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On uniform exponential stability of evolution families (**)

1 - Introduction

Let X be a real or a complex Banach space. The norm on X and on the space $\mathcal{L}(X)$ of all bounded linear operators from X into itself will be denoted by $\|\cdot\|$.

We recall that a family $\boldsymbol{\Phi} = \{\Phi(t, s)\}_{t \geq s \geq 0}$ of bounded linear operators is called an *evolution family* if the following properties are satisfied:

e_1) $\Phi(t, t) = I$, the identity operator on X;

e_2) $\Phi(t, s)\, \Phi(s, t_0) = \Phi(t, t_0)$, for all $t \geq s \geq t_0 \geq 0$;

e_3) for every $x \in X$ and every $t \geq 0$ the function $\Phi(t, \cdot)\, x$ is continuous on $[0, t]$ and the function $\Phi(\cdot, t)\, x$ is continuous on $[t, \infty)$;

e_4) there exist $M \geq 1$, $\omega \geq 0$ such that

$$\|\Phi(t, s)\| \leq M e^{\omega(t-s)}, \qquad \text{for all } t \geq s \geq 0 . \tag{1}$$

An evolution family $\boldsymbol{\Phi}$ is said to be *uniformly exponentially stable* if there exists $N, \nu > 0$ such that

$$\|\Phi(t, s)\| \leq N e^{-\nu(t-s)}, \qquad \text{for all } t \geq s \geq 0 .$$

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Let $\tau > 0$. The evolution family $\boldsymbol{\Phi}$ is said to be *τ-periodic* if

$$\Phi(t+\tau, s+\tau) = \Phi(t, s), \qquad \text{for all } t \geqslant s \geqslant 0$$

and it is called *periodic* if there is $\tau > 0$ such that $\boldsymbol{\Phi}$ is τ-periodic.

If $\boldsymbol{T} = \{T(t)\}_{t \geqslant 0}$ is a C_0-semigroup on the Banach space X then:

$$\Phi(t, s) = T(t-s), \qquad \text{for all } t \geqslant s \geqslant 0$$

is a τ-periodic evolution family for all $\tau > 0$.

Important attempts in the study of uniform exponential stability have been made in the papers [**1**], [**6**], [**12**]. A remarkable result has been obtained by R. Datko ([**3**]) and it is given by the following theorem:

Theorem 1.1. *(Datko) Let $\boldsymbol{\Phi} = \{\Phi(t, s)\}_{t \geqslant s \geqslant 0}$ be an evolution family on the Banach space X. Then $\boldsymbol{\Phi}$ is uniformly exponentially stable if and only if for every $x \in X$ there exists $M(x) > 0$ such that*

$$\int_{t_0}^{\infty} \|\Phi(t, t_0)\, x\|^2 \, dt \leqslant M(x), \qquad \textit{for all } t_0 \geqslant 0 .$$

Datko's result has been generalized by Rolewicz in [**13**] as follows:

Theorem 1.2. *(Rolewicz) Let $\varphi : \boldsymbol{R}^* \times \boldsymbol{R} \to \boldsymbol{R}$ be a function with the following properties:*

(i) for every $t > 0$, $s \to \varphi(t, s)$ is a continuous, non-decreasing function with $\varphi(t, 0) = 0$ and $\varphi(t, s) > 0$, for all $s > 0$;
(ii) for every $s \geqslant 0$, $t \to \varphi(t, s)$ is non-decreasing.

Let X be a Banach space and let $\boldsymbol{\Phi} = \{\Phi(t, s)\}_{t \geqslant s \geqslant 0}$ be an evolution family on X. If for every $x \in X$, there is $\alpha(x) > 0$ such that

$$\sup_s \int_s^{\infty} \varphi(\alpha(x), \|\Phi(t, s)\, x\|)\, dt < \infty$$

then $\boldsymbol{\Phi}$ is uniformly exponentialy stable.

Recently, for the case of C_0-semigroups, Neerven gave the following characterization ([**10**]):

Theorem 1.3. *(Neerven) Let* $\boldsymbol{T} = \{T(t)\}_{t \geq 0}$ *be a* C_0*-semigroup on Banach space* X *and let* B *be a Banach function space over* $\boldsymbol{R}_+$ (see Section 2) *with*

$$\lim_{t \to \infty} F_B(t) = \infty .$$

If for each $x \in X$ *the map* $t \mapsto \|T(t)\,x\|$ *belongs to* B, *then* $\boldsymbol{T}$ *is uniformly exponentially stable.*

In this paper we shall extend Neerven's result for the case of evolution families in general and for periodic ones in particular. We shall also obtain characterizations of Rolewicz's type for evolution families.

2 - Banach function spaces

In this section we recall some facts about Banach function spaces over $\boldsymbol{R}_+$. For the proofs we refer to [**7**] and [**14**].

