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A Steffensen's type method with modified functions (**)

1 - Introduction

Solving a nonlinear equation $f(x) = 0$ using iterative method is a classical problem in numerical analysis. The most useful and studied scheme is Newton's method (second order for simple zeros)

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \quad n = 0, 1, 2, \ldots .$$

In [**2**], Ben-Israel analyzes Newton's method when it is applied to modified functions. For instance, if we apply Newton's method to the modified equation $\hat{f}(x) = \dfrac{f(x)}{f'(x)} = 0$ we obtain a second order method for zeros of any multiplicity. On the other hand, if we consider $\hat{f}(x) = \dfrac{f(x)}{\sqrt{f'(x)}}$ the new iterative method is, the well know, Halley's method (third order for simple roots)

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n) - \dfrac{f''(x_n)}{2f'(x_n)} f(x_n)}, \ n = 0, 1, 2, \ldots$$

In this study, we analyze an improvement of the Steffensen's method that re-

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quires only two function values in each step. We are interested to obtain similar behavior as Newton's method. We propose also a quasi-Halley's method without using any derivative. From the convergence properties and the numerical results the introduced methods are good alternatives to the classical methods.

The paper is organized as follows: in section 2, we introduce the basic ingredients of the modified functions, some secant type conditions and an improved Steffensen's method. Its convergence analysis are studied in section 3. In section 4, a quasi-Halley iterative method is described. Finally, a numerical experiment and conclusions are presented in section 5.

2 - The basic ingredients

The above mentioned modified functions are special cases of

$$\hat{f}(x) = e^{-\int a(x)dx} f(x), \tag{1}$$

with a suitable integrand $a(x)$.

In this paper, as in [**2**], we study the special case of (1)

$$\hat{f}(x) = (x-\theta)^{\beta} f(x), \quad \theta, \beta \in \mathbb{R} \tag{2}$$

corresponding to the selection of $a(x)$ as

$$a(x) = -\frac{\beta}{x-\theta}. \tag{3}$$

Applying Newton's method to (2), we get

$$x_{n+1} = x_n - \frac{(x_n-\theta)f(x)}{(x_n-\theta)f'(x_n)+\beta f(x_n)}, \; n = 0, 1, 2, \ldots \tag{4}$$

The classical Steffensen's method can be considered as a modification of Newton's method where $f'(x_n)$ is approximated by $\delta f(x_n, x_n + f(x_n))$, where

$$\delta f(x, y) := \frac{f(x) - f(y)}{x - y}.$$

Our iterative procedure would be considered as a new approach based in a better approximation to the derivative $f'(x_n)$ from x_n and $x_n + f(x_n)$ in each iteration. We consider

$$\delta f(x_n, \tilde{x}_n) \tag{5}$$

where $\tilde{x}_n = x_n + a_n f(x_n)$.

These parameters $a_n \in \mathbb{R}$ will be a control of the good approximation to the derivative. Theoretically, if $a_n \to 0$, then

$$\delta f(x_n, \tilde{x}_n) \to f'(x_n).$$

In order to control the stability, but having a good resolution at every iteration, the parameters a_n can be computed such that

$$tol_c << |a_n f(x_n)| \leq tol_u,$$

where tol_c is related with the computer precision and tol_u is a free parameter for the user.

At practice, some advantage will present this modified Steffensen's method. Since in general $\delta f(x_n, \tilde{x}_n)$ is a better approximation to the derivative $f'(x_n)$ than $\delta f(x_n, x_n + f(x_n))$ the convergence will be faster (the first iterations will be better). Moreover, the size of the neighborhood can be higher, that is, we can consider worse starting points x_0 (taking a_n sufficiently small), as we will see at the numerical experiments.

Thus, we are interested to study the behavior of the iterative method given by

$$x_{n+1} = x_n - \frac{(x_n - \theta) f(x_n)}{(x_n - \theta)\delta f(x_n, \tilde{x}_n) + \beta f(x_n)}, \quad n = 0, 1, 2, \ldots \tag{6}$$

First, as in [2] for Newton's method, we obtain a geometric interpretation of the method (6).

Definition 1. *Two functions f and g are called secant at two points x_n and y_n $(x_n \neq y_n)$ if $f(x_n) = g(x_n)$ and $\delta f(x_n, y_n) = \delta g(x_n, y_n)$.*

Clearly, f and g are secant at x_n and $x_n + a_n f(x_n)$ if and only if $f \cdot h$ and $g \cdot h$ are secant at x_n and $x_n + a_n f(x_n)$, whenever $h(x_n)$ and $h(x_n + a_n f(x_n))$ are different to zero.

