Riv. Mat. Univ. Parma (7) **4*** (2005), 127-139

G. Bouchitté, T. Champion and C. Jimenez (*)

Completion of the space of measures in the Kantorovich norm (**)

Contents

1 - Introduction and notations

A lot of work has been devoted to duality principles in the Monge-Kantorovich theory of mass transport. In particular L. Hanin wrote a series of papers on the characterization of the dual of Lipschitz classes (see [**6**], [**7**]). Very recently a lot of attention has been focused on infinite sum of dipoles as elements of the dual of $Lip(X)^*$ (X beeing a complete metric space). Such dipoles appear as natural objects in the description of singularities of maps occuring in the theory of liquid crystals (see [**5**], [**10**]). As emphasized by H. Brezis, these infinite sums are not measures of finite total variation but can be viewed as particular distributions of order one.

(*) ANLA, UFR Sciences, Université de Toulon et du Var, BP 132, 83957 La Garde Cedex, France; e-mail: bouchitte@univ-tl.fr
(**) Received 21^{st} April 2005. AMS classification 49J45, 46F05, 46E15, 46N10.

In order to simplify the presentation, we will work on a compact subset $K = \overline{\Omega}$ where Ω is a bounded connected open subset of $\mathbb{R}^d$ with $C^{0,1}$ boundary and we will denote by

- $|\cdot|$ the Euclidean norm in $\mathbb{R}^d$, $d(x,y)$ the associated geodesic distance on Ω extended to K, $\mathbb{1}_A$ the characteristic function of a subset A in $\mathbb{R}^d$.
- $C^0(K)$ the space of continuous functions on K endowed with the sup norm $\|\cdot\|_\infty$,
- $Lip(K)$ the Banach space of Lipschitz functions on K endowed with the norm $\mathrm{Lip}\,(\varphi) := \sup\left\{\dfrac{\varphi(y)-\varphi(x)}{d(x,y)}, (x,y) \in K^2, x \neq y\right\}$,
- $Lip_0(K) := Lip(K)/\mathbb{R}$ its quotient by the constants,
- $Lip_1(K) := \{u \in Lip(K) \ : \ |\nabla u| \leq 1 \text{ a.e. on } \Omega\}$,
- $\mathcal{M}(K;\mathbb{R}^d)$ (resp $\mathcal{M}_+(K)$) the space of $\mathbb{R}^d$-valued (resp. positive) Borel measures on $\mathbb{R}^d$ compactly supported in K. Every element $\lambda \in \mathcal{M}(K;\mathbb{R}^d)$ can written as $\lambda = \sigma\mu$ where $\mu \in \mathcal{M}^+(K)$, $\sigma \in L^1_\mu(K;\mathbb{R}^d)$. In particular we may choose $|\sigma| = 1$ μ-a.e., so that $\mu = |\lambda|$ the total variation of λ and $\sigma = \dfrac{d\lambda}{d|\lambda|}$ (polar decomposition).
- $\mathcal{M}_0(K)$ the space of signed Radon measures μ supported in K such that $\displaystyle\int \mu = 0$.

  it is endowed with the Kantorovich norm defined by:

  $(K,d))$: $\|\mu\|_1 = \sup\left\{\displaystyle\int_K \varphi\, d\mu \ : \ \varphi \in Lip_1(K)\right\}$.

It is well known that on subsets of $\mathcal{M}_0(K)$ which are uniformly bounded in total variation, the topology induced by $\|\cdot\|_1$ is equivalent to the weak star topology on measures. It can be easily checked (see [**6**]) that the dual space $\mathcal{M}_0(K)^\star$ is isomorphic to $Lip_0(K)$ through the map: $L \in \mathcal{M}_0(K)^\star \mapsto \varphi$ where $\varphi(x) = L(\delta_x - \delta_a), a \in K$. Therefore $\mathcal{M}_0(K)$ can be identified with a subspace of its bidual, thus with a subspace of $Lip_0(K)^\star$.

Now we claim that the normed space $(\mathcal{M}_0(K), \|\cdot\|_1)$ is non complete: indeed let (a_n, b_n) a sequence in K^2 such that $\sum |b_n - a_n| < +\infty$ and $a_n \neq b_m, \forall (n,m)$. Clearly the finite sum of dipoles $\mu_n = \sum_1^n (\delta_{b_n} - \delta_{a_n})$ is a Cauchy sequence whose limit as an element of $Lip_0(K)^\star$ is not a measure of finite mass. Our aim is to characterize the completion of $\mathcal{M}_0(K)$ or equivalently the closure of $\mathcal{M}_0(K)$ as a subspace of $Lip_0(K)^\star$. As will be seen later, the main feature of an element f in this space is that the supremum in the dual Kantorovich problem

$$\sup\{\langle f, \varphi\rangle \ : \ \varphi \in Lip_1(K)\} \quad (= \|f\|_1)\,, \tag{1.1}$$

has at least one solution (called Monge-Kantorovich potential).

In all the paper we will denote by $\mathcal{M}_{0,1}(K)$ the completion of $\mathcal{M}_0(K)$ in $(Lip_0(K)^\star, \|\cdot\|_1)$. We emphasize that it is a strict subspace $Lip_0(K)^\star$ since otherwise $Lip_0(K)$ would be a reflexive Banach space. To our knowledge a characterization of $\mathcal{M}_{0,1}(K)$ is known only for $d = 1$. This appears in [**6**] where the case $d > 1$ is presented as an open problem.

