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Diophantine equations for orthogonal polynomials (**)

1 - Introduction

Let $P(x), Q(x) \in \mathbb{Q}[x]$ be two given polynomials of degree m, n, respectively. Consider the Diophantine equation

$$P(x) = Q(y) \tag{1.1}$$

in unknowns $x, y \in \mathbb{Z}$. We are interested in the following question.

Are there finitely or infinitely many $(x, y) \in \mathbb{Z}^2$ such that (1.1) holds?

Siegel's Theorem (see Theorem 1.3 below) allows to decide this question. Denote by K an arbitrary number field with $[K : \mathbb{Q}] < \infty$. Further let S be the finite set of places of K containing the infinite ones, $\mathcal{O}_S$ the ring of S-integers of K and R an arbitrary commutative integral domain.

Definition 1.1. *Let X be an affine absolutely irreducible curve over K and denote by $\overline{X}$ its projective completion. Then X is called* exceptional *if $\overline{X}$ has genus $\boldsymbol{g} = 0$ and at most two algebraic points at infinity.*

Definition 1.2. *Let $F(x, y) \in K[x, y]$. The Diophantine equation $F(x, y) = 0$ is said to have* infinitely many solutions with a bounded R-denominator *if the-*

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re exists $\Delta \in R$ with $\Delta \neq 0$, such that $F(x, y) = 0$ has infinitely many solutions $(x, y) \in K \times K$ with $\Delta x, \Delta y \in R$.

Theorem 1.3 (Siegel [**9**], 1929). *Let $F(x, y) \in K[x, y]$ be an absolutely irreducible polynomial. If the equation $F(x, y) = 0$ has infinitely many solutions with a bounded $\mathcal{O}_S$-denominator, then the polynomial $F(x, y)$ is exceptional.*

Thus, in view of equation (1.1) Siegel's Theorem gives the following algorithm.

Algorithm 1. First decompose $F(x, y) = P(x) - Q(y) = 0$ into $\mathbb{Q}$-irreducible factors. Secondly, for those factors which are not $\overline{\mathbb{Q}}$-reducible determine $\boldsymbol{g}$ and the number of points at infinity of the corresponding plane curve. Finally, for the factors with $\boldsymbol{g} = 0$ and number of points at infinity $\leqslant 2$ determine whether the associated equation has finitely or infinitely many integral solutions.

If the two polynomials $P(x)$ and $Q(x)$ are given explicitly, this procedure always leads to an answer. But, if one allows $P(x)$ and $Q(x)$ to be arbitrary members of a fixed polynomial family, general answers are out of reach. Recent progress in making Siegel's Theorem more suitable for applications, in particular the algorithmic criterion of Bilu and Tichy [**3**], has made this problem accessible.

1.1 - *Problem and examples*

Denote by $p_n(x)$ and $p_m(x)$ two members of a fixed polynomial family $\{p_k(x)\}$ with $p_k(x) \in \mathbb{Q}[x]$ and $\deg p_k(x) = k$ for $k \geqslant 1$. We raise the general question in a little provocative way:

What are the conditions for $\mathcal{A}, \mathcal{B}, \mathcal{C} \in \mathbb{Q}$, $\mathcal{A}\mathcal{B} \neq 0$ and $m, n \geqslant 2$ with $m \neq n$ such that the number of integral solutions (x, y) of the Diophantine equation

$$\mathcal{A}\, p_m(x) + \mathcal{B}\, p_n(y) = \mathcal{C} \tag{1.2}$$

is finite?

It is hoped for that an answer can be given without any additional requirements on the parameters $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$. To begin with, the general form (1.2) inherits interesting examples as special cases:

(i) Are there finitely many Pochhammer symbols $(x)_m$ and $(y)_n$ of fixed length m and n starting from x and y, respectively, such that they are equal?

