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Adjoint groups of radical rings (**)

1 - Introduction

In the mid-1940's, Nathan Jacobson noted an important connection between rings and semigroups or groups. For any ring R define the *circle* or *adjoint* operation on R via $a \circ b = a + b - ab$, for each $a, b \in R$. Then $(R, \circ)$ is a group if and only if R is a Jacobson radical ring. This group (semigroup) is called the *adjoint* or *circle* group (semigroup) of the ring. This paper discusses the interplay between a radical ring and its adjoint group, giving both an exposition of the development of the theory and some new results. Here ring will mean an associative ring, with no unity assumed except where specifically noted, and R will always be a ring. The Jacobson radical of R is denoted by $J(R)$; so when $R = J(R)$, then R is a (Jacobson) radical ring.

The earliest date I can find for explicit recognition of the adjoint semigroup or group, or even of $\circ$ as a binary operation, is July 1946, see [**14**, p. 481], although these concepts are implicit in Jacobson's seminal paper on the radical written in 1944, [**16**]. (While certain authors have stated that the adjoint group/semigroup concept can be found in earlier work, e.g., [**16**], [**25**], a careful reading of those papers reveals no evidence of this explicitly, although quasi-regular elements and quasi-inverses are discussed.

The theory of adjoint groups has developed along three main lines:

(1) implications of ring theoretic conditions on the adjoint group;

(2) implications of group theoretic conditions (imposed on the adjoint group) on the ring;

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(3) the implications of mixed conditions, group and ring theoretic, on the ring and on its adjoint group.

Each of these have at times connections with the additive group of the ring. Other allied lines of investigation are: connections between a ring and its group of quasi-regular elements, $(q(R), \circ)$; the interplay between R, its Lie ring, and $(q(R), \circ)$; and investigations of conditions on the semigroup $(R, \circ)$ that imply R is a radical ring.

This paper focuses on (1), (2), and (3), with some brief comments on the other lines of attack.

2 - Preliminaries

Two different operations have been used for the adjoint group (semigroup), the $\circ$ defined previously and also the operation $a \times b = a + b + ab$. The latter was the one used by Perlis [**25**] and Jacobson [**16**] in their discussions of quasi-regular elements. It is easy to see that $(R, \circ)$ and $(R, \times)$ are isomorphic semigroups and $(q(R), \circ)$ and $(q(R), \times)$ are isomorphic groups. (The earlies published use of this $\times$ operation appears to be by Hille in 1948; see [**14**, p. 681)]. Several other similar operations give such isomorphisms also; (see [**22**, p. 11]). A more general formulation is the following.

Proposition 2.1 ([**10**]). *Let α be an endomorphism on the bimodule ${}_R R_R$. Define $s *_\alpha t$ on R via: $s *_\alpha t = s + t + \alpha(st)$, for each $s, t \in R$. Then $(R, *_\alpha)$ is a monoid and $\alpha : (R, *_\alpha) \to (R, \circ)$ is a homomorphism. If α is an automorphism, then $(R, *_\alpha)$ and $(R, \circ)$ are isomorphic.*

Observe that for $\alpha = 1_R$ one gets the operation $\times$, and for $\alpha = -1_R$, the associated operation is $\circ$.

It was recognized by Jacobson, [**16**], that every nilpotent element is quasi-regular. Consequently, the set $N(R)$, of nilpotent elements in R, is a subset of $q(R)$. Implicit in [**25**] and [**16**] is that if R is nil, then $(R, \circ)$ is a group.

Observe that if $f: R \to S$ is a ring homomorphism, then $f: (R, \circ) \to (S, \circ)$ is a monoid homomorphism. The center of $(R, \circ)$ is equal to the center of R. We use $Z(R)$ for this mutual center. (Here, whenever a property is ascribed to R, it is a ring property, whereas group or semigroup properties will be ascribed to $(R, \circ)$).