Given β and θ, we consider parameters a and b such that

$$F(x, \theta, \beta) := \frac{a + b(x - \theta)}{(x - \theta)^\beta} \tag{7}$$

is secant to f at x_n and $\tilde{x}_n$.

If $\theta \neq x_n$, then f and $F(x, \theta, \beta)$ are secant at x_n and $\tilde{x}_n$ if and only if $(x - \theta)^\beta f(x)$ and $a + b(x - \theta)$ are secant at x_n and $\tilde{x}_n$.

We summarize:

Theorem 1. *Let the nonlinear equation $f(x) = 0$, let θ, β fixed, and let $x_n \neq \theta$ and $\tilde{x}_n \neq \theta$ be points where $(x_n - \theta)\delta f(x_n, \tilde{x}_n) + \beta f(x_n) \neq 0$. The function*

$$F(x, \theta, \beta) = \frac{a + b(x - \theta)}{(x - \theta)^\beta}$$

secant to f at x_n and $\tilde{x}_n$ has a zero at

$$x_{n+1} = x_n - \frac{(x_n - \theta)f(x)}{(x_n - \theta)\delta f(x_n, x_n + a_n f(x_n)) + \beta f(x_n)}, \quad n = 0, 1, 2, \ldots \tag{8}$$

which is the zero of the affine function

$$l(x) = a + b(x - \theta)$$

secant at x_n and $\tilde{x}_n$ to

$$\hat{f}(x) = (x - \theta)^{\beta} f(x).$$

This theorem states that the iterative method (6) is equivalent to finding the zero of the function (7) which is secant to f at x_n and $\tilde{x}_n$.

3 - Convergence analysis

Assuming convergence we analyze the order of (6).

The iterative method (6) can be written as

$$x_{n+1} = x_n - \frac{f(x_n)}{g(x_n)}$$

with $g(x_n) = \dfrac{f(x_n + a_n f(x_n)) - f(x_n)}{a_n f(x_n)} + \dfrac{\beta}{(x_n - \theta)} f(x_n)$.

By Taylor series,

$$f(x_n + a_n f(x_n)) = f(x_n) + \frac{f'(x_n)}{1!} a_n f(x_n) + \frac{f''(x_n)}{2!} (a_n f(x_n))^2 + O(a_n f(x_n)^3),$$

then

$$g(x_n) = f'(x_n)\left(1 - \frac{1}{2} h_n a_n f''(x_n) + O\left(\frac{a_n^2 f(x_n)^2}{f'(x_n)}\right)\right) + \frac{\beta}{(x_n - \theta)} f(x_n),$$

where $h_n = -\dfrac{f(x_n)}{f'(x_n)}$.

Thus,

$$x_{n+1} = x_n + h_n \left(1 + \frac{1}{2} h_n a_n f''(x_n) + \frac{\beta}{(x_n - \theta)} f(x_n) f'(x_n) + O(f(x_n)^2)\right).$$

On the other hand, since

$$0 = f(x^*) = f(x_n) + (x^* - x_n) f'(x_n) + \frac{1}{2}(x^* - x_n)^2 f''(\phi)$$

$\phi \in int(x^*, x_n)$, we deduce that

$$h_n = (x^* - x_n) + \frac{1}{2}(x^* - x_n)^2 \frac{f''(\phi)}{f'(x_n)}.$$

Finally, we obtain

$$\lim_{n\to+\infty} \frac{x_{n+1} - x^*}{(x_n - x^*)^2} = \frac{1}{2}\frac{f''(x^*)}{f'(x^*)}\Big(1 + f'(x^*) \lim_{n\to+\infty} a_n\Big),$$

and the method has order two of convergence for simple roots, as Newton [**2**] and Steffensen [**3**] methods.

3.1 - *Monotone convergence*

As Newton's method, the monotone convergence of the proposed iterative method (6) is related to a notion of convexity.

All functions in this subsection are twice continuously differentiable in a real interval I.

Definition 2. *A function f is supported by g at x_0 if $f(x_0) = g(x_0)$ and $f(x) \geq g(x)$ for all $x \in I$. And supported strictly if the above inequality is strict for all $x \neq x_0$.*

Let $\mathcal{F}$ be a family of functions : $I \to \mathbb{R}$.

