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Maged G. Bin-Saad and Salem S. Al-Qasemi Barahamh (*)

Derivation of generating relations for certain functions of three variables by fractional calculus method (**)

1 - Introduction and statement of main results

In [**13**], Chapter 5, Srivastava and Manocha presented a systematic introduction to the application of the fractional derivative method of obtaining generating relations involving Gaussian hypergeometric function ${}_2F_1$, the generalized hypergeometric function ${}_pF_q$ and Appell's functions F_1, F_2, F_3, and F_4 [**13**], Chapter 1. Also recently Banerji et al. [**2**] have introduced various generating relations for Appell's functions F_1, F_2, and F_3, obtained by using Nishimoto fractional calculus (cf. [**5**], [**6**]).

Motivated by the a aforementioned work of Srivastava and Manocha and the results of Banerji et al., we aim here at applying the concept of Nishimoto fractional calculus to obtain generating relations involving the triple hypergeometric function $F_G, F_M, F_N, F_P, F_K, F_S$ and F_T of Lauricella and G_A and G_B of Pandey (see e.g. [**3**], [**7**], [**9**], see also [**12**], p. 42-45). Making use of Nishimoto fractional calculus formulas (cf. [**5**], [**6**])

$$(1.1)\qquad (z^{\beta})_{\alpha(z)} = e^{-i\pi\alpha}\frac{\Gamma(\alpha-\beta)}{\Gamma(-\beta)}z^{\beta-\alpha}, \qquad \left|\frac{\Gamma(\alpha-\beta)}{\Gamma(-\beta)}\right| < \infty,$$

$$(1.2)\qquad \Big((z-1)^{\beta}\Big)_{\alpha(z)} = e^{-i\pi\alpha}\frac{\Gamma(\alpha-\beta)}{\Gamma(-\beta)}(z-1)^{\beta-\alpha}, \qquad \left|\frac{\Gamma(\alpha-\beta)}{\Gamma(-\beta)}\right| < \infty, \;\; z \neq 1.$$

(*) Department of Mathematics, Aden University, Khormakssar, P. O. Box 6014, Aden, Yemen Republic; e-mail: mgbinsaad@yahoo.com

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Let $u = u(z)$ and $\upsilon = \upsilon(z)$ be analytic and one valued functions, then

$$(u \cdot \upsilon)_a = \sum_{n=0}^{\infty} \frac{\Gamma(a+1)}{\Gamma(a-n+1)\Gamma(n+1)}\ u_{a-n}\upsilon_n, \tag{1.3}$$

and the technique illustrated fairly fully in [**13**], Chapter 5, Section 5.1, we can easily derive the following fractional derivative formulas, those are required in our investigation:

$$\left(x^{\beta}\left(1-x-\frac{z}{x}\right)^{-\gamma}\left(1-\frac{w}{x}\right)^{-\delta}\right)_{a(x)} = e^{-i\pi a}x^{\beta-a}\frac{\Gamma(a-\beta)}{\Gamma(-\beta)} \times G_A[a-\beta,\gamma,\delta;-\beta;x,z/x,w/x], \tag{1.4}$$

$$\left(x^{\beta}(x-1)^{-(1+\beta)}\left(1-\frac{z(x-1)}{x}\right)^{-\delta}\left(1-\frac{w(x-1)}{x}\right)^{-\gamma}\right)_{a(x)} = e^{-i\pi a}x^{\beta-a}(x-1)^{-(1+\beta)}\frac{\Gamma(a-\beta)}{\Gamma(-\beta)} \times G_B\left[a-\beta,-a,\delta,\gamma;-\beta;\frac{x}{x-1},\frac{z}{x}\,(x-1),\frac{w}{x}\,(x-1)\right], \tag{1.5}$$

$$\left(x^{\beta}(x-1)^{\gamma}\left(1-\frac{z}{x-1}\right)^{-\delta}\left(1-\frac{w}{x-1}\right)^{-\tau}\right)_{a(x)} = e^{-i\pi a}x^{\beta-a}(x-1)^{\gamma}\frac{\Gamma(a-\beta)}{\Gamma(-\beta)} \times F_G\left[-\gamma,-\gamma,-\gamma,-a,\delta,\tau;1-a+\beta,-\gamma,-\gamma;\frac{x}{x-1},\frac{z}{x-1},\frac{w}{x-1}\right], \tag{1.6}$$

