| \langle A\theta_n, \zeta\rangle | \leqslant (1 + D(\theta_s)\Big(\int_Q | \nabla\theta_n |^2 dxdt\Big)^{1/2}\Big(\int_Q | \nabla\zeta |^2 dxdt\Big)^{1/2}$$

$$+ \Big(\int_{\Sigma_a} a(x) | D_n(\theta_n) |^2 dSdt\Big)^{1/2}\Big(\int_{\Sigma_a} a(x) | \zeta |^2 dSdt\Big)^{1/2}$$

$$\leqslant \|\zeta\|_V \Big[(1 + D(\theta_s))\Big(\int_Q | \nabla\theta_n |^2 dxdt\Big)^{1/2} + (1 + D(\theta_s)) \int_{\Sigma_a} a(x) | \theta_n |^2 dSdt)^{1/2}\Big]$$

$$\leqslant \|\zeta\|_V((1 + D(\theta_s))\|\theta_n\|_V)$$

so that

$$\|A\theta_n\|_* \leqslant (1 + D(\theta_s))\|\theta_n\|_V$$

and the hemicontinuity emerges from a result of [**19**], Theorems 2.1 and 2.3.

ii)

$$\langle A\theta^1_n - A\theta^2_n, \theta^1_n - \theta^2_n\rangle = \int_Q D'_n(w) | \nabla(\theta^1_n - \theta^2_n) |^2 dxdt$$

$$+ \int_{\Sigma_a} a(x)(D_n(\theta^1_n) - D_n(\theta^2_n))(\theta^1_n - \theta^2_n)dSdt$$

$$= \int_Q D_n'(w) \mid \nabla(\theta_n^1 - \theta_n^2) \mid^2 dxdt + \int_{\Sigma_a} a(x)(\overline{D}(\theta_n^1) - \overline{D}(\theta_n^2))(\theta_n^1 - \theta_n^2) dSdt$$

$$+ \frac{1}{n} \int_{\Sigma_a} a(x) \mid \theta_n^1 - \theta_n^2 \mid^2 dSdt \geqslant 0$$

because of the monotonicity of $\overline{D}(s)$.

iii)

$$\langle A\theta_n, \theta_n \rangle = \int_Q D_n'(w) \mid \nabla\theta_n \mid^2 dxdt + \int_{\Sigma_a} a(x) D_n(\theta_n)\theta_n dSdt$$

$$\geqslant \frac{1}{n} \int_Q \mid \nabla\theta_n \mid^2 dxdt + \frac{1}{n} \int_{\Sigma_a} a(x) \mid \theta_n \mid^2 dSdt$$

$$\geqslant \frac{1}{n} \|\theta_n\|_V^2.$$

Hence,

$$\frac{\langle A\theta_n, \theta_n \rangle}{\|\theta_n\|_V} \geqslant \frac{1}{n} \|\theta_n\|_V \to +\infty, \text{ as } \|\theta_n\|_V \to +\infty.$$ ■

Finally, let

$$G : V \to V^*$$

be defined by

$$\langle G, \zeta \rangle := \int_Q f(x,t)\zeta dxdt - \int_{\Sigma_\varphi} \varphi(x,t)\zeta dSdt$$

$$- \int_{\Sigma_a} f_0(x,t)\zeta dSdt - \int_Q K(w) \frac{\partial}{\partial x_3} \zeta dxdt, \ \forall \zeta \in V.$$

Then, problem (3.5) can be reformulated as an abstract problem of the form

(3.6) $$L\theta_n + A\theta_n = G$$

to which we apply Theorem 3.1.

Hence, we can state the main result of the section.

Proposition 3.3. *Given a* $w \in L^2(Q)$, *assuming* H_D)-H_φ) *and* H_a), H_n) *the problem* (3.6) *admits a unique weak periodic solution.*

Proof. The existence of weak periodic solutions is a consequence of Theorem 3.1, while the uniqueness comes from the strict monotonicity. ■

4 - A fixed point argument

In this section our interest is focussed on the research of fixed points for an operator equation.

