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The Ramanujan property and some of its connections with diophantine geometry (**)

1 - Introduction

Let $\overline{\mathbb{Q}}$ be the field of algebraic numbers over $\mathbb{Q}$. Let us choose right away an embedding of $\overline{\mathbb{Q}}$ in the field of complex numbers $\mathbb{C}$.

Let E_2, E_4, E_6 be the series:

$$E_i(z) = 1 + b_i \sum_{n=1}^{\infty} \sigma_{i-1}(n)\, z^n, \quad i = 2, 4, 6, \quad b_2 = -24,\, b_4 = 240,\, b_6 = -504,$$

converging in the complex open disk $B = \{z \in \mathbb{C} \text{ such that } |z| < 1\}$.

The functions defined by these series satisfy $E_i(e^{2\pi i t}) = e_i(t)$, where e_2, e_4, e_6 are the classical *elliptic* normalised Eisenstein series of weights $2, 4, 6$ for $\mathbf{SL}_2(\mathbb{Z})$, defined over the complex upper half-plane $\mathcal{H}$.

It is well known that E_2, E_4, E_6 are algebraically independent over $\mathbb{C}(z)$, but we have much more:

THEOREM 1. *For every choice of $q \in B \setminus \{0\}$, three among the four complex numbers $q, E_2(q), E_4(q), E_6(q)$ are algebraically independent over $\mathbb{Q}$.*

This result was discovered by Nesterenko (see [8]; see also the contributions

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of Nesterenko in [**9**]). A corollary of theorem 1 is the algebraic independence of the three complex numbers π, e^{π} and $\Gamma(1/4)$.

Let p be a prime number; the series E_2, E_4, E_6 also converge in the p-adic open disk $B_p \subset \mathbb{C}_p$ of center 0 and radius 1. Let us fix an embedding of $\overline{\mathbb{Q}}$ in the field of p-adic complex numbers $\mathbb{C}_p$.

The latter corollary is also contained in the following result, proved by Philippon ([**17**], theorem 3, p. 3).

Theorem 2. *Let q be a non-zero element of B (resp. of B_p), if*

$$\frac{E_4(q)^3}{E_4(q)^3 - E_6(q)^2} \in \overline{\mathbb{Q}}, \tag{1}$$

then the degree of transcendence over $\mathbb{Q}$ of the subfield of $\mathbb{C}$ (resp. $\mathbb{C}_p$):

$$\mathbb{Q}(q, E_2(q), E_4(q), E_6(q))$$

is equal to 3.

The complex case of theorem 1 contains theorem 2. However, theorem 2 has a p-adic counterpart which does not appear in theorem 1. Moreover, condition (1) implies $\overline{\mathbb{Q}}$-rationality of a certain elliptic curve, ensuring that theorem 2 already contains most of the significant corollaries of algebraic independence of «classic constants» of theorem 1.

Let $q \in B \setminus \{0\}$. To prove theorem 1, Nesterenko applied, among other features, an algebraic independence criterion of Philippon [**15**]. He has constructed an infinite sequence of non-zero functions

$$G_N \in \mathcal{R}_1 := \mathbb{C}[q, E_2(q), E_4(q), E_6(q)]$$

such that, for all $N \in \mathbb{N}$, G_N can be written as a polynomial

$$P_N(X_1, X_2, X_3, X_4)$$

in four indeterminates with rational integer coefficients of degree N (these coefficients are not too big in absolute value), and such that for all N big enough, there exists a natural number M_N with the property that:

$$e^{-c_1 M_N} \leqslant |G_N(q)| \leqslant e^{-c_2 M_N}, \tag{2}$$

where c_1, c_2 are two strictly positive real numbers which do not depend on N. Moreover, it is required that $\limsup_N M_N = \infty$. In practice, the only way to construct the P_N's is to use the box principle, and it turns out very naturally that M_N is also equal to the order of vanishing at 0 of the function G_N.

To prove the lower bound of (2), one can advantageously apply an interpolation lemma [**16**], after having noticed that the series E_2, E_4, E_6 have rational integer coefficients, while to prove the upper bound of (2), one simply applies a Schwarz lemma in disks contained in B, whose radius depends solely on q, taking into account the multiplicity M_N at 0.

