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Singular perturbation methods for the solution of some nonlinear boundary value problems (**)

1 - Introduction

Several examples of nonlinear boundary value problems, exhibiting solutions of boundary layer type, arise in many areas, including chemical-reactor theory, optimal control theory, fluid dynamics and the physical theory of semiconducting devices.

The mathematical model for such problems is given by a differential system of the form

$$\boldsymbol{A}_\varepsilon \frac{d\boldsymbol{y}}{dx} = \boldsymbol{g}(x, \boldsymbol{y}, \varepsilon), \qquad x \in I = [0, 1] \tag{1}$$

where $\boldsymbol{y} : I \to \mathbb{R}^n$ is a n-dimensional unknown vector, $\boldsymbol{A}_\varepsilon$ is a $n \times n$ diagonal matrix, with non null elements $a_{ii} = \varepsilon^{q_i}$, which are integer powers of a small parameter $\varepsilon : 0 \leq \varepsilon < \varepsilon_0 \ll 1$, and where $\boldsymbol{g} = \{g_i\}$, $i = 1, \ldots, n$ is a function that is nonlinear in x, $\boldsymbol{y}$ and ε. The solution $\boldsymbol{y}(x, \varepsilon)$ is subjected to separated boundary conditions of type:

$$\begin{aligned} &y_r(0) = \alpha_r(y_s(0), \varepsilon) \qquad r = 1, \ldots, m; \ s = m+1, \ldots, n \\ &y_s(1) = \beta_{s-m}(y_r(1), \varepsilon) \end{aligned} \tag{2}$$

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where α_r, β_{s-m} are known invertible functions. In addition g_i, α_r, β_{s-m} have the asymptotic expansions

$$\begin{bmatrix} g_i(x, \boldsymbol{y}, \varepsilon) \\ \alpha_r(y_s(0), \varepsilon) \\ \beta_{s-m}(y_r(1), \varepsilon) \end{bmatrix} \sim \sum_{k=0}^{\infty} \begin{bmatrix} g_{ik}(x, \boldsymbol{y}) \\ \alpha_{rk}(y_s(0)) \\ \beta_{s-m,k}(y_r(1)) \end{bmatrix} \varepsilon^k. \tag{3}$$

If $q_i = 0$ for some value of index i, the corresponding solution components $y_i(x, \varepsilon)$ of the boundary problem (1-2) (called *slow variables*) will converge uniformly in $[0, 1]$ as $\varepsilon \to 0$. On the contrary, the other *fast variables*, which have derivatives multiplied by a positive power q_i of the small parameter ε, have strong variation at the ε^{q_i} wide boundary layers resulting at the right or left endpoint of I, as a consequence of the boundary conditions (2). Because of the nonuniform convergence of the fast variables as $\varepsilon \to 0$, the solution is constructed by means of composite asymptotic expansions. This classical method, originated from Prandtl's papers and developed by many Authors, see [**1**], [**2**], [**3**], [**4**] and related bibliography, allows to determine suitable boundary layer corrections, derived by rescaling the independent variable in the *stretched* variables

$$\tau = \frac{x}{\varepsilon^{q_i}}, \qquad \sigma = \frac{1-x}{\varepsilon^{q_i}}$$

and having asymptotic expansions as $\varepsilon \to 0$ with terms tending to zero as τ, $\sigma \to \infty$. In this paper the conditions to be satisfied by functions g_i and the structure of the composite solution to obtain a uniformly valid solution of problem (1-2) are studied. The ε-order approximate solution is explicitly determined for some problems with one or two initial and/or terminal boundary layer. A formal solution for boundary layers problems of type (1-2), using the aforementioned composite expansions method, is derived by O'Malley in [**5**], only when the function $\boldsymbol{g}$ is *quasilinear*, and precisely when it is linear in the *fast variables*.

Here $\boldsymbol{g}$ is a nonlinear function of any component of the variable $\boldsymbol{y}$: in this sense the following analysis may be considered as a generalization of the formal results presented in book [**5**]. Explicit results of the proposed analysis will be given in the last Section.

In particular, in Section **2** the case $q_1 = 0$, $q_2 = 1$, in which the state vector $\boldsymbol{y}$ is composed by a slow variable and a fast one, is considered; in Section **3** the more general case $q_1 = q_2 = 1$ with ε-order wide boundary layers at both endpoints is examined. In order to avoid inessential difficulties of treatment, in this paper the case $n = 2$ will be considered. In the last Section the above techniques are applied to determine approximate analytical solutions of two boundary value problems arising from Boltzmann like models.

2 - Problems with a single boundary layer

Calling for simplicity

$$y_1 = u, \quad y_2 = v, \quad g_1 = f, \quad g_2 = g,$$

consider the first-order system (1) rewritten here in the bidimensional form

$$\begin{cases} \dfrac{du}{dx} = f(x, u, v, \varepsilon) \\ \varepsilon \dfrac{dv}{dx} = g(x, u, v, \varepsilon) \end{cases} \tag{4}$$

for the unknown functions $u, v : I \to \mathbb{R}$ subjected to given boundary conditions

$$u(0) = \alpha(v(0), \varepsilon), \qquad v(1) = \beta(u(1), \varepsilon). \tag{5}$$

2.1 - *Initial boundary layer*

Let us seek for a uniformly valid solution of problem (4-5) for $x \in [0, 1]$ and $\varepsilon \to 0$ in the composite form:

$$\begin{aligned} u(x, \varepsilon) &= U(x, \varepsilon) + \xi(\tau, \varepsilon) \\ v(x, \varepsilon) &= V(x, \varepsilon) + \eta(\tau, \varepsilon) \end{aligned} \tag{6}$$

where (U, V) is the *outer* solution, *i.e.* the solution outside the initial boundary-layer region, and the functions (ξ, η) of the streched variable $\tau = x/\varepsilon$ are the *inner* solution, *i.e.* the boundary layer correction terms which occur at the left end point, vanishing outside the boundary layer region and satisfying the matching conditions

$$\lim_{\tau \to \infty} \xi(\tau, \varepsilon) = 0, \tag{7}$$

$$\lim_{\tau \to \infty} \eta(\tau, \varepsilon) = 0. \tag{8}$$

We assume that the functions (U, V), (ξ, η) have asymptotic expansions as $\varepsilon \to 0$

of the type of Eq. (3):

$$\begin{bmatrix} U(x, \varepsilon) \\ V(x, \varepsilon) \\ \xi(\tau, \varepsilon) \\ \eta(\tau, \varepsilon) \end{bmatrix} \sim \sum_{k=0}^{\infty} \begin{bmatrix} U_k(x) \\ V_k(x) \\ \xi_k(\tau) \\ \eta_k(\tau) \end{bmatrix} \varepsilon^k. \tag{9}$$

In the sequel we will omit, unless strictly necessary, the arguments of the functions U_k, V_k, ξ_k and η_k. Thus, the solution to problem (4-5) can be written in the form

$$u(x, \varepsilon) = \sum_{k=0}^{\infty} [U_k + \xi_k]\, \varepsilon^k, \qquad v(x, \varepsilon) = \sum_{k=0}^{\infty} [V_k + \eta_k]\, \varepsilon^k. \tag{10}$$

The problem of convergence of such a procedure is widely treated in the literature quoted in the Introduction, specially in the book [**4**]. Here we will observe that it is assured if the perturbations U_k, ξ_k, V_k, η_k, $k \geq 1$, are bounded $\forall x \in I$. In fact, in the applications shown in Section **4** this property is satisfied.