Let

$$\Phi : \ L^2(Q) \to \ L^2(Q)$$

be the mapping defined by

$$\Phi(w) = \theta_n$$

where θ_n is the unique weak periodic solution of (3.1)-(3.4). The mapping Φ is well-defined. In order to prove its continuity, we will prove some crucial estimates and convergences, useful to utilize the Schauder fixed point theorem. Let $w_k \in L^2(Q)$ be a sequence such that $w_k \to w$ strongly in $L^2(Q)$, we denote with θ_{nk} the weak periodic solution of

$$\int_Q \partial_t \theta_{nk} \zeta dx dt + \int_Q D'_n(w_k) \nabla \theta_{nk} \nabla \zeta dx dt + \int_{\Sigma_a} a(x) D_n(\theta_{nk}) \zeta dS dt \tag{4.1}$$

$$= \int_Q K(w_k) \frac{\partial \zeta}{\partial x_3} dx dt + \int_Q f(x,t) \zeta dx dt$$

$$- \int_{\Sigma_\varphi} \varphi(x,t) \zeta dS dt - \int_{\Sigma_a} f_0(x,t) \zeta dS dt, \ \forall \zeta \in V.$$

Chosen $\zeta = \theta_{nk}$ as a test function in (4.1), the periodicity of θ_{nk}, implies that

$$\int_Q D'_n(w_k) \mid \nabla \theta_{nk} \mid^2 dx dt + \int_{\Sigma_a} a(x) D_n(\theta_{nk}) \theta_{nk} dS dt$$

$$= - \int_Q K(w_k) \frac{\partial \theta_{nk}}{\partial x_3} dx dt + \int_Q f(x,t) \theta_{nk} dx dt$$

$$- \int_{\Sigma_\varphi} \varphi(x,t) \theta_{nk} dS dt - \int_{\Sigma_a} f_0(x,t) \theta_{nk} dS dt$$

and the Young inequality yields

$$\frac{1}{n}\int_Q |\nabla\theta_{nk}|^2\,dxdt+\frac{1}{n}\int_{\Sigma_a} a(x)\,|\theta_{nk}|^2\,dSdt$$

$$\leqslant\frac{1}{2\varepsilon}\int_Q |K(w_k)|^2\,dxdt+\frac{\varepsilon}{2}\int_Q |\nabla\theta_{nk}|^2\,dxdt$$

$$+\frac{1}{2\varepsilon}\int_Q |f(x,t)|^2\,dxdt+\frac{\varepsilon}{2}\int_Q |\theta_{nk}|^2\,dxdt$$

$$+\frac{1}{2\varepsilon}\int_{\Sigma_\varphi} |\varphi(x,t)|^2\,dSdt+\frac{\varepsilon}{2}\int_{\Sigma_\varphi} |\theta_{nk}|^2\,dSdt$$

$$+\frac{1}{2\varepsilon}\int_{\Sigma_a} |f_0(x,t)|^2\,dSdt+\frac{\varepsilon}{2}\int_{\Sigma_a} |\theta_{nk}|^2\,dSdt.$$

Therefore,

$$\Big(\frac{1}{n}-\frac{\varepsilon}{2}\Big)\Big(\int_Q |\nabla\theta_{nk}|^2\,dxdt+\int_{\Sigma_a} a(x)\,|\theta_{nk}|^2\,dSdt\Big)$$

$$\leqslant\frac{1}{2\varepsilon}\int_Q |K(w_k)|^2\,dxdt+\frac{1}{2\varepsilon}\int_Q |f(x,t)|^2\,dxdt$$

$$+\frac{\varepsilon}{2}\int_Q |\theta_{nk}|^2\,dxdt+\frac{1}{2\varepsilon}\int_{\Sigma_\varphi} |\varphi(x,t)|^2\,dSdt$$

$$+\frac{1}{2\varepsilon}\int_{\Sigma_a} |f_0(x,t)|^2\,dSdt+\frac{\varepsilon}{2}\int_{\Sigma_\varphi} |\theta_{nk}|^2\,dSdt+\frac{\varepsilon}{2}\int_{\Sigma_a} |\theta_{nk}|^2\,dSdt.$$

Recalling that

$$\|s\|_{L^2(P;L^2(\Gamma_\varphi))}\leqslant c_1\|s\|_V,\ \|s\|_{L^2(P;W^{1,2}(\Omega))}\leqslant c_2\|s\|_V,\ \|s\|_{L^2(P;L^2(\Gamma_a))}\leqslant c_3\|s\|_V$$

on account of the equivalence of the norms in V , we have

$$\Big(\frac{1}{n}-\frac{\varepsilon(1+c_1^2+c_2^2+c_3^2)}{2}\Big)\Big(\int_Q |\nabla\theta_{nk}|^2\,dxdt+\int_{\Sigma_a} a(x)\,|\theta_{nk}|^2\,dSdt\Big)$$

$$\leqslant\frac{k_M^2}{2\varepsilon}\,|Q|+\frac{1}{2\varepsilon}\int_Q |f(x,t)|^2\,dxdt$$

$$+\frac{1}{2\varepsilon}\int_{\Sigma_\varphi} |\varphi(x,t)|^2\,dSdt+\frac{1}{2\varepsilon}\int_{\Sigma_a} |f_0(x,t)|^2\,dSdt\leqslant C'.$$