But the inequalities of (2) alone are not enough to apply the mentioned criterion of Philippon: indeed it is also required (among other conditions that we shall not describe here) that the sequence $(M_N)_{N \in \mathbb{N}}$ do not tend to infinity for $N \to \infty$, too rapidly. In other words, Nesterenko needed a *multiplicity estimate*. This is the most difficult part of the proof. He proved:

Theorem 3. *Any element F of $\mathcal{R}_1$ which may be written as a non-vanishing polynomial $P(X_1, X_2, X_3, X_4)$ in four indeterminates of degree N, has its order of multiplicity at 0 at most $c_3 N^4$, for an absolute constant $c_3 > 0$.*

In particular, the sequence $(M_N)_N$ satisfies

$$M_N \leqslant c_3 N^4,$$

and (2) becomes:

$$e^{-c_4 N^4} \leqslant |G_N(q)| \leqslant e^{-c_5 N^4},$$

ready to be employed in the criterion of Philippon in [**15**].

The structure of the proof of theorem 2 is different: Philippon does not need any multiplicity estimate, and does not apply the result in [**15**]. He uses instead a measure of algebraic independence of «elliptic nature»: [**14**].

In the following, however, we will focus on problems which are more closely related to multiplicity estimates, particularly like theorem 3. In effect, we would like to convince the reader that the *differential structure* of certain rings (example: $\mathcal{R}_1$, but other more general examples will be introduced later) is deeply related to the arithmetic of «classical constants».

Multiplicity estimates such as theorem 3 are likely to suggest some kind of information about the transcendence degree expected for fields of finite type over $\mathbb{Q}$, generated by special values of the underlying functions.

2 - The Ramanujan property and the classical quasi-modular forms

We cannot sketch the proof of theorem 3 here, so let us come directly to the main difficulties that one has to overcome.

We want to find an upper bound for the multiplicity M at 0 of a function $G \in \mathcal{R}_1$ which is obtained as a non-zero irreducible polynomial of degree at most N, evaluated in the four functions z, $E_2(z)$, $E_4(z)$, $E_6(z)$ (In the following we will often identify $\mathcal{R}_1$ with $\mathbb{C}[X_1, X_2, X_3, X_4]$, since these rings are isomorphic over $\mathbb{C}$): we can deduce this estimate by computing the «multiplicity» of the ideal (G, DG) of $\mathcal{R}_1$ (to compute the «multiplicity» at 0 of an ideal of $\mathcal{R}_1$, one uses the notion of *distance*; see **[9]**, p. 88), provided that this ideal is not principal (3). But in fact, if the order of G at 0 is small, we do not need to know whether DG is in (G) at all: the problem only appears when the order of G at 0 is too big, and in this case we need to check a new property which says that if the multiplicity of G at 0 is too big, then the principal ideal generated by G does not contain DG.

Thus, it seems that it is unavoidable to use the fact that the ring $\mathcal{R}_1$, endowed with the derivation $D = q(d/dq)$, is a differential ring which satisfies the following *Ramanujan property*.

Definition. We say that a differential ring $(\mathcal{A}, \mathcal{D})$ (where $\mathcal{D} = \{D_1, \ldots, D_n\}$ is a set of derivations) satisfies the *Ramanujan property* if there exists an element $\kappa \in \mathcal{A} \setminus \{0\}$ such that $\kappa \in \mathcal{P}$ for every non-zero prime ideal $\mathcal{P}$ satisfying $D_i \mathcal{P} \subset \mathcal{P}$ for all $i = 1, \ldots, n$ (we will refer to such an ideal as a *differentially stable prime ideal*) (4).

First of all, the ring

$$\mathcal{R} = \mathbb{C}[E_2, E_4, E_6],$$

endowed with the derivation D, is a differential ring. Indeed, we have the relations:

$$(3) \quad DE_2 = \frac{1}{12}(E_2^2 - E_4), \quad DE_4 = \frac{1}{3}(E_2 E_4 - E_6), \quad DE_6 = \frac{1}{2}(E_2 E_6 - E_4^2),$$

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(3) More precisely, the estimate of theorem 3 is obtained as a corollary of a more general theorem that controls the multiplicity at 0 of a general unmixed ideal of $\mathcal{R}_1$, and is proved by induction on the dimension of the latter.