Let $(\boldsymbol{R}_+, \mathcal{L}, m)$ where $\mathcal{L}$ is the σ-algebra of all Lebesgue measurable sets $A \subset \boldsymbol{R}_+$ and m the Lebesgue measure. We shall denote by $\mathcal{M}$ the linear space of all m-measurable functions $f : \boldsymbol{R}_+ \to \boldsymbol{C}$, identifying functions which are equal a.e.

A *Banach function norm* is a function $N : \mathcal{M} \to \overline{\boldsymbol{R}}_+ = [0, \infty]$ with the following properties:

n_1) $N(f) = 0$ if and only if $f = 0$ a.e.;
n_2) if $|f| \leq |g|$ a.e. then $N(f) \leq N(g)$;
n_3) $N(\alpha f) = |\alpha| N(f)$, for all scalars $\alpha \in \boldsymbol{C}$ and all f with $N(f) < \infty$;
n_4) $N(f+g) \leq N(f) + N(g)$, for all $f, g \in \mathcal{M}$.

Let $B = B_N$ be the set defined by

$$B := \{ f \in \mathcal{M} : |f|_B := N(f) < \infty \} .$$

It is easy to see that $(B, |\cdot|_B)$ is a normed linear space. If B is complete, then B is called *Banach function space over* $\boldsymbol{R}_+$.

Remark 2.1. B is an ideal in $\mathcal{M}$, i.e. if $|f| \leq |g|$ a.e. with $g \in B$, then also $f \in B$ and $|f|_B \leq |g|_B$.

Remark 2.2. If $f_n \to f$ in B, then there is a subsequence (f_{k_n}) converging to f pointwise a.e. (see [**7**]).

For a Banach function space B over $\boldsymbol{R}_+$ we define

$$F_B\colon \boldsymbol{R}_+ \to \overline{\boldsymbol{R}}_+, \qquad F_B(t) := \begin{cases} |\chi_{[0,t)}|_B, & \text{if } \chi_{[0,t)} \in B \\ \infty, & \text{if } \chi_{[0,t)} \notin B \end{cases}$$

where $\chi_{[0,t)}$ denotes the characteristic function of the interval $[0, t)$. The function F_B is called *the fundamental function* of the Banach function space B.

Remark 2.3. F_B is a non-decreasing function.

In what follows we denote by $\mathcal{B}(\boldsymbol{R}_+)$ the set of all Banach function spaces B with the property

$$\lim_{t\to\infty} F_B(t) = \infty$$

and with $\mathcal{E}(\boldsymbol{R}_+)$ the set of all Banach function spaces $B \in \mathcal{B}(\boldsymbol{R}_+)$ with the property that there exists a strictly increasing sequence $(t_n)_n \subset \boldsymbol{R}_+$ such that

$$\sup_{n\in\boldsymbol{N}} (t_{n+1} - t_n) < \infty \qquad \text{and} \qquad \inf_{n\in\boldsymbol{N}} |\chi_{[t_n, t_{n+1})}|_B > 0 .$$

Example 2.1. We consider the Banach function norm $N : \mathcal{M} \to \overline{\boldsymbol{R}}_+$ defined by

$$N(f) = \sum_{n=1}^{\infty} \frac{1}{n} \int_{n-1}^{n} |f(s)|\, ds .$$

We observe that if $B = B_N$ then

$$F_B(n) = \sum_{j=1}^{n} \frac{1}{j}, \qquad \text{for all } n \in \boldsymbol{N}^*,$$

so $B \in \mathcal{B}(\boldsymbol{R}_+)$, but $B \notin \mathcal{E}(\boldsymbol{R}_+)$.

Example 2.2. For every $p \in [1, \infty)$ the space $L^p(\boldsymbol{R}_+, \boldsymbol{C})$ with respect to the norm

$$|f|_p := \left(\int_0^\infty |f(t)|^p\, dt \right)^{\frac{1}{p}}$$

is a Banach function space. It is easy to see that $F_{L^p}(t) = t^{1/p}$, for all $t > 0$ and for $t_n = n$, $|\chi_{[n, n+1)}|_p = 1$, for all $n \in \boldsymbol{N}$. So we obtain that $L^p(\boldsymbol{R}_+, \boldsymbol{C})$ belongs to $\mathcal{E}(\boldsymbol{R}_+)$.

Example 2.3. (*Orlicz spaces*). Let $\varphi : \boldsymbol{R}_+ \to \overline{\boldsymbol{R}}_+$ be a non-decreasing and left-continuous function which is not identically 0 or ∞ on $(0, \infty)$. The *Young function* associated to φ is given by

$$Y_\varphi(t) := \int_0^t \varphi(s)\, ds .$$

Let $f : \boldsymbol{R}_+ \to \boldsymbol{C}$ be a measurable function. We define

$$M_\varphi(f) := \int_0^\infty Y_\varphi(|f(s)|)\, ds .$$

The set L_φ of all f with the property that there exists a $k > 0$ such that $M_\varphi(kf) < \infty$ is easily checked to be a linear space. With respect to the norm

$$|f|_\varphi := \inf\left\{ k > 0 : M_\varphi\left(\frac{1}{k} f\right) \leqslant 1 \right\}$$

$(L_\varphi, |\cdot|_\varphi)$ is a Banach function space over $\boldsymbol{R}_+$ called the *Orlicz space* associated to φ.

