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On double generating functions of single hypergeometric polynomials (**)

1 - Introduction

Two interesting generating functions for the confluent hypergeometric function ${}_1F_1$ and generalized hypergeometric function ${}_AF_B$ [7], p. 73 (2) are given by Exton [5], p. 7 (4.9) and p. 11 (6.5).

$$\sum_{m,n=0}^{\infty} \frac{(\alpha+1)_{m+n} x^m (-x)^n}{m!\,n!}\, {}_1F_1\begin{bmatrix} -m; & \\ \alpha+1; & z\end{bmatrix} {}_1F_1\begin{bmatrix} -n; & \\ \alpha+1; & y\end{bmatrix} = \exp(xy - xz)\, {}_0F_1\begin{bmatrix} --; & \\ \alpha+1; & -x^2 yz\end{bmatrix} \tag{1.1}$$

$$\sum_{m,n=0}^{\infty} \frac{((d))_{m+n}((u))_{m+n}(v)_{m+n} x^m (-x)^n}{((g))_{m+n}((e))_{m+n}((f))_{m+n}\, m!n!}\; {}_{E+V+Q+1}F_{U+F+P}\begin{bmatrix} 1-(e)-m-n, (v)+m+n, (q), -m; & \\ 1-(u)-m-n, (f)+m+n, (p); & (-1)^{1+E-U} y\end{bmatrix}$$
$$= {}_{D+2V+P}F_{G+2F+P}\begin{bmatrix} (d), \left(\frac{v}{2}\right), \left(\frac{v}{2}\right)+\frac{1}{2}, (q); & \\ (g), \left(\frac{f}{2}\right), \left(\frac{f}{2}\right)+\frac{1}{2}, (p); & 4^{V-F} xy\end{bmatrix} \tag{1.2}$$

where for brevity (q) denotes Q parameters $q_1 \ldots q_Q$, with similar interpretation

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for $(d), (g)$ and so on. Also $((q))_m$ stands for $\Pi_{j=1}^{Q}(q_j)_m$ and so on. The purpose of this work is to introduce equations (1.1) and (1.2) as a main working tools to develop a theory of generating relations of special functions which are double generating functions of single polynomials.

2 - Generating relations

In (1.1), if we replace z by zt, multiply both sides by $t^{l-1}e^{-pt}$ and take their Laplace transform with the help of the result [**12**], p. 219 (6).

$$\int_0^\infty t^{\lambda-1}e^{-st}\,{}_AF_B\left[\begin{matrix}(a);\\(b);\end{matrix}\; zt\right] = \Gamma\lambda\, s^{-\lambda}\,{}_{A+1}F_B\left[\begin{matrix}(a),\lambda;\\(b);\end{matrix}\; \frac{z}{s}\right] \tag{2.1}$$

$(Re(\lambda) > 0, A \leqslant B;\ Re(s) > 0$ if $A < B;\ Re(s) > Re(z)$ if $A = B)$, we obtain

$$\begin{aligned}\sum_{m,n=0}^{\infty}\frac{(\alpha+1)_{m+n}}{m!\,n!}x^m(-x)^n\,{}_2F_1\left[\begin{matrix}-m,\ l;\\\alpha+1;\end{matrix}\; z\right]{}_1F_1\left[\begin{matrix}-n;\\\alpha+1;\end{matrix}\; y\right]\\ = \exp(xy)(1+xz)^{-l}\,{}_1F_1\left[\begin{matrix}l;\\\alpha+1;\end{matrix}\; \frac{-x^2zy}{(1+xz)}\right].\end{aligned} \tag{2.2}$$

Next, if in (2.2), we replace y by yt, multiply both sides by $t^{r-1}e^{-Pt}$ and take Laplace transform with the help of (2.1), we get

$$\begin{aligned}\sum_{m,n=0}^{\infty}\frac{(\alpha+1)_{m+n}}{m!\,n!}x^m(-x)^n\,{}_2F_1\left[\begin{matrix}-m,\ l;\\\alpha+1;\end{matrix}\; z\right]{}_2F_1\left[\begin{matrix}-n,\ r;\\\alpha+1;\end{matrix}\; y\right]\\ = (1-xy)^{-r}(1+xz)^{-l}\,{}_2F_1\left[\begin{matrix}l,\ r;\\\alpha+1;\end{matrix}\; \frac{-x^2zy}{(1+xz)(1-xy)}\right].\end{aligned} \tag{2.3}$$

Now starting from (2.3) and making use of the Laplace and inverse Laplace transform [**2**], p. 297 (1).

$$\mathcal{L}^{-1}\left\{s^{-\lambda}\,{}_AF_B\left[\begin{matrix}(a);\\(b);\end{matrix}\; \frac{z}{s}\right]: t\right\} = \frac{t^{\lambda-1}}{\Gamma\lambda}\,{}_AF_{B+1}\left[\begin{matrix}(a);\\(b),\lambda;\end{matrix}\; zt\right], \tag{2.4}$$

$$(Re(\lambda) > 0, A \leqslant B+1),$$

it is not difficult to show by induction that (cf. [**10**], p. 305-311)

$$\sum_{m,n=0}^{\infty} \frac{(\alpha+1)_{m+n}}{m!\,n!} x^m(-x)^n {}_{1+L}F_{K+1}\left[\begin{matrix}-m,(l); \\ \alpha+1,(k);\end{matrix}\; z\right]$$

$$(2.5)\qquad {}_{1+R}F_{W+1}\left[\begin{matrix}-n,(r); \\ \alpha+1,(w);\end{matrix}\; y\right] = \sum_{S=0}^{\infty} \frac{((l))_s((r))_s(-x^2yz)^s}{((k))_s((w))_s(\alpha+1)_s s!}$$

$$ {}_LF_K\left[\begin{matrix}(l)+s; \\ (k)+s;\end{matrix}\; -xz\right] {}_RF_W\left[\begin{matrix}(r)+s; \\ (w)+s;\end{matrix}\; xy\right].$$