(4) In other words, a differential ring $(\mathcal{A}, \mathcal{D})$ satisfies the Ramanujan property when its *differential nilradical* is non-zero.

(see [**7**], theorem 5.3). So is $\mathcal{R}_1$, because $Dz = z$. The ring

$$\mathcal{R}_2 = \mathbb{C}[\log(z), z, E_2(z), E_4(z), E_6(z)]$$

is also closed under the action of the derivation D, because $D\log(z) = 1$; its field of fractions has transcendence degree 4 over $\mathbb{C}(z)$, and we have:

Theorem 4. *The differential ring* $(\mathcal{R}_2, D)$ *satisfies the Ramanujan property. Its differential nilradical is generated by the element* $\kappa = q\Delta$, *where* $\Delta = E_4^3 - E_6^2$.

This implies that $(\mathcal{R}, D)$ and $(\mathcal{R}_1, D)$ satisfy the Ramanujan property.

Sketch of proof. The detailed proof will appear in [**13**]. We have $Dz = z$, and we see from the relations (3) that $D\Delta = E_2\Delta$: the ideals (q) and (Δ) are prime and differentially stable.

Let $\mathcal{P}$ be a differentially stable non-zero prime ideal of $(\mathcal{R}_2, D)$; here we only consider two cases.

(1). The ideal $\mathcal{P}$ is principal: thus $\mathcal{P} = (P)$ for some irreducible $P \in \mathcal{R}_2$. We have that $DP = FP$, for $F \in \mathcal{R}_2$, non-zero.

The ring $\mathcal{R}_2$ is graded, by setting the degree of E_i equal to i $(i = 2, 4, 6)$, and the degree of z and $\log(z)$ equal to 0.

The derivation D sends homogeneous elements of $\mathcal{R}$ into homogeneous elements. Studying the influence of D on degrees, it is easy to see that:

$$DP = (\lambda_1 + \lambda_2 E_2)\,P, \qquad \text{for some } \lambda_1, \lambda_2 \in \mathbb{C}\,. \tag{4}$$

We can identify P with a formal series:

$$P = \sum_{n=m}^{\infty} a_n Z^n,$$

where Z is an indeterminate, and $a_m, a_{m+1}, \dots$ are polynomials in $\mathbb{C}[z, \log(z)]$ with $a_m \neq 0$. Let D' be the derivation $Z(d/dZ)$, such that $D'Z = Z$ and $D'(z) = D'(\log(z)) = 0$; we easily obtain that $D'P = \lambda_2(1 - 24Z - \dots)\,P$ so that, comparing the coefficients of formal series, $ma_m = \lambda_2 a_m$. Since $a_m \neq 0$, we obtain $\lambda_2 \in \mathbb{Z}$.

In a similar way, one proves that $\lambda_1 \in \mathbb{Z}$. Hence, $\lambda_1, \lambda_2 \in \mathbb{Z}$ in (4). Thus, there exists a non-trivial $\mathbb{Z}$-linear dependence relation between $1 = (Dz/z)$, $E_2 = (D\Delta/\Delta)$ and DP/P. Equivalently, the elements $q, \Delta, P \in \mathcal{R}$ are multiplicatively dependent modulo $\mathbb{C}^\times$; but P is irreducible, thus $P = \alpha z$ or $P = \beta\Delta$, with $\alpha, \beta \in \mathbb{C}^\times$.

(2). If $\mathcal{P}$ is not principal, an elementary argument involving resultants reduces this case to the case (**1**), or to the following case: $\mathcal{P}$ contains a homogeneous non-zero element F of $\mathbb{C}[E_4, E_6]$ (thus, $F(e^{2\pi i\tau})$ is a modular form).

Let F, G be homogeneous elements of $\mathbb{C}[E_4, E_6]$, of degree f, g. The *Rankin bracket* of F and G is the homogeneous element:

$$[F, G] = fFDG - gGDf \in \mathbb{C}[E_4, E_6].$$

Clearly, if $F \in \mathcal{P}$ and $G \in \mathcal{R}$, then $[F, G] \in \mathcal{P}$.