The paper is organized as follows: in section 2, we give a characterization of $\mathcal{M}_{0,1}(K)$ (see Theorem 2.4); in section 3, we give an alternative representation by introducing a suitable class of tangential vector measures in $\mathcal{M}(K;\mathbb{R}^d)$.

2 - The Banach space $\mathcal{M}_{0,1}(K)$

First we show that $\mathcal{M}_{0,1}(K)$ can be identified to a subspace of $\mathcal{D}'_1(K)$ the distributions on $\mathbb{R}^d$ of order one supported in K. For every $f \in \mathcal{M}_{0,1}(K)$, the bracket $\langle f, \varphi\rangle$ is well defined for every $\varphi \in Lip(K)$. We denote by T_f the distribution obtained by setting $\langle T_f, \varphi\rangle = \langle f, \varphi\rangle$ for every $\varphi \in C^\infty(K)$.

Lemma 2.1. *The map: $f \in \mathcal{M}_{0,1}(K) \mapsto T_f \in \mathcal{D}'_1(K)$ is injective. Furthermore a distribution $T \in \mathcal{D}'_1(K)$ is of the form T_f for a suitable $f \in \mathcal{M}_{0,1}(K)$ if and only if, for every sequence $\{\varphi_n\}$ in $C^\infty(K)$, such that the following implication holds:*

$$\varphi_n \to c \quad \textit{uniformly} \quad (c \in \mathbb{R}), \quad \sup_K |\nabla\varphi_n| \le C \quad \Longrightarrow \quad \langle T, \varphi_n\rangle \to 0\,. \tag{2.1}$$

Proof. Let $f \in \mathcal{M}_{0,1}(K)$ and let us show that T_f satisfies (2.1). By definition, for every $\delta > 0$, there exists a measure $f_\delta \in \mathcal{M}_0(K)$ $\|f - f_\delta\|_1 < \delta$. Let φ_n be as in the left hand side of (2.1). Then:

$$|\langle T_f, \varphi_n\rangle| \le |\langle f_\delta, \varphi_n\rangle| + \delta\,\|\nabla\varphi_n\|_\infty \le |\langle f_\delta, \varphi_n\rangle| + C\,\delta.$$

Since $\langle f_\delta, c\rangle = 0$, the conclusion follows by letting $n \to \infty$, then $\delta \to 0$.

In fact the same argument works if we assume simply that $\{\varphi_n\}$ is a equi-Lipschitz family in $Lip(K)$ converging to a constant: we still obtain that $\langle f, \varphi_n\rangle \to 0$. Therefore if $T_f = 0$ and $\varphi \in Lip(K)$, by applying the previous property to $\{\varphi_n - \varphi\}$ where φ_n is a smooth approximation of φ, we infer that $\langle f, \varphi\rangle = 0$. The injectivity of the map $f \mapsto T_f$ follows.

Conversely we need to show that if $T \in \mathcal{D}_1(K)$ is such that (2.1) holds true, then $T = T_f$ for a suitable $f \in \mathcal{M}_{0,1}(K)$. First by (2.1), T can be extended in a unique way to $Lip(K)$ and the resulting linear form on $Lip(K)$ can be identified as an element $f \in (\mathrm{Lip}_0(K))^*$ such that $T = T_f$. To prove that this element f belongs to $\mathcal{M}_{0,1}(K)$ we use the following

Claim: There exists a sequence of linear operators: $S_\delta : C^0(K) \mapsto C^\infty(K)$ and suitable constants C, C_δ such that

i) $\mathrm{Lip}\,(S_\delta\varphi) \leq C_\delta \|\varphi\|_\infty$, for every $\varphi \in C^0(K)$,

ii) $\mathrm{Lip}\,(S_\delta\varphi) \leq C\,\mathrm{Lip}\,(\varphi)$ for every $\varphi \in Lip(K)$,

iii) $\|S_\delta\varphi - \varphi\|_\infty \leq \delta C\,\mathrm{Lip}\,(\varphi)$ for every $\varphi \in Lip(K)$.

Then we set $f_\delta := T \circ S_\delta$. By i) and ii), it is an element of $\mathcal{M}_0(K)$. Choose φ_δ in $Lip_1(K)$ so that $\langle f_\delta - f, \varphi_\delta\rangle > \|f_\delta - f\|_1 - \delta$. We may rewrite this inequality as

$$\|f_\delta - f\|_1 \leq \delta + \langle f, \Psi_\delta\rangle, \qquad \text{where} \quad \Psi_\delta := S_\delta\,\varphi_\delta - \varphi_\delta\,. \tag{2.2}$$

By ii) and iii), $\{\Psi_\delta\}$ is equi-Lipschitz and converges uniformly to 0 on K. It follows from (2.1) and (2.2) that $f_\delta \to f$ in $Lip_0(K)^\star$, and thus $f \in \mathcal{M}_{0,1}(K)$ (notice that by iii), S_δ is invariant over constant functions).

