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On totally umbilical holomorphic submanifolds (**)

1 - Introduction

The present paper deals with sectional and bisectional curvatures of an almost Hermitian manifold and its totally umbilical holomorphic submanifolds. Let $\widetilde{M}$ be the manifold and M a submanifold of $\widetilde{M}$ satisfying the above assumptions.

Some results are stated in Sections **3** and **4**.

Proposition 1 shows that the bisectional holomorphic curvature of M is greater or equal than the corresponding curvature of $\widetilde{M}$.

We consider then some expressions involving sectional and bisectional curvatures, known from the literature, and write the relations linking these expressions for M with the corresponding expressions for $\widetilde{M}$ (Proposition 2). When the above mentioned expressions are constant on $\widetilde{M}$, then at any point of M we derive upper and lower bounds for the corresponding expressions for the submanifold M (Corollary 2).

Similar to Corollary 2 is Corollary 1, where at any point of M upper and lower bounds for the holomorphic bisectional curvature of M are given.

Other results are stated in Section **5**.

In Proposition 3 we indicate hereditary properties concerning manifolds of constant type, in usual or weak sense. In Proposition 4 we point out that similar properties hold true for the known classes of AH_2-manifolds and of AH_3-manifolds, while an analogous property for AH_1-manifolds (parakähler manifolds) does not hold in general.

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2 - Preliminaries

Let $\widetilde{M} = \widetilde{M}(g)$ be an $\widetilde{m}$-dimensional *Riemanian manifold* and g its metric. Let M be an m-dimensional *submanifold* of $\widetilde{M}$, with induced metric, still denoted by g.

For the basic facts about the geometry of submanifols we refer to [**1**] Ch. 2, [**8**] Ch. 7 and [**20**] Ch. 2. In the sequel B denotes the *second fundamental form* and $H = \dfrac{1}{m}$ trace B the *mean curvature vector field* of M.

We recall that M is a *totally umbilical* submanifold of $\widetilde{M}$ if and only if at any point x of M we have

$$B(X, Y) = g(X, Y)\, H \tag{1}$$

for any pair of vectors X, Y of $T_x(M)$. Of course the form B and the vector field H, occurring in (1), have to be evalued at the point x.

We know that for a totally umbilical submanifold M of $\widetilde{M}$ we have the *fundamental relations*

$$\widetilde{\chi}_{rs} = \chi_{rs} - g(H, H) \cos rs \tag{2}$$

$$\widetilde{K}_r = K_r - g(H, H) \tag{3}$$

linking the bisectional and the *sectional curvatures of M* with the corresponding curvatures of $\widetilde{M}$ (see (8), (10) at p. 117 of [**16**]).

The above relations are true at any point x of the totally umbilical submanifold M and for any pair r, s of oriented planes of $T_x(M)$. Of course in (2), (3) $g(H, H)$ has to be evalued at x. For the definitions of bisectional curvature and of angle of two oriented planes see for example (1), (4) at p. 148-149 of [**11**].

We consider now in particular the case when $\widetilde{M} = \widetilde{M}(g, J)$ is an *almost Hermitian manifold.* A submanifold M of $\widetilde{M}$ is called *holomorphic* (or *J-invariant*), if and only if we have $J(T_x(M)) = T_x(M)$ at any point x of M. A holomorphic submanifold M of $\widetilde{M}$ is also an almost Hermitian manifold with the induced structures; so we can write $M = M(g, J)$.

3 - Some results

From now on we assume that $\widetilde{M} = \widetilde{M}(g, J)$ is an *almost Hermitian manifold*, that $M = M(g, J)$ is a *totally umbilical holomorphic submanifold* of $\widetilde{M}$ and that $\widetilde{m} > m \geqslant 4$.

Let $\tilde{x}$ be a point of $\widetilde{M}$, $\tilde{r}$ a plane of $T_{\tilde{x}}(\widetilde{M})$ and $\delta_{\tilde{r}}$ $(0 \leqslant \delta_{\tilde{r}} \leqslant \pi)$ its *holomorphic deviation.* The special cases when $\tilde{r}$ is *holomorphic* or *antiholomorphic* occur in the sequel. Holomorphic planes are always assumed to be *canonically oriented.* For the above notions we refer to [**12**] Sec. **2** and to [**14**] Sec. **6**.