In other words [**14**],

$$m!\binom{x}{m} - n!\binom{y}{n} = 0\,,$$

i.e. $p_k(x) = \binom{x}{k}$ and $(\mathcal{A}, \mathcal{B}, \mathcal{C}) = (m!,\ -n!,\ 0)$.

(ii) Given distinct positive integers m and n, how often can two octahedrons of dimension m and n, respectively, contain equally many integral points? The associated Diophantine equation [**2**], [**5**], [**12**] here reads

$$M_m^{(1,\,-1)}(x) = M_n^{(1,\,-1)}(y)\,,$$

where $\{M_k^{(\beta,\,c)}(x)\}$ denotes the Meixner polynomials defined in the Askey-scheme [**6**] by

$$M_k^{(\beta,\,c)}(x) = {}_2F_1\left[\begin{matrix}-k,\ -x\\ \beta\end{matrix};\ 1-\frac{1}{c}\right]. \tag{1.3}$$

(iii) Let $\{P_k^{(\alpha,\,\beta)}(x)\}$ be the Jacobi polynomials which are orthogonal with respect to the weight $(1-x)^\alpha(1+x)^\beta$. Consider

$$\mathcal{A}\,P_m^{(\alpha,\,\beta)}(x) + \mathcal{B}\,P_n^{(\alpha,\,\beta)}(y) = \mathcal{C}\,.$$

Note that there are two more parameters here to deal with [**13**].

2 - Methods

The proof of Bilu-Tichy's criterion (see Theorem 2.2) is based on Siegel's Theorem and so it is ineffective, too. It does not yield an explicit upper bound for $\max(|x|,\ |y|)$, where $(x,\,y)$ is an integral solution of (1.1). However, if $\min(m,\,n) = 2$ we are in the favourable position to use an effective result due to Baker on elliptic and hyperelliptic equations. By simple algebraic manipulation – constructing a perfect square on one side – one can translate (1.1) into an equation of shape given in the following theorem.

Theorem 2.1 (Baker [**1**], 1975). *Let $f(x) \in \mathbb{Q}[x]$ be a polynomial having at least three simple zeroes. Further let $b \in \mathbb{Q}\backslash\{0\}$. Then the Diophantine equation*

$$f(x) = by^2$$

has at most finitely many integral solutions $(x, y) \in \mathbb{Z}^2$ *which can be computed effectively.*

Hence, for the «small» case $\min(m, n) = 2$ of equation (1.2) we have the following algorithm.

Algorithm 2. Check whether there exist three simple zeroes of $f(x)$ in the equation obtained by transforming equation (1.2). Most probably, conditions relating $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ will appear.

What about the general case $\min(m, n) \geq 3$? In order to formulate the criterion given by Bilu and Tichy [3] we first introduce some notation. Let $\gamma, \delta \in \mathbb{Q}\backslash\{0\}$, $q, s, t \in \mathbb{Z}_{>0}$, $r \in \mathbb{Z}_{\geq 0}$ and let $v(x) \in \mathbb{Q}[x]$ be a non-zero polynomial (which may be constant). Recall the definition of the *Dickson* polynomials,

$$D_s(x, \gamma) = \sum_{i=0}^{\lfloor s/2 \rfloor} d_{s,i} x^{s-2i} \quad \text{with} \quad d_{s,i} = \frac{s}{s-i} \binom{s-i}{i} (-\gamma)^i.$$

The pair $(f(x), g(x))$ is called a *standard pair over* $\mathbb{Q}$ if it or the switched pair $(g(x), f(x))$ is one of the pairs listed in the following table. In particular, we call (f, g) a standard pair of the *first*, *second*, *third*, *fourth* and *fifth kind*, respectively, depending on the place of the entry.