We will show that solution (6) holds, and it can be approximated to the first order with respect to ε, if the problem (4-5) satisfies the following conditions:

• **C1.** *A function $\varphi(u, x)$ exists, that is continuous and differentiable in a neighbourhood of $U_0(x)$, such that*

$$g(x, u, \varphi(u, x), 0) = 0, \qquad 0 \leq x \leq 1 \tag{11}$$

and such that a solution $U_0(x)$ to the following nonlinear problem exists:

$$\frac{dU_0}{dx} = f(x, U_0, \varphi(U_0, 0), 0), \qquad U_0(1) = \lambda \tag{12}$$

with λ root of the algebraic equation

$$U_0(1) = \beta^{-1}(\varphi(U_0(1), 1)). \tag{13}$$

• **C2.** *A constant $c > 0$ exists, such that*

$$g_{0v} \equiv \frac{\partial g}{\partial v}(x, U_0, V_0, 0) \leq -c \tag{14}$$

for $0 \leq x \leq 1$, with $V_0(x) = \varphi(U_0(x), x)$.

- **C3.** *For the same value of c also the following applies*

$$g_{0v}(0,\ U_0(0),\ \delta,\ 0) \leqslant -c \tag{15}$$

for all values of δ between $V_0(0)$ and $v(0) = \alpha^{-1}(U_0(0),\ 0)$.

In the above conditions, α^{-1} and β^{-1} are the inverse functions of those defined by Eq. (5), and computed for $\varepsilon = 0$.

By using the asymptotic expansions (3), (9) the solution can be approximated to the first order with respect to ε, i.e. by a truncation at $N = 1$ of Eq. (10), as it follows. Taking into account that the outer functions U, V satisfy the system

$$\begin{cases} \dfrac{dU}{dx} = f(x,\ U,\ V,\ \varepsilon) \\ \varepsilon\dfrac{dV}{dx} = g(x,\ U,\ V,\ \varepsilon) \end{cases} \tag{16}$$

the first two coefficients of their expansions satisfy respectively:

$$\frac{dU_0}{dx} = f_0 = f(x,\ U_0,\ V_0,\ 0) \tag{17}$$

$$0 = g_0 = g(x,\ U_0,\ V_0,\ 0) \tag{18}$$

$$\frac{dU_1}{dx} = f_1 = f_{0u}\,U_1 + f_{0v}\,V_1 + f_{0\varepsilon} \tag{19}$$

$$\frac{dV_0}{dx} = g_1 = g_{0u}\,U_1 + g_{0v}\,V_1 + g_{0\varepsilon} \tag{20}$$

where f_{0u}, $\ldots$, $g_{0\varepsilon}$ are partial derivatives calculated for $\varepsilon = 0$:

$$f_{0u} = \frac{\partial f}{\partial u}(x,\ U_0,\ V_0,\ 0), \quad \ldots, \quad g_{0\varepsilon} = \frac{\partial g}{\partial \varepsilon}(x,\ U_0,\ V_0,\ 0).$$

Now we require that the composite solution satisfies the system (4). In a right interval of $x = 0$, the boundary layer correction functions $\xi(\tau,\ \varepsilon)$, $\eta(\tau,\ \varepsilon)$ must satisfy the system

$$
(21)\qquad
\begin{aligned}
\frac{d\xi}{d\tau} &= \varepsilon\{f(\varepsilon\tau, U(\varepsilon\tau,\varepsilon)+\xi(\tau,\varepsilon), V(\varepsilon\tau,\varepsilon)+\eta(\tau,\varepsilon),\varepsilon)-f(\varepsilon\tau, U(\varepsilon\tau,\varepsilon), V(\varepsilon\tau,\varepsilon),\varepsilon)\}\\
\frac{d\eta}{d\tau} &= g(\varepsilon\tau, U(\varepsilon\tau,\varepsilon)+\xi(\tau,\varepsilon), V(\varepsilon\tau,\varepsilon)+\eta(\tau,\varepsilon),\varepsilon)-g(\varepsilon\tau, U(\varepsilon\tau,\varepsilon), V(\varepsilon\tau,\varepsilon),\varepsilon)\equiv\bar{g}(\tau)\,.
\end{aligned}
$$

Inserting expansions (3), (9) into (21) and equating coefficients of like powers of ε for finite values of τ, give a hierarchy of equations for the boundary layer corrections ξ_k, η_k. For $k=0$ we have the differential equations

$$
(22)\qquad \frac{d\xi_0}{d\tau}=0
$$

$$
(23)\qquad \frac{d\eta_0}{d\tau}=g(0, U_0(0), V_0(0)+\eta_0, 0)-g(0, U_0(0), V_0(0), 0)=g(0, U_0(0), V_0(0)+\eta_0, 0).
$$

Taking into account condition (7), Eq. (22) gives

$$
\xi_0(\tau)=\text{const}=\xi_0(0)=0
$$

which shows that the solution $u(x, \varepsilon)$ for the *slow* variable, uniformly converges in I for $\varepsilon\to 0$. For $k=1$ we have the linear system

$$
(24)\qquad \frac{d\xi_1}{d\tau}=f(0, U_0(0), V_0(0)+\eta_0, 0)-f(0, U_0(0), V_0(0), 0)
$$

$$
(25)\qquad \frac{d\eta_1}{d\tau}=\frac{d\bar{g}(\tau)}{d\varepsilon}\bigg|_{\varepsilon=0}=g_{0v}(0, U_0(0), V_0(0)+\eta_0, 0)\cdot\eta_1+A_0(\tau)
$$

where the exponentially decaying term $A_0(\tau)$ is given by

$$
(26)\qquad
\begin{aligned}
A_0(\tau) &= \tau[g_{0x}(0, U_0(0), V_0(0)+\eta_0(\tau), 0)-g_{0x}(0, U_0(0), V_0(0), 0)]\\
&+[\tau U'(0)+U_1(0)][g_{0u}(0, U_0(0), V_0(0)+\eta_0(\tau), 0)-g_{0u}(0, U_0(0), V_0(0), 0)]\\
&+[\tau V'(0)+V_1(0)][g_{0v}(0, U_0(0), V_0(0)+\eta_0(\tau), 0)-g_{0v}(0, U_0(0), V_0(0), 0)]\\
&+\xi_1(\tau)\, g_{0u}(0, U_0(0), V_0(0)+\eta_0(\tau), 0)+g_{0\varepsilon}(0, U_0(0), V_0(0)+\eta_0(\tau), 0)\\
&-g_{0\varepsilon}(0, U_0(0), V_0(0), 0)
\end{aligned}
$$

where the primes denote partial derivatives with respect to $\varepsilon\tau=x$. By using the expansions (3) the boundary conditions (5) become

$$U_0(0) + \varepsilon U_1(0) + \ldots + \varepsilon \xi_1(0) + \ldots \tag{27}$$
$$= \alpha_0(V_0(0), \eta_0(0)) + \varepsilon \alpha_1(V_0(0), \eta_0(0), V_1(0), \eta_1(0)) + \ldots$$

$$V_0(1) + \varepsilon V_1(1) + \ldots + \eta_0(1/\varepsilon) + \varepsilon \eta_1(1/\varepsilon) + \ldots \tag{28}$$
$$= \beta_0(U_0(1)) + \varepsilon \beta_1(U_0(1), U_1(1)) + \ldots$$

being

$$\alpha_0 = \alpha(V_0(0), \eta_0(0), 0),$$

$$\beta_0 = \beta(U_0(1), 0),$$

$$\alpha_1 = \alpha_v(V_0(0) + \eta_0(0), 0)[V_1(0) + \eta_1(0)] + \alpha_\varepsilon(V_0(0) + \eta_0(0), 0)$$

$$\beta_1 = \beta_u(U_0(1), 0)\, U_1(1) + \beta_\varepsilon(U_0(1), 0)$$

where α_v, β_u, α_ε, β_ε are partial derivatives at $\varepsilon = 0$. The other terms of higher order α_k, β_k, $k = 2, 3\ldots$, have analogous expressions, and are linear with respect to $V_k(0) + \eta_k(0)$ and $U_k(1)$. Condition (8) requires that, when $\varepsilon \to 0$, the terms η_k vanish in the first member of Eq. (28), i.e.