For the remainder of this paper R will *always* be a Jacobson radical ring. The quasi-inverse of $x \in R$ is denoted by x', and $[x, y] = x' \circ y' \circ x \circ y$, for $x, y \in R$. If A and B are nonempty subsets of R, then we use $[A, B]$ for the subgroup of

$(R, \circ)$ generated by $\{[a, b]\colon a \in A, b \in B\}$. Then $R' = R^{(1)} = [R, R]$, the commutator subgroup of $(R, \circ)$, and $R^{(n)} = [R^{(n-1)}, R^{(n-1)}]$, the n-th derived subgroup of $(R, \circ)$, for $n \in \mathbb{N}$. (Here $\mathbb{N}$ is the set of all natural numbers and $R^{(0)} = R$). Also, $\gamma_0(R) = R$, $\gamma_1(R) = R'$, and $\gamma_n(R) = [\gamma_{n-1}(R), R]$, for all $n \in \mathbb{N}$, the n-th term of the lower central series for $(R, \circ)$. We use $\langle A, B\rangle$ for the ideal of R generated by $\{ab - ba\colon a \in A, b \in B\}$, the Lie ideal of R, and $R/\langle R, R\rangle$ is commutative. The next lemma will be useful in establishing some crucial connections between group and ring properties.

Lemma 2.1. *Let $x, y \in R$ and let I and K be ideals of R.*

(i) $[x, y] = (xy' - y'x) - x'(xy' - y'x) - (xy' - y'x)y + x(xy' - y'x)y$;
(ii) $[I, I] \subseteq \langle I, I\rangle$;
(iii) $[I, K] = [K, I] \subseteq IK + KI$
(iv) $R^{(n)} \subseteq R^{2^n}$, *for each* $n \in \mathbb{N}$;
(v) $\gamma_n(R) \subseteq R^{n+1}$, *for each* $n \in \mathbb{N}$.

Proof. Part (i) follows from a routine calculation, and (ii) is an immediate consequence of (i). Using (i), observe that $[I, K] \subseteq IK + KI + IKI + KIK + IKIK \subseteq IK + KI \subseteq I \cap K$. Since $x \in I$, $y \in K$ implies $[y', x'] = [x, y]' \in [I, K]' \subseteq [I, K]$, we have $[K, I] \subseteq [I, K]$ and hence $[K, I] = [I, K]$. Routine induction arguments then establish (iv) and (v).

Since each ring endomorphisms is also an endomorphism on $(R, \circ)$, the fully invariant subgroup $R^{(n)}$, $n \in \mathbb{N}$, is invariant under every ring endomorphism on R. Similarly, the characteristic subgroup $\gamma_n(R)$ is invariant under every ring automorphism on R.

The next result is implicit in some earlier papers, e.g., [**34**], but seems to have been first explicitly stated in [**22**]. The proof is immediate.

Lemma 2.2. *Every subring (respectively: one-sided ideal, two-sided ideal) of R is a subsemigroup (respectively: subgroup, normal subgroup) of $(R, \circ)$.*

Observe that if I is an ideal of R, then the natural ring homomorphism, $\eta\colon R \to R/I$, is also a group homomorphism of $(R, \circ)$ onto $(R/I, \circ)$, and furthermore $(R, \circ)/I = (R/I, \circ)$.

Subrings of radicals need not be subgroups of the adjoint group. For example, the subring $S = \{2x\colon x \in \mathbb{Z}\}$ of the radical ring $R = \left\{\dfrac{2x}{2y+1}\colon x, y \in \mathbb{Z}\right\}$ is not a subgroup of $(R, \circ)$ since the quasi-inverse of 6 is $\dfrac{6}{5}$. (Note that in [**22**, p. 12]), it is claimed that subrings of radical rings are subgroups, even though the counter example just given is discussed on the same page).

The next two results give conditions for a subring to be a subgroup of the adjoint group. Here for $x \in R$, we use $x^{[n]}$ for the n-th power of x in $(R, \circ)$. Observe that for $n > 0$, $x^{[n]} = \sum_{j=1}^{n} \binom{n}{j} (-1)^j x^j$.

Proposition 2.3. *Let S be a subring of R.*
If any one of the following hold, then S is a subgroup of $(R, \circ)$:

(i) $S'S \subseteq S$;
(ii) $SS' \subseteq S$;
(iii) *S is nil*;
(iv) *every element in S has finite order in $(R, \circ)$.*

Proof. Let $s \in S$. In light of Lemma 2.2 (i), we need only show that $s' \in S$. Since $ss' = s + s' = s's$, observe that $s' \in S$ if and only if either $ss' \in S$ or $s's \in S$. Then (i) and (ii) follow immediately. If $s^n = 0$, for $n > 1$, then $s' = -(s + s^2 + \ldots + s^{n-1}) \in S$. Finally, if s has order $m > 1$ in $(R, \circ)$, then $s' = s^{[m-1]} = \sum_{j=1}^{m-1} \binom{m-1}{j} (-1)^j s^j \in S$.