Special types of bisectional curvatures occur in the literature. For instance the *holomorphic bisectional curvature* (shortly *biholomorphic curvature*) $\tilde{\chi}_{\tilde{h}_1\tilde{h}_2}$, where $\tilde{h}_1$, $\tilde{h}_2$ are holomorphic planes of $T_{\tilde{x}}(\widetilde{M})$ (see [**5**]), and the *mixed curvature* $\tilde{\chi}_{\tilde{r}J\tilde{r}}$, where $\tilde{r}$ is a plane of $T_{\tilde{x}}(\widetilde{M})$ (see [**12**]).

Moreover, some expressions involving sectional and bisectional curvatures for a plane $\tilde{r}$ of $T_{\tilde{x}}(\widetilde{M})$ have been considered in [**12**], [**14**], [**15**]. Namely

$$\widetilde{\mathfrak{M}}_{\tilde{r}} = \frac{1}{2}(\widetilde{K}_{\tilde{r}} + \widetilde{K}_{J\tilde{r}}) \tag{4}$$

$$\tilde{\mathcal{E}}^{+}_{\tilde{r}} = \widetilde{K}_{\tilde{r}} + 2\,\tilde{\chi}_{\tilde{r}J\tilde{r}} + \widetilde{K}_{J\tilde{r}} \qquad \tilde{\mathcal{E}}^{-}_{\tilde{r}} = \widetilde{K}_{\tilde{r}} - 2\,\tilde{\chi}_{\tilde{r}J\tilde{r}} + \widetilde{K}_{J\tilde{r}}. \tag{5}$$

We are able now to give some results linking the above mentioned curvatures and expressions of the submanifold M with the corresponding curvatures and expressions of the manifold $\widetilde{M}$.

Proposition 1. *At any point x of $M \subset \widetilde{M}$ and for any pair of holomorphic planes h_1, h_2 of $T_x(M) \subset T_x(\widetilde{M})$ we have*

$$\chi_{h_1 h_2} \geqslant \tilde{\chi}_{h_1 h_2}. \tag{6}$$

Proposition 2. *At any point x of $M \subset \widetilde{M}$ and for any plane r of the tangent space $T_x(M) \subset T_x(\widetilde{M})$ we have*

$$\tilde{\chi}_{rJr} = \chi_{rJr} - g(H, H)\cos^2\delta_r \tag{7}$$

$$\widetilde{\mathfrak{M}}_r = \mathfrak{M}_r - g(H, H) \tag{8}$$

$$\tilde{\mathcal{E}}^{+}_{r} = \mathcal{E}^{+}_{r} - 2g(H, H)(1 + \cos^2\delta_r) \tag{9}$$

$$\tilde{\mathcal{E}}^{-}_{r} = \mathcal{E}^{-}_{r} - 2g(H, H)(1 - \cos^2\delta_r) \tag{10}$$

where the function $g(H, H)$ is evalued at point x.

Consequently we have

$$\chi_{rJr} \geqslant \tilde{\chi}_{rJr} \qquad \mathcal{M}_r \geqslant \widetilde{\mathcal{M}}_r \tag{11}$$

$$\mathcal{E}_r^+ \geqslant \tilde{\mathcal{E}}_r^+ \qquad \mathcal{E}_r^- \geqslant \tilde{\mathcal{E}}_r^- . \tag{12}$$

The proof of Proposition 1 is short. Let X, JX, and Y, JY be orthonormal bases of the holomorphic planes h_1, h_2 of $T_x(m) \subset T_x(\widetilde{M})$. Then

$$\cos h_1 h_2 = \det \begin{pmatrix} g(X, Y) & g(X, JY) \\ g(JX, Y) & g(JX, JY) \end{pmatrix} = (\, g(X, Y)\,)^2 + (\, g(X, JY)\,)^2 \geqslant 0 \, .$$

Since at the point x we have $g(H, H) \geqslant 0$, from (2) we immediately derive (6).