It remains to prove the claim. By a very nice result to be found in [11], Theorem 3 p. 174, there exists a linear continuous extension operator $\Xi : C^0(K) \mapsto C^0(\mathbb{R}^d)$ which also maps continuously $Lip(K)$ into $Lip(\mathbb{R}^d)$. Notice that a priori no regularity assumption is needed for the existence of such extension provided K is metrized by the Euclidian norm. In our case we are allowed to choose the geodesic distance which is equivalent since ∂K has been assumed to be Lipschitz. Note also that in case K is convex, the map $\Xi : \varphi \mapsto \varphi \circ p$ where p is the orthogonal projector on K fits to our purpose. Now we set $S_\delta\,\varphi$ to be the restriction to K of $\Xi(\varphi) \star \rho_\delta$ where ρ_δ is a usual convolution kernel with support in $B(0, \delta)$. It is easy to check that S_δ as a linear operator satisfies i), ii) and iii). ∎

Remark 2.2. The fact that $\mathcal{M}_{0,1}(K)$ is strictly embedded in $(Lip_0(K))^\star$ implies that there are elements $f \in (Lip_0(K))^\star$ such that $f \neq 0$ and $T_f = 0$ (this not surprising since C^∞ functions are not dense in $Lip(K)$). The required property (2.1) for T in order to be in $\mathcal{M}_{0,1}(K)$ is nothing else but the continuity with respect to the weak star topology of $W^{1,\infty}(\Omega)$.

The existence of a Monge potential associated with $f \in \mathcal{M}_{0,1}(K)$ is obtained in

Lemma 2.3. *Let $f \in \mathcal{M}_{0,1}(K)$. Then there exists a Lipschitz function $u \in Lip_1(K)$ maximizing the problem* (1.1).

Proof. Let $\{\varphi_n\}$ be a maximizing sequence in $Lip_1(K)$. As $\langle f, 1\rangle = 0$, it is no restrictive to assume that $\varphi_n(x_0) = 0$ at some point $x_0 \in \Omega$. By Ascoli's theorem, there exists $\varphi \in Lip_1(K)$ and a subsequence $\{\varphi_{n_k}\}$ such that $\varphi_{n_k} \to \varphi$ uniformly on K. By applying the property (2.1) established in Lemma 2.1 to $\{\varphi_{n_k} - \varphi\}$, we derive that

$$\langle f, \varphi\rangle \;=\; \lim_{k\to\infty} \langle f, \varphi_{n_k}\rangle \;=\; \sup\{(1.1)\}\,.$$ ∎

In the following, given an element $T \in \mathcal{D}'(\mathbb{R}^d;\mathbb{R}^d)$, we denote by $\operatorname{div} T$ the distribution defined by $\langle \operatorname{div} T, \varphi\rangle = -\sum_i \left\langle T_i, \dfrac{\partial \varphi}{\partial x_i}\right\rangle$. Obviously if T is compactly supported in K, so is $\operatorname{div} T$. Thus the divergence operator maps $\mathcal{D}'(K;\mathbb{R}^d)$ into a subspace of $\mathcal{D}'(K)$. We will denote by $\operatorname{div}\lambda$ (resp $\operatorname{div}\sigma$) the divergence of T if T is associated with a vector measure $\lambda \in \mathcal{M}(K;\mathbb{R}^d)$ (resp a vector density $\sigma \in L^1(\Omega;\mathbb{R}^d)$).

We recall (see for example the proof of Lemma 2.5) that for every $f \in \mathcal{M}_0(K)$, the problem

$$\inf\left\{\int |\lambda| \quad : \quad \lambda \in \mathcal{M}(K;\mathbb{R}^d)\ ,\ \operatorname{div}\lambda = \mu\right\}, \tag{2.3}$$

has at least a solution (optimal transport flux) and moreover we have

$$\inf(2.3) \;=\; \sup\{(1.1)\} \;= W_1(f_+, f_-)\ ,$$

where $W_1(f_+, f_-)(= \|f\|_1)$ denotes the Monge-Kantorovitch distance on (K,d) from the non-negative part to the non-positive part of f.

We are now in position to state the main result of this section.

Theorem 2.4. *The following equality holds between subsets of $\mathcal{D}'(K)$:*

$$\{T_f\ :\ f \in \mathcal{M}_{0,1}(K)\} \;=\; \left\{-\operatorname{div}\sigma\ :\ \sigma \in L^1(\Omega;\mathbb{R}^d)\right\}.$$

Futhermore, if V_0 denotes the closed subspace $V_0 := \{\sigma \in L^1(\Omega;\mathbb{R}^d)\ :\ \operatorname{div}\sigma = 0\}$, the linear map: $\sigma \in L^1(\Omega;\mathbb{R}^d)/V_0 \mapsto -\operatorname{div}\sigma \in \mathcal{M}_{0,1}(K)$ is an isometry, i.e.:

$$\|\sigma\|_{L^1(\Omega;\mathbb{R}^d)/V_0} \;=\; \|\operatorname{div}\sigma\|_1\ .$$

The proof relies on the following approximation result

Lemma 2.5. *Let $\mu \in \mathcal{M}_0(K)$ and $\varepsilon > 0$. Then there exists $\sigma \in L^1(\Omega;\mathbb{R}^d)$ such that*

$$-\ \operatorname{div}\sigma \;=\; \mu\,, \qquad \int_K |\sigma|\,dx \;\le\; \|\mu\|_1 \;+\; \varepsilon\,. \tag{2.4}$$