Similarly, in case of equation (1.2), if we use the same method of proof of formula (2.5), we get

$$\sum_{m,n=0}^{\infty} \frac{((d))_{m+n}((u))_{m+n}((v))_{m+n}((h))_{m+n}x^m(-x)^n}{((g))_{m+n}((e))_{m+n}((f))_{m+n}((t))_{m+n}m!n!}$$

$$(2.6)\quad {}_{E+V+Q+A+1}F_{U+F+P+B}\left[\begin{matrix}1-(e)-m-n,(v)+m+n,(q),(a),(-m); \\ 1-(u)-m-n,(f)+m+n,(p),(b);\end{matrix}\; (-1)^{1+E-V}y\right]$$

$$= {}_{D+2V+Q+A+H}F_{G+2F+P+B+T}\left[\begin{matrix}(d),\left(\frac{v}{2}\right),\left(\frac{v}{2}\right)+\frac{1}{2},(q),(a),(h); \\ (g),\left(\frac{f}{2}\right),\left(\frac{f}{2}\right)+\frac{1}{2},(p),(b),(t);\end{matrix}\; 4^{V-F}xy\right].$$

Now we mention some interesting special cases of the equations (2.3), (2.5) and (2.6). On setting $r=\alpha+1$, in (2.3), we get

$$(2.7)\qquad \sum_{m=0}^{\infty} \frac{(\alpha+1)_m}{m!}\left[\frac{x}{1+x(1-y)}\right]^m {}_2F_1\left[\begin{matrix}-m,\ l; \\ \alpha+1;\end{matrix}\; z\right]$$
$$= (1-xy)^{l-(\alpha+1)}(1-xy+xz)^{-l}(1+x-xy)^{\alpha+1}$$

which for $y=1$ reduces to a known result [**12**], p. 293 (12). For $y=0$, (2.5) reduces to

$$(2.8)\quad (1+x)^{-\alpha-1}\sum_{m=0}^{\infty} \frac{(\alpha+1)_m}{m!}\left[\frac{x}{1+x}\right]^m {}_{1+L}F_{K+1}\left[\begin{matrix}-m,(l); \\ \alpha+1,(k);\end{matrix}\; z\right] = {}_LF_K\left[\begin{matrix}(l); \\ (k);\end{matrix}\; -xz\right].$$

If in (2.8), we put $L=L'+1$, $l_{L+1}=\alpha+1$, we then have a known result [**4**], p. 267. Further, if in (2.5), we put $K=W=0$, $L=R=1$, $l_1=r_1=\frac{1}{2}$ and $\alpha=0$, then it reduces to another known result due to Exton [**5**], p. 9 (5.10) in its corrected form

$$\sum_{m,\,n=0}^{\infty} \frac{(m+n)!}{m!\,n!}\left[\frac{x}{z+(z^2-1)^{\frac{1}{2}}}\right]^m\left[\frac{-x}{y+(y^2-1)^{\frac{1}{2}}}\right]P_m(z)\,P_n(y) = (1+x\zeta_1)^{\frac{-1}{2}}(1-x\zeta_2)^{\frac{-1}{2}}\,\frac{2}{\pi}K(\sqrt{\xi}) \tag{2.9}$$

where $\zeta_1 = \dfrac{2(z^2-1)^{\frac{1}{2}}}{z+(z^2-1)^{\frac{1}{2}}}$, $\zeta_2 = \dfrac{2(y^2-1)^{\frac{1}{2}}}{y+(y^2-1)^{\frac{1}{2}}}$, $P_m(x)$ is Legendre polynomials defined by [**5**], p. 8 (5.6)

$$P_m(x) = [x+(x^2-1)^{\frac{1}{2}}]^m\,{}_2F_1\left[-m,\ \frac{1}{2}\,;\,1;\,\zeta\right], \qquad \zeta = \frac{2(x^2-1)^{\frac{1}{2}}}{x+(x^2-1)^{\frac{1}{2}}},$$

$K(\xi)$ is the complete elliptic integral of first kind [**3**], p. 318 (5) and $\xi = \dfrac{-x^2\zeta_1\zeta_2}{(1+x\zeta_1)(1-x\zeta_2)}$. If we replace x by $\dfrac{x}{(1-x)}$ in (2.6) together with $V = U = E = F = D = G = Q = P = T = 0$ and $H = 1$, we get

$$\sum_{m=0}^{\infty} \frac{(h)_m}{m!}\,{}_{1+A}F_B\left[\begin{matrix}-m,(a); \\ (b);\end{matrix}\ y\right]x^m = (1-x)^{-h}\,{}_{1+A}F_B\left[\begin{matrix}h,(a); \\ (b);\end{matrix}\ \frac{-xy}{(1-x)}\right] \tag{2.10}$$

$|x| < 1$, which is a result due to Chaundy [**12**], p. 138 (8).