To prove Proposition 2, recall first that we have $\cos rJr = \cos^2 \delta_r$ (see e.g. (2) of [**12**]). Then, starting from relations (2), (3) and taking account of definitions (4), (5), we obtain (7), (8), (9), (10). At this point, it is elementary to prove inequalities (11), (12).

4 - Corollaries and remarks

We consider now the special cases when the almost Hermitian manifold $\widetilde{M} = \widetilde{M}(g, J)$ has constant biholomorphic curvature, constant mixed curvature, respectively. Two more special cases arise when the expressions $\widetilde{\mathcal{M}}_{\tilde{r}}$, $\tilde{\mathcal{E}}_{\tilde{r}}^+$ are assumed to be constant at any point $\tilde{x}$ of $\widetilde{M}$ and for any plane $\tilde{r}$ of $T_{\tilde{x}}(\widetilde{M})$.

All these manifolds belong to the widely studied class of the *manifolds of constant holomorphic curvature.* To prove this fact, consider any holomorphic plane $\tilde{h}$ of $T_{\tilde{x}}(\widetilde{M})$. Since we have $\tilde{h} = J\tilde{h}$, taking $\tilde{h}_1 = \tilde{h}_2 = \tilde{r} = \tilde{h}$ we can write

$$\tilde{\chi}_{\tilde{h}_1 \tilde{h}_2} = \tilde{\chi}_{\tilde{r} J \tilde{r}} = \widetilde{\mathcal{M}}_{\tilde{r}} = \frac{1}{4}\, \tilde{\mathcal{E}}_{\tilde{r}}^+ = \widetilde{K}_{\tilde{h}}$$

and the conclusion follows immediately.

We consider now the classes of the above introduced manifolds and give some information on the bisectional curvatures of the submanifolds.

Corollary 1. *Let $\widetilde{M} = \widetilde{M}(g, J)$ be a manifold of constant biholomorphic curvature c. Then at any point x of $M \subset \widetilde{M}$ and for any pair of holomorphic* *planes* h_1, h_2 *of* $T_x(m) \subset T_x(\widetilde{M})$ *we have*

$$c \leqslant \chi_{h_1 h_2} \leqslant c + g(H, H) \tag{13}$$

where the function $g(H, H)$ *is evalued at* x.

The minimun and the maximum of $\chi_{h_1 h_2}$ *at* x *are attained if and only if the holomorphic planes* h_1, h_2 *are orthogonal, coincide, respectively. The biholomorphic curvature of* M *has a minimum on* M; *namely the constant* c.

Corollary 2. *If at any point* $\tilde{x}$ *of* $\widetilde{M} = \widetilde{M}(g, J)$ *we have respectively*

$$\tilde{\chi}_{\tilde{r}J\tilde{r}} = c_1 \qquad \widetilde{\mathcal{M}}_{\tilde{r}} = c_2 \qquad \tilde{\mathcal{E}}^+_{\tilde{r}} = c_3$$

where the c_i*'s are constants on* $\widetilde{M}$, *then at any point* x *of* $M \subset \widetilde{M}$ *and for any plane* r *of* $T_x(M) \subset T_x(\widetilde{M})$ *we have, respectively*

$$c_1 \leqslant \chi_{rJr} \leqslant c_1 + g(H, H) \tag{14}$$

$$\mathcal{M}_r = c_2 + g(H, H) \tag{15}$$

$$c_3 + 2g(H, H) \leqslant \mathcal{E}^+_r \leqslant c_3 + 4g(H, H) \tag{16}$$

where the function $g(H, H)$ *is evalued at* x.

The minimum and the maximum of χ_{rJr} *and of* $\mathcal{E}^+_r$ *at* x *are attained if and only if the plane* r *is antiholomorphic, holomorphic, respectively. The mixed curvature of* M *has a minimum on* M; *namely the constant* c_1.

Corollary 1 is an immediate consequence of (2) of Sec. **2**, since we have $\cos h_1 h_2 \geqslant 0$ (Sec. **3**). Corollary 2 follows from (7), (8), (9) of Proposition 2, since we have $\delta_r = 0, \pi$; $\delta_r = \frac{\pi}{2}$ iff r is holomorphic, antiholomorphic, respectively ([**12**], p. 179).

We end the section with some remarks.