Trivial examples of Orlicz spaces are $L^p(\boldsymbol{R}_+, \boldsymbol{C})$, $1 \leqslant p \leqslant \infty$. They are obtained for

$$\varphi(t) = pt^{p-1}, \text{ for } 1 \leqslant p < \infty \text{ and } \varphi(t) = \begin{cases} 0, & 0 \leqslant t \leqslant 1 \\ \infty, & t > 1 \end{cases} \quad \text{for } p = \infty .$$

Proposition 2.1. *If $0 < \varphi(t) < \infty$ for all $t > 0$ then the Orlicz space L_φ has the following properties*

i) the Young function Y_φ is bijective;

ii) the fundamental function F_{L_φ} can be expressed in terms of the Y_φ^{-1} by

$$F_{L_\varphi}(t) = \frac{1}{Y_\varphi^{-1}\left(\dfrac{1}{t}\right)}, \qquad \textit{for all } t > 0 ;$$

iii) $\lim_{t \to \infty} F_{L_\varphi}(t) = \infty$ *and hence* $L_\varphi \in \mathcal{B}(\boldsymbol{R}_+)$;

iv) $L_\varphi \in \mathcal{E}(\boldsymbol{R}_+)$.

Proof. i) It is easy to see that Y_φ is strictly increasing, continuous with $Y_\varphi(0)=0$ and $Y_\varphi(t) \geqslant (t-1)\varphi(1)$, for all $t>1$, so $\lim\limits_{t\to\infty} Y_\varphi(t) = \infty$. Hence Y_φ is bijective.

ii) Let $t>0$. Since

$$M_\varphi\left(\frac{1}{k}\chi_{[0,\,t)}\right) = tY_\varphi\left(\frac{1}{k}\right), \qquad \text{for all } k>0\,,$$

it follows that $M_\varphi\left(\frac{1}{k}\chi_{[0,\,t)}\right) \leqslant 1$ if and only if $1/Y_\varphi^{-1}\left(\frac{1}{t}\right) \leqslant k$. So

$$F_{L_\varphi}(t) = \frac{1}{Y_\varphi^{-1}\left(\dfrac{1}{t}\right)}\,, \qquad \text{for all } t>0\,.$$

iii) Since $Y_\varphi^{-1}(0)=0$, using (ii) it follows that $\lim\limits_{t\to\infty} F_{L_\varphi}(t) = \infty$.

iv) We observe that for every $n \in \boldsymbol{N}$

$$|\chi_{[n,\,n+1)}|_\varphi = \frac{1}{Y_\varphi^{-1}(1)}\,. \qquad \blacksquare$$

3 - Preliminary results

We start with following

Lemma 3.1. *Let $\boldsymbol{\Phi} = \{\Phi(t,s)\}_{t\geqslant s\geqslant 0}$ be an evolution family on the Banach space X. If there exist $\delta>0$ and $c\in(0,1)$ such that:*

$$\|\Phi(t_0+\delta,\, t_0)\| < c\,, \qquad \textit{for all } t_0 \geqslant 0$$

then $\boldsymbol{\Phi}$ is uniformly exponentially stable.

Proof. Let $\nu>0$ such that $c=e^{-\nu\delta}$. For $t\geqslant t_0\geqslant 0$ there exist $n\in\boldsymbol{N}$ and $r\in[0,\delta)$ such that $t=t_0+n\delta+r$. Then we have

$$\begin{aligned}\|\Phi(t,\,t_0)\| &\leqslant \|\Phi(t,\,t_0+n\delta)\|\,\|\Phi(t_0+n\delta,\,t_0)\| \\ &\leqslant Me^{\omega\delta}e^{-\nu n\delta} \leqslant Me^{(\omega+\nu)\delta}e^{-\nu(t-t_0)},\end{aligned}$$

where M,ω are given by (1). It follows that $\boldsymbol{\Phi}$ is uniformly exponentially stable. $\blacksquare$

Lemma 3.2. *Let B be a Banach function space over $\boldsymbol{R}_+$. If $S : \boldsymbol{R}_+ \to \mathcal{L}(X)$ is a mapping such that for all $x \in X$ the function*

$$S_x : \boldsymbol{R}_+ \to \boldsymbol{R}_+, \qquad S_x(t) = \|S(t)\, x\|$$

defines an element of B, then there exists $M > 0$ such that

$$|S_x|_B \leq M\|x\|, \qquad \text{for all } x \in X.$$

Proof. Let M_B be the set of all measurable mappings $f : \boldsymbol{R}_+ \to X$ with $\|f\| \in B$. In M_B we identify the functions which are equal a.e. Thus M_B is a Banach space with respect to the norm

$$|f|_{M_B} = |\|f\||_B.$$

We consider the map $\tilde{S} : X \to M_B$ defined by

$$\tilde{S}(x)(t) = S(t)\, x, \qquad \text{for all } t \geq 0.$$

Using the closed graph theorem, it is sufficient to show that the linear map $\tilde{S}$ is closed.