$$\left(x^{\beta}(x-1)^{\gamma}\left(1-\frac{z}{x-1}\right)^{-\delta}(1-wx)^{-\tau}\right)_{a(x)} = e^{-i\pi a}x^{\beta-a}(x-1)^{\gamma}\frac{\Gamma(a-\beta)}{\Gamma(-\beta)} \times F_N\left[\delta,\tau,-a,-\gamma,1+\beta,-\gamma;-\gamma,1-a+\beta,1-a+\beta;\frac{z}{x-1},\,wx,\frac{x}{x-1}\right], \tag{1.7}$$

$$\left(x^{\beta}(x-1)^{\gamma}\left(1-zx\frac{w}{x-1}\right)^{-\delta}\right)_{a(x)} = e^{-i\pi a}x^{\beta-a}(x-1)^{\gamma}\frac{\Gamma(a-\beta)}{\Gamma(-\beta)} \times F_P\left[-\gamma,1+\beta,-\gamma,\delta,\delta,-a;-\gamma,1-a+\beta,1-a+\beta;\frac{w}{x-1},\,zx,\frac{x}{x-1}\right], \tag{1.8}$$

$$(1.9)\quad \left(x^{\beta}(x-1)^{\gamma}(1-zx)^{-\delta}(1-wx)^{-\tau}\right)_{a(x)} = e^{-i\pi a}x^{\beta-a}(x-1)^{\gamma}\frac{\Gamma(a-\beta)}{\Gamma(-\beta)}$$
$$\times F_S\left[-a,1+\beta,1+\beta,-\gamma,\delta,\tau;1-a+\beta,1-a+\beta,1-a+\beta;\frac{x}{x-1},\,zx,\,wx\right],$$

$$(1.10)\quad \left(x^{\beta}(x-1)^{\gamma}(1-zx)^{-\delta}\left(1-\frac{wx}{x-1}\right)^{\gamma}\right)_{a(x)} = e^{-i\pi a}x^{\beta-a}(x-1)^{\gamma}\frac{\Gamma(a-\beta)}{\Gamma(-\beta)}$$
$$\times F_T\left[\delta,-\gamma,-\gamma,1+\beta,-a,1+\beta;1-a+\beta,1-a+\beta,1-a+\beta;zx,\frac{x}{x-1},\frac{wx}{x-1}\right].$$

By proceeding in a fashion similar to that in [**13**], Chapter 5 and using (1.4) to (1.10), we aim in this work at establishing the following linear and bilinear generating relations.

$$(1.11)\quad \sum_{n=0}^{\infty}\frac{(\lambda)_n}{n!}G_A\left[a+\beta,\gamma,\lambda+n;-\beta;x,\frac{w}{x},\frac{z}{x}\right]t^n$$
$$=(1-t)^{-\lambda}G_A\left[a+\beta,\gamma,\lambda;-\beta;x,\frac{w}{x(1-t)},\frac{z}{x}\right],$$

$$(1.12)\quad \sum_{n=0}^{\infty}\frac{(\lambda)_n}{n!}G_B\left[a+\beta,a,\delta,\lambda+n;\beta;\frac{x}{x-1},\frac{z}{x}(x-1),\frac{w}{x}(x-1)\right]t^n$$
$$=(1-t)^{-\lambda}G_B\left[a+\beta,a,\delta,\lambda;\beta;\frac{x}{x-1},\frac{z(x-1)}{x},\frac{w(x-1)}{x(1-t)}\right],$$

$$(1.13)\quad \sum_{n=0}^{\infty}\frac{(\lambda)_n}{n!}F_G\left[\gamma,\gamma,\gamma,a,\delta,\lambda+n;1+a+\beta,\gamma,\gamma;\frac{x}{x-1},\frac{z}{x-1},\frac{w}{x-1}\right]t^n$$
$$=(1-t)^{-\lambda}F_G\left[\gamma,\gamma,\gamma,a,\delta,\lambda;1+a+\beta,\gamma,\gamma;\frac{x}{x-1},\frac{z}{x-1},\frac{w}{(x-1)(1-t)}\right],$$