Corollary 2.4. *If either of the following hold, then every subring of R is a subgroup of $(R, \circ)$:*

(i) *R is nil*;
(ii) *$(R, \circ)$ is torsion.*

3 - Implications of ring conditions for the adjoint group

An obvious relation between ring T and its adjoint semigroup $(T, \circ)$ is that the centers, $Z(T)$, of the two systems coincide and that for $s, t \in T$, then s and t commute in the ring if and only if they commute in the adjoint in the semigroup. In 1965 Watters observed a more subtle relation: if $R^n = 0$, then $(R, \circ)$ is nilpotent of class at most $n - 1$; see [**3**]. (Recall here R is always a Jacobson radical ring). Having a nilpotent adjoint group does not imply the ring is nilpotent, as was observed by Watters [**34**]. The following example is a generalization of the example given by Watters.

Example 3.1. Let D be an integral domain (commutative ring with unity and no divisors of zero) and let S be a nonzero subring of D with $1 \notin S$. Let

$W = \left\{ \dfrac{x}{1+y} : x,\, y \in I \right\}$, a subring of the ring of quotients for D. Then $\dfrac{x}{1+y}$ has quasi-inverse $\dfrac{-x}{y-x+1}$, and hence $W = J(W)$. This yields a ring without zero divisors with an abelian (and hence nilpotent) adjoint group.

We next give an extension of Watters result on nilpotency of the adjoint group, as well as an analogous result on solvability.

Proposition 3.2. (i) *If $R^n \subseteq Z(R)$, for some $n \in \mathbb{N}$, then $(R, \circ)$ is nilpotent of class n.*

(ii) *If R^{2^n} is commutative, for some $n \in \mathbb{N}$, then $(R, \circ)$ is solvable of length $n+1$.*

Proof. (i) The case $n = 1$ is trivial, so consider $n > 1$. Since $\gamma_{n-1}(R) \subseteq R^n$ and $(R^n, \circ)$ is in the center of $(R, \circ)$, we have $\gamma_n(R) = 0$.

(ii) The hypothesis and Lemma 2.1 (iv) yield $(R^{(n)}, \circ)$ is abelian, and hence $R^{(n+1)} = 0$.

As an immediate corollary of Proposition 3.2 (i), we obtain a result originally due to Ault and Watters, [**3**]: if R is nilpotent of index n, then $(R, \circ)$ is nilpotent of class $n-1$. Proposition 3.2 (ii) was first noted in [**10**, Theorem 4], albeit in a different form.

The next result extends Proposition 3.2 (i).

Corollary 3.3. *Let I be an ideal of R with $I \subseteq Z(R)$. If $(R/I)^n \subseteq Z(R/I)$, for some $n \in \mathbb{N}$, then $(R, \circ)$ is nilpotent of class $n+1$.*

Proof. Observe that $\gamma_n(R/I) = 0$; so $\gamma_n(R) \subseteq I \subseteq Z(R)$, and hence $\gamma_{n+1}(R) = 0$.

More success can be had extending Proposition 3.2(ii), the reason being that a solvable-by-solvable extension yields a solvable group, which has no direct nilpotent analog. (However, some results can be obtained using Hall's theorem as will be shown in a subsequent paper). First some terminology is given.

A ring T is said to *satisfy a permutation identity* (or be a *permutation identity ring*) if there is a non-identity permutation π on n symbols, $n > 1$, such that $t_1 \cdots t_n = t_{\pi_1} \cdots t_{\pi_n}$, for each $t_1, \cdots t_n \in T$. Basic properties of permutation identity rings and numerous examples of such rings can be found in [**4**], [**5**]. Crucial for our purposes is that if T is a permutation identity ring, then $\langle T, T \rangle$ is nilpotent, [**5**].

Proposition 3.4. *If I is an ideal of R such that I^n and $(R/I)^m$ are permutation identity rings, for some $n, m \in \mathbb{N}$, then $(R, \circ)$ is solvable.*

Proof. Observe that $(R/\langle R, R\rangle, \circ)$ is always abelian. First consider the case where R is a permutation identity ring. Then $\langle R, R\rangle$ is a nilpotent ring and hence $(\langle R, R\rangle, \circ)$ is a nilpotent group. Consequently, $(R, \circ)$ is solvable. Next, take I^n and $(R/I)^m$ to be permutation identity rings. Use $\overline{R} = R/I$. Since $\overline{R}^m$ and I^n are also Jacobson radical rings, we have that $(\overline{R}^m, \circ)$ and $(I^n, \circ)$ are solvable. Each of the rings I/I^n and $\overline{R}/\overline{R}^m$ are nilpotent; so $(I/I^n, \circ)$ and $(\overline{R}/\overline{R}^m, \circ)$ are nilpotent groups. Thus $(I, \circ)$ and $(\overline{R}, \circ)$ are solvable, and consequently $(R, \circ)$ is solvable.