A specific value of ε, gives us

$$\int_Q |\nabla\theta_{nk}|^2\, dxdt + \int_{\Sigma_a} a(x)\,|\theta_{nk}|^2\, dSdt \leqslant C_n \tag{4.2}$$

where C_n is a positive constant independent of k.

From (4.1) and the energy estimate (4.2), it follows that $\partial_t\theta_{nk}$ is bounded in the V^* norm. Therefore θ_{nk} lies in a bounded set of $\mathcal{D}$, namely

$$\|\theta_{nk}\|_{\mathcal{D}} \leqslant C_n,\ \forall k \in N.$$

Thus, we can select a subsequence, still denoted by θ_{nk}, such that

$$\theta_{nk} \rightharpoonup \theta_n \text{ in } \mathcal{D} \text{ when } k \to +\infty.$$

By a result of [**20**], Theorem 5.1, the sequence θ_{nk} is precompact in $L^2(Q)$ that is

$$\theta_{nk} \to \theta_n \text{ in } L^2(Q) \text{ and a.e. in } Q.$$

Lemma 4.1. *The mapping Φ is continuous.*

Proof. The convergences

$$\begin{aligned} \theta_{nk} &\to \theta_n \text{ in } L^2(Q) \text{ and a.e. in } Q \\ \nabla\theta_{nk} &\rightharpoonup \nabla\theta_n \text{ in } L^2(P;(L^2(\Omega))^n) \\ \theta_{nk} &\rightharpoonup \theta_n \text{ in } L^2(\Gamma_a) \\ w_k &\to w \text{ in } L^2(Q) \\ D_n'(w_k) &\to D_n'(w) \text{ in } L^2(Q) \end{aligned}$$

enable us to conclude that $\Phi(w_k) = \theta_{nk}$ converges strongly to $\Phi(w) = \theta_n$ in $L^2(Q)$. ■

Lemma 4.2. *There exists a constant $R > 0$ such that*

$$\|\Phi(w)\|_{L^2(Q)} \leqslant R,\ \forall w \in L^2(Q).$$

Proof. The assertion of lemma is obtained letting $k \to +\infty$ in (4.1). ■

Since $\Phi(L^2(Q)) \subset \mathcal{D}$ and the embedding $\mathcal{D} \hookrightarrow L^2(Q)$ is compact, the operator Φ is compact from $L^2(Q)$ into itself.

Then,

Theorem 4.3. *If* H_D)-H_φ) *and* H_a), H_n) *hold, there exists at least a weak periodic solution θ_n of* (3.1)-(3.4).

Proof. As a consequence of Lemmas 4.1 and 4.2, the mapping Φ is both continuous and compact. Therefore, by the Schauder fixed point theorem Φ has a fixed point which is a weak periodic solution to (3.1)-(3.4) i.e.

$$\int_Q \theta_{nt}\zeta dxdt + \int_Q D'_n(\theta_n)\nabla\theta_n\nabla\zeta dxdt - \int_Q K(\theta_n)\frac{\partial}{\partial x_3}\zeta dxdt + \int_{\Sigma_a} a(x)D_n(\theta_n)\zeta dSdt = \int_Q f(x,t)\zeta dxdt - \int_{\Sigma_\varphi}\varphi(x,t)\zeta dSdt - \int_{\Sigma_a} f_0(x,t)\zeta dSdt, \forall\zeta\in V. \tag{4.3}$$