For $E = U = V = F = 0$, the left hand side of (2.6) becomes separable in the form

$$\begin{aligned}\sum_{m=0}^{\infty} \frac{((d))_m((h))_m}{((g))_m((t))_m}\,&{}_{D+H}F_{G+T}\left[\begin{matrix}(d)+m,(h)+m; \\ (g)+m,(t)+m;\end{matrix}\ -x\right] \\ &{}_{Q+A+1}F_{P+B}\left[\begin{matrix}(q),(a),(-m); \\ (p),(b);\end{matrix}\ -y\right]\frac{x^m}{m!} \\ &= {}_{D+Q+A+H}F_{G+P+B+T}\left[\begin{matrix}(d),(q),(a),(h); \\ (g),(p),(b),(t);\end{matrix}\ xy\right].\end{aligned} \tag{2.11}$$

Now, on putting $A = B = H = T = 0$, (2.11) reduces to a known result due to Exton [**5**], p. 11 (6.6)]. Next, we turn to some generating functions involving the Jacobi polynomials $P_n^{(\alpha,\,\beta)}(x)$ [**7**], p. 254 (1). On setting $B = E = U = V = Q = P = 0$,

$F = A = 1$, $a_1 = \lambda$ and replacing y by $-y$, equation (2.6) reduces to

(2.12)
$$\sum_{m,n=0}^{\infty} \frac{((d))_{m+n}((h))_{m+n}(f+m+n)_m x^m(-x)^n}{((g))_{m+n}((t))_{m+n}(f)_{2m+n}\, m!\, n!}\, {}_2F_1\left[\begin{matrix} -m,\ \lambda; \\ f+m+n; \end{matrix}\ y\right] = {}_{1+D+H}F_{G+T+2}\left[\begin{matrix} (d), (h),\ \lambda; \\ (g), (t),\ \frac{f}{2},\ \frac{f}{2}+\frac{1}{2}; \end{matrix}\ \frac{-xy}{4}\right].$$

For $f = -\alpha-\beta$, $\lambda = -\alpha$, equation (2.12) yields an interesring generating function for Jacobi polynomials given by

(2.13)
$$\sum_{m,n=0}^{\infty} \frac{((d))_{m+n}((h))_{m+n}(-x)^n}{((g))_{m+n}((t))_{m+n}(-\alpha-\beta)_{2m+n}} \left[\frac{2x}{1-y}\right]^m \boldsymbol{P}_m^{(\alpha-m,\,\beta-2m-n)}(y) = {}_{1+D+H}F_{G+T+2}\left[\begin{matrix} (d), (h),\ -\alpha; \\ (g), (t),\ \frac{(-\alpha-\beta)}{2},\ \frac{(-\alpha-\beta)}{2}+\frac{1}{2}; \end{matrix}\ \frac{-2x}{4(1-y)}\right]$$

which follow from [**9**], p. 593 (15)

(2.14)
$$\boldsymbol{P}_n^{(\alpha-n,\,\beta-n)}(x) = \binom{n-\alpha-\beta-1}{n}\left(\frac{1-x}{2}\right)^n {}_2F_1\left[\begin{matrix} -n,\ -\alpha; \\ -\alpha-\beta; \end{matrix}\ \frac{2}{1-x}\right].$$

On replacing z and y respectively by $\left(\frac{1-z}{2}\right)$ and $\left(\frac{1-y}{2}\right)$ in (2.3), setting $l = r = 1+\alpha+\beta$ and using the definition [**9**], p. 593 (20).

(2.15)
$$\boldsymbol{P}_n^{(\alpha,\beta)}(x) = \binom{\alpha+n}{n}\, {}_2F_1\left[\begin{matrix} -n,\ 1+\alpha+\beta+n; \\ 1+\alpha; \end{matrix}\ \frac{1-x}{2}\right],$$

we get

(2.16)
$$\sum_{m,n=0}^{\infty} \frac{(\alpha+1)_{m+n}x^m(-x)^n}{(\alpha+1)_m(\alpha+1)_n} \boldsymbol{P}_m^{(\alpha,\,\beta-m)}(z)\, \boldsymbol{P}_n^{(\alpha,\,\beta-n)}(y) = \left\{\left(1-\frac{x}{2}+\frac{xy}{2}\right)\left((1+\frac{x}{2}-\frac{xz}{2}\right)\right\}^{-(1+\alpha+\beta)} {}_2F_1\left[\begin{matrix} 1+\alpha+\beta,\ 1+\alpha+\beta; \\ \alpha+1; \end{matrix}\ \frac{-x^2(1-z)(1-y)}{(2+x-xz)(2-x+xy)}\right].$$

3 - Double generating relations for hypergeometric functions of several variables

We shall now generalize these relations of section **2** and we will show how Laplace and inverse Laplace transforms of equations (2.5) and (1.2) would yield a generating function of several variables, which are double generating relations. We recall the definition of the generalised Ka'mpe de Fe'riet function of several variables [**6**], p. 28 (1.4.3).

$$\begin{aligned}(3.1)\quad & \boldsymbol{F}^{A:\,B';\,\dots;\,B^{(n)}}_{C:\,D';\,\dots;\,D^{(n)}}[x_1,\,\dots,\,x_n] = \boldsymbol{F}^{A:\,B';\,\dots;\,B^{(n)}}_{C:\,D';\,\dots;\,D^{(n)}}\left[\begin{matrix}(a):(b');\,\dots;(b^{(n)});\\ (c):(d');\,\dots;(d^{(n)});\end{matrix}\; x_1,\,\dots,\,x_n\right]\\ &= \sum_{m_1,\,\dots,\,m_n=0}^{\infty} \frac{((a))_{m_1+\dots+m_n}((b'))_{m_1}\dots((b^{(n)}))_{m_n} x_1^{m_1}\dots\dots\dots x_n^{m_n}}{((c))_{m_1+\dots+m_n}((d'))_{m_1}\dots((d^{(n)}))_{m_n} m_1!\dots\dots\dots m_n!}\,.\end{aligned}$$

