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Derivations on k-th commutators in semiprime rings (**)

Several authors have studied commutativity in prime and semiprime rings admitting derivations which have zero or invertible values on appropriate subsets of a prime rings. In [**3**] Bergen, Herstein and Lanski proved that if R is a ring with 1 of characteristic not 2, d a non-zero derivation of R such that $d(x)$ is zero or invertible in R, for any $x \in R$, then there exists a division ring D such that either $R = D$ or $R = M_2(D)$, the ring of 2×2 matrices over D, or $R = D \oplus D$. In [**2**] Bergen and Carini generalized this result to the case of a Lie ideal for semiprime rings. Later in [**10**] Lee showed that the same conclusions can be obtained if R is a semiprime ring and $f(x_1, \ldots, x_n)$ a monic non-central multilinear polynomial in n non-commuting variables such that $d(f(r_1, \ldots, r_n))$ is zero or invertible in R, for any $r_1, \ldots, r_n \in R$. Recently in [**5**] we studied the case when the commutator $[d(f(x_1, \ldots, x_n)), f(x_1, \ldots, x_n)]$ has zero or invertible values in R. In the present paper we shall consider the case when the multilinear polynomial $f(x_1, \ldots, x_n)$ is replaced by the k-th commutator $[[x_1, x_2], [x_3, x_4]]_k$, which is not multilinear. More precisely we will prove the following:

THEOREM 1.1. *Let R be a 2-torsion free semiprime ring, d a non-zero derivation of R, U the left Utumi quotient ring of R. If $[d([[r_1, r_2], [r_3, r_4]]_k), [[r_1, r_2], [r_3, r_4]]_k] = 0$, for any $r_1, r_2, r_3, r_4 \in R$, then there exists a central idempotent element e of U such that in the sum decomposition $U = eU \oplus (1-e)\,U$, the derivation d vanishes identically on eU and $(1-e)\,U$ is commutative.*

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Theorem 2.1. *Let R a 2-torsion free semiprime ring, d a non-zero derivation of R and $a, b, c, d \in R$ such that $[d([[a, b],[c, d]]_k),[[a, b],[c, d]]_k] \neq 0$. If $[d([[r_1, r_2],[r_3, r_4]]_k),[[r_1, r_2],[r_3, r_4]]_k]$ is zero or invertible in R, for any $r_1, r_2, r_3, r_4 \in R$, then R is a division ring.*

As a consequence we will study the relationship between the structure of a prime ring of characteristic different from 2 and the behaviour of $[d([x_1, x_2]_k), [x_1, x_2]_k]$ in a non-central Lie ideal.

1 - Commutators zero-valued

In all that follows we will denote R a 2-torsion free semiprime ring, d a non-zero derivation of R, Q the Martindale quotient ring of R, $C = Z(Q)$ the extended centroid of R, $S = RC$ the central closure of R. Moreover we will introduce the left Utumi quotient ring U of R. Its axiomatic formulation, definition and main properties can be found in [**1**], [**6**] and [**9**].

In order to prove the main result of this section we will make use of the following facts:

Claim 1 [**12**]. If R is a semiprime ring and I_R a dense sub-module of U then I_R, Q and U satisfy the same differential identities.

(Notice that in the case R is prime, any two-sided ideal I of R is a dense sub-module of U).

Claim 2 [**8**]. Let R be a prime ring and $g(x_1, \dots, x_n, d(x_1), \dots, d(x_n))$ a differential identity of R. One of the following holds:

1) either d is an inner derivation in Q, in the sense that there exists $c \in Q$ such that $d = ad(c)$ and $d(x) = ad(c)(x) = [c, x] = cx - xc$, for all $x \in R$ and R satisfies the generalized polynomial identity

$$g(x_1, \dots, x_n, [c, x_1], \dots, [c, x_n]);$$

2) or R satisfies the generalized polynomial identity $g(x_1, \dots, x_n, y_1, \dots, y_n)$.

We begin by analysing the case when R is a prime ring. Hence we will extend the result to semiprime case, by using the theory of orthogonal completion (see [**1**], Chapter 3).

Lemma 1.1. *Let R be a prime ring of characteristic different from 2, I a non-zero two-sided ideal of R, d a non-zero derivation of R. If $[d([[r_1, r_2], [r_3, r_4]]_k), [[r_1, r_2], [r_3, r_4]]_k] = 0$, for any $r_1, r_2, r_3, r_4 \in I$, then R is commutative.*

Proof. Denote the differential polynomial

$$[d([[x_1, x_2],[x_3, x_4]]_k),[[x_1, x_2],[x_3, x_4]]_k] = g(x_1, \dots, x_4, d(x_1), .. d(x_4)).$$

Since I and R satisfy the same differential identities (see Claim 1), then

$$g(x_1, \dots, x_4, d(x_1), \dots, d(x_4)) = [d([[x_1, x_2],[x_3, x_4]]_k),[[x_1, x_2],[x_3, x_4]]_k]$$

$$= \left[d\left(\sum_h(-1)^h\binom{k}{h}[x_3, x_4]^h[x_1, x_2][x_3, x_4]^{k-h}\right),\right.$$

$$\left.\left(\sum_h(-1)^h\binom{k}{h}[x_3, x_4]^h[x_1, x_2][x_3, x_4]^{k-h}\right)\right]$$

is a differential identity on R.