■

5 - Existence of periodic solutions

The assumption

$H_{\overline{D}}$) $\overline{D}^{-1}(s)$ Hölder continuous of order $\gamma\in(0,1)$

shall play the leading role for the existence of weak periodic solutions. Taking $\zeta = D_n(\theta_n)$ as a test function in (4.3) and using a result of [**2**], we get

$$\int_0^\omega \frac{\partial}{\partial t}\int_\Omega\Big(\int_0^{\theta_n(x,t)} D_n(\tau)d\tau\Big)dxdt + \int_Q |\nabla D_n(\theta_n)|^2 dxdt + \int_{\Sigma_a} a(x)\,|D_n(\theta_n)|^2 dSdt$$
$$= \int_Q K(\theta_n)\frac{\partial}{\partial x_3}D_n(\theta_n)dxdt + \int_Q f(x,t)D_n(\theta_n)dxdt - \int_{\Sigma_\varphi}\varphi(x,t)D_n(\theta_n)dSdt - \int_{\Sigma_a} f_0(x,t)D_n(\theta_n)dSdt.$$

The periodicity of θ_n and the Young inequality lead to

$$\int_Q |\nabla D_n(\theta_n)|^2 dxdt + \int_{\Sigma_a} a(x)\,|D_n(\theta_n)|^2 dSdt \leqslant \frac{1}{2\varepsilon}\int_Q |K(\theta_n)|^2 dxdt + \frac{\varepsilon}{2}\int_Q |\nabla D_n(\theta_n)|^2 dxdt$$

$$+\frac{1}{2\varepsilon}\int_Q |\, f(x,t)\,|^2\, dxdt + \frac{\varepsilon}{2}\int_Q |\, D_n(\theta_n)\,|^2\, dxdt$$

$$+\frac{1}{2\varepsilon}\int_{\Sigma_\varphi} |\, \varphi(x,t)\,|^2\, dSdt + \frac{\varepsilon}{2}\int_{\Sigma_\varphi} |\, D_n(\theta_n)\,|^2\, dSdt$$

$$+\frac{1}{2\varepsilon}\int_{\Sigma_a} |\, f_0(x,t)\,|^2\, dSdt + \frac{\varepsilon}{2}\int_{\Sigma_a} |\, D_n(\theta_n)\,|^2\, dSdt.$$

Finally,

$$\int_Q |\, \nabla D_n(\theta_n)\,|^2\, dxdt + \int_{\Sigma_a} a(x)\,|\, D_n(\theta_n)\,|^2\, dSdt$$

$$\leqslant \frac{k_M^2}{2\varepsilon}\,|\, Q\,| + \frac{\varepsilon}{2}c_2^2\|D_n(\theta_n)\|_V^2 + \frac{1}{2\varepsilon}\int_Q |\, f(x,t)\,|^2\, dxdt$$

$$+\frac{\varepsilon}{2}c_2^2\|D_n(\theta_n)\|_V^2 + \frac{1}{2\varepsilon}\int_{\Sigma_\varphi} |\, \varphi(x,t)\,|^2\, dSdt$$

$$+\frac{\varepsilon}{2}c_1^2\|D_n(\theta_n)\|_V^2 + \frac{1}{2\varepsilon}\int_{\Sigma_a} |\, f_0(x,t)\,|^2\, dSdt + \frac{\varepsilon}{2}c_3^2\|D_n(\theta_n)\|_V^2\ ,$$

by which

$$\Big(1 - \frac{\varepsilon}{2}(c_1^2 + 2c_2^2 + c_3^2)\Big)\|D_n(\theta_n)\|_V^2 \leqslant k_1.$$

For a suitable choice of ε, we can obtain

$$\|D_n(\theta_n)\|_V^2 \leqslant k_2 \tag{5.1}$$

and

$$\|\nabla D_n(\theta_n)\|^2_{L^2(P;(L^2(\Omega))^n)} \leqslant k_3. \tag{5.2}$$

Thanks to (5.1),

$$\|\overline{D}(\theta_n)\|_V^2 + \frac{\|\theta_n\|_V^2}{n^2} \leqslant k_4 \tag{5.3}$$

and from (4.1), θ_{nt} is bounded in $L^2(P;(W^{1,2}(\Omega))^*)$ i.e.

$$\|\theta_{nt}\|_{V^*} \leqslant k_5$$

where k_i , $i = 1, 2, 3, 4, 5$ are positive constants independent of n.

By virtue of (5.3), $\overline{D}(\theta_n)$ is bounded in V and in $L^2(P;W^{s,2}(\Omega))$ $\forall s \in (0,1)$, because V is continuously embedded in $L^2(P;W^{s,2}(\Omega))$. The Hölder continuity of $\overline{D}^{\,-1}$ and $\overline{D}^{\,-1}(0)=0$ imply that $\theta_n \in W^{\gamma s,2/\gamma}(\Omega)$, for a.e. $t\in P$.