$$(1.14)\quad \sum_{n=0}^{\infty}\frac{(\lambda)_n}{n!}F_N\left[\delta,\lambda+n,a,\gamma,1+\beta,\gamma;\gamma,1+a+\beta,1+a+\beta;\frac{z}{x-1},wx,\frac{x}{x-1}\right]t^n$$
$$=(1-t)^{-\lambda}F_N\left[\delta,\lambda,a,\gamma,1+\beta,\gamma;\gamma,1+a+\beta,1+a+\beta;\frac{z}{x-1},\frac{wx}{1-t},\frac{x}{x-1}\right],$$

$$\sum_{n=0}^{\infty}\frac{(\lambda)_n}{n!}F_P\left[\gamma,1+\beta,\gamma,\lambda+n,\lambda+n,a;\gamma,1+a+\beta,1+a+\beta;\frac{w}{x-1},zx,\frac{x}{x-1}\right]t^n \tag{1.15}$$
$$=(1-t)^{-\lambda}F_P\left[\gamma,1+\beta,\gamma,\lambda,\lambda,a;\gamma,1+a+\beta,1+a+\beta;\frac{w}{(x-1)(1-t)},\frac{zx}{1-t},\frac{x}{x-1}\right],$$

$$\sum_{n=0}^{\infty}\frac{(\lambda)_n}{n!}F_S\left[a,1+\beta,1+\beta,\gamma,\delta,\lambda+n;1+a+\beta,1+a+\beta,1+a+\beta;\frac{x}{x-1},zx,wx\right]t^n \tag{1.16}$$
$$=(1-t)^{-\lambda}F_S\left[a,1+\beta,1+\beta,\gamma,\delta,\lambda;1+a+\beta,1+a+\beta,1+a+\beta;\frac{x}{x-1},zx,\frac{wx}{1-t}\right],$$

$$\sum_{n=0}^{\infty}\frac{(\lambda)_n}{n!}F_T\left[\lambda+n,\gamma,\gamma,1+\beta,a,1+\beta;1+a+\beta,1+a+\beta;zx,\frac{x}{x-1},\frac{wx}{x-1}\right]t^n \tag{1.17}$$
$$=(1-t)^{-\lambda}F_T\left[\lambda,\gamma,\gamma,1+\beta,a,1+\beta;1+a+\beta,1+a+\beta,1+a+\beta;\frac{zx}{1-t},\frac{x}{x-1},\frac{wx}{x-1}\right],$$

$$\sum_{n=0}^{\infty}\frac{(\lambda)_n}{n!}G_A[\mu+\beta,a,\lambda+n;\beta;x,w,/x,z/x]\times G_A[\upsilon+\gamma,\delta,\lambda+n;\gamma;y,v/y,u/y]t^n \tag{1.18}$$
$$=(1-t)^{-\lambda}\sum_{n=0}^{\infty}\frac{(\lambda)_n(\mu+\beta)_n(\upsilon+\gamma)_n}{n!(\beta)_n(\gamma)_n}\left(\frac{wvt}{xy(1-t)^2}\right)^n$$
$$\times G_A\left[\mu+\beta+n,\gamma,\lambda+n;n+\beta;x,\frac{w}{x(1-t)}z/x\right]G_A\left[\upsilon+\gamma+n,\delta,\lambda+n;n+\gamma;y,\frac{v}{y(1-t)},u/y\right],$$

$$\sum_{n=0}^{\infty}\frac{(\lambda)_n}{n!}G_B\left[\mu+\beta,\mu,\delta,\lambda+n;\beta;\frac{x}{x-1},\frac{z}{x}(x-1),\frac{w}{x}(x-1)\right] \tag{1.19}$$
$$G_B\left[\upsilon+\gamma,\upsilon,a,\lambda+n;\gamma;\frac{y}{y-1},\frac{u}{y}(y-1),\frac{v}{y}(y-1)\right]t^n$$
$$=(1-t)^{-\lambda}\sum_{n=0}^{\infty}\frac{(\lambda)_n(\mu+\beta)_n(\upsilon+\gamma)_n}{n!(\beta)_n(\gamma)_n}\left(\frac{wv(x-1)(y-1)t}{xy(1-t)^2}\right)^n$$
$$G_B\left[\mu+\beta+n,\mu,\delta,\lambda+n;n+\beta;\frac{x}{x-1},\frac{z}{x}(x-1),\frac{w(x-1)}{x(1-t)}\right]$$
$$G_B\left[\upsilon+\gamma+n,\upsilon,a,\lambda+n;n+\gamma;\frac{y}{y-1},\frac{u}{y}(y-1),\frac{v(y-1)}{y(1-t)}\right],$$