We next give as a corollary several interesting consequences of the previous proposition and its proof.

Corollary 3.5. *If any one of the following hold, then* $(R, \circ)$ *is solvable:*

(i) R *is a permutation identity ring;*
(ii) $\langle R, R\rangle$ *is nilpotent;*
(iii) $(\langle R, R\rangle)^n$ *satisfies a permutation identity, for some* $n \in \mathbb{N}$.

By sharpening the hypotheses in Corollary 3.5 (i) and (iii) some precise bounds on the solvable length are obtained. A ring T is *left* (*right*) *permutable* if $abc = bac$, ($abc = acb$), for each $a, b, c \in T$. If T is both left and right permutable, then T is said to be *permutable.* (See [**4**]).

Proposition 3.6. (i) *If* R *is left* (*right*) *permutable, then* $(R, \circ)$ *is metabelian.*

(ii) *If* R *is permutable, then* $(R, \circ)$ *is nilpotent of class two.*

(iii) *If* $(\langle R, R\rangle)^k$ *is commutative, for some* $k = 2^{n-1}$, $n \in \mathbb{N}$, *then* $(R, \circ)$ *is solvable of length at most* $n+1$.

Proof. Parts (i) and (ii) are proved in [**10**, Theorem 7]. To establish part (iii), use Lemma 2.1 (iii) to obtain $R^{(1)} \subseteq \langle R, R\rangle$. Then $R^{(2)} \subseteq \langle R, R\rangle, \langle R, R\rangle] \subseteq (\langle R, R\rangle)^2$. Repeat the process to obtain $R^{(n)} \subseteq (\langle R, R\rangle)^k$, where $k = 2^{n-1}$, for $n \geqslant 1$. So if $(\langle R, R\rangle)^k$ is commutative, then $R^{(n+1)} = 0$.

A brief excursion into transfinite versions of some of the previous results of this section are given next. Let α be an arbitrary infinite ordinal. Define $T^\alpha = \bigcap_{\beta<\alpha} T^\beta$, if α is a limit ordinal, and $T^\alpha = T^{\alpha-1}\cdot T$ otherwise. For a group G, define $G^{(\alpha)} = \bigcap_{\beta<\alpha} G^{(\beta)}$ and $\gamma_\alpha(G) = \bigcap_{\beta<\alpha} \gamma_\beta(G)$, if α is a limit ordinal, and $G^{(\alpha)} = (G^{(\alpha-1)})'$, $\gamma_\alpha(G) = [\gamma_{\alpha-1}(G), G]$, if α is not a limit ordinal. This defines the *generalized lower* central series, $\gamma_\alpha(G)$, and the *generalized derived series*, $G^{(\alpha)}$, [**29**, pp. 175, 215]. It is known that $\gamma_w(G) = 1$ if and only if G is residually nilpotent [**29**, p. 183]. In this case G is isomorphic to a subdirect product of nilpotent groups.

Proposition 3.7. (i) *If* $R^w = 0$, *then* $\gamma_w(R) = 0$ *and* $(R, \circ)$ *is a subdirect product of nilpotent groups.*

(ii) *If* $R^w \subseteq Z(R)$, *then* $\gamma_{w+1}(R) = 0$.

(iii) *If* R^w *is a permutation identity ring, then* $R^{(w+n)} = 0$, *for some* $n \in \mathbb{N}$, *and* $(R, \circ)$ *is hyperabelian. In particular, if* R^w *is commutative, then* $R^{(w+1)} = 0$.

Proof. Observe that $\gamma_w(R, \circ) = \bigcap_1^\infty \gamma_n(R) \subseteq \bigcap_1^\infty R^n = R^w$. If $R^w = 0$, then $(R, \circ)$ is residually nilpotent. If $R^w \subseteq Z(R)$, then $\gamma_w(R, \circ)$ is in the center of $(R, \circ)$ and hence $\gamma_{w+1}(R, \circ) = 0$.