kind	explicit form of $(f(x), g(x))$ or switched	parameter restrictions
first	$(x^q, \gamma x^r v(x)^q)$	with $0 \leq r < q$, $(r, q) = 1$, $r + \deg v > 0$
second	$(x^2, (\gamma x^2 + \delta) v(x)^2)$	—
third	$(D_s(x, \gamma^t), D_t(x, \gamma^s))$	with $(s, t) = 1$
fourth	$(\gamma^{-s/2} D_s(x, \gamma), -\delta^{-t/2} D_t(x, \delta))$	with $(s, t) = 2$
fifth	$((\gamma x^2 - 1)^3, 3x^4 - 4x^3)$	—

Theorem 2.2 (Bilu-Tichy [3], 2000). *Let* $P(x), Q(x) \in \mathbb{Q}[x]$ *be non-constant polynomials. Then the following two assertions are equivalent:*

(a) *The equation* $P(x) = Q(y)$ *has infinitely many rational solutions with a bounded denominator.*

(b) *We can express* $P \circ \kappa_1 = \varphi \circ f$ *and* $Q \circ \kappa_2 = \varphi \circ g$ *where* $\kappa_1, \kappa_2 \in \mathbb{Q}[x]$ *are linear,* $\varphi(x) \in \mathbb{Q}[x]$, *and* (f, g) *is a standard pair over* $\mathbb{Q}$.

Regarding (1.2) we now have the following algorithm.

Algorithm 3. Check whether $p_m(x)$ and $-\dfrac{\mathcal{B}}{\mathcal{A}}p_n(x)+\dfrac{\mathcal{C}}{\mathcal{A}}$ can be decomposed as demanded in (b) of Theorem 2.2. By writing down the possible decompositions in terms of coefficient equations, it may be possible to derive contradictions. In that case finiteness of number of integral solutions (x, y) holds by Theorem 2.2.

Note that this task is not as easy as it may seem at the first glance. For example, if one takes the standard pair of the first kind into account, then for every «new» coefficient equation there will be a «new» variable coming out as a coefficient of the arbitrary polynomial $v(x)$. By iteratively solving the «new» equations to the «new» variables one has no hope to end up with a contradiction. Therefore one also has to demand further properties of the polynomial family $\{p_k(x)\}$.

3 - Properties (P1) and (P2)

Two properties of $p_k(x)$ are of great use:

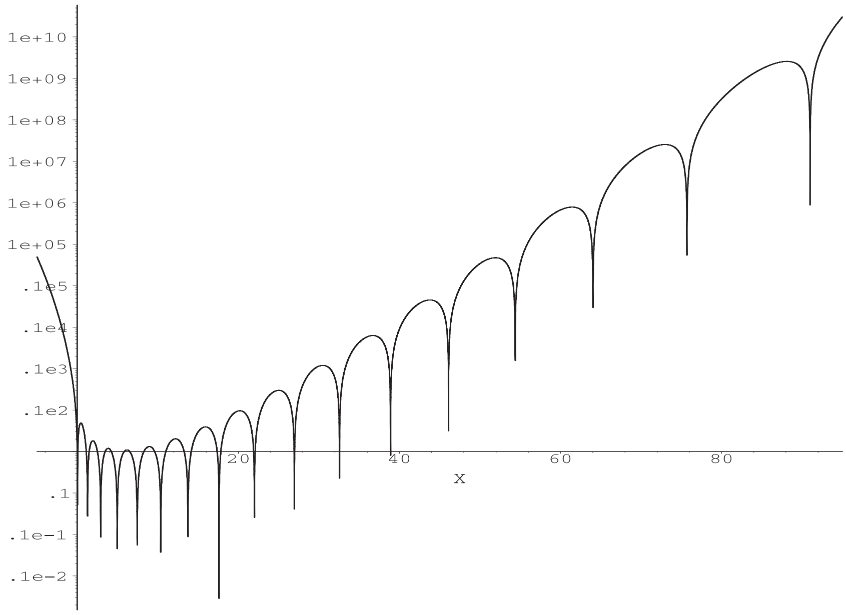
Property (P1): simple stationary points
Property (P2): two interval monotonicity

by (**P2**) we mean

Definition 3.1. *A real polynomial $p(x)$ is called* two interval monotone *if there exist two intervals I_1 and I_2 (one possibly empty) with $I_1 \cup I_2 = (-\infty, \infty)$ such that the local maxima of $|p(x)|$ are strictly decreasing on I_1 and strictly increasing on I_2.*