$$(1.20)\quad \sum_{n=0}^{\infty}\frac{(\lambda)_n}{n!}F_G\left[a,a,a,\mu,\tau,\lambda+n;1+\mu+\beta,a,a,a;\frac{x}{x-1},\frac{z}{x-1},\frac{w}{x-1}\right]$$
$$F_G\left[\delta,\delta,\delta,\upsilon,\eta,\lambda+n;1+\upsilon+\gamma,\delta,\delta;\frac{y}{y-1},\frac{u}{y-1},\frac{v}{y-1}\right]t^n$$
$$=(1-t)^{-\lambda}\sum_{n=0}^{\infty}\frac{(\lambda)_n}{n!}\left(\frac{wvt}{(x-1)(y-1)(1-t)^2}\right)^n$$
$$F_G\left[n+a,n+a,n+a,\mu,\tau,\lambda+n;1+\mu+\beta,n+a,n+a;\frac{x}{x-1},\frac{z}{x-1},\frac{w}{(1-t)(x-1)}\right]$$
$$F_G\left[n+\delta,n+\delta,n+\delta,\upsilon,\eta,\lambda+n;1+\upsilon+\gamma,n+\delta,n+\delta;\frac{y}{y-1},\frac{u}{y-1},\frac{v}{(1-t)(y-1)}\right],$$

$$(1.21)\quad \sum_{n=0}^{\infty}\frac{(\lambda)_n}{n!}F_N\left[\tau,\lambda+n,\mu,a,1+\beta,a;a,1+\mu+\beta,1+\mu+\beta;\frac{z}{x-1},wx,\frac{x}{x-1}\right]$$
$$\times F_N\left[\eta,\lambda+n,\upsilon,\delta,1+\gamma,\delta;\delta,1+\upsilon+\gamma,1+\upsilon+\gamma;\frac{u}{y-1},vy,\frac{y}{y-1}\right]t^n$$
$$=(1-t)^{-\lambda}\sum_{n=0}^{\infty}\frac{(\lambda)_n(1+\beta)_n(1+\gamma)_n}{n!(1+\mu+\beta)_n(1+\upsilon+\gamma)_n}\left(\frac{wvxt}{(1-t)^2}\right)^n$$
$$F_N\left[\tau,\lambda+n,\mu,a,1+\beta+n,a;a,1+\mu+\beta+n,1+\mu+\beta+n;\frac{z}{x-1},\frac{wx}{1-t},\frac{x}{x-1)}\right]$$
$$\times F_N\left[\eta,\lambda+n,\upsilon,\gamma,1+\gamma+n,\gamma;\gamma,1+\upsilon+\gamma+n,1+\upsilon+\gamma+n;\frac{u}{y-1},\frac{vy}{1-t},\frac{y}{y-1}\right],$$

$$(1.22)\quad \sum_{n=0}^{\infty}\frac{(\lambda)_n}{n!}F_S\left[\mu,1+\beta,1+\beta,a,\tau,\lambda+n;1+\mu+\beta,1+\mu+\beta,1+\mu+\beta;\frac{x}{x-1},zx,wx\right]$$
$$F_S\left[\upsilon,1+\gamma,1+\gamma,\delta,\eta,\lambda+n;1+\upsilon+\gamma,1+\upsilon+\gamma,1+\upsilon+\gamma;\frac{y}{y-1},uy,vy\right]t^n$$
$$=(1-t)^{-\lambda}\sum_{n=0}^{\infty}\frac{(\lambda)_n(1+\beta)_n(1+\gamma)_n}{n!(1+\mu+\beta)_n(1+\upsilon+\gamma)_n}\left(\frac{wvxyt}{(1-t)^2}\right)^n$$
$$F_S\left[\mu,1+\beta+n,1+\beta+n,a,\tau,\lambda+n;1+\mu+\beta+n,1+\mu+\beta+n,1+\mu+\beta+n;\frac{x}{x-1},zx,\frac{wx}{1-t}\right]$$
$$F_S\left[\upsilon,1+\gamma+n,1+\gamma+n,\delta,\eta,\lambda+n;1+\upsilon+\gamma+n,1+\upsilon+\gamma+n,1+\upsilon+\gamma+n;\frac{y}{y-1},uy,\frac{vy}{1-t}\right],$$