Proof. Following [**2**], we use a p-Laplace approximation of the problem (1.1). For every $p > d$, we set

$$a_p := -\inf\left\{\frac{1}{p}\int_\Omega |\nabla u|^p\,dx - \int_K u\,d\mu \quad : \quad u \in W^{1,p}(\Omega)\right\}.$$

It turns out that the infimum in the right hand side is attained at a unique point u_p such that $\int_K u_p dx = 0$. Set:

$$\sigma_p := |\nabla u_p|^{p-2}\nabla u_p \text{ on } \Omega, \quad \sigma_p = 0 \text{ on } \mathbb{R}^d \setminus \Omega \,.$$

There holds:

$$-\operatorname{div}\sigma_p = \mu\,, \quad a_p = \frac{1}{p'}\int_\Omega |\sigma_p|^{p'}\,dx \quad \text{(p' conjugate of p).} \tag{2.5}$$

Then by applying [2], Theorem 4.2 and (4.15), we have that:

- $\lim_{p\to\infty} a_p = \|\mu\|_1$.
- For a suitable subsequence $\{p_k\}$, $u_{p_k} \to u$ in $C^0(K)$ and $\sigma_{p_k} \to \lambda$ weakly (star) in $\mathcal{M}(K;\mathbb{R}^d)$, where u, λ solve (1.1) and (2.3) respectively.

In particular, as $p \to +\infty$ (thus $p' \to 1$), we deduce from (2.5) and Hölder inequality that

$$\limsup_{p\to\infty}\int_\Omega |\sigma_p|dx \;\le\; \limsup_{p\to\infty}\left(\int_\Omega |\sigma_p|^{p'}dx\right)^{1/p'} |\Omega|^{1/p} \;=\; \|\mu\|_1 \,.$$

Therefore σ_p satisfies (2.4) for large p. ■

Remark 2.6. According to Theorem 2.3, the conclusions of Lemma 2.5 can be extended to all $\mu \in \mathcal{M}_{0,1}(K)$.

Proof of Theorem 2.4. We first prove the inclusion

$$\left\{T_f \;:\; f \in \mathcal{M}_{0,1}(K)\right\} \;\subset\; \left\{-\operatorname{div}\sigma \;:\; \sigma \in L^1(\Omega;\mathbb{R}^d)\right\} \,.$$

Let $f \in \mathcal{M}_{0,1}(K)$ and let $\{f_n, n \ge 1\}$ be a sequence in $\mathcal{M}_0(K)$ such that $\|f_n - f\|_1 \to 0$. Possibly after extracting a subsequence, we may assume that $\varepsilon_n := \|f_{n+1} - f_n\|_1$ satisfies $\sum \varepsilon_n < +\infty$. Thus setting $f_0 = 0$ and $\mu_n := f_{n+1} - f_n$, we have written f under the form

$$f = \sum_{n=0}^{\infty} \mu_n \qquad \text{where} \qquad \mu_n \in \mathcal{M}_0(K)\,, \qquad \|\mu_n\| = \varepsilon_n \,.$$

Now by applying Lemma 2.5 to μ_n, we find $\xi_n \in L^1(\Omega;\mathbb{R}^d)$ such that

$$-\operatorname{div}\xi_n \;=\; \mu_n \qquad \text{and} \qquad \int_K |\xi_n|\,dx \;\le\; 2\,\varepsilon_n \,. \tag{2.6}$$

Then we define $\sigma := \sum_0^\infty \xi_n$. By (2.6), it is an absolutely convergent series in $L_1(\Omega; \mathbb{R}^d)$. As the divergence operator is continuous with respect to the convergence in the sense of distributions, we deduce immediately from (2.6) that

$$- \operatorname{div} \sigma = \sum_0^\infty \mu_n = f .$$

This finishes the proof of the first inclusion; the reverse inclusion is a straightforward application of Lemma 2.1 and the first assertion of Theorem 2.4 follows. It is easy to check that the linear map $\mathcal{L} : \sigma \in L_1(\Omega; \mathbb{R}^d) \mapsto \operatorname{div} \sigma \in (Lip_0(K))^\star$ is continuous. More precisely:

$$\|\operatorname{div} \sigma\|_1 = \sup \left\{ \int_K (\sigma \cdot \nabla\varphi)\, dx \ , \ \varphi \in Lip_1(K) \right\} \leq \|\sigma\|_{L^1(\Omega;\mathbb{R}^d)} . \tag{2.7}$$

Now as shown before, the range of $\mathcal{L}$ is exactly $\mathcal{M}_{0,1}(K)$ which by construction is a closed subspace of $(Lip_0(K))^\star$. By the closed graph theorem, we induce that there exists a suitable constant $C > 0$ such that

$$\|\sigma\|_{L^1(\Omega;\mathbb{R}^d)/V_0} \leq C \, \| \operatorname{div} \sigma\|_1 . \tag{2.8}$$