By using Claim 2, one of the following holds:

1) either d is an inner derivation in Q, induced by $c \in Q$ and R satisfies the generalized polynomial identity

$$g(x_1, .. x_4,[c, x_1], \dots,[c, x_4]);$$

2) or R satisfies the generalized polynomial identity $g(x_1, \dots, x_4, y_1, \dots, y_4)$.

In this last case in particular R satisfies the identity

$$g(x_1, . x_2, x_3, x_4, 0, y_2, 0, 0) = [[[x_1, y_2],[x_3, x_4]]_k,[[x_1, x_2],[x_3, x_4]]_k]$$

that is for any $r_1, r_2, r_3, r_4, r_5 \in R$

$$[[[r_1, r_5],[r_3, r_4]]_k,[[r_1, r_2],[r_3, r_4]]_k] = 0 .$$

Since R is a P.I. ring then there exists a field F such that R and $M_t(F)$, the ring of $t \times t$ matrices over F, satisfy the same polynomial identities.

Suppose $t \geqslant 2$ and choose $r_1 = e_{22}$, $r_2 = e_{21}$, $r_3 = e_{21}$, $r_4 = e_{12}$, $r_5 = e_{12}$. Then we obtain the following contradiction

$$0 = [[-e_{12}, e_{22} - e_{11}]_k,[e_{21}, e_{22} - e_{11}]_k] = [-(2^k)\, e_{12},(-2)^k e_{21}]$$

$$= (-1)^{k+1}2^{2k}(e_{11} - e_{22}) \neq 0 .$$

Therefore must be $t = 1$ and so R is commutative.

Now let d be the inner derivation induced by an element $c \in Q$. Thus

$$0 = [[c, [[r_1, r_2], [r_3, r_4]]_k], [[r_1, r_2], [r_3, r_4]]_k] = [[c, [[r_1, r_2], [r_3, r_4]]_k]_2,$$

for any $r_1, r_2, r_3, r_4 \in R$, i.e. R satisfies a non-trivial generalized polynomial identitiy. By [**13**] it follows that $S = RC$ is a primitive ring with $soc(R) = H \neq 0$ and eHe is a simple central algebra finite dimensional over C, for any minimal idempotent element $e \in S$. Moreover we may assume H non-commutative, otherwise also R must be commutative. Notice that H satisfies $[[c, [[x_1, x_2], [x_3, x_4]]_k], [[x_1, x_2], [x_3, x_4]]_k]$ (see for example [**10**, proof of Theorem 1]).

Since H is a simple ring then one of the following holds: either H does not contain any non-trivial idempotent element or H is generated by its idempotents.

In this last case, suppose that H contains three minimal orthogonal idempotent elements e, f, g. Thus eH, fH, gH are isomorphic H-modules and there are

$$fbg,\ gaf \in H \ \text{such that}\ fbgaf = f, \qquad gafbg = g$$

$$epg,\ gde \in H \ \text{such that}\ gdepg = g,\ epgde = e\,.$$

Let

$$[r_1, r_2] = [fbg, g] = fbg \quad [r_3, r_4] = [gde, epg] = g - e$$

$$[[r_1, r_2], [r_3, r_4]]_k = [fbg, g - e]_k = fbg\,.$$

By the hypotesis $[c, fbg]_2 = 0$, that is $(-2)\, fbgcfbg = 0$. Right multiplying by af and left multiplying by ga we have $gcf = 0$.

This implies that, for any orthogonal idempotent element of rank 1, g and f, $gcf = 0$. Hence $[c, g] = 0$, for any idempotent of rank 1, and $[c, H] = 0$, since H is generated by these idempotent elements. This argument gives the contradiction that $c \in C$ and $d = 0$.

Therefore H cannot contain three minimal orthogonal idempotent elements and so $H = M_2(D)$, for a suitable division ring D finite dimensional over its center. This implies that $Q = H$ and $c \in H$. By [**15**, Theorem 2.3.29, p. 131] (see also [**10**, Lemma 2]), there exists a field F such that $H \subseteq M_n(F)$ and $M_n(F)$ satisfies $[c, [[x, y], [z, t]]_k]_2 = 0$, for F a field. As we have just seen, if $n \geq 3$ then $c \in C$ and $d = 0$. If $n = 1$ then $R \subseteq F$ and we are also done, thus we say $H \subseteq M_2(F)$. We want to prove that, in this case, we have a contradiction.

Since for any $u \in [H, H]$, $u^2 \in F$, then

$$[c, u]_2 = cu^2 + u^2 c - 2ucu = 2(u^2 c - ucu).$$

Fix i, j and denote e_{ij} the matrix unit with 1 in (i, j)-entry and zero elsewhere.