$$\lim_{\varepsilon \to 0} \eta_k(1/\varepsilon) = 0, \qquad k = 0, 1, \ldots$$

Therefore, equating in Eqs. (27) and (28) the coefficients of like powers of ε gives

$$U_0(0) = \alpha(V_0(0) + \eta_0(0), 0) \tag{29}$$

$$V_0(1) = \beta(U_0(1), 0)) \tag{30}$$

and for sufficiently small ε:

$$U_1(0) + \xi_1(0) = \alpha_v(V_0(0) + \eta_0(0), 0)[V_1(0) + \eta_1(0)] + \alpha_\varepsilon(V_0(0) + \eta_0(0), 0) \tag{31}$$

$$V_1(1) = \beta_u(U_0(1), 0)\, U_1(1) + \beta_\varepsilon(U_0(1), 0). \tag{32}$$

Equations (17-20) and (22-25), when solved with the conditions provided by Eqs. (29) and (32), supply the unknown functions of the composite solution in the ε-order approximation. Thus the solution can be determined as follows.

Following condition **C1**, Eq. (18) can be written as a (unique) function $V_0(x) = \varphi(U_0(x), x)$, and owing to condition (30) the terminal value of $U_0(x)$ must satisfy Eq. (13). The solution of (17) is therefore

$$U_0(x) = \lambda - \int_x^1 f(s,\, U_0(s),\, \varphi(U_0(s),\, s),\, 0)\, ds\,. \tag{33}$$

The initial value $U_0(0)$ obtained from (33), allows to determine from condition (29) the initial value $\eta_0(0)$ of the boundary layer correction of the fast variable $v(x,\, \varepsilon)$. Integrating (23) with such initial data we find

$$\begin{aligned}\eta_0(\tau) &= \alpha^{-1}(U_0(0)) - \varphi(U_0(0),\, 0) \\ &+ \int_0^\tau [g(0,\, U_0(0),\, \varphi(U_0(0),\, 0) + \eta_0(s),\, 0) - g(0,\, U_0(0),\, \varphi(U_0(0),\, 0),\, 0)]\, ds\end{aligned} \tag{34}$$

that, if condition **C3** is satisfied, is monotonically decreasing to zero for $\tau \to \infty$ (crf. [3], pp. 83-84) as prescribed by Eq. (8). Note that (34) is an integral equation for η_0 which could presumably be solved by successive approximations, but not explicitely. The solution of the *reduced* problem is then determined, and it can be used to obtain the higher-order terms as it will be explained below.

With regard to the outer solution of order ε, we first observe that

$$\frac{dV_0}{dx} = \frac{\partial \varphi}{\partial U_0}\, \frac{dU_0}{dx} + \frac{\partial \varphi}{\partial x} = -\, \frac{g_{0u}}{g_{0v}} f_0 + \varphi_x .$$

Replacing in (20) gives

$$V_1 = \frac{1}{g_{0v}} \left[-\, \frac{g_{0u}}{g_{0v}} f_0 + \varphi_x - g_{0u} U_1 - g_{0\varepsilon} \right]. \tag{35}$$

The terminal value $V_1(1)$ follows condition (32); so $U_1(1)$ must satisfy the algebraic equation

$$\frac{1}{g_{0v}} \left[-\, \frac{g_{0u}}{g_{0v}} f_0 + \varphi_x - g_{0u} U_1 - g_{0\varepsilon} \right]_{x=1} = \beta_u(U_0(1),\, 0)\, U_1(1) + \beta_\varepsilon(U_0(1),\, 0)\,. \tag{36}$$

Eq. (36) is a linearized version of (13), so it will be uniquely solvable if the solution of (13) is locally unique. If μ is a solution of Eq. (36), it determines the terminal condition to assign to the differential equation (19). By substituting (35) we find the linear problem

$$\frac{dU_1}{dx} = a(x,\, U_0)\, U_1 + b(x,\, U_0), \qquad U_1(1) = \mu \tag{37}$$

with

$$a(x,\, U_0) = f_{0u} - \frac{f_{0v}\, g_{0u}}{g_{0v}}, \qquad b(x,\, U_0) = \frac{f_{0v}}{g_{0v}}\left[-\frac{g_{0u}}{g_{0v}} f_0 + \varphi_x - g_{0\varepsilon}\right] + f_{0\varepsilon} \tag{38}$$

that has the following solution:

$$U_1(x) = -\exp\left[\int_x^1 a(s,\, U_0(s))\, ds\right]\left\{\int_x^1 \exp\left[-\int_s^1 a(t,\, U_0(t))\, dt\right]\cdot b(s,\, U_0(s))\, ds - \mu\right\}.$$

Finally let us consider the ε-order boundary layer correction terms. As ξ_1 must vanish when $\tau \to \infty$, we find from (24):

$$\xi_1(\tau) = -\int_\tau^\infty [f(0,\, U_0(0),\, V_0(0) + \eta_0(s),\, 0) - f(0,\, U_0(0),\, V_0(0),\, 0)]\, ds\,. \tag{39}$$

Once $U_1(0)$ and $\xi_1(0)$ are known, the boundary condition (31) gives the initial data of the differential equation (25) for the boundary layer correction term η_1, namely:

$$\eta_1(0) = \frac{U_1(0) + \xi_1(0) - \alpha_\varepsilon(V_0(0) + \eta_0(0),\, 0)}{\alpha(V_0(0),\, \eta_0(0),\, 0)} - V_1(0)$$

and, if joined to Eq. (25), defines an initial value linear problem with the solution

$$\eta_1(\tau) = \exp\left[\int_0^\tau g_{0v}(0,\, U_0(0),\, V_0(0) + \eta_0(s),\, 0)\, ds\right]$$

$$\cdot\left\{\int_0^\tau A_0(s)\exp\left[-\int_0^s g_{0v}(0,\, U_0(0),\, V_0(0) + \eta_0(t),\, 0)\, dt\right] ds + \eta_1(0)\right\}$$

that, considering the hypotheses stated above, decreases exponentially to zero when $\tau \to \infty$, as it is shown in ref. [3]. In conclusion, the ε-order approximated solution of problem (4-5) is:

$$
\begin{aligned}
u^1(x,\varepsilon) &= U_0(x) + \varepsilon\left[U_1(x) + \xi_1\left(\frac{x}{\varepsilon}\right)\right] \\
v^1(x,\varepsilon) &= \varphi(U_0(x), x) + \eta_0\left(\frac{x}{\varepsilon}\right) + \varepsilon\left[V_1(x) + \eta_1\left(\frac{x}{\varepsilon}\right)\right],
\end{aligned} \tag{40}
$$

and the higher-order terms can be found by proceeding as shown above. Generally the solution is not unique, and its multiplicity depends on the number of solutions of the algebraic equations (13) and (36).

2.2 - *Terminal boundary layer*

In this subsection we derive conditions under which the fast variable $v(x,\varepsilon)$ has a *terminal* boundary layer in a left neighbourhood of $x = 1$. These conditions may be stated by introducing the simple transformation $z = 1 - x$ of the space variable and by applying the calculations developed in § **2.1** in order to obtain analogous results. However, since they will be used in the application which follows in Section **4.2**, for sake of clarity we prefer to avoid this transformation and show the new conditions in terms of the original space variable x. Thus, let us modify conditions **C1**...**C3** as follows:

- **C1'-C2'**. *A function* $\varphi(u,x)$ *exists, that satisfy* (11) *and such that a solution to the following problem exists*

$$\frac{dU_0}{dx} = f(x, U_0, \varphi(U_0, 1), 0), \qquad U_0(0) = \lambda$$

with initial condition defined by a solution λ *to the following algebraic equation*

$$U_0(0) = \alpha(\varphi(U_0(0), 1), 0)$$

and such that the derivative g_{0v} *in* (14) *is strictly positive*: $g_{0v}(x, U_0, \varphi(U_0, 1), 0) \geq c$, with $c \in \mathbb{R}_+$.