Let us suppose that $F \in \mathcal{P}$ has minimal non-zero degree in E_4. Then:

$$\begin{aligned}[F, E_6] &= \frac{\partial F}{\partial E_2}[E_2, E_6] + \frac{\partial F}{\partial E_4}[E_4, E_6] + \frac{\partial F}{\partial E_6}[E_6, E_6] \\ &= \frac{\partial F}{\partial E_4}[E_4, E_6].\end{aligned}$$

Since $(\partial F/\partial E_4)$ is non-zero and does not belong to $\mathcal{P}$ by hypothesis, we have $[E_4, E_6] \in \mathcal{P}$. Now, the differential relations (3) imply

$$\begin{aligned}[E_4, E_6] &= 4E_4DE_6 - 6E_6DE_4 \\ &= 2E_4(E_2E_6 - E_4^2) - 2E_6(E_2E_4 - E_6) \\ &= 2(E_6^2 - E_4^3) = -2\Delta,\end{aligned} \tag{5}$$

and $\Delta \in \mathcal{P}$. If F has degree 0 in E_4, then $E_6 \in \mathcal{P}$ and $[E_6, E_4] = -[E_4, E_6] \in \mathcal{P}$: in all cases, $\Delta \in \mathcal{P}$.

Nesterenko has already proved a weaker result: he shows that every non-zero stable prime ideal $\mathcal{P}$ of $\mathcal{R}_1$ vanishing at $q = 0$, contains κ: our proof of the first part of theorem 4 is an extension of his lemma 5.2 p. 161 in [**9**].

This kind of consideration on principal differentially stable ideals, already appears in the literature. For example, lemma 5.2 of loc. cit. is a generalisation of lemma 3 p. 211 of [**19**], where the problem is to determine arithmetic conditions over parameters of a certain linear differential equation of second order (equation (9) p. 209), so that its solutions are not algebraic, and not solutions of a first order Riccati equation.

Similar arguments also appear in the different context of Painlevé equations; the problem that often appears this time, is to find conditions over the parameters so that the solutions of a given Painlevé equation are not algebraic, and do not satisfy a first order Riccati equation: see for example, [**20**], proposition 2.2 p. 161.

More generally, theorem 4 underlines a profound problem in our study: the classification of «differential factor rings», which has been considered as a central problem also in the theories of Siegel's E and G-functions since their beginning.

The second part in our proof of theorem 4 is new: it totally replaces the arguments of section **5** pp. 162-165 of [**9**].

We can improve theorem 4 (see [**13**]) and give a full description of the differentially stable ideals of $\mathcal{R}_2$. In fact, the most interesting ideals are the stable prime ideals of $\mathcal{R}$.

Theorem 5. *For every $c \in \mathbb{C}$, we have the tower of stable prime ideals of $\mathcal{R}$:*

$$(E_2 - c, E_4 - c^2, E_6 - c^3) \supset (E_2^2 - E_4, E_2^3 - E_6) \supset (\Delta).$$

Moreover, the set of all these prime ideals is equal to the set of the stable prime ideals of $\mathcal{R}$.

Remark. It is easy to see that the ideal $(E_2^2 - E_4, E_2^3 - E_6)$ is also the Jacobson differential radical of $\mathcal{R}$.

In [**11**], thanks to the techniques sketched here, we have proved an extension of the multiplicity estimate (3) to the differential ring generated by z, e^z, and three algebraically independent quasi-modular (non-holomorphic) forms of weight 2 with respect to a chosen co-compact Schwarz triangular subgroup of $\mathbf{SL}_2(\mathbb{R})$.

3 - Hilbert modular and quasi-modular forms

It would be of great interest to extend the above framework to a more general class of diophantine problems. Indeed, if we go back to the algebraically independent numbers π, e^π, $\Gamma(1/4)$, we see that they are (roughly speaking) connected with the periods of extensions of the elliptic curve E with a Weierstrass model $y^2 = 4x^3 - 4x$, by algebraic groups $\mathbb{G}_m^b \times \mathbb{G}_a^c$ with $b, c \in \mathbb{N}$: the interpretation of these numbers as special values of quasi-modular forms (in the sense of [**6**]) and the function $e^{2\pi i\tau}$ explains why so many «classical» constants are found to be algebraically independent in theorem 1.