Of course, (**P1**) is satisfied for all polynomials having only simple real zeroes, e.g. for orthogonal polynomials. The difficult point lies behind (**P2**). Mention only that for a lot of discrete orthogonal polynomials (for instance Meixner, Meixner-Pollaczek, Krawtchouk, Charlier etc.) there is a great numerical evidence that they fulfill (**P2**). However, the proofs are missing. It seems that (**P2**) is a consequence of a certain «well-distribution» of the zeroes.

Example. The discrete Meixner polynomials $M_k^{(\beta, c)}(x)$ (see (1.3)) are orthogonal for $\beta > 0$ and $0 < c < 1$. Maple gives the following *logplot* for $M_{17}^{(6, 1/2)}(x)$:



What are (**P1**) and (**P2**) good for? Suppose all members of $\{p_k(x)\}$ satisfy (**P1**) and (**P2**). We may assume that $m > n$. Further write $p_k(x) = x^i + k_{i-1}^{(i)} x^{i-1} + k_{i-2}^{(i)} x^{i-2} + k_{i-3}^{(i)} x^{i-3} + \ldots + k_0^{(i)}$ and let $\mathcal{A}' = -\dfrac{\mathcal{B}}{\mathcal{A}}$, $\mathcal{B}' = \dfrac{\mathcal{C}}{\mathcal{A}}$. In view of Theorem 2.1 (Baker) and Theorem 2.2 (Bilu-Tichy) we have the following simplifications [**13**], [**14**]:

- [Baker] Let $\delta \in \mathbb{Q}$. Then $p_m(x) + \delta$ has at least three simple zeroes for $m \geqslant 7$.
- [Bilu-Tichy] We have $m = 2n$ and

$$p_m(x) = \mathcal{A}' p_n(v_2 x^2 + v_1 x + v_0) + \mathcal{B}' . \tag{3.1}$$

Further,

$$6 k_{m-1}^{(m)} k_{m-2}^{(m)} n(1-n) + (k_{m-1}^{(m)})^3 (2n^2 - 3n + 1) + 6 k_{m-3}^{(m)} n^2 = 0 . \tag{3.2}$$

Having at disposal (**P1**) and (**P2**), Algorithm 2 and Algorithm 3 can now be replaced by

ALGORITHM 2′. Only check the existence of three simple zeroes for $m = 3, 4, 5$ and 6.

ALGORITHM 3′. In the best case use (3.2) to come to a contradiction. Otherwise extract more coefficient equations from (3.1).

In [**13**] we applied these algorithms to the classes of continuous classical orthogonal polynomials. Property (**P2**), being a «continuous property», follows from an old result of Szegö on solutions of Sturm-Liouville differential equations.

4 - Sturm-Liouville differential equations

It is well-known [8] that

$$(4.1)\qquad (ax^2+bx+c)\,y_n''(x)+(dx+e)\,y_n'(x)-\lambda_n y_n(x)=0,\quad n\in\mathbb{Z}_{\geqslant 0},\ a,b,c,d,e\in\mathbb{C}$$

with $\lambda_n = n[(n-1)\,a + d]$ has a finite or infinite chain of polynomial solutions $\{y_n(x)\}$. The following lemma relies on some nice idea used by Szegö [**15**].

Lemma 4.1. *Let $y_n(x)$ be a polynomial solution of the differential equation*

$$\sigma(x)\,y_n''(x) + \tau(x)\,y_n'(x) - \lambda_n y_n(x) = 0\,,$$

with $\sigma(x) = ax^2 + bx + c$ and $\tau(x) = dx + e$. Furthermore, suppose that $\sigma'(x) - 2\tau(x)$ does not vanish identically. Then $y_n(x)$ satisfies (**P2**).