Remark 1. In Corollaries 1, 2 the assumptions on $\widetilde{M}$ can be weakened. In effect you may limit yourself to consider only planes tangent to M.

Remark 2. In Corollary 1 you may assume that the biholomorphic curvature of $\widetilde{M}$ is only *point-wise constant*; i.e. c is a function on $\widetilde{M}$. Then, in (13) c has to be evalued at x and the last statement is no more true in general. Similar remarks for Corollary 2.

Remark 3. To complete the present section we should consider also the special case when the expression $\tilde{\mathcal{E}}_{\tilde{r}}^{-}$, defined by (5), is constant. This constant however is necessarily zero, since at any point $\tilde{x}$ of $\widetilde{M}$ and for any holomorphic plane $\tilde{h}$ of $T_{\tilde{x}}(\widetilde{M})$ we have $\tilde{\mathcal{E}}_{\tilde{h}}^{-}=0$.

5 - Hereditary properties

We consider here other classes of almost Hermitian manifolds, which are known in the literature.

We recall first that $\widetilde{M}=\widetilde{M}(g,J)$ has *pointwise constant type α in a weak sense*, if and only if at any point $\tilde{x}$ of $\widetilde{M}$ and for any antiholomorphic plane $\tilde{a}$ of $T_{\tilde{x}}(\widetilde{M})$ we have

$$\tilde{\mathcal{E}}_{\tilde{a}}^{-}=2\alpha \tag{17}$$

where α is a function on $\widetilde{M}$, evalued at $\tilde{x}$ (cf. [**15**], p. 175). In particular, $\widetilde{M}$ has *pointwise constant type α, if we have*

$$\widetilde{K}_{\tilde{a}}-\tilde{\chi}_{\tilde{a}J\tilde{a}}=\alpha \tag{18}$$

(cf. [**3**], p. 288, [**19**], p. 488).

By Remark 3 the manifolds $\widetilde{M}=\widetilde{M}(g,J)$ with $\tilde{\mathcal{E}}_{\tilde{r}}^{-}$ constant are manifolds of constant type 0 in a weak sense.

We recall also that three identities for the curvature tensor field $\widetilde{R}$ of the manifold $\widetilde{M}=\widetilde{M}(g,J)$ play a special role in the literature (see for example [**18**], p. 368). The first one is the *Kähler identity*, the third one expresses the fact that the curvature tensor field $\widetilde{R}$ is *J-invariant*, the second one is intermediate.

We denote now by AH_1, AH_2, AH_3, respectively, the classes of almost Hermitian manifolds with curvature tensor field satisfying the above identities. AH_1-manifolds are also known as *parakähler manifolds* ([**10**], p. 51) as well as *F-spaces* (see [**17**]). AH_3-manifolds are called also *RK*-manifolds ([**19**], p. 487).

Finally, it is worth remarking that by virtue of Theorem 2 at p. 327 of [**13**] the three curvature identities can be expressed in terms of bisectional curvature.

We are able to give some results that justify the title of the section.

Proposition 3. *If $\widetilde{M}=\widetilde{M}(g,J)$ has point-wise constant type α in a weak sense, then the submanifold M has point-wise constant type $\alpha+g(H,H)$ in a weak sense. Similarly in the special case of point-wise constant type manifolds.*

Proposition 4. *If $\widetilde{M} = \widetilde{M}(g, J)$ is an AH_3-manifold, an AH_2-manifold, then the submanifold M is an AH_3-manifold, an AH_2-manifold, respectively. The analogous property for AH_1-manifolds, is true if and only if M is totally geodesic.*

6 - Proofs and remarks

The proof of Proposition 3 is short. Let x be a point of $M \subset \widetilde{M}$ and a be any antiholomorphic plane of $T_x(M) \subset T_x(\widetilde{M})$. We have $\delta_a = \dfrac{\pi}{2}$. Now, using (17), (9) we prove the first part of the statement. For the second part, just use (18), (2), (3).

To prove Proposition 4, we recall that the classes AH_1, AH_2, AH_3 can be defined in terms of bisectional curvature by conditions (15), (16), (17) of [**13**].