Let $x_n \to x$ in X and $\tilde{S}(x_n) \to f$ in M_B. By Remark 2.2 it follows that there exists a subsequence (x_{k_n}) such that $\tilde{S}(x_{k_n}) \to f$ a.e.. Since for every $t \geq 0$ we have

$$\tilde{S}_{x_{k_n}}(t) = S(t)\, x_{k_n} \to S(t)\, x = \tilde{S}_x(t),$$

it follows that $\tilde{S}_x = f$ a.e., which proves the closedness of $\tilde{S}$. ■

Lemma 3.3. (*Müller*) *Suppose that A is a bounded operator on a Banach space X whose spectral radius satisfies $r(A) \geq 1$. Then for every $\varepsilon \in (0, 1)$ and for every decreasing sequence of positive real numbers (α_n) with $\alpha_n \to 0$ there exists $x \in X$ with $\|x\| = 1$ such that*

$$\|A^n x\| \geq \varepsilon \alpha_n, \qquad \text{for all } n \in \boldsymbol{N}.$$

Proof. See [**9**] or [**11**]. ■

Lemma 3.4. *Let $\boldsymbol{\Phi} = \{\Phi(t, s)\}_{t \geq s \geq 0}$ be a τ-periodic evolution family and $A = \Phi(\tau, 0)$. Then $\boldsymbol{\Phi}$ is uniformly exponential stable if and only if $r(A) < 1$.*

Proof. We observe that $A^n = \Phi(n\tau, 0)$ for all $n \in \boldsymbol{N}^*$. Hence, it follows that if $\boldsymbol{\Phi}$ is uniformly exponentially stable then $r(A) < 1$.

Conversely, if $r(A) < 1$ then there exists $\nu > 0$ such that $r(A) < e^{-\nu\tau}$ and there exists $n_0 \in \boldsymbol{N}^*$ with

$$\|A^n\| \leq e^{-n\nu\tau}, \qquad \text{for all } n \geq n_0.$$

For the begining we prove that there exists $K > 0$ with

$$\|\Phi(t, 0)\| \leq K e^{-\nu t}, \qquad \text{for all } t \geq 0 .$$

We denote by $M = \sup\{\|\Phi(t, s)\| : t, s \in [0, n_0\tau], t \geq s\}$. For $t \in [0, n_0\tau]$ we have that

$$\|\Phi(t, 0)\| \leq M \leq M e^{n_0\nu\tau} e^{-\nu t}. \tag{2}$$

Let $t > n_0\tau$, $t = n\tau + r$ with $n \in \boldsymbol{N}$ and $r \in [0, \tau)$. Then:

$$\begin{aligned} \|\Phi(t, 0)\| &\leq \|\Phi(t, n\tau)\| \|\Phi(n\tau, 0)\| = \|\Phi(r, 0)\| \|\Phi(n\tau, 0)\| \\ &\leq M e^{-n\nu\tau} \leq M e^{\nu\tau} e^{-\nu t}. \end{aligned} \tag{3}$$

Denoting by $K = M e^{n_0\nu\tau}$ from the relations (2) and (3) we obtain that

$$\|\Phi(t, 0)\| \leq K e^{-\nu t}, \qquad \text{for all } t \geq 0 .$$

Let now $t \geq s \geq 0$, $t = n\tau + r$, $s = k\tau + u$ with $n \geq k$ and $r, u \in [0, \tau)$. If $n = k$ then:

$$\|\Phi(t, s)\| = \|\Phi(r, u)\| \leq M \leq M e^{\nu\tau} e^{-\nu(t-s)}$$

and if $n \geq k + 1$ then:

$$\|\Phi(t, s)\| \leq \|\Phi(t, (k+1)\tau)\| \|\Phi((k+1)\tau, s)\|$$

$$\leq \|\Phi(t - (k+1)\tau, 0)\| \|\Phi(\tau, u)\| \leq M K e^{\nu\tau} e^{-\nu(t-s)}.$$

It follows that $\boldsymbol{\Phi}$ is uniformly exponentially stable. ■

4 - The main results

In this section we shall give necessary and sufficient conditions for uniform exponential stability of evolution families in Banach spaces.