On multiplying both sides of equation (2.5) by

$$t^{a_1-1}e^{-st}\boldsymbol{F}^{A:\,B';\,\dots;\,B^{(n)}}_{C:\,D';\,\dots;\,D^{(n)}}[z_1 t,\,\dots,\,z_n t]$$

replacing z by zt and taking Laplace transform with the help of (2.1), we obtain

$$\begin{aligned}(3.2)\quad & \sum_{m,\,n=0}^{\infty}\frac{(\alpha+1)_{m+n}x^m(-x)^n}{m!\,n!}\sum_{m_1,\,\dots,\,m_n=0}^{\infty}\Omega(m_1,\,\dots,\,m_n)\\ & {}_{2+L}F_{K+1}\left[\begin{matrix}a_1+m_1+\dots+m_n,(l),\;-m;\\ \alpha+1,(k);\end{matrix}\; z\right]{}_{1+R}F_{W+1}\left[\begin{matrix}-n,(r);\\ \alpha+1,(w);\end{matrix}\; y\right]\\ &= \sum_{m_1,\,\dots,\,m_n=0}^{\infty}\Omega(m_1,\,\dots,\,m_n)\sum_{s=0}^{\infty}\frac{(a_1+m_1+\dots+m_n)_s((l))_s((r))_s(-x^2yz)^s}{((k))_s((w))_s(\alpha+1)_s s!}\\ & {}_{1+L}F_K\left[\begin{matrix}a_1+m_1+\dots+m_n+s,(l)+s;\\ (k)+s;\end{matrix}\; -xz\right]{}_RF_W\left[\begin{matrix}(r)+s;\\ (w)+s;\end{matrix}\; xy\right]\end{aligned}$$

where $\Omega(m_1,\,\dots,\,m_n) = \dfrac{((a))_{m_1+\dots+m_n}((b'))_{m_1}\dots((b^{(n)}))_{m_n} z_1^{m_1}\dots\dots\dots z_n^{m_n}}{((c))_{m_1+\dots+m_n}((d'))_{m_1}\dots((d^{(n)}))_{m_n} m_1!\dots\dots\dots m_n!}$.

Now, replacing y by yt in (3.2) and multiplying both the sides by

$$t^{u_1-1}e^{-st}\boldsymbol{F}^{U:\,E';\,\dots;\,E^{(r)}}_{V:\,F';\,\dots;\,F^{(r)}}[y_1 t,\,\dots,\,y_r t]$$

and taking the Laplace transform with the help of (2.1), we get

$$\sum_{m,\,n=0}^{\infty} \frac{(\alpha+1)_{m+n}x^m(-x)^n}{m!\,n!} \sum_{m_1,\,\ldots,\,m_n,\,k_1,\,\ldots,\,k_r=0}^{\infty} \Omega(m_1,\,\ldots,\,m_n)\,\Lambda(k_1,\,\ldots,\,k_r)$$

$${}_{2+L}F_{K+1}\left[\begin{matrix} a_1+m_1+\ldots+m_n,\ -m,(l); \\ \alpha+1,(k); \end{matrix}\ z\right]{}_{2+R}F_{W+1}\left[\begin{matrix} u_1+k_1+\ldots+k_r,\ -n,(r); \\ \alpha+1,(w); \end{matrix}\ y\right]$$

$$(3.3)\quad = \sum_{m_1,\,\ldots,\,m_n,\,k_1,\,\ldots,\,k_r=0}^{\infty} \Omega(m_1,\,\ldots,\,m_n)\,\Lambda(k_1,\,\ldots,\,k_r)\sum_{s=0}^{\infty}\frac{(a_1+m_1+\ldots+m_n)_s}{((k))_s((w))_s}$$

$$\frac{(u_1+k_1+,\,\ldots,\,k_r)_s((l))_s((r))_s(-x^2yz)^s}{(\alpha+1)_s\,s!}$$

$${}_{1+L}F_K\left[\begin{matrix} a_1+m_1+\ldots+m_n+s,(l)+s; \\ (k)+s; \end{matrix}\ -xz\right]{}_{1+R}F_W\left[\begin{matrix} u_1+k_1+\ldots+k_r+S,(r)+s; \\ (w)+s;\quad xy \end{matrix}\right]$$

where $\Lambda(k_1,\,\ldots,\,k_r) = \dfrac{((u))_{k_1+\ldots+k_r}((e'))_{k_1}\ldots((e^{(r)}))_{k_r}y_1^{k_1}\ldots\ldots\ldots y_r^{k_r}}{((v))_{k_1+\ldots+k_r}((f'))_{k_1}\ldots((f^{(r)}))_{k_r}k_1!\ldots\ldots\ldots\ldots k_r!}$.

Now starting from (3.3) and making use of inverse Laplace and Laplace transform with the help of the results (2.1) and (2.4), the method of mathematical induction and the relation $(a)_{m+n} = (a)_m(a+m)_n$, we obtain on obvious simplification

$$\sum_{m,\,n=0}^{\infty} \frac{(\alpha+1)_{m+n}x^m(-x)^n}{m!\,n!}$$

$$\boldsymbol{F}^{A:\,B';\,\ldots;\,B^{(n)};\,L+1}_{C:\,D';\,\ldots;\,D^{(n)};\,K+1}\left[\begin{matrix} (a):(b');\,\ldots;(b^{(n)});(l),\ -m; \\ (c):(d');\,\ldots;(d^{(n)});(k),\ \alpha+1; \end{matrix}\ z_1,\,\ldots,\,z_n,\,z\right]$$