Similarly, $R^{(w)} = \bigcap_1^\infty R^{(n)} \subseteq \bigcap_1^n R^{2n} = R^w$. If R^w is a permutation identity ring, then $(R^w, \circ)$ is solvable. Consequently, $R^{(w+n)} = 0$, for some $n \in \mathbb{N}$.

In contrast to the situation for finite exponents, $T^w = 0$ does not imply that $J(T) = T$. For example, if E is the ring of even integers, then $E^w = 0$.

Corollary 3.8. *Let* I *be an ideal of* R.

(i) *If* $(R/I)^w = 0$ *and* $I \subseteq Z(R)$, *then* $\gamma_{w+1}(R) = 0$.

(ii) *If* $(R/I)^w \subseteq Z(R/I)$ *and* $I \subseteq Z(R)$, *then* $\gamma_{w+2}(R) = 0$.

(iii) *Let* $(R/I)^w$ *and* I^n, *for some* $n \in \mathbb{N}$, *be permutation identity rings. Then* $(R, \circ)$ *is hyperabelian of length less than* $2w$.

(iv) *If* $(\langle R, R\rangle)^w$ *satisfies a permutation identity, then* $(R, \circ)$ *is hyperabelian of length less than* $2w$.

Proof. (i) Use Proposition 3.7 (i) to obtain $\gamma_w(R/I) = 0$. So $\gamma_w(R) \subseteq I \subseteq Z(R)$, and hence $\gamma_{w+1}(R) = 0$.

(ii) Proceed similarly using Proposition 3.7 (ii).

(iii) Use Proposition 3.7 (iii) to obtain $(R/I)^{(w+k)} = 0$, for some $k \in \mathbb{N}$. Then $R^{(w+k)} \subseteq I$. By Corollary 3.5 (i), there exists $m \in \mathbb{N}$ such that $I^{(m)} = 0$. Then $R^{(w+k+m)} = 0$.

(iv) Let $L = \langle R, R\rangle$. Use Proposition 3.7 (iii) to obtain $L^{(w+n)} = 0$, for some $n \in \mathbb{N}$. Since $(R/L, \circ)$ is abelian, we have $R' \subseteq L$, and hence $R^{(j)} \subseteq L^{(j+1)}$, for each $j \in \mathbb{N}$. So $R^{(w)} \subseteq L^{(w)}$, yielding $R^{(w+n)} = 0$.

A more extensive development of generalized series for adjoint groups will be given in a subsequent paper. For allied results see [**2**].

Relationships between commutativity with respect to the circle operation and commutativity with respect to the Lie commutator operation in a ring have been investigated by Laue [**23**], Du [**8**], and Catino [**6**]. We adopt the notation used in those papers. Recursively define the sequences $Z_n = Z_n(R)$, $Y_n = Y_n(R)$, and $C_n = C_n(R)$ as follows:

(i) $Z_0 = 0,\ Z_{n+1} = \{a \in R\colon ax - xa \in Z_n,\ \text{for each } x \in R\}$.
(ii) $Y_0 = 0,\ Y_{n+1} = \{a \in R\colon [a, x] \in Y_n,\ \text{for each } x \in R\}$.
(iii) $C_0 = 0,\ C_{n+1} = \{a \in R\colon x' \circ a \circ - a \in C_n,\ \text{for each } x \in R\}$.

It is immediate that $Z_1(R) = Y_1(R) = C_1(R) = Z(R)$. Du [**8**] showed that $Z_n(R) = Y_n(R)$, for each $n \in \mathbb{N}$, and Catino [**6**] established that $C_n(R) = Z_n(R)$, for each $n \in \mathbb{N}$.

Earlier, Jennings [**18**] proved that the associated Lie ring of a radical ring is nilpotent if and only if the adjoint group is nilpotent. It is safe to say that the full relevance of the Du-Catino identities for adjoint groups is yet to be fully realized. Recently, Riley and Tasic investigated relationships between Lie identities in a nil ring and commutator properties in the adjoint group, [**26**].

4 - Connections between the additive group and the adjoint group

In general there is no close connection between the order of elements in $(R, \circ)$ and the order of elements in $(R, +)$. However, for nil rings satisfactory relations have been established by Amberg and Dickenscheid, [**1**, Lemma 2.4], and in a somewhat different form and proof scheme by Shan [**30**, Chapter 3]. We adopt the notation $\tau(G)$ for the set of all elements of finite order in a group G, and R_p^+ and R_p° for the sets of all elements a power of a prime p in $(R, +)$ and $(R, \circ)$, respetively.