Theorem 4.1. *Let* $\boldsymbol{\Phi} = \{\Phi(t, s)\}_{t \geq s \geq 0}$ *be an evolution family on the Banach space X. Then* $\boldsymbol{\Phi}$ *is uniformly exponentially stable if and only if there exists a Banach function space* $B \in \mathcal{E}(\boldsymbol{R}_+)$ *such that:*

(*i*) *for every* $x \in X$ *and* $s \geq 0$ *the function*

$$f_{s,x} : \boldsymbol{R}_+ \to \boldsymbol{R}_+, \qquad f_{s,x}(t) = \|\Phi(t+s, s)\, x\|$$

belongs to B;

(*ii*) *there exists a function* $K : X \to (0, \infty)$ *such that*

$$|f_{s,x}|_B \leq K(x), \qquad \text{for all } x \in X \quad \text{and all } s \geq 0 .$$

Proof. *Necessity.* Let $N, \nu > 0$ such that

$$\|\Phi(t, s)\| \leq N e^{-\nu(t-s)}, \qquad \text{for all } t \geq s \geq 0$$

and $B = L^p(\boldsymbol{R}_+, \boldsymbol{C})$ where $p \in [1, \infty)$. Then for every $x \in X$ and $s \geq 0$ we have that

$$|f_{s,x}|_p \leq \frac{N}{(\nu p)^{1/p}} \|x\| .$$

Sufficiency. Since $B \in \mathcal{E}(\boldsymbol{R}_+)$ there exists a strictly increasing sequence $(t_n) \subset (0, \infty)$ such that

$$\delta = \sup_n (t_{n+1} - t_n) < \infty \qquad \text{and} \qquad c = \inf_n |\chi_{[t_n, t_{n+1})}|_B > 0 . \tag{4}$$

Let $n \in \boldsymbol{N}$ and $s \geq 0$. For every $t \in [t_n, t_{n+1})$ we have that:

$$\|\Phi(t_{n+1} + s, s)\, x\| \leq M e^{\omega\delta} \|\Phi(t+s, s)\, x\|$$

where M and ω are given by the relation (1). It follows that:

$$\chi_{[t_n, t_{n+1})}(t) \|\Phi(t_{n+1} + s, s)\, x\| \leq M e^{\omega\delta} \|\Phi(t+s, s)\, x\|$$

for all $t \geq 0$. Using the relation (4) and the hypothesis we obtain that:

$$c\|\Phi(t_{n+1} + s, s)\, x\| \leq M e^{\omega\delta} K(x)$$

for every $x \in X$, $n \in \boldsymbol{N}$ and $s \geq 0$. By the uniform boundedness principle it results that there exists $L_1 > 0$ such that

$$\|\Phi(t_{n+1} + s, s)\| \leq L_1$$

for all $n \in \boldsymbol{N}$ and $s \geq 0$.

Let $s \geq 0$ and $t \geq t_1$. Then there exists an unique $n \in \boldsymbol{N}^*$ such that $t_n \leq t < t_{n+1}$. Hence we deduce that:

$$\|\Phi(s+t, s)\| \leq \|\Phi(s+t, s+t_n)\| \|\Phi(s+t_n, s)\| \leq Me^{\omega\delta} L_1.$$

Denoting by $L = max\{Me^{\omega\delta}L, Me^{\omega t_1}\}$ we obtain that

$$\|\Phi(t+s, s)\| \leq L, \qquad \text{for all } t, s \geq 0.$$

Let $x \in X$ and $n \in N^*$. For $t \in [0, n]$ we have:

$$\|\Phi(n+s, s)\, x\| \leq L \|\Phi(t+s, s)\, x\|$$

so

$$\chi_{[0,n]}(t) \|\Phi(n+s, s)\, x\| \leq L \|\Phi(s+t, s)\, x\|$$

for all $t \geq 0$. It follows that

$$F_B(n) \|\Phi(n+s, s)\, x\| \leq L |f_{s,x}|_B \leq LK(x).$$

By the uniform boundedness principle we obtain that there exists $K > 0$ such that

$$F_B(n) \|\Phi(n+s, s)\| \leq K, \qquad \text{for all } s \geq 0 \text{ and } n \in \boldsymbol{N}^*.$$

Since $B \in \mathcal{E}(\boldsymbol{R}_+)$ there exists $n_0 \in \boldsymbol{N}$ with $F_B(n_0) > 2K$. Then we deduce that

$$\|\Phi(n_0 + s, s)\| \leq \frac{1}{2}, \qquad \text{for all } s \geq 0.$$

Using Lemma 3.1 it follows that $\boldsymbol{\Phi}$ is uniformly exponentially stable. ∎