$$\boldsymbol{F}^{U:\,E';\,\ldots;\,E^{(r)};\,R+1}_{V:\,F';\,\ldots;\,F^{(r)};\,W+1}\left[\begin{matrix} (u):(e');\,\ldots;(e^{(r)}),(r)^{-n}; \\ (v):(f');\,\ldots;(f^{(r)});(w),\ \alpha+1; \end{matrix}\ y_1,\,\ldots,\,y_r,\,y\right]$$

$$(3.4)\quad = \sum_{s=0}^{\infty}\frac{((l))_s((r))_s((a))_s((u))_s(-x^2yz)^s}{((k))_s((w))_s((c))_s((v))_s(\alpha+1)_s\,s!}$$

$$\boldsymbol{F}^{A:\,B';\,\ldots;\,B^{(n)};\,L}_{C:\,D';\,\ldots;\,D^{(n)};\,K}\left[\begin{matrix} (a)+s:(b');\,\ldots;(b^{(n)});(l)+s; \\ (c)+s:(d');\,\ldots;(d^{(n)});(k)+s; \end{matrix}\ z_1,\,\ldots,\,z_n,\ -xz\right]$$

$$\boldsymbol{F}^{U:\,E';\,\ldots;\,E^{(r)};\,R}_{V:\,F';\,\ldots;\,F^{(r)};\,W}\left[\begin{matrix} (u)+s:(e');\,\ldots;(e^{(r)}),(r)+s; \\ (v)+s:(f');\,\ldots;(f^{(r)});(w)+s; \end{matrix}\ y_1,\,\ldots,\,y_r,\,xy\right].$$

Similarly on multiplying both the sides of (1.2) by

$$t^{a_1-1}e^{-st}\boldsymbol{F}^{A:\,B';\,\ldots;\,B^{(n)}}_{C:\,D';\,\ldots;\,D^{(n)}}[x_1t,\,\ldots,\,x_nt],$$

replacing y by yt using Laplace transforms formula (2.1)and evaluting the integrals on both the sides and then multiplying both the sides of the resulting expression by $t^{h-1}e^{-st}$ replacing x by xt and again taking the Laplace transform with the help of (2.1), we obtain

$$\sum_{m,n=0}^{\infty}\frac{((d))_{m+n}((u))_{m+n}((v))_{m+n}(h)_{m+n}x^m(-x)^n}{((g))_{m+n}((e))_{m+n}((f))_{m+n}m!\,n!}\sum_{m_1,\ldots,m_n=0}^{\infty}\Phi(m_1,\ldots,m_n)$$

$$(3.5)\quad {}_{E+V+Q+2}F_{U+F+P}\left[\begin{matrix}1-(e)-m-n,(v)+m+n,(q),a_1+m_1+\ldots+m_n,-m;\\ \qquad (-1)^{1+E-U}y\\ 1-(u)-m-n,(f)+m+n,(p);\end{matrix}\right]$$

$$=\sum_{m_1,\ldots,m_n=0}^{\infty}\Phi(m_1,\ldots,m_n)$$

$${}_{D+2V+Q+2}F_{G+2F+P}\left[\begin{matrix}(d),\left(\frac{v}{2}\right),\left(\frac{v}{2}\right)+\frac{1}{2},(q),a_1+m_1+\ldots+m_n,h;\\ \qquad 4^{V-F}xy\\ (g),\left(\frac{f}{2}\right),\left(\frac{f}{2}\right)+\frac{1}{2},(p);\end{matrix}\right]$$

where $\Phi(m_1,\ldots,m_n)=\dfrac{((a))_{m_1+\ldots+m_n}((b'))_{m_1}\ldots((b^{(n)}))_{m_n}x_1^{m_1}\ldots\ldots x_n^{m_n}}{((c))_{m_1+\ldots+m_n}((d'))_{m_1}\ldots((d^{(n)}))_{m_n}m_1!\ldots\ldots m_n!}$.

Again starting from (3.5) and making use of Laplace and inverse Laplace transform techniques with the help of the result (2.1) and (2.4) and the method of mathematical induction, we obtain

$$\sum_{m,n=0}^{\infty}\frac{((d))_{m+n}((u))_{m+n}((v))_{m+n}(h)_{m+n}x^m(-x)^n}{((g))_{m+n}((e))_{m+n}((f))_{m+n}((t))_{m+n}m!\,n!}$$

$$(3.6)\quad \boldsymbol{F}^{A:B';\ldots;B^{(n)};E+V+Q+1}_{C:D';\ldots;D^{(n)};U+F+P}\left[\begin{matrix}(a):(b');\ldots;(b^{(n)});1-(e)-m-n,(v)+m+n,(q),-m;\\ \qquad x_1,\ldots,x_n,(-1)^{1+E-U}y\\ (c):(d');\ldots;(d^{(n)});1-(u)-m-n,(f)+m+n,(p);\end{matrix}\right]$$

$$=\boldsymbol{F}^{A:B';\ldots;B^{(n)};D+2V+Q+H}_{C:D';\ldots;D^{(n)};G+2F+P+T}\left[\begin{matrix}(a):(b');\ldots;(b^{(n)});(d),\left(\frac{v}{2}\right),\left(\frac{v}{2}\right)+\frac{1}{2},(q),(h);\\ \qquad x_1,\ldots,x_n,(4)^{V-F}xy\\ (c):(d');\ldots;(d^{(n)});(g),\left(\frac{f}{2}\right),\left(\frac{f}{2}\right)+\frac{1}{2},(p),(t);\end{matrix}\right].$$