It is well-known (Shoda's Theorem) that in any ring T, then $\tau(T, +)$ is the ring direct sum of all the T_p^+. Since any ring direct sum of ideals yields a corresponding direct product of normal subgroups in the adjoint group, we have $\tau(R, \circ)$ is the direct product of all the R_p°. However, the group decomposition need not be the analogous one of the torsion subgroup into Sylow p-subgroups, since $\tau(R, +)$ need not be $\tau(R, \circ)$ and the R_p^+ need not be the R_p°, as the following examples illustrate.

Example 4.1. In Example 3.1 let $D = \mathbb{Z}_2[x]$ and let S be the ideal generated by x. So $W = \left\{ \dfrac{xf(x)}{1 + xg(x)} : f(x),\ g(x) \in \mathbb{Z}_2[x] \right\}$. Observe that while x has order two in $(W, +)$, it does not have finite order in $(W, \circ)$.

Example 4.2. In Example 3.1 let $D=\mathbb{Z}$ and let $S=2\mathbb{Z}$. Then $W=\left\{\dfrac{x}{1+y}: x, y \in S\right\}$. A routine calculation shows that $\tau(W, \circ)=\{0, -2\}$. Of course, $\tau(W, +)=0$. Shan has shown that W satisfies the maximum condition on ideals, [**30**, pp. 10-11].

These examples make the results for nil rings all the more interesting. The next proposition serves as a synopsis of the results on the order of elements in $(R, +)$ and $(R, \circ)$ for nil rings that are given in [**11**] and [**30**].

Proposition 4.3. *Let R be nil. Then*:

(i) $\tau(R, +)=\tau(R, \circ)$, *and the latter is a fully invariant subgroup of* $(R, \circ)$;

(ii) *for each prime $p\in\mathbb{N}$, $R_p^+=R_p^\circ$, and each R_p° is a fully invariant subgroup of* $(R, \circ)$;

(iii) $\tau(R, \circ)$ *is the internal direct product of the R_p°, where p ranges over all primes in* $\mathbb{N}$;

(iv) $(R, +)$ *is torsion* (*torsion-free*) *if and only if* $(R, \circ)$ *is torsion* (*torsion-free*).

Recall that a ring T has *bounded index of nilpotence* if there exists $n\in\mathbb{N}$ such that whenever $t\in T$ and t is nilpotent, then $t^n=0$. Nilpotent rings have bounded index of nilpotence, but so do many nil rings that are not nilpotent.

Corollary 4.4. *Let R be a nil ring of bounded index of nilpotence. If $(R, +)$ has bounded order, then $(R, \circ)$ has bounded order* (*finite exponent*).

Some interesting results have been obtained relating chain conditions (finiteness conditions) in $(R, +)$ with chain conditions in $(R, \circ)$. The first such results were given by Watters.

Proposition 4.5 ([**34**]). *The following are equivalent*:

(i) $(R, \circ)$ *satisfies the maximum condition on subgroups*;

(ii) $(R, \circ)$ *satisfies the maximum condition on abelian subgroups*;

(iii) $(R, \circ)$ *is a finitely generated nilpotent group*;

(iv) $(R, +)$ *is finitely generated.*

This proposition and Example 4.2 illustrate the dramatic difference in the maximum condition on subgroups of $(R, \circ)$ and the maximum condition on ideals (or one-sided ideals) in R. It is worth noting that if $(R, \circ)$ satisifes the maximum condition on subgroups (normal subgroups), then R satisfies the maximum condition on one-sided ideals (ideals) in R. A strictly analogous result holds for miminal conditions. Since a Jacobson radical ring with minimum condition on left (right) ideals must be nilpotent, [**17**, p. 38], if $(R, \circ)$ satisifes the minimum condition on subgroups, then R is nilpotent.

Amberg and Dickenscheid have dramatically extended Watter's results (Proposition 4.5, above). We next give the relevant parts of their development. A group G is called a *minimax group* if G has a subnormal series of finite length each of whose factors satisfies either the maximum condition or the minimum condition on subgroups.

Proposition 4.6 ([**1**]). The adjoint group $(R, \circ)$ is a minimax group if and only if $(R, +)$ is a minimax group. In this case R is nilpotent.