A theorem of Rolewicz's type is given by

Theorem 4.2. *Let $\boldsymbol{\Phi} = \{\Phi(t, s)\}_{t \geq s \geq 0}$ be an evolution family on the Banach space X. Then $\boldsymbol{\Phi}$ is uniformly exponentially stable if and only if there exist a non-decreasing function $\varphi : \boldsymbol{R}_+ \to \boldsymbol{R}_+$ and $K > 0$ such that:*

(i) $\varphi(0) = 0$ and $\varphi(t) > 0$ for all $t > 0$;

(ii) for every $x \in X$ *with* $\|x\| \leq 1$ *and every* $s \geq 0$

$$\int_s^\infty \varphi(\|\Phi(t, s)\, x\|)\, dt \leq K .$$

Proof. *Necessity.* Let $\varphi(t) = t$ for all $t \geq 0$. Let $N, \nu \in (0, \infty)$ such that

$$\|\Phi(t, s)\| \leq N e^{-\nu(t-s)}, \qquad \text{for all } t \geq s \geq 0 .$$

Then for every $x \in X$ with $\|x\| \leq 1$ and $s \geq 0$ we have that

$$\int_s^\infty \|\Phi(t, s)\, x\|\, dt \leq \frac{N}{\nu} .$$

Sufficiency. Let M, ω given by relation (1), $t_0 > 0$ such that $K < t_0 \varphi(1)$ and $\delta = 1/M e^{\omega t_0}$.

Let $x \in X$ with $\|x\| \leq 1$, $t \geq t_0$ and $s \geq 0$. We have that:

$$\|\Phi(s+t, s)\, \delta x\| \leq \|\Phi(u, s)\, x\|, \qquad \text{for all } u \in [s+t-t_0, s+t] .$$

Since φ is non-decreasing using the relation from above it follows that

$$t_0 \varphi(\|\Phi(s+t, s)\, \delta x\|) \leq \int_{s+t-t_0}^{s+t} \varphi(\|\Phi(u, s)\, x\|)\, du \leq K .$$

Taking in account the way that t_0 was chosen, from the last inequality we obtain

$$\|\Phi(s+t, s)\, \delta x\| \leq 1, \qquad \text{for all } t \geq t_0 \text{ and } s \geq 0$$

so

$$\|\Phi(s+t, s)\| \leq \frac{1}{\delta}, \qquad \text{for all } t \geq t_0 \text{ and } s \geq 0 . \tag{5}$$

Denoting by $L = \dfrac{1}{\delta} = M e^{\omega t_0}$ and using the relation (5) it follows that:

$$\|\Phi(s+t, s)\| \leq L, \qquad \text{for all } t, s \geq 0 .$$

Without lost of generality we can suppose that φ is left-continuous (if not we consider the function $\tilde{\varphi}(t) = \lim\limits_{s \nearrow t} \varphi(s)$ for $t > 0$ and the proof is the same).

Let L_φ be the Orlicz space associated to φ. For every $x \in X$ and $s \geqslant 0$ let

$$f_{s,x} \colon \boldsymbol{R}_+ \to \boldsymbol{R}_+, \qquad f_{s,x}(t) = \|\Phi(t+s,s)\,x\|.$$

If $x \in X \setminus \{0\}$ and $\tilde{x} = \dfrac{x}{(K+1)\,L\|x\|}$ we have:

$$Y_\varphi(f_{s,\tilde{x}}(t)) = Y_\varphi(\|\Phi(s+t,s)\,\tilde{x}\|) \leqslant \|\Phi(t+s,s)\,\tilde{x}\|\,\varphi(\|\Phi(t+s,s)\,\tilde{x}\|)$$

$$\leqslant \frac{1}{K+1}\,\varphi(\|\Phi(t+s,s)\,\tilde{x}\|).$$

It follows that

$$M_\varphi(f_{s,\tilde{x}}) < 1$$

so $f_{s,\tilde{x}} \in L_\varphi$ and $|f_{s,\tilde{x}}|_\varphi \leqslant 1$. Because L_φ is a linear space and $f_{s,\tilde{x}} = \dfrac{1}{(K+1)\,L\|x\|} f_{s,x}$ it results that $f_{s,x} \in L_\varphi$ and

$$|f_{s,x}|_\varphi \leqslant (K+1)\,L\|x\|.$$

By applying Proposition 2.1 and Theorem 4.1 we conclude that $\boldsymbol{\Phi}$ is uniformly exponentially stable. ■

Remark 4.1. In Theorem 4.2 for $\varphi(t) = t^2$ we obtain the theorem of Datko.

The version of Theorem 4.1 for periodic evolution families is given by:

Theorem 4.3. *Let $\boldsymbol{\Phi} = \{\Phi(t,s)\}_{t \geqslant s \geqslant 0}$ be a periodic evolution family on the Banach space X. Then $\boldsymbol{\Phi}$ is uniformly exponentially stable if and only if there exists $B \in \mathcal{B}(\boldsymbol{R}_+)$ such that for every $x \in X$ the function*

$$f_x \colon \boldsymbol{R}_+ \to \boldsymbol{R}_+, \qquad f_x(t) = \|\Phi(t,0)\,x\|$$

belongs to B.