We claim that in (2.8) we can take $C = 1$, so that combining with (2.7) we deduce the required equality. To prove the claim for a given $\sigma \in L^1(\Omega; \mathbb{R}^d)$, we set $f := - \operatorname{div} \sigma$ and we consider any approximating sequence $f_n \in \mathcal{M}_0(K)$ such that $\|f_n - f\|_1 \leq \frac{1}{n}$. Owing to Lemma 2.5, for every $\varepsilon > 0$, there exists $\sigma_n \in L^1(\Omega; \mathbb{R}^d)$ such that

$$- \operatorname{div} \sigma_n = f_n \, , \qquad \int_K |\sigma_n| \, dx \leq \|f\|_1 + \varepsilon . \tag{2.9}$$

On the other hand, by (2.8), there exists $\xi_n \in L^1(\Omega; \mathbb{R}^d)$ such that

$$- \operatorname{div} \xi_n = f - f_n \, , \qquad \int_K |\xi_n| \, dx \leq C \, \|f - f_n\|_1 + \varepsilon . \tag{2.10}$$

Therefore, from (2.9) and (2.10) it follows that $\operatorname{div} (\sigma_n + \xi_n) = \operatorname{div} \sigma$ so that

$$\|\sigma\|_{L^1(\Omega;\mathbb{R}^d)/V_0} \leq \|\sigma_n + \xi_n\|_{L^1(\Omega;\mathbb{R}^d)} \leq \|f\|_1 + C \, \|f - f_n\|_1 + 2\varepsilon .$$

The claim hence the proof of Theorem 2.4 is achieved by letting $n \to \infty$ and $\varepsilon \to 0$.

■

3 - Optimal transport through tangential vector measures

In this section, we want to answer the following question: let λ be a vector measure in $\mathcal{M}(K;\mathbb{R}^d)$, do we have that $-\operatorname{div}\lambda$, as a distribution on $\mathbb{R}^d$, belongs to $\mathcal{M}_{0,1}(K)$? Owing to our Theorem 2.4, this is cleary true for those λ which are absolutely continuous with respect to the Lebesgue measure on Ω. However, it cannot be true in general as shown in the following example:

Example 3.1. Let $d=2$, $K=[-2,2]^2$ and $S_0=[-1,1]\times\{0\}$. Then $\lambda:=(0,1)\,\mathcal{H}^1 \llcorner S_0$ is an element of $\mathcal{M}(K;\mathbb{R}^2)$ but $-\operatorname{div}(\lambda)$ does not belong to $\mathcal{M}_{0,1}(K)$. Indeed the condition (2.1) in Lemma 2.1 is violated if we choose the sequence in $C^\infty(K)$ defined by $\varphi_n(x,y)=\dfrac{1}{n}\sin(ny)$. Clearly this sequence converges uniformly to 0 on K, satisfies the upperbound $\sup_K|\nabla\varphi_n|\le 1$, whereas $\langle\lambda,\varphi_n\rangle=2$ for every $n\ge 1$. We notice that, in this example, the direction of the measure λ is orthogonal to the segment S_0 where it is supported. In contrast choosing a measure like $\lambda=(1,0)\,\mathcal{H}^1\llcorner S_0$ would lead to the conclusion that $-\operatorname{div}(\lambda)$ does belong to $\mathcal{M}_{0,1}(K)$. This latter fact falls in the framework of tangential measures described below.

Take $\lambda\in\mathcal{M}(K,\mathbb{R}^d)$ and consider a decomposition $\lambda=\sigma\mu$ with $\mu\in\mathcal{M}^+(K)$ and $\sigma\in L^1_\mu(K,\mathbb{R}^d)$. As noticed in the above example, if $-\operatorname{div}(\lambda)\in\mathcal{M}_{0,1}(K)$ then Lemma 2.1 implies that for any equi-Lipschitz sequence (φ_n) in $C^\infty(K)$ such that $\varphi_n\to 0$ uniformly on K one has

$$\langle -\operatorname{div}(\lambda),\varphi_n\rangle \;=\; \langle\lambda,\nabla\varphi_n\rangle=\int_K \sigma\cdot\nabla\varphi_n\,d\mu \;\to\; 0\,. \tag{3.1}$$

This suggests the introduction of the following set:

$$\mathcal{N}:=\Big\{\xi\in L^\infty_\mu(K,\mathbb{R}^d):\ \exists(u_n)_n,\ u_n\in C^\infty(K),\ u_n\to 0 \text{ uniformly},\ Du_n\to\xi \ in\ \sigma(L^\infty_\mu,L^1_\mu)\Big\}. \tag{3.2}$$

The orthogonal of $\mathcal{N}$ in $L^1_\mu(K,\mathbb{R}^d)$ defined by

$$\mathcal{N}^\perp \;:=\; \left\{\eta\in L^1_\mu(K,\mathbb{R}^d)\ :\ \int_K \eta\cdot\xi\;\mu(dy)\;=\;0 \qquad \text{for all } \xi\in\mathcal{N}\right\},$$

is a closed vector subspace of $L^1_\mu(K,\mathbb{R}^d)$. Following **[1]**, **[3]**, **[4]**, we introduce the notion of tangent space T_μ to the measure μ through the following local characterization of $\mathcal{N}^\perp$ (see **[9]** for further details related to the L^∞-case under consideration here):