4 - Special cases

It is easy to observe that the equations (3.4) and (3.6) give a large number of generating functions, new as well as known. In this section, we will mention only some special cases of our formulas (3.4) and (3.6). The well known Ka'mpe de Fe'riet function of two variables [1] is defined and represented in the following manner

$$F^{(2)}\begin{bmatrix}(a):(b);(b');\\(c):(d);(d');\end{bmatrix} x, y\Bigg] = \sum_{m,n=0}^{\infty} \frac{((a))_{m+n}((b))_m((b'))_n x^m y^n}{((c))_{m+n}((d))_m((d'))_n m!\,n!} \tag{4.1}$$

where $A+B \leqslant C+D+1$, $A+B' \leqslant C+D'+1$, and the equalities hold when $(|x|+|y| < \min(1, 2^{C-A+1}))$. By specializing the various parameters in (3.4) and (3.6) to suit case (4.1) above and letting $z_i = y_j = x_q = 0$, $i = 2, 3, \dots, n$, $j = 2, 3, \dots, r$ and $q = 2, 3, \dots, n$, we obtain the generating functions

$$\begin{aligned}&\sum_{m,n=0}^{\infty} \frac{(\alpha+1)_{m+n} x^m (-x)^n}{m!n!}\\ &F^{(2)}\left[\begin{matrix}(a):(l);\ -m;(b);\\(c):(k),\ 1+\alpha;(d);\end{matrix}\ z, t\right] F^{(2)}\left[\begin{matrix}(u):(r);\ -n;(e);\\(v):(w),\ 1+\alpha;(f);\end{matrix}\ y, \omega\right]\\ &= \sum_{s=0}^{\infty} \frac{((l))_s((r))_s((a))_s((u))_s(-x^2yz)^s}{((k))_s((w))_s((c))_s(\alpha+1)_s s!}\\ &F^{(2)}\left[\begin{matrix}(a)+s:(l)+s;(b);\\(c)+s:(k)+s;(d);\end{matrix}\ -xz, t\right] F^{(2)}\left[\begin{matrix}(u)+s:(r)+s;(e);\\(v)+s:(w)+s;(f);\end{matrix}\ xy, \omega\right]\end{aligned} \tag{4.2}$$

$$\begin{aligned}&\sum_{m,n=0}^{\infty} \frac{((d))_{m+n}((u))_{m+n}((v))_{m+n}((h))_{m+n} x^m(-x)^n}{((g))_{m+n}((e))_{m+n}((f))_{m+n}((t))_{m+n} m!\,n!}\\ &F^{(2)}\left[\begin{matrix}(a):\ 1-(e)-m-n,(v)+m+n,(q),\ -m;(b')\\(c):\ 1-(u)-m-n,(f)+m+n,(p);(d');\end{matrix}\ (-1)^{1+E-U}y, \omega\right]\\ &= F^{(2)}\left[\begin{matrix}(a):(d),\left(\frac{v}{2}\right),\left(\frac{v}{2}\right)+\frac{1}{2},(q),(h);(b');\\(c):(g),\left(\frac{f}{2}\right),\left(\frac{f}{2}\right)+\frac{1}{2},(p),(t);(d');\end{matrix}\ 4^{V-E}xy, \omega\right].\end{aligned} \tag{4.3}$$

Note that, when $y, \omega \to 0$, $x = \dfrac{x}{1-x}$, $L = L'+1$ and $l_{L+1} = \alpha+1$, formula (4.2) would reduces to a result due to Srivastava [8], p. 94 (7.1). Also for $D = G = U = V = E = F = T = 0$, $H = 1$, (4.3) reduces to the same result of Srivastava mentioned above. It may be of interest to remark that the relations (3.4) and (3.6) can also be specialized fairly easily to yield a large number of result involving series of the type.

$$\sum_{m,\,n=0}^{\infty} \frac{(\alpha+1)_{m+n}\, x^m(-x)^n}{m!\, n!}\, \phi(\omega,\, y)\, \phi^*(u,\, v) \tag{4.4}$$

$$\sum_{m,\,n=0}^{\infty} \frac{(\alpha+1)_{m+n}\, x^m(-x)^n}{m!\, n!}\, \Phi(\omega,\, y,\, z)\, \phi^*(u,\, v,\, s) \tag{4.5}$$

$$\sum_{m,\,n=0}^{\infty} \frac{(\alpha+1)_{m+n}\, x^m(-x)^n}{m!\, n!}\, \Omega(x_1,\, \ldots,\, x_n)\, \Omega^*(y_1,\, \ldots,\, y_r) \tag{4.6}$$

$$\sum_{m,\,n=0}^{\infty} \frac{C(m+n)\, x^m(-x)^n}{m!\, n!}\, \phi(\omega,\, y) \tag{4.7}$$

$$\sum_{m,\,n=0}\, ^{\infty} \frac{C(m+n)\, x^m(-x)^n}{m! n!}\, \Psi(\omega,\, y,\, z) \tag{4.8}$$

$$\sum_{m,\,n=0}^{\infty} \frac{C(m+n)\, x^m(-x)^n}{m!\, n!}\, \Omega(x_1,\, \ldots,\, x_n) \tag{4.9}$$

where $\phi(\omega,\, y)$ and $\phi^*(u,\, v)$ are one or the other of the Appell's functions F_1, F_2 and F_3, [**12**], p. 53, $\Psi(\omega,\, y,\, z)$ and $\Psi^*(u,\, v,\, s)$ are hypergeometric functions of three variables $F^{(3)}$[**12**], p. 69 (39), $\Omega(x_1,\, \ldots,\, x_n)$ and $\Omega^*(y_1,\, \ldots,\, y_r)$ are one or the other of the Lauricella's functions $F_A^{(n)}$, $F_B^{(n)}$ and $F_D^{(n)}$[**12**], p. 60 and $C(m+n)$ is function of $(m+n)$ only and is independent of any variable.