Proof. *Necessity.* If $\boldsymbol{\Phi}$ is uniformly exponentially stable and $p \in [1,\infty)$ then for every $x \in X$ the map f_x belongs to $L^p(\boldsymbol{R}_+, \boldsymbol{C})$.

Sufficiency. By Lemma 3.2 applied for

$$S : \boldsymbol{R}_+ \to \mathcal{L}(X), \qquad S(t) = \Phi(t,0)$$

there exists $M > 0$ such that

(6) $$|f_x|_B \leqslant M\|x\|, \qquad \text{for all } x \in X.$$

Suppose that $\boldsymbol{\Phi}$ is τ-periodic. To prove that $\boldsymbol{\Phi}$ is uniformly exponentially stable, according to Lemma 3.4 it is sufficient to prove that $r(A) < 1$ where $A = \Phi(\tau, 0)$.

Suppose the contrary, i.e. $r(A) \geqslant 1$. For every $p \in \boldsymbol{N}^*$ we consider the sequence

$$\alpha_n^p := \begin{cases} 1, & \text{if } n \leqslant p \\ 0, & \text{if } n \geqslant p+1. \end{cases}$$

An application of Lemma 3.3 for the sequence (α_n^p) and $\varepsilon = \dfrac{1}{2}$ concludes that there exists $x_p \in X$ with $\|x_p\| = 1$ and

(7) $$\|A^n x_p\| \geqslant \frac{1}{2}, \qquad \text{for all } n \in \boldsymbol{N}^* \text{ with } n \leqslant p.$$

If we denote by

$$L := \sup\{\|\Phi(t, s)\| : t, s \in [0, \tau], t \geqslant s\}$$

then for every $n \in \boldsymbol{N}$, $x \in X$ and $t \in [n\tau, (n+1)\tau)$ we have

(8) $$\|\Phi((n+1)\tau, 0)x\| \leqslant \|\Phi((n+1)\tau, t)\| \|\Phi(t, 0)x\| \leqslant L\|\Phi(t, 0)x\|.$$

From (7) and (8) we obtain that

$$\chi_{[0, p\tau)}(t) \leqslant 2L\|\Phi(t, 0)x_p\| = 2Lf_{x_p}(t), \qquad \text{for all } t \geqslant 0 \text{ and } p \in \boldsymbol{N}^*.$$

Using the relation (6) it follows that

$$F_B(p\tau) \leqslant 2ML, \qquad \text{for all } p \in \boldsymbol{N}^*.$$

This last inequality contradicts the assumption that $B \in \mathcal{B}(\boldsymbol{R}_+)$. ■

Theorem 4.4. *Let* $\boldsymbol{\Phi} = \{\Phi(t, s)\}_{t \geqslant s \geqslant 0}$ *be a periodic evolution family on the Banach space X. Then* $\boldsymbol{\Phi}$ *is uniformly exponentially stable if and only if there exists a non-decreasing function* $\varphi : \boldsymbol{R}_+ \to \boldsymbol{R}_+$ *with* $\varphi(0) = 0$ *and* $\varphi(t) > 0$ *for*

$t > 0$ *such that:*

$$\int_0^\infty \varphi(\|\Phi(t, 0)\, x\|)\, dt < \infty\,, \qquad \textit{for all } x \in X\,.$$

Proof. *Necessity.* It results for $\varphi(t) = t$, for all $t \geqslant 0$.

Sufficiency. Without lost of generality we may assume that φ is left-continuous (if not we can use $\tilde{\varphi}(t) = \lim_{s \nearrow t} \varphi(s)$ and the proof is the same).

For the begining we prove that

$$\lim_{t \to \infty} \|\Phi(t, 0)\, x\| = 0\,, \qquad \text{for all } x \in X\,.$$

Let $x \in X \setminus \{0\}$. Assume that $\|\Phi(t, 0)\, x\| \nrightarrow 0$ for $t \to \infty$. Hence there exists $\varepsilon > 0$ and $t_n \to \infty$ such that:

$$\|\Phi(t_n, 0)\, x\| \geqslant \varepsilon\,, \qquad \text{for all } n \in \boldsymbol{N}\,.$$

Without lost of generality we may assume that $t_0 \geqslant 1$ and $t_{n+1} - t_n \geqslant 1$ for all $n \in \boldsymbol{N}$. Let $M, \omega > 0$ given by the relation (1). If $n \in \boldsymbol{N}$ and $t \in [t_n - 1, t_n]$ then using

$$\|\Phi(t_n, 0)\, x\| \leqslant \|\Phi(t_n, t)\|\, \|\Phi(t, 0)\, x\| \leqslant Me^{\omega}\, \|\Phi(t, 0)\, x\|$$

we obtain that

$$\|\Phi(t, 0)\, x\| \geqslant \frac{\varepsilon}{Me^{\omega}}\,, \qquad \text{for all } t \in [t_n - 1, t_n] \text{ and } n \in \boldsymbol{N}\,.$$