By extending the previous work of Bochner [**4**], Lesky [**8**] showed that the famous three continuous classical orthogonal families (Laguerre, Jacobi, Hermite) are the only positive definite orthogonal polynomial solutions of (4.1) up to linear transformation $x \mapsto Ax + B$. The parameters of the «standard» forms of these polynomials are listed in the following table (see [**13**]).

Standard classes of continuous classical orthogonal polynomials

name	$y_n(x)$	condition	a	b	c	d	e	λ_n
Laguerre	$L_n^{(\alpha)}(x)$	$\alpha > -1$	0	1	0	-1	$\alpha+1$	$-n$
Jacobi	$P_n^{(\alpha,\beta)}(x)$	$\alpha, \beta > -1/2$	-1	0	1	$-(\alpha+\beta+2)$	$\beta-\alpha$	$-n(n+\alpha+\beta+1)$
- Gegenbauer	$C_n^{(\lambda)}(x)$	$\lambda > -1/2,\ \lambda \neq 0$	-1	0	1	$-(2\lambda+1)$	0	$-n(n+2\lambda)$
- Legendre	$P_n(x)$	—	-1	0	1	-2	0	$-n(n+1)$
- Chebyshev	$T_n(x)$	—	-1	0	1	-1	0	$-n^2$
Hermite	$H_n(x)$	—	0	0	1	-2	0	$-2n$

Note that Gegenbauer polynomials $\alpha=\beta=\lambda-1/2$, Legendre polynomials $\alpha=\beta=0$ and Chebyshev polynomials $\alpha=\beta=-1/2$ are just special cases of the Jacobi polynomial class. In our Diophantine context we can restrict ourselves to the three standard classes because every linear contribution in the variable as well as every multiplicative factor can be handled in (3) by appropriately choosing the parameters v_2, v_1, v_0, $\mathcal{A}'$ and $\mathcal{B}'$. Due to this fact, we are also able to give results for slightly 'modified' orthogonal polynomials, i.e. classical orthogonal polynomials affected by a linear transformation in the variable [**11**].

4.1 - *Finiteness results*

By using ALGORITHM $2'$, ALGORITHM $3'$ and Maple calculations we obtained

Theorem 4.2 (Stoll-Tichy [**13**], 2003). *Let $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ denote arbitrary rational numbers with $\mathcal{A}\mathcal{B}\neq 0$ and $m>n\geq 4$ arbitrary rational integers. Then the number of integral (x, y) satisfying* (1.1) *is finite, if $\{p_k(x)\}$ is one of the following families,*

- Laguerre *polynomials, $\{L_k^{(\alpha)}(x)\}$ with $\alpha>1$,*
- Jacobi *polynomials $\{P_k^{(\alpha,\beta)}(x)\}$ with $\alpha,\beta>-1$ and $\alpha\neq\beta$,*
- Hermite *polynomials $\{H_k(x)\}$,*
- Gegenbauer *polynomials $\{C_k^{(\lambda)}(x)\}$ with $\lambda\neq 0$ and $\lambda>-1/2$ (thus also including the* Legendre *polynomials $\{P_k(x)\}$ with $\lambda=1/2$).*

Let $\{p_k(x)\}$ denote the Chebyshev *polynomials $\{T_k(x)\}$. Then under the above requirements the number of rational (x, y) with bounded denominator satisfying* (1.1) *is infinite.*

Note that the classical orthogonal Chebyshev polynomials $T_k(x)$ indeed do not satisfy (**P2**) as the local maxima of $|T_k(x)|$ are all of equal value.
As an example of the «small cases» we give