Proposition 3.2. i) *There exists a μ-measurable multifunction T_μ from K to the subspaces of $\mathbb{R}^d$ such that:*

$$\xi \in \mathcal{N}^\perp \iff \xi(x) \in T_\mu(x) \quad \mu\text{-a.e. } x \in \mathbb{R}^d .$$

ii) *The linear operator $u \in C^1(K) \mapsto P_\mu(x)\nabla u(x) \in L^\infty_\mu(K;\mathbb{R}^d)$ where $P_\mu(x)$ denotes the ortogonal projector on $T_\mu(x)$ can be extended in a unique way as a linear continuous operator*

$$\nabla_\mu : \mathrm{Lip}\,(K) \mapsto \nabla_\mu u \in L^\infty_\mu(K;\mathbb{R}^d)$$

where $\mathrm{Lip}\,(K)$ is equipped with the uniform convergence on bounded subsets of $\mathrm{Lip}\,(K)$ and $L^\infty_\mu(K;\mathbb{R}^d)$ with the weak star topology.

Remark 3.3. The second assertion of Proposition 3.2 asserts that a Lipschitz function admits, for every measure μ, a tangential gradient defined μ-a.e. In the case where μ is the k-dimensional Hausdorff measure on a smooth k-dimensional manifold in $\mathbb{R}^d$, this tangential gradient coincides with the one which is obtained by using Rachemader Theorem on local charts representing the manifold.

Proof. i) We first show that $\mathcal{N}$ is a vector subspace of $L^\infty_\mu(K,\mathbb{R}^d)$ satisfying the property

$$\forall \xi \in \mathcal{N},\ \forall \varphi \in C^1(K),\ \xi\varphi \in \mathcal{N}. \tag{3.3}$$

Let $\xi \in \mathcal{N}$, and $\varphi \in C^1(K)$. There exists a sequence (u_n) in $C^1(K)$ such that:

$$u_n \to 0 \ \text{ uniformly}, \qquad Du_n \to \xi \ \ in\ \ \sigma(L^\infty_\mu, L^1_\mu).$$

Then one readily checks that the sequence (φu_n) satisfies

$$\varphi u_n \to 0 \quad \text{uniformly}, \qquad D(u_n\varphi) = \varphi Du_n + (D\varphi)u_n \ \to\ \varphi\xi \ \ in\ \ \sigma(L^\infty_\mu, L^1_\mu).$$

Thus $\varphi\xi \in \mathcal{N}$ and the claim (3.3) follows. We deduce immediately the following so called "decomposability property" of the space $\mathcal{N}^\perp$:

$$\forall \sigma \in \mathcal{N}^\perp, \qquad \forall A\ \mu\text{-measurable} \subset K, \qquad \sigma 1_A \in \mathcal{N}^\perp . \tag{3.4}$$

Indeed take a smooth sequence $(\varphi_n)_n$ converging to 1_A for the weak star topology of $L^\infty_\mu(K)$. For such a σ and for all $\xi \in \mathcal{N}$, we get thanks to (3.3)

$$\begin{aligned}\langle \sigma\, 1_A,\ \xi\rangle &= \int_K 1_A(y)\sigma(y)\cdot\xi(y)\ d\mu(y)\\ &= \lim_{n\to\infty}\int_K \varphi_n(y)\sigma(y)\cdot\xi(y)\ d\mu(y)\\ &= \lim_{n\to\infty}\ \langle \xi\,\varphi_n,\ \sigma\rangle \ = \ 0\end{aligned}$$

which concludes the proof of (3.4). Then by applying [8] (Theorem 3.1 p. 158), there exists a closed valued $\mu-$ measurable multifunction T_μ such that:

$$\mathcal{N}^\perp \;=\; \Big\{\sigma \in \mathrm{L}^1_\mu(K)\,; \qquad \sigma(x) \in T_\mu(x) \quad \mu\text{-a.e.}\Big\}\,.$$

Clearly $T_\mu(x)$ is a vector subspace of $\mathbb{R}^d$ (tangent space at x to the measure μ).

ii) We have to show that given an equi-Lipchitz sequence (u_n) in $C^1(K)$ converging uniformly to u, then the sequence $(P_\mu \nabla u_n)$ does converge weakly star in L^∞_μ. Since the set $\{\nabla u_n\}$ is uniformly bounded in $\mathbb{R}^d$, it is enough to check that all its clusters points in L^∞_μ weak-star share the same orthogonal projection on T_μ. Given ξ_1, ξ_2 two such points, we clearly have by (3.2) that $\xi = \xi_2 - \xi_1$ belongs to $\mathcal{N}$ and then by i) $P_\mu \xi = 0$. Thus $P_\mu \xi_1 = P_\mu \xi_2$ μ-a.e. ■

Recalling (3.1) (3.2) and in view of the Proposition 3.2, it is natural to consider measures $\lambda = \sigma\mu$ such that $\sigma \in \mathcal{N}^\perp$. Therefore we introduce *the space of tangential measures on* K defined as follows

$$\mathcal{M}_T(K, \mathbb{R}^d) := \big\{\lambda = \sigma\,\mu \;\;:\;\; \mu \in \mathcal{M}^+(K),\; \sigma(x) \in T_\mu(x)\;\; \mu\text{-}a.e.\big\}\,. \tag{3.5}$$

It can be shown that the property $\lambda \in M_T(K, \mathbb{R}^d)$ is independant of the chosen decomposition $\lambda = \sigma\mu$ (see for instance [**1**]).