Hence we have

$$\int_0^\infty \varphi(\|\Phi(t, 0)\, x\|)\, dt \geqslant \sum_{n=0}^{\infty} \int_{t_n - 1}^{t_n} \varphi(\|\Phi(t, 0)\, x\|)\, dt \geqslant \sum_{n=0}^{\infty} \varphi\left(\frac{\varepsilon}{Me^{\omega}}\right) = \infty$$

which is a contradiction.

Using the uniform boundedness principle it follows that there exists $L > 0$ such that

$$\|\Phi(t, 0)\, x\| \leqslant L\|x\|\,, \qquad \text{for all } t \geqslant 0 \text{ and } x \in X\,.$$

Let L_φ be the Orlicz space associated to φ and for every $x \in X$ let

$$f_x \colon \boldsymbol{R}_+ \to \boldsymbol{R}_+\,, \qquad f_x(t) = \|\Phi(t, 0)\, x\|\,.$$

For all $t \geqslant 0$ we have that:

$$Y_\varphi(f_x(t)) = \int_0^{f_x(t)} \varphi(s)\, ds \leqslant f_x(t)\, \varphi(f_x(t)) \leqslant L\|x\|\, \varphi(f_x(t)).$$

Then we obtain that

$$M_\varphi(f_x) = \int_0^\infty Y(f_x(t))\, dt \leqslant L\|x\| \int_0^\infty \varphi(\|\Phi(t,\, 0)\, x\|)\, dt < \infty .$$

It follows that $f_x \in L_\varphi$ for all $x \in X$. Using Theorem 4.3 and Proposition 2.1 it results that $\boldsymbol{\Phi}$ is uniformly exponentially stable. ■

Remark 4.2. Generally, if the periodic evolution family $\boldsymbol{\Phi} = \{\Phi(t,\, s)\}_{t \geqslant s \geqslant 0}$ is uniformly exponentially stable and $\varphi : \boldsymbol{R}_+ \to \boldsymbol{R}_+$ is a non-decreasing function with $\varphi(0) = 0$ and $\varphi(t) > 0$, for $t > 0$ it does not result that

$$\int_0^\infty \varphi(\|\Phi(t,\, 0)\, x\|)\, dt < \infty$$

for $x \in X \setminus \{0\}$. This fact is illustrated by the following example

Example 4.1. Let $X = \boldsymbol{R}$ and

$$\Phi(t,\, s)\, x = e^{-(t-s)} x\,, \qquad \text{for all } t \geqslant s \geqslant 0 \text{ and all } x \in \boldsymbol{R}\,.$$

It is obviously that $\boldsymbol{\Phi}$ is a periodic evolution family which is uniformly exponentially stable.

We consider

$$\varphi : \boldsymbol{R}_+ \to \boldsymbol{R}_+, \qquad \varphi(t) = \begin{cases} 0\,, & t = 0 \\ -\dfrac{1}{\ln t}\,, & t \in \left(0,\, \dfrac{1}{e}\right] \\ et\,, & t > \dfrac{1}{e} \end{cases}.$$

Then φ is a continuous non-decreasing function with $\varphi(0) = 0$ and $\varphi(t) > 0$, for $t > 0$.

Let $x \in \boldsymbol{R}^*$. Then

$$\int_0^\infty \varphi(|\Phi(t, 0)\, x|)\, dt = \int_0^\infty \varphi(e^{-t}|x|)\, dt$$

$$= \int_0^{|x|} \frac{\varphi(y)}{y}\, dy \geqslant \int_0^{\alpha_x} \frac{\varphi(y)}{y}\, dy\,,$$

where $\alpha_x = \min\left\{|x|\,,\ \dfrac{1}{e}\right\}$. Since for every $\varepsilon \in (0,\, \alpha_x)$ we have

$$\int_\varepsilon^{\alpha_x} \frac{\varphi(y)}{y}\, dy = \int_\varepsilon^{\alpha_x} \frac{1}{y ln y}\, dy = ln\left(ln\frac{1}{\varepsilon}\right) - ln\left(ln\frac{1}{\alpha_x}\right)$$

it follows that

$$\int_0^\infty \varphi(|\Phi(t, 0)\, x|)\, dt = \infty\,, \qquad \text{for all } x \in \boldsymbol{R}^*.$$

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Abstract

The aim of this paper is to obtain necessary and sufficient conditions for uniform exponential stability of evolution families. We obtain generalizations of some theorems due to Datko, Neerven and Rolewicz.

* * *