Theorem 4.3 (Stoll [**10**], 2003). *The Diophantine equations*

(i) $\mathcal{A}\,P_m(x)+\mathcal{B}\,P_2(y)=\mathcal{C}$,

(ii) $\mathcal{A}\,H_m(x)+\mathcal{B}\,H_2(y)=\mathcal{C}$

with $m \geq 3$ *only admit finitely many integral solutions* (x, y) *with exception of*

$$ad\ \text{(i)}: \ m = 4, \quad \frac{\mathcal{C}}{\mathcal{A}} - \frac{\mathcal{B}}{3\,\mathcal{A}} \in \left\{ -\frac{24}{245}, \frac{3}{35} \right\},$$

$$ad\ \text{(ii)}: \ m = 4, \quad \frac{\mathcal{C}}{\mathcal{A}} - \frac{\mathcal{B}}{2\,\mathcal{A}} \in \left\{ -\frac{3}{2}, \frac{2}{3} \right\}.$$

Moreover, the solutions satisfy $\max(|x|, |y|) < C_1 = C_1(\mathcal{A}, \mathcal{B}, \mathcal{C}, m)$.

5 - Discrete orthogonal polynomials

For orthogonal polynomials whose orthogonality relation holds with respect to a discrete measure nothing is known about (**P2**). However, in very particular cases one can use the special form of the leading coefficient of $p_k(x)$ in order to come to a contradiction. Without going into detail we just mention that such contradictions can be obtained whenever the leading coefficient is built up by factorials, Pochhammer symbols and/or powers. For example, this is the case for the Meixner polynomials $M_k^{(\beta, c)}(x)$, whose leading coefficient is $2^n/(\beta)_n$. This surprisingly suffices to show

Theorem 5.1 (Stoll [**10**], 2003). *Let* m *and* n *be distinct integers satisfying* $m, n \geq 3$, *further let* $\beta \in \mathbb{Z} \setminus \{0, -1, -2, -\max(n, m) + 1\}$. *Then the equation*

$$M_m^{(\beta, -1)}(x) = M_n^{(\beta, -1)}(y) \tag{5.1}$$

has only finitely many solutions in integers (x, y).

Moreover, let $\{K_k^{(p, N)}(x)\}$ denote the class of classical Krawtchouk polynomials [**6**]. Because of $K_n^{(p, N)}(x) = M_n^{(-N, p/(p-1))}(x)$ we immediately have

Corollary 5.2. *Let* m *and* n *be distinct integers satisfying* $m, n \geq 3$, *further let* $N \geq \max(m, n)$. *Then the equation*

$$K_m^{(1/2, N)}(x) = K_n^{(1/2, N)}(y) \tag{5.2}$$

has only finitely many solutions in integers (x, y).

Extensions which allow to consider Diophantine equations of the form $M_m^{(\beta, c_1)}(x) = M_n^{(\beta, c_2)}(y)$ and $K_m^{(p_1, N)}(x) = K_n^{(p_2, N)}(x)$ have recently been obtained by Tichy and the author [**12**].

6 - Open problems

Of course, while looking at the general form of equation (1.2) for Meixner polynomials, the considerations of Section **5** are not sufficient at all. So, an interesting problem remains to study (**P2**) for Meixner polynomials. We are also interested in (**P2**) for orthogonal polynomials whose orthogonality relation holds with respect to a quasi-definite moment functional, so as for instance Bessel polynomials, Pseudo-Jacobi polynomials etc.

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Abstract

We give a short survey about results obtained by Tichy and the author in the last year. They take part of the doctoral thesis [**10**] *of the speaker; the original papers we refer to are* [**12**], [**13**]. *Both papers deal with Diophantine equations of the form* $\mathcal{A}p_m(x) + \mathcal{B}p_n(x) = \mathcal{C}$ *where* $p_m(x)$ *and* $p_n(x)$ *are different members of a fixed orthogonal polynomial family* $\{p_k(x)\}$. *Finiteness results are established by means of methods based on theorems of Baker and of Bilu and Tichy.*

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