$$(1.23)\quad \sum_{n=0}^{\infty}\frac{(\lambda)_n}{n!}F_T\left[\lambda+n,a,a,1+\beta,\mu,1+\beta;1+\mu+\beta,1+\mu+\beta,1+\mu+\beta;\frac{zx}{x-1},\frac{x}{x-1},wx\right]$$

$$F_T\left[\lambda+n,\delta,\delta,1+\gamma,\upsilon,1+\gamma;1+\upsilon+\beta,1+\upsilon+\gamma,1+\upsilon+\gamma;\frac{uy}{y-1},\frac{y}{y-1},vy\right]t^n$$

$$=(1-t)^{-\lambda}\sum_{n=0}^{\infty}\frac{(\lambda)_n(1+\beta)_n(1+\gamma)_n}{n!(1+\mu+\beta)_n(1+\upsilon+\gamma)_n}\left(\frac{wvxyt}{(1-t)^2}\right)^n$$

$$F_T\left[\lambda+n,a,a,1+\beta+n,\mu,1+\beta+n;1+\mu+\beta+n,1+\mu+\beta+n,1+\mu+\beta+n;\frac{zx}{x-1},\frac{x}{x-1},\frac{wx}{1-t}\right]$$

$$F_T\left[\lambda+n,\delta,\delta,1+\gamma+n,\upsilon,1+\gamma+n;1+\upsilon+\gamma+n,1+\upsilon+\gamma+n,1+\upsilon+\gamma+n;\frac{uy}{y-1},\frac{y}{y-1},\frac{vy}{1-t}\right].$$

2 - Derivation of main results

First , we proceed to the proof of the linear generating relations (1.11) to (1.17). Consider the elementary identity (cf. [**13**], p. 291])

$$(2.1)\qquad [(1-x)-t]^{-\lambda}=(1-t)^{-\lambda}\left[1-\frac{x}{1-t}\right]^{-\lambda},$$

which can be expressed in the form

$$(2.2)\qquad \sum_{n=0}^{\infty}\frac{(\lambda)_n}{n!}(1-x)^{-(\lambda+n)}t^n=(1-t)^{-\lambda}\left[1-\frac{x}{1-t}\right]^{-\lambda},\qquad |t|<|1-x|.$$

Next, replace x by w/x in (2.2), multiply both sides by $x^{\beta}(1-x-z/x)^{-\gamma}$, take the fractional differentiation of both sides of order a with respect to x and employ the fractional formula (1.4), we are led finally to the generating function (1.11). The derivation of the assertions (1.12) to (1.17) runs parallel to that of (1.11) and we skip the details.

In order to proof the bilinear generating relations (1.18) to (1.23), we need the elementary identity (cf. [**13**], p. 297)

$$(2.3)\qquad [(1-x)(1-y)-t]^{-\lambda}=(1-t)^{-\lambda}\left[\left(1-\frac{x}{1-t}\right)\left(1-\frac{y}{1-t}\right)-\frac{xyt}{(1-t)^2}\right]^{-\lambda},$$

where $|t/(1-x)(1-y)|<1$ and $|xyt/(1-x-t)(1-y-t)|<1$, which can be rewritten it in the form

$$\sum_{n=0}^{\infty}\frac{(\lambda)_n}{n!}(1-x)^{-(\lambda+n)}(1-y)^{-(\lambda+n)}t^n \tag{2.4}$$
$$=(1-t)^{-\lambda}\sum_{n=0}^{\infty}\frac{(\lambda)_n}{n!}\left(1-\frac{x}{1-t}\right)^{-(\lambda+n)}\left(1-\frac{y}{1-t}\right)^{-(\lambda+n)}\left(\frac{xyt}{(1-t)^2}\right)^n.$$

If in (2.4), we replace x and y by w/x and v/y respectively, multiply both sides by

$$x^{\beta}y^{\gamma}(1-x-z/x)^{-a}(1-y-u/y)^{-\delta},$$

apply the fractional operator of order μ with respect to x and order υ with respect to y and make use of (1.4), we get formula (1.18). In a similar manner, one can proof the relations (1.19) to (1.23).