C3'. *With the same value of* c *and with* δ *between* $V_0(1)$ *and* $v(1) = \beta(U_0(1), 0)$ *we have also*

$$g_{0v}(1, U_0(1), \delta, 0) \geq c\,.$$

Under these conditions the composite solution of the problem (4-5) has to be found in the following form

$$u(x,\ \varepsilon) = U(x,\ \varepsilon) + \chi(\sigma,\ \varepsilon)$$
$$v(x,\ \varepsilon) = V(x,\ \varepsilon) + \zeta(\sigma,\ \varepsilon) \tag{41}$$

where $\chi(\sigma,\ \varepsilon)$, $\zeta(\sigma,\ \varepsilon)$ are boundary layer corrections depending on the stretched variable $\sigma = (1-x)/\varepsilon \in [0,\ +\infty)$, endowed with asymptotic expansions for $\varepsilon \to 0$ and such that vanish outside the terminal boundary layer.

By proceeding in a similar way as above, with obvious modifications, problem (4-5) has in this case the following solution, approximated to ε-order terms:

$$u^1(x,\ \varepsilon) = U_0(x) + \varepsilon\left[U_1(x) + \chi_1\left(\frac{1-x}{\varepsilon}\right)\right]$$
$$v^1(x,\ \varepsilon) = \varphi(U_0(x),\ x) + \zeta_0\left(\frac{1-x}{\varepsilon}\right) + \varepsilon\left[V_1(x) + \zeta_1\left(\frac{1-x}{\varepsilon}\right)\right]$$

where:

$$U_0(x) = \lambda + \int_0^x f(s,\ U_0(s),\ \varphi(U_0(s),\ s),\ 0)\, ds$$
$$U_1(x) = \exp\left[\int_0^x a(s,\ U_0(s))\, ds\right]\left\{\int_0^x \exp\left[-\int_0^s a(t,\ U_0(t))\, dt\right]\cdot b(s,\ U_0(s))\, ds + \mu\right\} \tag{42}$$

with $\mu = U_1(0)$ root of

$$\mu = \alpha_{0\varepsilon} + \left[\frac{\alpha_{0v}}{g_{0v}}\left(-\frac{g_{0u}}{g_{0v}} f_0 + \varphi_x - g_{0u}\mu - g_{0\varepsilon}\right)\right]_{x=0}$$

and moreover:

$$\chi_1(\sigma) = \int_\sigma^\infty [f(1,\ U_0(1),\ V_0(1) + \zeta_0(s),\ 0) - f(1,\ U_0(1),\ V_0(1),\ 0)]\, ds \tag{43}$$

$$\zeta_0(\sigma) = \beta(U_0(1),\ 0) - \varphi(U_0(1),\ 1)$$
$$- \int_0^\sigma [g(1,\ U_0(1),\ \varphi(U_0(1),\ 1) + \zeta_0(s),\ 0) - g(1,\ U_0(1),\ \varphi(U_0(1),\ 1),\ 0)]\, ds \tag{44}$$

$$\zeta_1(\sigma) = \exp\left[-\int_0^\sigma g_{0v}(1, U_0(1), V_0(1) + \zeta_0(s), 0)\, ds\right]$$
$$\cdot\left\{\int_0^\sigma B_0(s) \exp\left[\int_0^s g_{0v}(1, U_0(1), V_0(1) + \zeta_0(t), 0)\, dt\right] ds + \zeta_1(0)\right\} \tag{45}$$

with:

$$\zeta_1(0) = \beta_u(U_0(1), 0)[U_1(1) + \chi_1(0)] + \beta_\varepsilon(U_0(1), 0) - V_1(1),$$

$$B_0(\sigma) = \sigma[g_{0x}(1, U_0(1), V_0(1) + \zeta_0(\sigma), 0) - g_{0x}(1, U_0(1), V_0(1), 0)]$$
$$+ [\sigma U_0'(1) + U_1(1)][g_{0u}(1, U_0(1), V_0(1) + \zeta_0(\sigma), 0) - g_{0u}(1, U_0(1), V_0(1), 0)]$$
$$+ [\sigma V_0'(1) + V_1(1)][g_{0v}(1, U_0(1), V_0(1) + \zeta_0(\sigma), 0) - g_{0v}(1, U_0(1), V_0(1), 0)]$$
$$+ \chi_1(\sigma)\, g_{0u}(1, U_0(1), V_0(1) + \zeta_0(\sigma), 0) + g_{0\varepsilon}(1, U_0(1), V_0(1) + \zeta_0(\sigma), 0)$$
$$- g_{0\varepsilon}(1, U_0(1), V_0(1), 0).$$

3 - Problems with boundary layers at both endpoints

In this Section we consider the case: $n = 2$, $q_1 = q_2 = 1$ where Eq. (1) is written in the form

$$\varepsilon \frac{d\boldsymbol{y}}{dx} = \boldsymbol{g}(x, \boldsymbol{y}, \varepsilon), \qquad x \in I = [0, 1] \tag{46}$$

where the small parameter ε multiplies both the components of the derivative of the unknown vector $\boldsymbol{y} = (y_1, y_2)^T$. To satisfy the conditions

$$\begin{aligned} y_1(0) &= \alpha(y_2(0), \varepsilon) \\ y_2(1) &= \beta(y_1(1), \varepsilon), \end{aligned} \tag{47}$$

both the components y_1 and y_2 may undergo strong changes in an initial or terminal boundary layer. A suitable approximation to the solution of problem (46-47), uniformly valid in I for $\varepsilon \to 0$, must be searched in a composite form consisting of an outer approximation (Y_1, Y_2) valid outside the boundary layer regions for $0 < x < 1$, plus initial or terminal boundary layer correction terms (ξ, η) or (χ, ζ) which are to be defined on the basis of properties of the nonlinear function $\boldsymbol{g}$

$= (g_1, g_2)$. Namely, by assuming that g_i, Y_i have asymptotic expansions:

$$i=1,2: \quad Y_i(x,\varepsilon) \sim \sum_{k=0}^{\infty} Y_{ik}(x)\,\varepsilon^k, \quad g_i(x,\boldsymbol{Y},\varepsilon) \sim \sum_{k=0}^{\infty} g_{ik}(x,\boldsymbol{Y})\,\varepsilon^k, \quad \boldsymbol{Y} = (Y_1, Y_2)$$

and by assuming the existence of the outer solution $(Y_{10}(x), Y_{20}(x))$ of the *reduced problem* that is defined by Eq. (46) with $\varepsilon = 0$, we will show that the boundary layer functions have to be choosen on the basis of the following

Proposition 1. *If the jacobian matrix $\boldsymbol{g}_y$ is non-singular in I, and two positive constants c_1, c_2 exist such that for $0 \leqslant x \leqslant 1$ it results*

$$\frac{\partial g_1}{\partial y_1}(x, Y_{10}, Y_{20}, 0) \geqslant c_1; \qquad \frac{\partial g_2}{\partial y_2}(x, Y_{10}, Y_{20}, 0) \leqslant -c_2 \tag{48}$$

with

$$\frac{\partial g_1}{\partial y_1}(1, \delta_1, Y_{20}(1), 0) \geqslant c_1$$

for all values of δ_1 between $Y_{10}(1)$ and $\beta^{-1}(Y_{20}(1), 0)$ and

$$\frac{\partial g_2}{\partial y_2}(0, Y_{10}(0), \delta_2, 0) \leqslant -c_2$$

for all values of δ_2 between $Y_{20}(0)$ and $\alpha^{-1}(Y_{10}(0), 0)$, then the solution of problem (46-47) *has the form*

$$y_1(x,\varepsilon) = \sum_{k=0}^{\infty}\left[Y_{1k}(x) + \chi_k\left(\frac{1-x}{\varepsilon}\right)\right]\varepsilon^k, \qquad y_2(x,\varepsilon) = \sum_{k=0}^{\infty}\left[Y_{2k}(x) + \eta_k\left(\frac{x}{\varepsilon}\right)\right]\varepsilon^k \tag{49}$$

where $y_1(x,\varepsilon)$ and $y_2(x,\varepsilon)$ are, for $\varepsilon \to 0$, non uniformly convergent respectively in a terminal and initial boundary layer.