For instance, in terms of Lauricella functions $F_A^{(n)}$ and $F_D^{(n)}$, equation (3.4) would give us the following special cases of type (4.6).

$$\begin{aligned}
&\sum_{m,n=0}^{\infty} \frac{(\alpha+1)_{m+n}\, x^m(-x)^n}{m!\, n!}\, F_A^{(n+1)}[a, b_1, \ldots\ldots, b_n, -m; d_1, \ldots, d_n, k; z_1, \ldots, z_n, z] \\
&\qquad F_A^{(r+1)}[u,\, e_1, \ldots\ldots,\, e_r,\, -n;\, f_1,\, \ldots,\, f_r,\, w;\, y_1,\, \ldots,\, y_r,\, y] \\
&\qquad = \sum_{s=0}^{\infty} \frac{(a)_s(u)_s(\alpha+1)_s(-x^2yz)^s}{(k)_s(w)_s s!} \\
&\quad F_A^{(n+1)}[a+s, b_1, \ldots\ldots,\, b_n,\, 1+\alpha+s;\, d_1,\, \ldots,\, d_n,\, k+s;\, z_1,\, \ldots,\, z_n,\, -xz] \\
&\quad F_A^{(r+1)}[u+s, e_1, \ldots\ldots,\, e_r,\, 1+\alpha+s;\, f_1,\, \ldots,\, f_r,\, w+s;\, y_1,\, \ldots,\, y_r,\, xy]
\end{aligned} \tag{4.10}$$

$$\begin{aligned}
&\sum_{m,n=0}^{\infty} \frac{(\alpha+1)_{m+n} x^m (-x)^n}{m!\,n!} F_D^{(n+1)}[a, b_1, \ldots, b_n, -m; c; z_1, \ldots, z_n, z]\\
&F_D^{(r+1)}[u, e_1, \ldots, e_r, -n; v; y_1, \ldots, y_r, y]\\
(4.11)\qquad &= \sum_{s=0}^{\infty} \frac{(a)_s (u)_s (\alpha+1)_s (-x^2 yz)^s}{(c)_s (v)_s s!}\\
&F_D^{(n+1)}[a+s, b_1, \ldots, b_n, 1+\alpha+s; c+s; z_1, \ldots, z_n, -xz]\\
&F_D^{(r+1)}[u+s, e_1, \ldots, e_r, 1+\alpha+s; v+s; y_1, \ldots, y_r, xy].
\end{aligned}$$

If in (4.10), we set $y_1 = \ldots = y_r = 0$, and $u = w$, we obtain

$$\begin{aligned}
&[1+x(1-y)]^{-\alpha-1} \sum_{m=0}^{\infty} \frac{(\alpha+1)_m}{m!} \left[\frac{x}{1+x(1-y)}\right]^m\\
(4.12)\qquad &F_A^{(n+1)}[a, b_1, \ldots, b_n, -m; d_1, \ldots, d_n, k; z_1, \ldots, z_n, z]\\
&= (1-xy)^{-\alpha-1} \sum_{s=0}^{\infty} \frac{(a)_s (\alpha+1)_s}{(k)_s s!} \left[\frac{-x^2 yz}{1-xy)}\right]^s\\
&F_A^{(n+1)}[a+s, b_1, \ldots, b_n, 1+\alpha+s; d_1, \ldots, d_n, k+s; z_1, \ldots, z_n, -xz].
\end{aligned}$$

For $y = 1$, (4.12) reduces to a multivariable extension of a known result [**12**], p. 293 (12), (see (2.7)).

On taking $z_2 = \ldots = z_n = y_1 = \ldots = y_r = y = 0$, (4.10) reduces to

$$\begin{aligned}
&\sum_{m=0}^{\infty} \frac{(\alpha+1)_m x^m}{m!} F_2[a, -m, b; k, d; z, \omega]\\
(4.13)\qquad &= (1-x)^{-(\alpha+1)} F_2\left[a, 1+\alpha, b; k, d; \frac{-xz}{1-x}, \omega\right].
\end{aligned}$$

Now, on replacing $\alpha + 1$ and x by λ and $\dfrac{x}{\lambda}$ in (4.13) respectively, letting $\lambda \to \infty$ and using the results [8], p. 94 (7.3)

$$(4.14)\qquad \lim_{\lambda\to\infty}\left(1-\frac{z}{\lambda}\right)^{-\lambda} = e^z \quad \text{and} \quad \lim_{\lambda\to\infty}(\lambda)\left(\frac{z}{\lambda}\right)^n = z^n$$

equation (4.13) reduces to a known result of Srivastava [8], p. 94 (7.5). On setting $z_1 = \ldots = z_n = z$, $y_1 = \ldots = y_r = y$ and using the reduction formula [**11**], p. 34 (6)

$$F_D^{(n)}[a, b_1, \ldots, b_n; c; x, \ldots, x] = {}_2F_1[a, b_1 + \ldots + b_n; c; x]$$

equation (4.11) reduces to

$$\sum_{m,\,n=0}^{\infty}\frac{(\alpha+1)_{m+n}\,x^{m}(-x)^{m}}{m!\,n!}$$

$$(4.15)\qquad {}_2F_1\left[\begin{matrix}b_1+\ldots+b_n-m,\ a;\\ c;\end{matrix}\ z\right]{}_2F_1\left[\begin{matrix}e_1+\ldots+e_r-n,\ u;\\ v;\end{matrix}\ y\right]$$