It would be very interesting to check the algebraic independence over $\mathbb{Q}$ of the numbers π, e^π, $e^{\sqrt{5}\pi}$, $\Gamma(1/5)$, $\Gamma(2/5)$: they are connected to «periods» of extensions of the jacobian A of the curve $y^2 = 4x^5 - 4$ of genus 2, by an algebraic group $\mathbb{G}_m^b \times \mathbb{G}_a^c$. The transcendence degree over $\mathbb{Q}$ of the field generated by these numbers is $\geqslant 2$: see [**2**] and [**3**].

This naturally leads to ask similar questions about Hilbert modular and quasi-modular forms: notice indeed that the abelian surface A has «real multiplication» by $\mathbb{Q}(\sqrt{5})$, so that the Hilbert modular group associated to $\mathbb{Q}(\sqrt{5})$ has to play a priviledged role.

3.1 - *Basic properties of Hilbert quasi-modular forms*

In this section we shall describe a few properties of the differential structure of rings of *Hilbert quasi-modular forms*, in connection to the Ramanujan property.

Let K be a totally real number field of degree n over $\mathbb{Q}$, let $\sigma_1, \dots, \sigma_n$ be its embeddings in $\mathbb{R}$, let $\mathcal{O}_K$ be its ring of integers. If $a \in K$, we also write $a_i = \sigma_i(a)$. The Hilbert modular group $\Gamma_K := \mathbf{SL}_2(\mathcal{O}_K)$ acts on $\mathcal{H}^n$ by homographic transformations in the usual way: let $t = (t_1, \dots, t_n) \in \mathcal{H}^n$ and $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_K$. Then:

$$\gamma(t) = \left(\frac{a_1 t_1 + b_1}{c_1 t_1 + d_1}, \dots, \frac{a_n t_n + b_n}{c_n t_n + d_n} \right).$$

Note that if $n = 1$, then $K = \mathbb{Q}$ and $\Gamma_{\mathbb{Q}} = \mathbf{SL}_2(\mathbb{Z})$.

Definition. Let $F : \mathcal{H}^n \to \mathbb{C}$ be a holomorphic function. We say that F is a *Hilbert quasi-modular form of weight* $(k_1, \dots, k_n) \in \mathbb{Z}^n$ *and depth* $s \in \mathbb{Z}$ if:

(i) $n = 1$ and F is a quasi-modular form in the sense of Kaneko-Zagier [**6**] or

(ii) $n > 1$, and there exists a polynomial $P \in \mathrm{Hol}(\mathcal{H}^n)[X_1, \dots, X_n]$ of total degree s in $X_1, \dots, X_n$ (whose coefficients are holomorphic functions on $\mathcal{H}^n$), such that:

$$F(\gamma(t)) := \prod_{i=1}^{n} (c_i t_i + d_i)^{k_i} P\left(\frac{c_1}{c_1 t_1 + d_1}, \dots, s, \frac{c_n}{c_n t_n + d_n} \right). \tag{6}$$

A Hilbert quasi-modular form is *parabolic* if it vanishes at the cusps of Γ_K.

A Hilbert quasi-modular form of weight $(k_1, \dots, k_n)$ is said to have *parallel weight* k, if $k_1 = \dots = k_n = k$.

Clearly, a Hilbert quasi-modular form of depth 0 is a Hilbert modular form. The constant term of P with respect to $X_1, \dots, X_n$ is equal to F. Moreover, the analytic behaviour required in [**6**] in the case of $n = 1$, or the Koecher principle in the case $n > 1$, provide that if F is a Hilbert quasi-modular form of weight $(k_1, \dots, k_n)$, then $k_i \in \mathbb{N}$ for all $i = 1, \dots, n$, and if $k_1 = \dots = k_n = 0$, then $F \in \mathbb{C}$. More precisely:

Lemma 1. *Let F be a Hilbert quasi-modular form of weight $(k_1, \dots, k_n)$*

and depth s. Let

$$P_0 = \sum_{s_1 + \ldots + s_n = s} c_{s_1, \ldots, s_n}(t)\, X_1^{s_1} \ldots X_n^{s_n}$$

be the sum of the homogeneous monomials of degree s of the polynomial P associated to F in (6). *Then, for all $s_1, \ldots, s_n$ such that $s_1 + \ldots + s_n = s$, the function $c_{s_1, \ldots, s_n}(t)$ is a Hilbert modular forms of weight $(k_1 - 2s_1, \ldots, k_n - 2s_n)$.*

Proof. This will appear in [**12**].