This Proposition may be replaced by the following equivalent one.

Proposition 2. *If the jacobian matrix $\boldsymbol{g}_y$ is non-singular in I and two positive costants c_1, c_2 exist such that for $0 \leqslant x \leqslant 1$ it results*

$$\frac{\partial g_1}{\partial y_1}(x, Y_{10}, Y_{20}, 0) \leqslant -c_1; \qquad \frac{\partial g_2}{\partial y_2}(x, Y_{10}, Y_{20}, 0) \geqslant c_2 \tag{50}$$

with

$$\frac{\partial g_1}{\partial y_1}(0,\,\delta_1,\,Y_{20}(0),\,0) \leqslant -c_1$$

for all values of δ_1 *between* $Y_{10}(0)$ *and* $\alpha(Y_{20}(0),0)$, *and in addition*

$$\frac{\partial g_2}{\partial y_2}(1,\,Y_{10}(1),\,\delta_2,\,0) \geqslant c_2$$

for all values of δ_2 *between* $Y_{20}(1)$ *and* $\beta(Y_{10}(1),0)$, *then* $y_1(x,\varepsilon)$ *and* $y_2(x,\varepsilon)$ *have respectively an initial and terminal boundary layer, and the solution must be sought in the form*

$$y_1(x,\varepsilon)=\sum_{k=0}^{\infty}\left[Y_{1k}(x)+\xi_k\left(\frac{x}{\varepsilon}\right)\right]\varepsilon^k, \qquad y_2(x,\varepsilon)=\sum_{k=0}^{\infty}\left[Y_{2k}(x)+\zeta_k\left(\frac{1-x}{\varepsilon}\right)\right]\varepsilon^k. \tag{51}$$

Equations (48) and (50) represent the hyperbolicity conditions of the jacobian of $\boldsymbol{g}(x,\boldsymbol{Y},\varepsilon)$, if calculated in terms of the outer solution of the *reduced problem*, and are analogous to the ones introduced in [**3**] to study singularly perturbed linear and quasi-linear systems.

Proofs. The terms of the outer solution must satisfy, for $0 \leqslant x \leqslant 1$, the equations

$$0=\boldsymbol{g}(x,\,\boldsymbol{Y}_0,\,0) \tag{52}$$

$$\frac{d\boldsymbol{Y}_0}{dx}=\boldsymbol{g}_{y0}\boldsymbol{Y}_1+\boldsymbol{g}_{\varepsilon 0} \tag{53}$$

where $\boldsymbol{g}_{y0}$ is the jacobian matrix of $\boldsymbol{g}$ calculated for $\varepsilon=0$ and $\boldsymbol{g}_{\varepsilon 0}$ is the vector of partial derivatives with respect to ε, calculated for $\varepsilon=0$. The zero and ε-order boundary layer corrections must satisfy:

$$\frac{d\chi_0}{d\sigma}=-g_1(1,\,Y_{10}(1)+\chi_0,\,Y_{20}(1),\,0)+g_1(1,\,Y_{10}(1),\,Y_{20}(1),\,0) \tag{54}$$

$$\frac{d\eta_0}{d\tau}=g_2(0,\,Y_{10}(0),\,Y_{20}(0)+\eta_0,\,0)-g_2(0,\,Y_{10}(0),\,Y_{20}(0),\,0) \tag{55}$$

$$\frac{d\chi_1}{d\sigma}=-\frac{\partial g_1}{\partial y_1}(1,\,Y_{10}(1)+\chi_0,\,Y_{20}(1),\,0)\cdot\chi_1+B_0(\sigma) \tag{56}$$

$$\frac{d\eta_1}{d\tau}=\frac{\partial g_2}{\partial y_2}(0,\,Y_{10}(0),\,Y_{20}(0)+\eta_0,\,0)\cdot\eta_1+A_0(\tau) \tag{57}$$

where $A_0(\tau)$ and $B_0(\sigma)$ are the following known functions of the zero-order solution:

$$(58)\quad \begin{aligned} A_0(\tau) = \tau &\left\{ \frac{\partial g_2}{\partial x}(0, Y_{10}(0), Y_{20}(0) + \eta_0(\tau), 0) - \frac{\partial g_2}{\partial x}(0, Y_{10}(0), Y_{20}(0),0) \right\} \\ + &\left[\tau \frac{dY_{10}}{dx}(0) + Y_{11}(0) \right] \left\{ \frac{\partial g_2}{\partial y_1}(0, Y_{10}(0), Y_{20}(0) + \eta_0(\tau),0) - \frac{\partial g_2}{\partial y_1}(0, Y_{10}(0), Y_{20}(0),0) \right\} \\ + &\left[\tau \frac{dY_{20}}{dx}(0) + Y_{21}(0) \right] \left\{ \frac{\partial g_2}{\partial y_2}(0, Y_{10}(0), Y_{20}(0) + \eta_0(\tau),0) - \frac{\partial g_2}{\partial y_2}(0, Y_{10}(0), Y_{20}(0),0) \right\} \\ &+ \frac{\partial g_2}{\partial \varepsilon}(0, Y_{10}(0), Y_{20}(0) + \eta_0(\tau),0) - \frac{\partial g_2}{\partial \varepsilon}(0, Y_{10}(0), Y_{20}(0),0) \end{aligned}$$

$$(59)\quad \begin{aligned} B_0(\sigma) = \sigma &\left\{ \frac{\partial g_1}{\partial x}(1, Y_{10}(1) + \chi_0(\sigma), Y_{20}(1),0) - \frac{\partial g_1}{\partial x}(1, Y_{10}(1), Y_{20}(1),0)\} \right\} \\ + &\left[\sigma \frac{dY_{10}}{dx}(1) + Y_{11}(1) \right] \left\{ \frac{\partial g_1}{\partial y_1}(1, Y_{10}(1) + \chi_0(\sigma), Y_{20}(1),0) - \frac{\partial g_1}{\partial y_1}(1, Y_{10}(1), Y_{20}(1),0) \right\} \\ + &\left[\sigma \frac{dY_{20}}{dx}(1) + Y_{21}(1) \right] \left\{ \frac{\partial g_1}{\partial y_2}(1, Y_{10}(1) + \chi_0(\sigma), Y_{20}(1),0) - \frac{\partial g_1}{\partial y_2}(1, Y_{10}(1), Y_{20}(1),0) \right\} \\ &+ \frac{\partial g_1}{\partial \varepsilon}(1, Y_{10}(1) + \chi_0(\sigma), Y_{20}(1)),0) - \frac{\partial g_1}{\partial \varepsilon}(1, Y_{10}(1), Y_{20}(1),0)\,. \end{aligned}$$