3 - Special cases and observations

It is easy to observe that the main results (1.11) to (1.23) give a number of generating functions of two variables involving, for example, Horn's functions G_2 and H_2 and Appell's functions F_1, F_2, and F_3 ([**12**], p. 22-24). In this section, we will mention only some special cases. First, if in (1.12), we let $z \to 0$ and use the identity

$$(\lambda)_n = (-1)^n/(1-\lambda)_n, \tag{3.1}$$

we shall obtain the formula

$$\sum_{n=0}^{\infty}\frac{(\lambda)_n}{n!}G_2\left[\lambda+n, a, 1-\beta, a+\beta; \frac{w}{x}(1-x), \frac{x}{1-x}\right]t^n \tag{3.2}$$
$$=(1-t)^{-\lambda}G_2\left[\lambda, a, 1-\beta, a+\beta; \frac{w(1-x)}{x(1-t)}, \frac{x}{1-x}\right].$$

Next, on replacing w and v by $wx/(x-1)$ and $vy/(y-1)$ respectively in formula (1.19), taking $x=y=z=u=0$ and making the following parameter changes $\mu \to \mu-\beta$ and $\upsilon \to \upsilon-\gamma$, formula (1.19) would yields the result

$$\sum_{n=0}^{\infty}\frac{(\lambda)_n}{n!}\,{}_2F_1[\lambda+n,\mu;\beta;a+\beta;w]\times{}_2F_1[\lambda+n,\upsilon;\gamma;v]t^n \tag{3.3}$$
$$=(1-t)^{-\lambda}\sum_{n=0}^{\infty}\frac{(\lambda)_n(\mu)_n(\upsilon)_n}{n!(\beta)_n(\gamma)_n}\left(\frac{wvt}{(1-t)^2}\right)^n$$
$$\times{}_2F_1[\lambda+n,\mu+n;\beta+n;w/(1-t)]\times{}_2F_1[\lambda+n,\upsilon+n;\gamma+n;v/(1-t)],$$

which is a known result of Meixner (see [**13**], p. 298 (9)).

Alternatively, in terms of Appell functions F_1, F_2 and F_3, the formulas (1.13), (1.15), (1.20) and (1.21) would give us the following four special cases:

$$\sum_{n=0}^{\infty}\frac{(\lambda)_n}{n!}F_1[\gamma,\delta,\lambda+n;a;z,w]t^n=(1-t)^{-\lambda}F_1[\gamma,\delta,\lambda;a;z,w/(1-t)], \tag{3.4}$$

$$\sum_{n=0}^{\infty}\frac{(\lambda)_n}{n!}F_2[\gamma,a,\lambda+n;\beta,\gamma;x,w]t^n=(1-t)^{-\lambda}F_2[\gamma,a,\lambda;\beta,\gamma;x,w/(1-t)], \tag{3.5}$$

note that equations (3.3) and (3.4) are known results of Anandani ([**1**], pp, 51-52; also [**11**], p. 74);

$$\begin{aligned}\sum_{n=0}^{\infty}\binom{\lambda}{n}&F_2[a,n-\lambda,\mu;a,\beta;w,x]\\ \times F_2[\delta,n-\lambda,\upsilon;\delta,\gamma;v,y]t^n&=(1+t)^{\lambda}\sum_{n=0}^{\infty}\binom{\lambda}{n}\left(\frac{wvy}{(1+t)^2}\right)^n\\ \times F_2\left[a+n,n-\lambda,\mu;a+n,\beta;\frac{w}{1+t},x\right]&\times F_2\left[\delta+n,n-\lambda,\upsilon;\delta+n,\gamma;\frac{v}{1+t},y\right],\end{aligned} \tag{3.6}$$

which except for some notional differences, a special case of a known result of Srivastava (see [**11**], p.82 (3.12); also [**13**], p. 309, 12 (iii)); and