Remark 3.4. If $\sigma \in L^1(\Omega, \mathbb{R}^d)$, then the measure $\sigma\mathcal{L}^d \llcorner K$ is an element of $\mathcal{M}_T(K, \mathbb{R}^d)$ since $T_{\mathcal{L}^d}(x) = \mathbb{R}^d$ a.e. on Ω. On the other hand, if $\lambda \in M_T(K, \mathbb{R}^d)$, the condition $\dfrac{d\lambda}{d|\lambda|} \in T_{|\lambda|}(x)$, $|\lambda|$-a.e implies that $\dim(T_{|\lambda|}(x)) \geq 1$, $|\lambda|$-a.e.

As a consequence elements of $M_T(K, \mathbb{R}^d)$ are atomless.

The following proposition gives an equivalent definition for $\mathcal{M}_T(K, \mathbb{R}^d)$ and solves the question raised in the introduction of this section:

Proposition 3.5. *Let* $\lambda \in \mathcal{M}(K, \mathbb{R}^d)$*, then* $-\operatorname{div}(\lambda) \in \mathcal{M}_{0,1}(K)$ *if and only if* $\lambda \in \mathcal{M}_T(K, \mathbb{R}^d)$*. In this case, writing* $\lambda = \sigma\,\mu$*, we have for every* $u \in \operatorname{Lip}(K)$*:*

$$\langle -\operatorname{div}\lambda, u\rangle \;=\; \int_K \sigma\cdot\nabla_\mu u\, d\mu\,. \tag{3.6}$$

Proof. Let $\lambda \in \mathcal{M}(K, \mathbb{R}^d)$ such that $-\operatorname{div}(\lambda) \in \mathcal{M}_{0,1}(K)$. Let $\mu \in \mathcal{M}^+(K)$ and $\sigma \in L^1(K, \mathbb{R}^d)$ such that $\lambda = \sigma\mu$. We are going to show that σ is in $\mathcal{N}^\perp$. For any ξ in $\mathcal{N}$, it exists a sequence $(u_n)_n$ in $C^\infty(K)$ such that

$$u_n \to 0 \quad uniformly, \quad Du_n \to \xi \quad in\ \sigma(L^\infty_\mu, L^1_\mu).$$

As $-\operatorname{div}(\lambda)$ is in $\mathcal{M}_{0,1}(K)$, according to Lemma 2.1, the condition (2.1) is satisfied for $(u_n)_n$, so we get:

$$\int_K \sigma \cdot \xi \, d\mu = \lim_{n\to\infty} \int_K \sigma \cdot \nabla u_n \, d\mu = \lim_{n\to\infty} \langle -\operatorname{div}(\lambda), u_n\rangle = 0.$$

As the above equality holds for any ξ in $\mathcal{N}$, σ belongs to $\mathcal{N}^\perp$ and thus, by Proposition 3.2 i), $\sigma(x) \in T_\mu(x)$ μ-a.e. that is $\lambda \in \mathcal{M}_T(K, \mathbb{R}^d)$.

Conversely, let λ in $\mathcal{M}_T(K, \mathbb{R}^d)$, $\mu \in \mathcal{M}^+(K)$ and $\sigma \in L^1(K, \mathbb{R}^d)$ such that $\lambda = \sigma\mu$. Then by definition, there holds $\sigma(x) \in T_\mu(x)$, μ-a.e. Now if the sequence $(u_n)_n$ is equilipschitz and converges uniformly to some u, by Proposition 3.2 ii), the projected gradients $P_\mu \nabla u_n$ do converge weakly star to $\nabla_\mu u$ and therefore:

$$\lim_{n\to\infty} \langle -\operatorname{div}(\lambda), u_n\rangle = \lim_{n\to\infty} \int_K \sigma \cdot \nabla u_n \, d\mu = \lim_{n\to\infty} \int_K \sigma \cdot \nabla_\mu u_n \, d\mu = \int_K \sigma \cdot \nabla_\mu u \, d\mu \,. \tag{3.7}$$

So if u is a constant, the previous limit vanishes and the implication (2.1) holds true. By Lemma 2.1 it follows that λ belongs to $\mathcal{M}_{0,1}(K)$. Eventually, if u is an arbitrary Lipschitz function, we see that the left hand side limit in (3.7) agrees with $\langle -\operatorname{div}(\lambda), u\rangle$. That yields (3.6). ■

We are now in position to state the main result of this section:

Theorem 3.6. *The following equality holds between subsets of* $\mathcal{D}'(K)$*:*

$$\{T_f : f \in \mathcal{M}_{0,1}(K)\} \;=\; \{-\operatorname{div}\lambda : \lambda \in \mathcal{M}_T(K; \mathbb{R}^d)\}\,.$$