Taking into account that terms χ_k, η_k vanish outside the corresponding boundary layers, the boundary conditions (47) give for $\varepsilon \to 0$:

$$(60)\qquad Y_{10}(0) = \alpha(Y_{20}(0) + \eta_0(0),\ 0)$$

$$(61)\qquad Y_{20}(1) = \beta(Y_{10}(1) + \chi_0(0),\ 0)$$

$$(62)\qquad Y_{11}(0) = \frac{\partial \alpha}{\partial y_2}(Y_{20}(0) + \eta_0(0),\ 0)\cdot[Y_{21}(0) + \eta_1(0)] + \alpha_{\varepsilon 0}$$

$$(63)\qquad Y_{21}(1) = \frac{\partial \beta}{\partial y_1}(Y_{10}(1) + \chi_0(0),\ 0)\cdot[Y_{11}(1) + \chi_1(0)] + \beta_{\varepsilon 0}.$$

According to the hypotheses, since the jacobian is non-singular two differentiable functions φ_1, φ_2 exist that satisfy Eq. (53) and represent the outer solution of the problem obtained by (46-47) for $\varepsilon \to 0$:

$$
\begin{aligned}
Y_{10}(x) &= \varphi_1(x) \\
Y_{20}(x) &= \varphi_2(x)\,.
\end{aligned} \tag{64}
$$

Eqs. (64) do not necessarily satisfy the boundary conditions. Therefore we must correct the solution of the reduced problem with boundary layer terms described by the differential equations (54-55). These equations, subject to initial conditions $\chi_0(0)$, $\eta_0(0)$ deduced from (60) and (61), yield:

$$
\begin{aligned}
&\chi_0(\sigma) = \beta^{-1}(\varphi_2(1), 0) - \varphi_1(1) \\
&- \int_0^\sigma \{g_1(1, \varphi_1(1) + \chi_0(s), \varphi_2(1), 0) - g_1(1, \varphi_1(1), \varphi_2(1), 0))\}\, ds
\end{aligned} \tag{65}
$$

$$
\begin{aligned}
&\eta_0(\tau) = \alpha^{-1}(\varphi_1(0), 0) - \varphi_2(0) \\
&+ \int_0^\tau \{g_2(0, \varphi_1(0), \varphi_2(0) + \eta_0(s), 0), 0) - g_2(0, \varphi_1(0), \varphi_2(0), 0)\}\, ds
\end{aligned} \tag{66}
$$

which decay to zero at infinity owing to the hypotheses stated in Proposition 1, and determine the boundary layer correction terms to account the non-uniform convergence of Y_{10} and Y_{20} near the end points.

As regard to the ε-order terms, we observe that equation (53) supplies the outer solution:

$$
\boldsymbol{Y}_1 \equiv \begin{bmatrix} Y_{11} \\ Y_{21} \end{bmatrix} = \boldsymbol{g}_{y0}^{-1}\left(\frac{d\varphi}{dx} - \boldsymbol{g}_{\varepsilon 0}\right) \tag{67}
$$

where $d\varphi/dx$ is the vector of the derivatives of the known functions $\varphi_1(x)$, $\varphi_2(x)$. Inserting the terminal values of Y_{11}, Y_{21} in Eqs. (62) and (63), the above outer solution can be solved with respect to $\chi_1(0)$ and $\eta_1(0)$:

$$
\begin{aligned}
\chi_1(0) &= \frac{Y_{21}(1) - \beta_{\varepsilon 0}}{\dfrac{\partial \beta}{\partial y_1}(Y_{10}(1) + \chi_0(0), 0)} - Y_{11}(1) \\
\eta_1(0) &= \frac{Y_{11}(0) - \alpha_{\varepsilon 0}}{\dfrac{\partial \alpha}{\partial y_2}(Y_{20}(0) + \eta_0(0), 0)} - Y_{21}(0)
\end{aligned} \tag{68}
$$

in order to determine (if the partial derivatives appearing in the right sides do not vanish) the initial data for the linear equations (56-57) whose solutions $\xi_1(\sigma)$, $\eta_1(\tau)$, according to the hypotheses, are exponentially vanishing for $\sigma, \tau \to \infty$. In this way the ε-order approximate solution of problem (46-47) is determined; its component $y_1(x, \varepsilon)$ has a terminal boundary layer, while $y_2(x, \varepsilon)$ has an initial boundary layer.

If on the contrary the function $\boldsymbol{g}(x, \boldsymbol{Y}, \varepsilon)$ satisfies the hypotheses of Proposition 2 with regard to the signs of the partial derivatives $\partial g_i/\partial y_j$ defined in (50), by an analogous treatment we can construct a solution in the form (51) where the regions of non-uniform convergence are inverted, i.e. y_1 is characterized by an *initial* boundary layer, while y_2 has a *terminal* boundary layer. To complete the composite solution (51), we can obtain the boundary layer corrections terms by solving the following problems:

$$\frac{d\xi_0}{d\tau} = g_1(0, Y_{10}(0) + \xi_0, Y_{20}(0), 0) - g_1(0, Y_{10}(0), Y_{20}(0), 0) \tag{69}$$

$$\frac{d\zeta_0}{d\sigma} = -g_2(1, Y_{10}(1), Y_{20}(1) + \zeta_0, 0) + g_2(1, Y_{10}(1), Y_{20}(1), 0) \tag{70}$$

$$\frac{d\xi_1}{d\tau} = \frac{\partial g_1}{\partial y_1}(0, Y_{10}(0) + \xi_0, Y_{20}(0), 0)\cdot\xi_1 + A_0(\tau) \tag{71}$$

$$\frac{d\zeta_1}{d\sigma} = -\frac{\partial g_2}{\partial y_2}(1, Y_{10}(1), Y_{20}(1) + \zeta_0, 0)\cdot\zeta_1 + B_0(\sigma) \tag{72}$$

where $A_0(\tau)$ and $B_0(\sigma)$ are derived from (58-59) by inverting the two components of $\boldsymbol{g}$, and the initial conditions are respectively

$$\xi_0(0) = \alpha(\varphi_2(0), 0) - \varphi_1(0) \tag{73}$$

$$\zeta_0(0) = \beta(\varphi_1(1), 0) - \varphi_2(1) \tag{74}$$

$$\xi_1(0) = \frac{\partial \alpha}{\partial y_2}(\varphi_2(0), 0)\cdot Y_{21}(0) + \alpha_{\varepsilon 0} - Y_{11}(0) \tag{75}$$

$$\zeta_1(0) = \frac{\partial \beta}{\partial y_1}(\varphi_1(1), 0)\cdot Y_{11}(1) + \beta_{\varepsilon 0} - Y_{21}(1). \qquad \blacksquare \tag{76}$$

Higher-order terms in the obtained composite solutions can be calculated, as usual, by using all the terms of lower order appearing in the asymptotic expansions. Obviously, for the convergence of the method, the perturbation terms must satisfy the properties already recalled in Section **2**.

4 - Applications

Singularly perturbed problems are well known in the context of kinetic theory and fluid dynamic equations. In particular boundary layer solutions for models of the Boltzmann equation can be found in [**7**]. In this Section the techniques here proposed are applied to determine approximate analytical solutions of two boundary value problems arising from stationary Boltzmann like equations. Contrary to ref. [**7**], these two problems are entirely treated at the microscopic scale. In particular, the theory developed in Section **3** is used to study a modified version of the well-known Ruijgrok-Wu model, and the techniques of Section **2** are applied, as a second example, to study a model for a mixture of reacting gases with two chemical species of disparate molecular masses.

4.1 - *A model of the Ruijgrok-Wu type*

Two-velocity models of the Boltzmann equation of the Ruijgrok-Wu type [**9**] have been studied by assuming a diffusive scaling of the kinetic equations describing the time and space evolution of a finite mass gas in a medium [**10**], [**8**]. In particular the model equations, at a molecular scale, will take into account:

- absorbtion and scattering effects of the gas particles by the host medium;
- particle source terms, distributed on the interval $[0, 1]$ of the x-axis;
- elastic collisions between the gas particles.