Let $\mathcal{Y}(K)$ be the multi-graded ring (by the weights) of all the Hilbert quasi-modular forms of every possible depth. It is easy to see that if

$$\mathcal{D} = \{D_1, \ldots, D_n\} = \{(2\pi i)^{-1}\, \partial/\partial t_1, \ldots, (2\pi i)^{-1}\, \partial/\partial t_n\},$$

then $(\mathcal{Y}(K), \mathcal{D})$ is a differential ring.

Lemma 2. *The field of fractions of $\mathcal{Y}(K)$ has transcendence degree $3n$ over $\mathbb{C}$.*

Proof. According to [**1**], the differential field $\mathcal{F}(K)$ generated by all the modular functions for Γ_K has transcendence degree $3n$ over $\mathbb{C}$. Now, it is not too difficult to prove that the algebraic closure of the field of fraction of $\mathcal{Y}(K)$ coincides with the algebraic closure of $\mathcal{F}(K)$.

Remark. By lemma 2, the degree of transcendence of $\mathcal{F}(\mathbb{Q})$ is 3. Moreover, in [**6**], it is proved that $\mathcal{Y}(\mathbb{Q}) = \mathbb{C}[e_2, e_4, e_6]$. Thus, we have an isomorphism of differential rings:

$$\mathcal{Y}(\mathbb{Q}) \cong \mathcal{R}\,. \tag{7}$$

The proof can also be performed by using lemma 2, the existence of e_2 together with its functional equation, and the differential isomorphism $\mathbb{C}[e_2, e_4, e_6] \cong \mathcal{R}$.

3.2 - *Degenerate Hilbert quasi-modular forms*

Following a suggestion of K. Buzzard, we begin with the *totally degenerate totally real «number field»* of degree n, with its trivial embedding in $\mathbb{R}^n$:

$$K_0 := \mathbb{Q} \oplus \ldots \oplus \mathbb{Q} \to \mathbb{R}^n.$$

The Hilbert modular group Γ_{K_0} is equal to a direct sum of n copies of $\mathbf{SL}_2(\mathbb{Z})$. By the isomorphism (7), the differential ring of Hilbert quasi-modular forms $\mathcal{Y}(K_0)$ is isomorphic to the differential n-fold tensor power:

$$(\mathcal{R}, D)\otimes\ldots\otimes(\mathcal{R}, D)$$
$$= (\mathbb{C}[E_2(z_1), E_4(z_1), E_6(z_1), \ldots, E_2(z_n), E_4(z_n), E_6(z_n)], \{D_1, \ldots, D_n\}),$$

where $D_i = z_i(d/dz_i)$. We see that the transcendence degree of $\mathcal{Y}(K_0)$ over $\mathbb{C}$ is equal to $3n$, (see also lemma 2). Inside $\mathcal{Y}(K_0)$, the reader can determine, as a simple exercice, the subring of Hilbert modular forms (transcendence degree $2n$), and the subring of Hilbert modular forms of parallel weight (of transcendence degree $n+1$). The reader can also determine the full structure of differentially stable prime ideals in $\mathcal{Y}(K_0)$ (that is, $\mathcal{D}$-stable, with $\mathcal{D} = (D_1, \ldots, D_n)$), applying theorem 5. In particular, every non-zero differentially stable prime ideal of $\mathcal{Y}(K_0)$ contains the element:

$$\kappa(K_0) := \prod_{i=1}^{n}(E_4^3(z_i) - E_6^2(z_i)). \tag{8}$$

3.3 - *Differential structure for Hilbert quasi-modular forms*

Let us examine $\mathcal{Y}(K)$, when K is a (non-degenerate) totally real number field. It would be very useful to study the structure of differentially stable prime ideals in this case, and especially, its differential nilradical, but this is probably a difficult task. First of all:

Theorem 6. *For $n>1$, the ring $\mathcal{Y}(K)$ is not finitely generated.*

The proof of this result will appear in [**12**], and is very similar to that of the lemma 16 of [**10**]. There, we have proved that if $n>1$, then the subring $\mathcal{L}(K)$ generated by the Hilbert modular forms is not finitely generated, and the main point in the proof was the well known fact that if F is a Hilbert modular form of weight $(k_1, \ldots, k_n)$, and if there exists an index i such that $k_i = 0$ and another index j such that $k_j \neq 0$, then $F = 0$.