By standard result (see [**10**], Lemma, p. 266), θ_n is bounded in $L^{2/\gamma}(P;W^{\gamma s,2/\gamma}(\Omega))$ because

$$\|\theta_n(t)\|^{1/\gamma}_{W^{\gamma s,2/\gamma}(\Omega)} \leqslant \|\overline{D}(\theta_n)\|_{W^{s,2}(\Omega)}\|\overline{D}^{\,-1}\|^{1/\gamma}_{H\ddot{o}lder}\ .$$

An integration of this inequality over P yields

$$\|\theta_n(t)\|^{2/\gamma}_{L^{2/\gamma}(P;W^{\gamma s,2/\gamma}(\Omega))} \leqslant \|\overline{D}(\theta_n)\|^2_{L^2(P;W^{s,2}(\Omega))}\|\overline{D}^{\,-1}\|^{2/\gamma}_{H\ddot{o}lder}\ .$$

Since

$$W^{\gamma s,2/\gamma}(\Omega)\subset L^{2/\gamma}(\Omega)\subset L^2(\Omega)\subset (W^{1,2}(\Omega))^*$$

with compact injection (see [**10**], Theorem 3, p. 266), by a compactness result given in [**21**] the injection of the set

$$\{\theta_n \in L^{2/\gamma}(P;W^{\gamma s,2/\gamma}(\Omega)) : \theta_{nt}\in L^2(P;(W^{1,2}(\Omega))^*)\}$$

is compact in $L^{2/\gamma}(P;L^{2/\gamma}(\Omega))$. Therefore,

$$\theta_n \to \theta \text{ in } L^2(Q) \text{ and a.e. in } Q \tag{5.4}$$

$$\theta_{nt} \rightharpoonup \theta_t \text{ in } L^2(P;(W^{1,2}(\Omega))^*). \tag{5.5}$$

Then (5.1) and $\|D_n(\theta_n)\|_{L^2(P;L^2(\Gamma_a))} \leqslant c_3\|D_n(\theta_n)\|_V$ yield

$$D_n(\theta_n)\rightharpoonup \chi \text{ in } L^2(P;W^{1,2}(\Omega)) \text{ and in } L^2(P;L^2(\Gamma_a)). \tag{5.6}$$

By (5.4) it follows that

$$\begin{aligned} |\,D_n(\theta_n)-\overline{D}(\theta)\,| &\leqslant |\,D_n(\theta_n)-\overline{D}(\theta_n)\,| + |\,\overline{D}(\theta_n)-\overline{D}(\theta)\,| \\ &\leqslant \frac{|\,\theta_n\,|}{n} + |\,\overline{D}(\theta_n)-\overline{D}(\theta)\,| \to 0 \end{aligned}$$

a.e. when n goes to infinity. Thus,

$$\overline{D}(\theta)=\chi\ .$$

Furthermore, for (5.2) we have

$$\nabla D_n(\theta_n)\rightharpoonup \mu \text{ in } L^2(P;(L^2(\Omega)^n)$$

hence,

$$\int_Q \nabla D_n(\theta_n)\zeta dxdt = \int_Q \nabla\overline{D}(\theta_n)\zeta dxdt + \frac{1}{n}\int_Q \nabla\theta_n\zeta dxdt\ ,\ \forall\zeta\in V.$$

Letting $n \to +\infty$, we infer from

$$\int_Q \mu\zeta dxdt = \int_Q \nabla\overline{D}(\theta)\zeta dxdt$$

that

$$\nabla D_n(\theta_n) \rightharpoonup \nabla\overline{D}(\theta) \text{ in } L^2(P;(L^2(\Omega))^n)\ . \tag{5.7}$$

Next we give the main result of the paper.

Theorem 5.1. *Assume* H_D)-$\mathrm{H}_{\overline{D}}$) *and* H_a), H_n). *Then there exists at least a weak periodic solution for* (1.1)-(1.4).

Proof. The existence of solutions is proven taking into account (5.4)-(5.7) and passing to the limit in (4.3). This concludes the proof. ■

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Abstract

The main interest in this paper is to prove the existence of weak periodic solutions for a degenerate rainfall infiltration into an unsaturated soil model which consists of Richards' equation with nonlinear flux boundary periodic conditions. The aim shall be achieved re-formulating the problem in abstract form in order to apply some general results of the maximal monotone mappings theory and the Schauder fixed point theorem.

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