Then in the stationary case the kinetic equations may be written as

$$(77) \qquad \begin{cases} \varepsilon \dfrac{dy_1}{dx} = g_1(x, y_1, y_2, \varepsilon) = -\alpha y_1 + \beta y_2 + \varepsilon\gamma F(y_1)\, G(y_2) - \delta_1 y_1 + s_1(x) \\[2ex] -\varepsilon \dfrac{dy_2}{dx} = -g_2(x, y_1, y_2, \varepsilon) = \alpha y_1 - \beta y_2 - \varepsilon\gamma F(y_1)\, G(y_2) - \delta_2 y_2 + s_2(x) \end{cases}$$

where $y_1(x)$, $y_2(x) \in \mathbb{R}_+$ are the number densities of the molecules moving at equal speeds $\pm c$ on the x-axis, respectively; $\varepsilon = 1/c$ is the small parameter; F and G are bounded functions taking into account particle scattering; $s_1(x)$ and $s_2(x)$ are $C^1[0, 1]$ positive functions representing sources; and α, β, γ, $\delta_{1,2}$ are non-negative real constants. In particular, α and β represent cross sections of the particle scattering with the host medium; γ is the cross section of elastic particle collisions, and $\delta_{1,2}$ are the total absorption cross sections.

Let us seek for a solution of Eq. (77) satisfying the following boundary conditions

$$y_1(0) = k_0 y_2(0) \tag{78}$$

$$y_2(1) = k_1 y_1(1) \tag{79}$$

describing either absorbtion of the gas molecules by the walls, if the non-negative constants k_0, k_1 are lower than unity, or pure specular reflection if $k_0 = k_1 = 1$.

Since the partial derivatives of g_1, g_2 satisfy hypothesis (50), the composite solutions to the problem (77)-(78, 79), truncated at the terms of order ε, will have the form

$$y_1(x, \varepsilon) = Y_{10}(x) + \xi_0(\tau) + \varepsilon[Y_{11}(x) + \xi_1(\tau)] + O(\varepsilon^2)$$

$$y_2(x, \varepsilon) = Y_{20}(x) + \zeta_0(\sigma) + \varepsilon[Y_{21}(x) + \zeta_1(\sigma)] + O(\varepsilon^2)$$

where $\xi(\tau, \varepsilon)$, $\zeta(\sigma, \varepsilon)$ are, respectively, initial and terminal boundary layer corrections to the outer solution $\boldsymbol{Y}(x, \varepsilon) = (Y_1, Y_2)$ for the two densities.

The latter one has zero-order terms that can be found as the unique solution of $\boldsymbol{g}(x, Y_1, Y_2, 0) = 0$. By setting $C = \alpha\delta_2 + \beta\delta_1 + \delta_1\delta_2 > 0$, it is given by

$$Y_{10}(x) = \varphi_1(x) = \frac{1}{C}[(\beta + \delta_2)\, s_1(x) + \beta s_2(x)] \tag{80}$$

$$Y_{20}(x) = \varphi_2(x) = \frac{1}{C}[\alpha s_1(x) + (\alpha + \delta_1)\, s_2(x)] \tag{81}$$

and is non-negative for all $x \in [0, 1]$, under the assumptions made for the above coefficients. Therefore, using the boundary conditions (73), (74) and integrating Eqs. (69), (70), the correction terms in the solution of the reduced problem can be found as

$$\xi_0(\tau) = [k_0 \varphi_2(0) - \varphi_1(0)]\, e^{-(\alpha + \delta_1)\tau} \tag{82}$$

$$\zeta_0(\sigma) = [k_1 \varphi_1(1) - \varphi_2(1)]\, e^{-(\beta + \delta_2)\sigma}. \tag{83}$$

The above zero-order solution is now used to determine the further terms in the composite expansion of the solution. The outer terms of order ε are given by Eq. (67), that for the present problem yields

(84) $$Y_{11}(x) = \frac{1}{C}[(\beta + \delta_2)\varphi_1'(x) - \beta\varphi_2'(x) - \gamma\delta_2 F(\varphi_1(x))\, G(\varphi_2(x))]$$

(85) $$Y_{21}(x) = \frac{1}{C}[\alpha\varphi_1'(x) - (\alpha + \delta_1)\varphi_2'(x) + \gamma\delta_1 F(\varphi_1(x))\, G(\varphi_2(x))]$$

where the primes denote derivatives with respect to x of the previous solution terms. The values at the boundaries of these outer terms satisfy:

$$Y_{11}(0) + \xi_1(0) = k_0 Y_{21}(0)$$

$$Y_{21}(1) + \zeta_1(0) = k_1 Y_{11}(1)$$

and determine the initial conditions that must be assigned to the linear differential equations (71) and (72) for the unknown correction terms $\xi_1(\tau)$, $\zeta_1(\sigma)$. In our problem these equations are rewritten as

(86) $$\frac{d\xi_1}{d\tau} = -(\alpha + \delta_1)\,\xi_1(\tau) + A_0(\tau)$$

(87) $$\frac{d\zeta_1}{d\sigma} = -(\beta + \delta_2)\,\zeta_1(\sigma) + B_0(\sigma),$$

where the non-homogeneous terms are known functions of the zero-order solution:

$$A_0(\tau) = \gamma[F(\varphi_1(0) + \xi_0(\tau)) - F(\varphi_1(0))]\, G(\varphi_2(0))$$

$$B_0(\sigma) = \gamma F(\varphi_1(1))[G(\varphi_2(1) + \zeta_0(\sigma)) - G(\varphi_2(1))].$$

Therefore, integrating Eqs. (86) and (87) one obtains

$$\xi_1(\tau) = e^{-(\alpha+\delta_1)\tau}\left[\int_0^\tau A_0(t)\, e^{(\alpha+\delta_1)t} dt + k_0 Y_{21}(0) - Y_{11}(0)\right]$$

$$\zeta_1(\sigma) = e^{-(\beta+\delta_2)\sigma}\left[\int_0^\sigma B_0(t)\, e^{(\beta+\delta_2)t} dt + k_1 Y_{11}(1) - Y_{21}(1)\right]$$

which, under wide conditions on the nonlinear functions $F(y_1)$ and $G(y_2)$, are asymptotically vanishing as $\tau, \sigma \to +\infty$, as prescribed by the matching method to all terms of the boundary layer corrections.

4.2 - *Mixture of reacting gases*

A two-velocity model for a finite mass mixture of gas particles has been proposed under suitable assumptions in ref. [**11**]. The gas mixture is formed by two chemical species of disparate molecular masses m_A, m_B with $m_A/m_B = \varepsilon \ll 1$, undergoing an irreversible chemical reaction (recombination) of type $A + B \rightarrow C$. If in addition source terms distributed on the interval $x \in [0, 1]$ are present, then the kinetic equations derived in [**11**] are modified, in the stationary case, into the following two subsystems:

$$\text{(88)} \qquad \begin{cases} \dfrac{du^+}{dx} = f_1(u^+, v^-) = \dfrac{1}{c}[-ku^+v^- + s_1(x)] \\ -\varepsilon\dfrac{dv^-}{dx} = -g_1(u^+, v^-) = \dfrac{1}{c}[-ku^+v^- + s_4(x)] \end{cases}$$

$$\text{(89)} \qquad \begin{cases} -\dfrac{du^-}{dx} = -f_2(u^-, v^+) = \dfrac{1}{c}[-ku^-v^+ + s_2(x)] \\ \varepsilon\dfrac{dv^+}{dx} = g_2(u^-, v^+) = \dfrac{1}{c}[-ku^-v^+ + s_3(x)] \end{cases}$$

where $u^+(x)$, $u^-(x)$: $C^0[0, 1] \rightarrow \mathbb{R}_+$ are the densities of the particles of species A moving, respectively, along the positive and the negative directions of the x-axis; $v^+(x)$, $v^-(x)$: $C^0[0, 1] \rightarrow \mathbb{R}_+$ are the densities of species B, defined analogously; $s_i(x)$ are, as in the previous application, $C^1[0, 1]$ positive functions; k is the collisional frequency to be considered as a known positive constant, and c is a reference velocity.