Futhermore, for any $f \in \mathcal{M}_{0,1}(K)$*, there exists* $\overline{\lambda} \in \mathcal{M}_T(\mathbb{R}^d; \mathbb{R}^d)$ *such that:*

$$\|f\|_1 \;=\; |\overline{\lambda}|(K) \;=\; \min_{\lambda \in \mathcal{M}_T(K;\mathbb{R}^d)} \left\{ \int |\lambda| : -\operatorname{div}\lambda = f \right\}.$$

Proof. As a consequence of Theorem 2.4 and Remark 3.4, we have:

$$\{T_f : f \in \mathcal{M}_{0,1}(K)\} \subset \{-\operatorname{div}\lambda : \lambda \in \mathcal{M}_T(K; \mathbb{R}^d)\}.$$

The reverse inclusion is a consequence of Proposition 3.5. It remains to show that for any $f \in M_{0,1}(K)$:

$$\|f\|_1 = \min_{\lambda \in \mathcal{M}_T(K;\mathbb{R}^d)} \left\{ \int |\lambda| : -\operatorname{div}\lambda = f \right\}.$$

Notice that by Proposition 3.5, the infimum in the right hand side could be taken as well over all $\mathcal{M}(K, \mathbb{R}^d)$. The existence of a minimal $\overline{\lambda}$ follows then by the lower semicontinuity of the total variation and the fact that the distributional divergence constraint is closed under the weak star convergence of measures. The value of the minimum $\overline{\lambda}(K)$ is clearly below the infimum taken over absolutely continuous measures $\lambda = \sigma\, dx$ (see remark 3.4) and by Theorem 2.3 this latter infimum agrees with $\|f\|_1$. To prove the converse inequality we simply notice that, for every $u \in \mathrm{Lip}_1(K)$, we have $|\nabla_\mu u| \leq 1$. Thus, recalling (3.6), it follows that for every admissible $\lambda = \sigma\,\mu$:

$$\int_K |\lambda| \;\geq\; \int_K (\sigma \cdot \nabla_\mu)\, d\mu \;=\; -\langle \mathrm{div}\,\lambda, u\rangle \ . \qquad \blacksquare$$

Example 3.7. Let us consider again the infinite sum of dipoles discussed in section 1, that is $f = \sum_n (\delta_{b_n} - \delta_{a_n})$. It is difficult to explicit a representation of f as established in Theorem 2.3. However it becomes straightfoward in the framework of lower dimensional tangential measures. Indeed consider any geodesic curve $S_n \subset K$ joining a_n to b_n. Then, if τ_n denotes the oriented tangent vector of S_n, we obtain an element $\lambda \in \mathcal{M}_T(K, \mathbb{R}^d)$ such that $-\,\mathrm{div}\,\lambda = f$ by setting

$$\lambda := \sum_n \tau_n\, \mathcal{H}^1 \llcorner S_n \ .$$

References

[1] G. Bouchitté, G. Buttazzo and P. Seppecher, *Energies with respect to a measure and applications to low dimensional structures*, Calc. Var. Partial Differential Equations **5** (1997), 37–54.

[2] G. Bouchitté, G. Buttazzo and G. De Pascale, *A p-laplacian approximation for some mass optimization problems*, J. Optim. Theory Appl. **118** (2003), 1-25.

[3] G. Bouchitté and I. Fragalà, *Variational theory of weak geometric structures,Variational theory of weak geometric structures*, The measure method and its applications, Progress in Nonlinear Differential Equations and Their Applications **51** (2002), 19-40.

[4] G. Bouchitté and I. Fragalà, *Second-order energies on thin structures: variational theory and non-local effects*, J. Funct. Anal. **204** (2003), 228-267.

[5] J. Bourgain, H. Brezis and P. Mironescu, *$H^{1/2}$ maps with values into the circle: minimal connections, lifting, and the Ginzburg-Landau equation*, Publ. Math. Inst. Hautes Études Sci. **99** (2004), 1-115.

[6] L. G. Hanin, *Duality for general Lipschitz classes and applications*, Proc. London Math. Soc. (3) **75** (1997), 134-156.

[7] L. G. Hanin, *An extension of the Kantorovich norm. Monge Ampère equation: applications to geometry and optimization* (Deerfield Beach, FL, 1997), 113-130, Contemp. Math., 226, Amer. Math. Soc., Providence, RI. 1999.

[8] F. Hiai and H. Umegaki, *Integrals, conditonal expectations and martingales functions*, J. Multivariate Anal. **7** (1977), 149-182.

[**9**] C. JIMENEZ, *Optimisation de problèmes de transport*, Ph. D. thesis, Laboratoire ANAM, Université du Sud-Toulon Var (2005).

[**10**] A. C. PONCE, *On the distributions of the form* $\sum_i (\delta_{p_i} - \delta_{n_i})$, J. Funct. Anal. **210** (2004), 391-435.

[**11**] E. M. STEIN, *Singular integrals and differentiability properties of functions*, Princeton Mathematical Series, 30, Princeton Univ. Press, Princeton 1970.

Abstract

In this note we propose a new characterization of the completion of the set of balanced bounded measures compactly supported in $\mathbb{R}^d$ *with respect to the Monge-Kantorovich norm. This extends the well known case* $d = 1$ (*see* [**6**]). *Different issues connected with the existence of Monge dual potentials and tangent spaces to measures* (*see* [**1**], [**3**], [**4**]) *are also discussed.*

* * *