(Note: Amberg and Dickenscheid state and prove a more general result than this in their Theorem A).

We conclude this section with some results due to Dickenscheid [**7**]. Let G be a group. Then:

(i) G has *finite torsion-free rank* if G has a subnormal series of finite length whose factors are either abelian or periodic; the number of such factors is an invariant of G and is denoted by $\boldsymbol{r}_0(G)$;

(ii) G has *finite Prüfer rank*, $\boldsymbol{r} = \boldsymbol{r}(G)$, if every finitely generated subgroup of G can be generated by $\boldsymbol{r}$ elements, but not by less than $\boldsymbol{r}$ elements;

(iii) G has *finite abelian subgroup rank* if each abelian subgroup of G has finite torsion-free rank and each abelian p-subgroup of G has finite Prüfer rank, for every prime p. (For details on these rank concepts see [**27**, Chapter 4]).

Proposition 4.7 ([**7**, Theorem A]). *Let R be a nil ring. Then*:

(i) *if $(R, +)$ has torsion-free rank $n < \infty$, then $\boldsymbol{r}_0(R, \circ) = n$*;

(ii) *if* $(R, +)$ has finite abelian subgroup rank, then so does $(R, \circ)$;

(iii) *if $(R, +)$ has finite Prüfer rank, then so does $(R, \circ)$, and $\boldsymbol{r}(R, \circ) \leqslant 3\boldsymbol{r}(R, +)$; if $(R, +)$ also contains no elements of order two, then $\boldsymbol{r}(R, \circ) \leqslant 2\boldsymbol{r}(R, +)$.*

Proposition 4.8 ([**7**], Theorem B). *Let $(R, +)$ be periodic (torsion). Then:*

(i) *if $(R, +)$ has finite abelian subgroup rank, then so does $(R, \circ)$;*

(ii) *if $(R, +)$ has finite Prüfer rank, then so does $(R, \circ)$, and $\boldsymbol{r}(R, \circ) \leqslant 3\boldsymbol{r}(R, +)$; if $(R, +)$ also contains no elements of order two, then $\boldsymbol{r}(R, \circ) \leqslant 2\boldsymbol{r}(R, +)$.*

5 - Further examples and constructions

Let $\mathfrak{J}$ be the class of all Jacobson radical rings. It is well-known that $\mathfrak{J}$ is closed under homomorphic images, direct sums, and direct products. However, $\mathfrak{J}$ is not closed under subdirect products, as illustrated by the ring E of all even integers. Then $\bigcap_1^\infty E_n = 0$, where $E_n = E/E^n$, $n = 2, 3, \ldots$, and each E_n is nilpotent. So E is a subdirect product of nilpotent rings, while $J(E) = 0$.

Let Λ be an arbitrary nonempty index set and let $R_\lambda \in \mathfrak{J}$, $\lambda \in \Lambda$. We use $\sum \oplus R_\lambda$ and $\prod R_\lambda$ for the direct sum and direct product of the R_λ, $\lambda \in \Lambda$, respectively. Following Robinson [**27**, pp. 21-22], for a set of groups, G_λ, $\lambda \in \Lambda$, we use $\underset{\lambda}{Dr}\, G_\lambda$ and $\underset{\lambda}{Cr}\, G_\lambda$ for the restricted and unrestricted direct products of the G_λ, respectively. It is routine to establish that $\left(\sum_\lambda \oplus R_\lambda, \circ\right) \approx \underset{\lambda}{Dr}(R_\lambda, \circ)$ and $\prod_\lambda (R_\lambda, \circ) \approx \underset{\lambda}{Cr}(R_\lambda, \circ)$.

Example 5.1. Let $(\Lambda, \leqslant)$ be a directed partially ordered set and let $R_\lambda \in \mathfrak{J}$, for $\lambda \in \Lambda$. Assume $(R_\lambda, f_\lambda^\mu, \Lambda)$ is a direct family in the category of rings and form the direct limit of this family, $\varinjlim R_\lambda$. Observe that each ring homomorphism $f_\lambda^\mu : R_\lambda \to R_\mu$, gives rise to a group homomorphism $f_\lambda^\mu : (R_\lambda, \circ) \to (R_\mu, \circ)$.

This yields the direct family $((R_\lambda, \circ), f_\lambda^\mu, \Lambda)$ in the category of groups, and the corresponding direct limit, $\varinjlim (R_\lambda, \circ)$. Shan has shown that $\left(\varinjlim R_\lambda, \circ\right)$ is isomorphic to $(R_\lambda, \circ)$, [**30**, pp. 34-35].