Now, it turns out that the same is true for Hilbert quasi-modular forms. For example:

Lemma 3. *If $n>1$, for every j, $0 \leq j \leq n-1$, there does not exist any non-zero Hilbert quasi-modular form of weight* $(\underbrace{0, \ldots, 0}_{j}, 2, \underbrace{0, \ldots, 0}_{n-j-1})$.

Proof. We only consider the case $n=2$, and to simplify matters, we also suppose that the class number of K is 1 (see [**12**] for a more general statement), so that there is only one cusp class for Γ_K. Let us assume by contradiction that there exists a non-zero Hilbert quasi-modular form F of weight $(2, 0)$. By lemma 1, its depth must be 1, because is is well known that non-zero Hilbert modular forms of weight $(2, 0)$ do not exist. The associated polynomial for which (6) is true, is necessarily:

$$P = F(t, t') + \lambda X_1,$$

with $\lambda \in \mathbb{C}^{\times}$. The holomorphic function $F(t, t)$ defined over $\mathcal{H}$ is clearly a quasi-modular form of weight 2 and depth 1. Using (7), we see that:

$$F(t, t) = \mu E_2(e^{2\pi i t}),$$

with $\mu \in \mathbb{C}$. But F is parabolic: the Fourier series of F at the cusp at infinity has a zero constant term because F has non parallel weight. Thus $\mu = 0$, because $E_2 = 1 - 24z - \ldots$ (that is, e_2 is not parabolic). Let $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbf{SL}_2(\mathbb{Z}) \subset \Gamma_K$. We have:

$$\begin{aligned} 0 &= F(\gamma(t, t)) \\ &= (ct+d)^2 \left(F(t, t) + \frac{\lambda c}{ct+d} \right) \\ &= \lambda c(ct+d). \end{aligned}$$

Since this relation holds for every $\gamma \in \mathbf{SL}_2(\mathbb{Z})$, we find that $\lambda = 0$, which provides the required contradiction.

Corollary 1. *If $n > 1$, there does not exist any non-zero differentially stable principal ideal in $\mathcal{Y}(K)$.*

Proof. If F is a Hilbert quasi-modular form such that $D_i F = A_i F$ with $A_i \in \mathcal{Y}(K)$ for $i = 1, \ldots, n$, then A_i is quasi-modular of weight $(\underbrace{0, \ldots, 0}_{i-1}, 2, 0, \ldots, 0)$, and by lemma 3, $A_i = 0$: this implies $F \in \mathbb{C}$. If F is any polynomial in $\mathcal{Y}$ (that is, not necessarily a quasi-modular form), then one easily reduces the proof to the case above, using the fact that the derivations of $\mathcal{D}$ are homogeneous of weight $(\underbrace{0, \ldots, 0}_{i-1}, 2, 0, \ldots, 0)$ for $i = 1, \ldots, n$.

Very few things are known at present about the differentially stable prime ideals of $\mathcal{Y}(K)$ for $n > 1$. For $n = 1$, or for $K = K_0$, the problem is completely solved because, as we said earlier, the differential nilradical always contains the element $\kappa(K_0)$ defined in (8).

In the particular case of $K = \mathbb{Q}(\sqrt{5})$, we have obtained some further information. The full structure of the ring of Hilbert modular forms of parallel weight for $\Gamma_{\mathbb{Q}(\sqrt{5})}$ is known (see [4], [5] or theorem 6.1 of [**10**]). In particular, there exists exactly one non-trivial parabolic modular form χ_{15} of parallel weight $(15, 15)$ (up to normalisation), such that $\chi_{15}(z, z') = \chi_{15}(z', z)$ (symmetric). In [**12**], we have proved:

Theorem 7. *Let $\mathcal{P}$ be a non-zero differentially stable prime ideal of the ring $\mathcal{Y}(\mathbb{Q}(\sqrt{5}))$. If $\mathcal{P}$ contains a non-zero Hilbert modular form, then $\chi_{15} \in \mathcal{P}$.*

Theorem 7 and other arguments contained in [**12**] lead to partial generalisations of theorem 3 for Hilbert quasi-modular forms associated with $K = \mathbb{Q}(\sqrt{5})$. The results that we have obtained are still far from optimal, so we do not quote them; they will appear elsewere.