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{(\lambda)_n}{n!}&F_3[\lambda+n,\mu,\beta,a;\sigma;w,x]\times F_3[\lambda+n,a,\upsilon,\gamma,\delta;\rho;v,y]t^n\\ &=(1-t)^{-\lambda}\sum_{n=0}^{\infty}\frac{(\lambda)_n(\beta)_n(\gamma)_n}{n!(\sigma)_n(\rho)_n}\left(\frac{wvy}{(1-t)^2}\right)^n\\ F_3\left[\lambda+n,\mu,\beta+n,a;\sigma+n\frac{w}{1-t},x\right]&\times F_3\left[\lambda+n,\upsilon,\gamma+n,\delta;\rho+n;\frac{v}{1-t},y\right],\end{aligned} \tag{3.7}$$

which is another known result of Srivastava (see [**11**], p. 83 (3.15)), respectively.

Some of our results in section 1, aided by appropriate fractional derivative operators, are useful instrumental in deriving bilateral generating functions. For instance, if (1.13), we let $z\to 0$, replace t by $t(1-y)$, multiply both the sides by y^{τ}, take the fractional derivative of both sides of the resulting expression of order μ with respect to y and adjust the parameters and variables, we get the following bilateral generating function

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{(\lambda)_n}{n!}&F_2[\gamma,a,\lambda+n;\beta,\gamma;x,w]\times{}_2F_1[-n,\tau;\mu;y]t^n\\ &=(1-t)^{-\lambda}F_K\left[\tau,\gamma,\gamma,\lambda,\lambda,a;\mu,\beta,\gamma;\frac{ty}{t-1},x,\frac{w}{1-t}\right].\end{aligned} \tag{3.8}$$

Similarly, starting from (1.17), replacing z and w by z/x and $w(x-1)/x$ respectively, letting $x \to 0$ and making use of the foregoing fractional method, we find that

$$\sum_{n=0}^{\infty} \frac{(\lambda)_n}{n!} F_1[\beta, \lambda+n, \gamma; a; z, w] \times_2 F_1[-n, \tau; \mu; y]\, t^n = (1-t)^{-\lambda} F_M\left[\tau, \beta, \beta, \lambda, \gamma, \lambda; \mu, a, a; \frac{ty}{t-1}, w, \frac{z}{1-t}\right]. \tag{3.9}$$

It may be of interest to remark that the generating relations (3.8) and (3.9) are also a known results of Srivastava (see [**11**], Equations (5.9) and (5.10)).

As a matter of fact, both of the results of Srivastava mentioned above were derived from the corresponding results of Manocha and Sharma (see [4], pp.25-26; also [**11**], Equations (5.3) and (5.4)), by using series rearrangement technique. We conclude this paper by recording a rather straightforward extension of a known result [8], p. 155 (4.13).

Indeed, upon replacing y by $y/(1-u)$ in (3.9), multiplying both sides by $u^{\eta-1}(1-u)^{-\tau}$ and applying the fractional derivative operator of order $\eta-\sigma$ with respect to u, the formula (3.9) readily yields a generating relation involving the quadruple hypergeometric function $F_{31}^{(4)}$ of Sharma and Parihar [**10**], p. 125 (2.31):

$$\sum_{n=0}^{\infty} \frac{(\lambda)_n}{n!} F_1[\beta, \lambda+n, \gamma; a; z, w] \times F_2[\tau-n, \eta; \mu, \sigma; y, u] t^n = (1-t)^{-\lambda} F_{31}^{(4)}\left[\beta, \beta, \tau, \tau, \lambda, \gamma, \lambda, \eta; a, a, \mu, \sigma; \frac{z}{1-t}, w, \frac{ty}{t-1}, u\right], \tag{3.10}$$

which for $w = z = 0,\ \lambda = \lambda + 1$ and $\tau = \lambda$ happens to be a known result of Pathan and Bin-Saad [8], p. 155 (4.13).

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Abstract

In this paper, we apply the concept of Nishimoto's fractional calculus to obtain some linear, bilinear and bilateral generating relations involving hypergeometric functions of three variables. A number of (known and new) results are shown to follow as special cases of our formulas.

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