Example 5.2 (Shan, [**30**, pp. 53-54]). There is a natural extension of Example 3.1 to a noncommutative setting. Let T be a ring with unity and let I be a proper nonzero ideal of R. Define $S = \{1 + x : x \in I\}$. Then S will be a left denominator set for T, yielding the ring of fractions $S^{-1}I$, and $J(S^{-1}I) = S^{-1}I$.

The next example arises in applied analysis.

Example 5.3. Let $\mathfrak{M}$ be the linear space of all complex-valued functions on the interval $[0, \infty)$, together with the Duhamel convolution operation: $(f * g)(t) = \int_0^t f(t-x)\, g(x)\, \mathrm{d}x$. With this multiplication $\mathfrak{M}$ is a commutative associative algebra over the complex number field, and $\mathfrak{M}$ has no nonzero divisors of zero. Huffman has shown that $\mathfrak{M} = J(\mathfrak{M})$, [**15**]. (A strictly analogous result holds for continuous real valued functions). Many natural subsets of $\mathfrak{M}$ are ideals in the algebra, and hence are themselves Jacobson radical rings. (See [**15**]).

It is of interest that using $J(\mathfrak{M}) = \mathfrak{M}$, novel new proofs were obtained for existence and uniqueness theorems for certain integral and integro-differential equations, [**11**], [**15**].

For more examples of exotic radical rings and adjoint groups, see [**12**].

6 - Other lines of investigation

In this section brief mention is made of some other approaches to investigating adjoint groups of radical rings. The question of which groups are isomorphic to adjoint groups has been addressed by Kaloujnine [**19**], Ault and Watters [**3**], Hales and Possi [**9**], Tahara and Hosomi [**32**], Amberg, Dickenscheid, and Sysak [**2**], and Shan [**30**]. In [**3**], [**9**], [**19**], [**24**], [**28**], [**31**], [**32**] various conditions are given which guarantee that a group is isomorphic to the adjoint group of a ring. Examplary of some recent developments along these lines is the following.

Proposition 6.1 ([**30**], [**31**]). *Let G be a nilpotent class two group. If any one of the following hold, then G is isomorphic to the adjoint group of a nilpotent ring:*

(i) *$Z(G)$ is uniquely 2-radicable*;
(ii) *$Z(G)$ is a p-group, for some odd prime p*;
(iii) *G/G' is uniquely 2-radicable and G' has no elements of order two*;
(iv) *$G/Z(G)$ is finitely generated*;
(v) *G satisfies the maximal condition on subgroups.*

(Recall that a group G is *uniquely* 2*-radicable* if for each $g \in G$ there exists a unique $b \in G$ such that $g = b^2$, [**33**]).

In [**2**] conditions were given which must be satisfied by every adjoint group. (However, see [**12**], [**30**] for comments on these conditions). The consequences of $(R, \circ)$ being solvable or nilpotent have been investigated by Kruse [**21**], [**22**]. Recently, [**13**], conditions on the adjoint semigroup of a ring T which force $(T, \circ)$ to be a group have been developed. To conclude the paper some of these results are given.

Proposition 6.1 ([**13**, Proposition 2.3]). *Let $(T, \circ)$ be simple. If any one of the following hold, then $(T, \circ)$ is a group*:

(i) *T has no non-zero nilpotent elements*;
(ii) *T has finite right (left) uniform dimension*;
(iii) *T satisfies the maximum condition on right (left) ideals*;
(iv) *T satisfies the minimal conditions on right (left) ideals.*
(Note: Under condition (iv) the ring T is nilpotent).

Proposition 6.2 ([**13**, Proposition 4.3]). *Let T have no zero divisors but not* *be a division ring. If* $(T, \circ)$ *is von Neumann regular, then* $(T, \circ)$ *is a group.*

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Summary

Let R be a Jacobson radical ring. This paper discusses the connections between a ring R and its adjoint group $(R, \circ)$, where $a \circ b = a + b - ab$. Both an exposition of the development of the theory and some new results are given. Conditions on R which guarantee that $(R, \circ)$ is solvable, nilpotent, or generalized solvable are considered. A new set of conditions for a group to be isomorphic to an adjoint group is given. Several diverse examples of Jacobson radical rings are exhibited.

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