Instead of this, we now explain why the Hilbert modular form χ_{15} is a first-rate candidate to be a «$\kappa(\mathbb{Q}(\sqrt{5}))$» for the Ramanujan property. Let us note the equality (5), which says that $\kappa(\mathbb{Q}) = \Delta$: we see that Δ is the Rankin bracket of the generators E_4, E_6 of $\mathbb{C}[E_4, E_6]$.

Here we consider Hilbert modular and quasi-modular forms associated with a real quadratic number field K (possibly degenerate). If F, G, H are Hilbert modular forms of parallel weight f, g, h, we define a multilinear bracket:

$$[F, G, H] = \det \begin{pmatrix} fF & gG & hH \\ \dfrac{\partial F}{\partial z} & \dfrac{\partial G}{\partial z} & \dfrac{\partial H}{\partial z} \\ \dfrac{\partial F}{\partial z'} & \dfrac{\partial G}{\partial z'} & \dfrac{\partial H}{\partial z'} \end{pmatrix}.$$

It is easy to check that $[F, G, H]$ is a Hilbert modular form of parallel weight $f + g + h + 2$. Let us examine the degenerate case of $K = K_0 = \mathbb{Q} \oplus \mathbb{Q}$. By using (3), one verifies:

$$\begin{aligned}
&[E_4(z_1)\, E_4(z_2),\, E_6(z_1)\, E_6(z_2),\, E_4(z_1)^3 E_6(z_2)^2] \\
&\qquad = 2E_4(z_1)^3 E_6(z_2)^2 (E_4(z_1)^3 - E_6(z_1)^2)(E_4(z_2)^3 - E_6(z_2)^2) \\
&\qquad = 2E_4(z_1)^3 E_6(z_2)^2 \kappa(\mathbb{Q} \oplus \mathbb{Q}).
\end{aligned}$$

Once again, we see that a bracket allows us to compute the element κ of the Ramanujan property; but this time it is a trilinear bracket.

Going back to the case $K = \mathbb{Q}(\sqrt{5})$, it is proved in loc. cit., that the ring of Hilbert modular forms of parallel weight associated to K is generated by χ_{15}, and three algebraically independent modular forms $\varphi_2, \chi_5, \chi_6$ (the subscripts are also the weights). In [**10**], we have proved that:

$$[\varphi_2, \chi_5, \chi_6] = \lambda \chi_{15},$$

for a non-zero complex number λ, so that we really feel now, that the hypothesis that $\kappa(\mathbb{Q}(\sqrt{5})) = \chi_{15}$ is reasonable. Of course, the fact that χ_{15} is a multilinear bracket does not completely explain why χ_{15} belongs to every differentially prime ideal also containing modular forms. This is such a nice coincidence, proved in [**12**], but we do not know why it happens yet. The following question arises:

Question. *Let K be a totally real number field of degree $n > 1$. Does $\mathcal{Y}(K)$ satisfy the Ramanujan property? Are the multilinear brackets defined above involved in the definition of $\kappa(K)$?*

In this direction, we mention that we have proved in [**12**], for $K = \mathbb{Q}(\sqrt{5})$, that there exists two non-zero Hilbert modular forms F_1, F_2 of weights $(7, 9)$, $(9, 7)$, such that the three ideals of $\mathcal{Y}(K)$:

$$(a\varphi_2^5 - \chi_5^2,\ b\varphi_2^3 - \chi_6,\ \chi_{15})$$

with

$$(a, b) = \left(\frac{1}{800000}, \frac{1}{800}\right), \left(\frac{1}{253125}, \frac{1}{675}\right), (0, 0),$$

have (geometric) height 2, and are $\{F_1 D_1, F_2 D_2\}$-stable. Since they all contain χ_{15}, this confirms once again, the hypothesis that $\kappa(\mathbb{Q}(\sqrt{5})) = \chi_{15}$.

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Abstract

The aim of this text is to introduce and describe certain differential rings of modular and quasi-modular forms, playing an important role in diophantine geometry. We first explain in an informal way how these properties are connected with the problem of multiplicity lemmas, then we describe a few properties of the structure of differential rings of Hilbert quasi-modular forms.

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