Definition 3. *A function f is called (strictly) $\mathcal{F}$-convex if at each point in I it is (strictly) supported by a member of $\mathcal{F}$.*

We use the family of functions (7)

$$\mathcal{F}_{\theta,\beta} = \left\{\frac{a + b(x-\theta)}{(x-\theta)^\beta} \ : \ a, b \in \mathbb{R}\right\}$$

where θ and β are given parameters.

We refer [**2**] and the references therein for more details.

Theorem 2. *Let f be strictly $\mathcal{F}_{\theta,\beta}$-convex in an interval I, $\theta \in \mathbb{R}\setminus I$ and $x_0, x^* \in I$, where $f(x^*) = 0$. Then, all iterates generated by*

$$x_{n+1} = x_n - \frac{(x_n - \theta) f(x)}{(x_n - \theta)\delta f(x_n, \tilde{x}_n) + \beta f(x_n)}, \ \ n = 0, 1, 2, \ldots \tag{9}$$

are in I and

(a) *If* $f(x_0) > 0$ *then* f *is positive for all points* x_n.

(b) *If* $f(x_0) < 0$ *then* $f(x_1) > 0$ *for* a_0 *sufficiently small and thereafter* f *is positive at all iterations.*

Proof. We use that the theorem holds for Newton's method (Theorem 2 of [**2**]).

Let us denote x_i^{Stef} and x_i^{New} the sequences defined from x_0 by the scheme (6) and Newton's method respectively.

(a) Since $f(x_0) > 0$ we have $\delta f(x_0, x_0 + a_0 f(x_0)) \geq f'(x_0)$, then

$$x_1^{Stef} = x_0 - \frac{(x_0-\theta)f(x_0)}{(x_0-\theta)\delta f(x_0,\tilde{x}_0)+\beta f(x_0)} \geq x_1^{New} = x_0 - \frac{(x_0-\theta)f(x_0)}{(x_0-\theta)f'(x_0)+\beta f(x_0)}.$$

(b) Given $\varepsilon < |f(x_1^{New})|$, there exists a_0 such that $|f(x_1^{New}) - f(x_1^{Stef})| < \varepsilon$.
Since $f(x_1^{New}) > 0$ we obtain $f(x_1^{Stef}) > 0$. □

4 - A quasi-Halley iterative method

In [**2**], it is proposed the following quasi-Halley method

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n) - \dfrac{f'(x_n)-f'(x_{n-1})}{2(x_n-x_{n-1})f'(x_n)} f(x_n)}, \quad n = 0,1,2,\ldots$$

where the original second derivative $f''(x_n)$ has been approximated by $\dfrac{f'(x_n)-f'(x_{n-1})}{(x_n-x_{n-1})}$.

Following a similar idea as before with Newton's method, we consider the new quasi-Halley method given by

$$x_{n+1} = x_n - \frac{f(x_n)}{\delta f(x_n - a_n f(x_n), x_n + a_n f(x_n)) - \dfrac{\gamma(x_n, a_n; f)}{2\delta f(x_n - a_n f(x_n), x_n + a_n f(x_n))} f(x_n)}, \tag{10}$$

$n = 0,1,2,\ldots$, where $\gamma(x_n, a_n; f) := [x_n - a_n f(x_n), x_n, x_n + a_n f(x_n); f]$ denotes a second divided difference of f

$$\gamma(x_n, a_n; f) = \frac{f(x_n + a_n f(x_n)) - 2f(x_n) + f(x_n - a_n f(x_n))}{(a_n f(x_n))^2}.$$

For simple roots of sufficiently smooth functions the quasi-Halley method proposed in [**2**] has order of convergence 2.41, our new scheme has order 3:

Theorem 3. *If the quasi-Halley scheme* (10) *converges to a simple root of a sufficiently smooth function, then it has order three.*

Proof. The iterative method (10) can be written as

$$x_{n+1} = x_n - \frac{f(x_n)}{g(x_n)}$$

with

$$g(x_n) = \delta f(x_n - a_n f(x_n), x_n + a_n f(x_n)) - \frac{\gamma(x_n, a_n; f)}{2\delta f(x_n - a_n f(x_n), x_n + a_n f(x_n))} f(x_n).$$