Let x be a point of $M \subset \widetilde{M}$ and r, s a pair of oriented planes of $T_x(M) \subset T_x(\widetilde{M})$. Further, let $\overset{*}{\sum_r}, \overset{*}{\sum_s}$ be the systems of antiholomorphic oriented planes, introduced in [**9**], Sec. 5 and let r^*, s^* be planes belonging to $\overset{*}{\sum_r}, \overset{*}{\sum_s}$ respectively. Then, taking into account relations (2) of Sec. **2**, we can write:

$$\tilde{\chi}_{rs} - \tilde{\chi}_{JrJs} = \chi_{rs} - \chi_{JrJs} - g(H, H)(\cos rs - \cos JrJs) \tag{19}$$

$$\begin{gathered}(\tilde{\chi}_{rs^*} - \tilde{\chi}_{rJs^*}) \sin \delta_s + (\tilde{\chi}_{r^*s} - \tilde{\chi}_{Jr^*s}) \sin \delta_r \\ (\chi_{rs^*} - \chi_{rJs^*}) \sin \delta_s + (\chi_{r^*s} - \chi_{Jr^*s}) \sin \delta_r \\ -g(H, H)[(\cos rs^* - \cos rJs^*) \sin \delta_s + (\cos r^*s - \cos Jr^*s) \sin \delta_r]\end{gathered} \tag{20}$$

$$\tilde{\chi}_{rs} - \tilde{\chi}_{rJs} = \chi_{rs} - \chi_{rJs} - g(H, H)(\cos rs - \cos rJs) \tag{21}$$

where the function $g(H, H)$ is evalued at point x.

We are now able to complete the proof.

Let X, Y and Z, W be ortonormal bases in the oriented planes r, s, respectively. It is immediate to check that $\cos rs = \cos JrJs$ (see (4) of [**11**]). Then, by virtue of (17) of [**13**], relation (19) leads to the result for AH_3-manifolds.

The proof in the second case is similar. For any pair r^*, s^* of oriented planes of $\overset{*}{\sum_r}, \overset{*}{\sum_s}$, respectively, we can choose orthonormal bases X, Y and Z, W in r, s, such that X, JY and Z, JW are bases for r^*, s^*, respectively (see [**13**], p. 328).

Using (4) of [**11**] for the cosines and (1) of [**12**] for the holomorphic deviations δ_r, δ_s, we can prove that the whole expression in brackets, occurring in (20), vanishes. Then, using relation (20) and condition (16) of [**13**], we derive the property, relative to AH_2-manifolds.

Finally, if $\widetilde{M}$ is a parakähler manifold (AH_1-manifold), then condition (15) of [**13**] is satisfied on $\widetilde{M}$ and relation (21) reduces to

$$\chi_{rs} - \chi_{rJs} = g(H, H)(\cos rs - \cos rJs) \tag{22}$$

In particular, for any antiholomorphic plane a of $T_x(M)$ we have

$$K_a - \chi_{aJa} = g(H, H). \tag{23}$$

Relation (23) shows that condition (15) of [**13**] is not satisfied for any pair of planes of $T_x(M)$, when H is different from zero at x. In other words, M is not a parakähler manifold (AH_1-manifold) in general. In the matter of fact, since M is totally umbilical (Sec. **3**), equations (22), (23) imply that M is a parakähler manifold (AH_1-manifold) if and only if M is totally geodesic.

The proof of Proposition 4 is now complete.

The paper ends with

Remark 4. In Proposition 4 the assumptions on $\widetilde{M}$ can be weakened, by requiring that conditions (15), (16), (17) of [**13**] are satisfied only by the planes tangent to the submanifold M. Similarly in Proposition 3 for conditions (19), (20) concerning antiholomorphic planes.

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Summary

Let M be a holomorphic totally umbilical submanifold of an almost Hermitian manifold $\widetilde{M}$. Some relations for biholomorphic and mixed curvatures of M and the corresponding curvatures of $\widetilde{M}$ are obtained. When these curvatures are assumed to be constant on $\widetilde{M}$, we derive lower and upper bounds for the corresponding curvatures of M. Similar results are true for some known expressions involving sectional and bisectional curvature. Hereditary properties for AH_2-manifolds, for AH_3 -manifolds, for almost constant type manifolds and for constant type manifolds are also derived.

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