Let us assume that the effects of the molecular interactions at the walls $x = 0$ and $x = 1$ can be described as

$$\text{(90)} \qquad u^+(0) = \alpha_0 v^-(0), \qquad v^-(1) = \beta_0 u^+(1)$$

$$\text{(91)} \qquad u^-(0) = \alpha_1 v^+(0), \qquad v^+(1) = \beta_1 u^-(1)$$

with $\alpha_0, \ldots, \beta_1 \leq 1$ are prescribed positive constants. If their value is chosen lower than unity, it denotes partial absorption by the walls, whereas $\alpha_0 = \ldots = \beta_1 = 1$ mean specular reflection of the two chemical species. Then Eq. (90) yields the boundary conditions for the subsystem (88), and Eq. (91) prescribes the boundary conditions for the subsystem (89).

Since in the kinetic equations we have

$$\frac{\partial g_1}{\partial v^-} = \frac{k}{c} u^+ > 0, \qquad \frac{\partial g_2}{\partial v^+} = -\frac{k}{c} u^- < 0,$$

it follows that as $\varepsilon \to 0$ the fast variables $v^-(x)$ and $v^+(x)$ have non uniform convergence, respectively, in a *terminal* and in an *initial* boundary layer. Therefore the N-order approximate solution to problem (88-90) must be searched in the form

$$u^+(x,\varepsilon) \simeq \sum_{k=0}^{N} [U_k^+(x) + \chi_k(\sigma)]\,\varepsilon^k, \quad v^-(x,\varepsilon) \simeq \sum_{k=0}^{N} [V_k^-(x) + \zeta_k(\sigma)]\,\varepsilon^k, \quad \sigma = \frac{1-x}{\varepsilon}$$

and the one related to problem (89-91) in the form

$$u^-(x,\varepsilon) \simeq \sum_{k=0}^{N} [U_k^-(x) + \xi_k(\tau)]\,\varepsilon^k, \quad v^+(x,\varepsilon) \simeq \sum_{k=0}^{N} [V_k^+(x) + \eta_k(\tau)]\,\varepsilon^k, \quad \tau = \frac{x}{\varepsilon}.$$

The solution to the *reduced problem* for (88-90) is determined as follows. Owing to (90) the zero-order terms must satisfy the boundary conditions

$$U_0^+(0) = \alpha_0 V_0^-(0), \qquad V_0^-(1) + \zeta_0(0) = \beta_0 U_0^+(1). \tag{92}$$

The equation $g_1(U_0^+, V_0^-) = 0$ is satisfied by

$$V_0^- = \varphi(U_0^+(x), x) = \frac{s_4(x)}{kU_0^+(x)} \tag{93}$$

where $U_0^+(0)$ is determined by Eq. (92) as

$$U_0^+(0) = \sqrt{\frac{\alpha_0 s_4(0)}{k}}.$$

This initial condition is used to calculate $U_0^+(x)$ from Eq. (42), with the result

$$U_0^+(x) = \sqrt{\frac{\alpha_0 s_4(0)}{k}} + \frac{1}{c}\int_0^x [s_1(t) - s_4(t)]\,dt. \tag{94}$$

Since it is required to be strictly positive, the assumed source terms must satisfy for each $x \in [0, 1]$ the additional condition

$$\int_0^x s_1(t)\,dt > \int_0^x s_4(t)\,dt - c\sqrt{\frac{\alpha_0 s_4(0)}{k}}. \tag{95}$$

The zero-order corrections in the terminal boundary layer are $\chi_0(\sigma) \equiv 0$ and

$\zeta_0(\sigma)$ whose initial condition, from Eq. (92), is

$$\zeta_0(0) = \beta_0 U_0^+(1) - \varphi(U_0^+(1), 1).$$

Therefore, integration of Eq. (44) yields

$$\zeta_0(\sigma) = [\beta_0 U_0^+(1) - \varphi(U_0^+(1), 1)] \exp\left(-\frac{kU_0^+(1)}{c}\sigma\right). \tag{96}$$

The reduced problem for (89-91) can be solved by satisfying the boundary conditions

$$U_0^-(0) = \alpha_1[V_0^+(0) + \eta_0(0)], \qquad V_0^+(1) = \beta_1 U_0^-(1). \tag{97}$$

By solving $g_2(U_0^-, V_0^+) = 0$ one has

$$V_0^+(x) = \varphi(U_0^-(x), x) = \frac{s_3(x)}{kU_0^-(x)} \tag{98}$$

where $U_0^-(x)$ is determined by using Eq. (33) and the second of Eqs. (97):

$$U_0^-(x) = \sqrt{\frac{s_3(1)}{k\beta_1}} - \frac{1}{c}\int_x^1 [s_3(t) - s_2(t)]\, dt\,. \tag{99}$$

Again positivity of $U_0^-(x)$ requires that

$$\int_x^1 s_2(t)\, dt > \int_x^1 s_3(t)\, dt - c\sqrt{\frac{s_3(1)}{k\beta_1}}\,. \tag{100}$$

The zero-order corrections in the initial boundary layer are $\xi(\tau) \equiv 0$ and $\eta(\tau)$ that must satisfy the initial condition prescribed in the first of Eq. (97). By also applying Eq. (34) one obtains

$$\eta_0(\tau) = \left[\frac{U_0^-(0)}{\alpha_1} - \varphi(U_0^-(0), 0)\right] \exp\left(-\frac{kU_0^-(0)}{c}\tau\right).$$

It follows that the zero-order solutions for the fast variables, which satisfy the prescribed boundary conditions and are uniformly valid in the interval $0 \leqslant x \leqslant 1$ under the action of source terms satisfying Eqs. (95), (100), are

$$v^-(x, \varepsilon) = \frac{s_4(x)}{kU_0^+(x)} + \left[\beta_0 U_0^+(1) - \frac{s_4(1)}{kU_0^+(1)}\right] \exp\left[-\frac{kU_0^+(1)}{\varepsilon c}(1-x)\right]$$

$$v^+(x, \varepsilon) = \frac{s_3(x)}{kU_0^-(x)} + \left[\frac{U_0^-(0)}{\alpha_1} - \frac{s_3(0)}{kU_0^-(0)}\right] \exp\left(-\frac{kU_0^-(0)\, x}{\varepsilon c}\right)$$

where $U_0^+(x)$, $U_0^-(x)$ are given by Eqs. (94) and (99). Once the (unique) solutions to the reduced problems have been determined, the $\varepsilon -$order terms and the further ones in the composite expansions can now be calculated by a straightforward application of the procedure given in Section **2**.

5 - Conclusions

This paper supplies a methodology which can be applied to determine an analytical, approximate solution to singularly perturbed boundary-value problems arising from the study of a wide class of nonlinear systems of ODE. In particular, in the present paper applications to Boltzmann's like models have been developed. Quantitative results can be obtained by solving numerically, if necessary, a pertinent set of nonlinear ordinary differential equations describing the boundary layer functions, i.e. Eqs. (34-44), (65-66) or (69-70) respectively. The above analysis also completes the treatment of singular perturbation techniques, that began with a previous paper [**12**] by considering initial–value problems for a similar class of systems of ODE, with application to extended kinetic theory.

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Abstract

This paper deals with some singular perturbation problems, described by two classes of nonlinear differential equations with nonlinear boundary conditions. Uniformly valid expansions composed of inner and outer solutions are used to solve initial and/or terminal boundary layer problems. In two examples, approximate solutions are explicitly derived for boundary value problems arising from stationary Boltzmann like model equations.

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