By Taylor series,

$$\delta f(x_n - a_n f(x_n), x_n + a_n f(x_n)) = f'(x_n) + O(f(x_n)^2)$$

$$\gamma(x_n, a_n; f) = f''(x_n) + O(f(x_n)^2)$$

$$g(x_n) = f'(x_n) - \frac{f''(x_n)}{2f'(x_n)} f(x_n) + O(f(x_n)^2).$$

Thus,

$$x_{n+1} = x_n + \frac{f(x_n)}{f'(x_n) - \dfrac{f''(x_n)f(x_n)}{2f'(x_n)}} + O(f(x_n)^3))$$

$$= x_{n+1}^{Halley} + O(f(x_n)^3))$$

and the method is of order three [**1**]. □

4.1 - *Comparison of the steps*

In this section, we compare the step of Halley's scheme [**2**]

$$h_n = \frac{f(x_n)}{f'(x_n) - \dfrac{f''(x_n)}{2f'(x_n)} f(x_n)}$$

and the step of the new quasi-Halley's scheme

$$q_n = \frac{f(x_n)}{\delta f(x_n - a_n f(x_n), x_n + a_n f(x_n)) - \dfrac{\gamma(x_n, a_n; f)}{2\delta f(x_n - a_n f(x_n), x_n + a_n f(x_n))} f(x_n)}.$$

We assume that f have continuous third derivative in the interval J_0 and that $|f'(x_n)|^2 \leq 2f(x_n)M$ where $M = \sup_{x \in J_0} |f''(x)|$.

Using the mean value theorem and algebraic computations we arrive to

$$|h_n - q_n| \leq \frac{2(f'(x_n))^2|a_n|\,|u_n|(|u_n|N + 4M + 2|u_n|\,|a_n|M^2)}{M\left(\left|4\,\frac{u_nM}{(f'(x_n))^2} + 2Ma_n^2u_n - \frac{1}{f'(x_n)}| - 4|a_n\right|\right)}$$

where $N = \sup_{x\in J_0}|f'''(x)|$ and $u_n = \frac{f(x_n)}{f'(x_n)}$.

5 - A numerical experiment and conclusions

In order to show the performance of the modified Steffensen's method, we have compared it with classical Steffensen's type methods. We consider $tol_u = 10^{-8} >> tol_c$. We would like to show the importance to consider the parameters a_n.

We consider the following modified (not-differentiable) function (in particular we can not apply the classical Newton's method),

$$\hat{f}(x) = \begin{cases} x^4 + x & \text{if} \quad x < 0 \\ -(x^3 + x) & \text{if} \quad x \geq 0 \end{cases} \tag{11}$$

where k is a real constant.

For $x_0 = 0.1$ and $k = 1$, all the iterative method are Q-quadratically convergent, see table 1. Nevertheless, for $a_n = \varepsilon$ fix and small the method has problems with the last iterations. If we consider a stop criterium in order to avoid this problems then we won't get full accuracy. Our scheme converges without stability problems.

Finally, in table 2 we take different initials guess. In this table, we do not write the results for Steffensen's because in all the cases, the method does not converge after 10^6 iterations. On the other hand, if ε is not small enough the convergence is slow, but if it is too small stability problems appear, as we said before. Our iterative method gives goods results in all the cases.

TABLE 1. – *Error, equation* (11), $x_0 = 0.1$

iter.	Steff.	$\varepsilon = 10^{-4}$	$\varepsilon = 10^{-8}$	$tol_u = 10^{-8}$
1	$1.38e-03$	$2.99e-04$	$2.99e-04$	$2.99e-04$
2	$5.09e-11$	$1.72e-13$	$5.21e-09$	$2.26e-14$
3	$0.00e+00$	NaN	NaN	$0.00e+00$

TABLE 2. – *Final iteration and error, equation* (11)

	$\varepsilon = 10^{-4}$		$\varepsilon = 10^{-8}$		$tol_u = 10^{-8}$	
x_0	iter.	error	iter.	error	iter.	error
4	13	$4.20e-13$	9	$6.14e-09$	9	$0.00e+00$
8	457095	$1.27e-13$	11	$2.42e-09$	12	$0.00e+00$
16	$> 10^6$	—	20	$3.46e-09$	14	$0.00e+00$
32	$> 10^6$	—	$> 10^6$	—	16	$0.00e+00$

Concluding:

We have studied a Steffensen's type method with modified functions. We have made an analysis of the convergence using generalized convexity. The new iterative method seems to work very well in our numerical results, since we have obtained full order of accuracy without using any derivative.

References

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Abstract

A generalization of the Steffensen's method with modified functions is studied. Our goal is to obtain similar properties as Newton's method, but without evaluating any derivative. A quasi-Halley's method with only function evaluations is also presented. Convergence analysis and numerical results are analyzed

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