$$=\sum_{s=0}^{\infty}\frac{(a)_s\,(u)_s\,(\alpha+1)_s\,(-x^2yz)^s}{(c)_s\,(v)_s\,s!}F_1\,[a+s,1+\alpha+s,b_1+.......+b_n;c+s;\,-xz,z]$$

$$F_1[u+s,\,1+\alpha+s,\,e_1+.......\,+e_r;\,v+s;\,xy,\,y]\,.$$

On putting $y\to 0,\ b_i=0,\ i=2,\,3,\,\ldots,\,n$, (4.15) evidently reduces to a known result [**12**], p. 150 (44). For $b_i=0,\ i=2,\,3,\,\ldots,\,n$ and $u=v$, formula (4.15) may atonce written in the form

$$(4.16)\qquad \sum_{m=0}^{\infty}\frac{(\alpha+1)_m}{m!}\left[\frac{x}{1+x(1-y)}\right]^m{}_2F_1\left[\begin{matrix}b-m,\ a;\\ c;\end{matrix}\ z\right]$$

$$=(1-xy)^{-\alpha-1}\,(1+x(1-y))^{\alpha+1}\,F^{(3)}\left[\begin{matrix}a::\alpha+1;\,-;\,-:-;\,-;b;\\ c::\,-;\,-;\,-:-;\,-;\,-;\end{matrix}\quad -xz,\ \frac{-x^2yz}{(1-xy)},\ z\right]$$

where $F^{(3)}[x,\,y,\,z]$ is Srivastava's triple hypergeometric series [**12**], p. 69 (39) and (40). It is important to note that, the left-hand side of equation (4.16) can be summed by using one or other of the results [**12**], p. 150 (43), (44) and p. 151 (45), to obtain some transformation formulas for $F^{(3)}$ (right- hand side of (4.16)) in the form of functions of Gaussain ${}_2F_1$, Appell F_1 and a special case of Ka'mpe de Fe'riet of two variables $F^{(2)}$. For example, if in (4.16) we set $\alpha+1=\lambda$, $t=\dfrac{x}{(1+x(1-y))}$ and use [**12**], p. 150 (44), we get the transformation formula

$$(4.17)\qquad F_1\left[a,b,\lambda;c;z,\ \frac{zx}{xy-1}\right]=F^{(3)}\left[\begin{matrix}a::\alpha+1;\,-;\,-:-;\,-;b;\\ c::\,-;\,-;\,-:-;\,-;\,-;\end{matrix}\quad -xz,\ \frac{-x^2yz}{(1-xy)},\ z\right].$$

Also a similar transformations can be obtained from the main result (3.4). Further, on setting $y_1=\ldots=y_r=y=z_2=\ldots=z_n=0$, in (4.11), replacing $\alpha+1$ and x by λ and $\dfrac{x}{\lambda}$ respectively, letting $\lambda\to\infty$ and using (4.14), formula (4.11) reduces to another known result [8], p. 94 (7.4).

On other hand, as particular cases of our result (4.3) we obtain three linear generating relations involving the Appell function F_2 and F_3. Indeed we have the following generating relations of the type (4.7).

$$\sum_{m,n=0}^{\infty} \frac{(u)_{m+n}(h)_{m+n}x^m(-x)^n}{(t)_{m+n}m!\,n!} F_2[a,\,-m,\,b;\,1-u-m-n,\,d;\,y,\,\omega] = F_2[a,\,h,\,b;\,t,\,d;\,xy,\,\omega] \tag{4.18}$$

$$\sum_{m,n=0}^{\infty} \frac{(d)_{m+n}(h)_{m+n}x^m(-x)^n}{(e)_{m+n}m!\,n!} F_3[-m,\,b_1,\,1-e-m-n,\,b_2;\,c;\,y,\,\omega] = F_3\left[d,\,b_1,\,h,\,b_2;\,c;\,\frac{xy}{4},\,\omega\right] \tag{4.19}$$

$$\sum_{m,n=0}^{\infty} \frac{(v)_{m+n}x^m(-x)^n}{m!\,n!} F_3[-m,\,b_1,\,v+m+n,\,b_2;\,c;\,y,\,\omega] = F_3\left[\frac{v}{2},\,b_1,\,\frac{v}{2}+\frac{1}{2},\,b_2;\,c;\,-4xy,\,\omega\right]. \tag{4.20}$$

Finally on setting $H = B^i = 1,\, D^i = 0,\, i = 1,\, \ldots,\, n$ and $E = F = V = U = T = Q = P = D = G = 0$ in (3.6) together with the reduction formula for the generalized Ka'mpe de Fe'riet series of several variables [**11**], p. 39 (32), we get a generalization of (2.10) given by

$$\sum_{m=0}^{\infty} \frac{(h)_m}{m!}\, {}_{A+1}F_B\left[\begin{matrix} b_1+\ldots+b_n-m,(a); \\ (c); \end{matrix}\; y\right]x^m = (1-x)^{-h}F^{(2)}\left[\begin{matrix} (a):\, b_1+\ldots+b_n;\, h; \\ (c):\, ---;\, -; \end{matrix}\; y,\, \frac{xy}{(x-1)}\right]. \tag{4.21}$$

Now, if in (4.21), we set $b_i = 0,\ i = 2,\, 3,\, \ldots,\, n$, then it reduces to a known result of Srivastava [**12**], p. 150 (43).

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Abstract

Our starting points are results due to Exton [5] *on Confluent and generalized hypergeometric functions whose applications give certain double generating functions for polynomials of Gauss and Jacobi and functions of several variables of Ka'mpe de Fe'riet, Lauricella and Appell.*

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