Let $u = [[[x, y], [z, t]]_{k-1}, [z, t]] = [[x, y], [z, t]]_k \in [H, H]$, and choose $x = e_{ii}$, $y = e_{ij}$, $z = e_{ij}$, $t = e_{ji}$. Thus $u = (-2)^k e_{ij}$. Moreover consider $c = \sum c_{rs} e_{rs}$, with $c_{rs} \in F$. Since $[c, u]_2 = 0$,

$$0 = [c, (-2)^k e_{ij}]_2 = (-2)^{2k} e_{ij} c e_{ij}$$

thus $c_{ji} = 0$, for $i \neq j$, that is c is a diagonal matrix in $M_2(F)$. Now choose

$$[x, y] = [e_{22}, e_{21} + e_{12}] = e_{21} - e_{12}$$

$$[z, t] = [e_{12}, e_{21}] = e_{11} - e_{22}.$$

If k is even then $[[x, y], [z, t]]_k = 2^k (e_{21} - e_{12})$ and so, since the characteristic of R is different from 2,

$$0 = [c, e_{21} - e_{12}]_2 = (-2c_{11} + 2c_{22})\, e_{11} + (2c_{11} - 2c_{22})\, e_{22}.$$

This means $c_{11} = c_{22}$ and so $c \in F$ and $d = 0$, which is a contradiction.

If k is odd then $[[x, y], [z, t]]_k = 2^k (e_{21} + e_{12})$ and

$$0 = [c, e_{21} + e_{12}]_2 = 2c - 2c_{22} e_{11} - 2c_{11} e_{22}.$$

Also in this case we have the contradiction $c_{11} = c_{22}$, $c \in F$ and $d = 0$.

On the other hand, if H does not contain any non-trivial idempotent element, then H is a finite dimensional division algebra over C and $c \in H = RC = Q$. If C is finite then H is a finite division ring, that is H is a commutative field and so R is commutative too.

If C is infinite then $H \otimes_C F \cong M_r(F)$, where F is a splitting field of H. In this case, a Vandermonde determinant argument shows that in $M_r(F)$ $[c, [[x_1, x_2], [x_3, x_4]]_k]_2 = 0$ is still an identiy. As above one can see that if $r \geq 3$ then c commutes with any idempotent element in $M_r(F)$ and also if $r = 2$ then $c \in F$. In any case we have the contradiction $d = 0$. ■

Now we extend the previous result to semiprime rings.

Theorem 1.1. *Let R be a semiprime 2-torsion free ring, d a non-zero derivation of R. If $[d([[r_1, r_2], [r_3, r_4]]_k), [[r_1, r_2], [r_3, r_4]]_k] = 0$, for any*

$r_1, r_2, r_3, r_4 \in R$, *then there exists a central idempotent element e of U such that in the sum decomposition* $U = eU \oplus (1-e)\,U$, *the derivation d vanishes identically in eU and* $(1-e)\,U$ *is commutative.*

Proof. It is well known that the derivation d can be uniquely extended to U and all the derivations in R will be implicitily defined on the whole U (see Lemma 2 in [**12**]). Moreover R and U satisfy the same differential identities (see Claim 1), thus

$$[d([[r_1, r_2], [r_3, r_4]]_k), [[r_1, r_2], [r_3, r_4]]_k] = 0, \qquad \forall r_1, r_2, r_3, r_4 \in U\,.$$

Let M be any maximal ideal of B, the boolean algebra of idempotents in C. We know that MU is a prime ideal of U and $\cap_M MU = 0$. Let $\overline{d}$ be the derivation induced by d in $\overline{U} = U/MU$, which is a prime ring of characteristic different from 2. Notice that $\overline{d}$ satisfies in $\overline{U}$ the same property of d in U.

Hence $[\overline{d}([[r_1, r_2], [r_3, r_4]]_k), [[r_1, r_2], [r_3, r_4]]_k] = 0$, for any $r_1, r_2, r_3, r_4 \in \overline{U}$.

Since $\overline{U}$ is prime and by previous lemma, either $\overline{d} = 0$ or $\overline{U}$ is commutative.

This implies that, for any M maximal ideal of B, either $d(U) \subseteq MU$ or $[U, U] \subseteq MU$.

In any case $d(U)\,U[U, U] \subseteq \bigcap_M MU = 0$. As a consequence of the theory of orthogonal completion for semiprime rings, there exists a central idempotent element e of U such that $d(eU) = 0$ and $(1-e)\,U$ is commutative (for more details see chapter 3 in [**1**]). If pose $U_1 = eU$, $U_2 = (1-e)\,U$, then $U = U_1 \oplus U_2$ as required. ■

Corollary 1.1. *Let R be a prime ring of characteristic different from* 2, *d a non-zero derivation of R, L a Lie ideal of R. If* $[d([u, v]]_k), [u, v]]_k] = 0$, *for any* $u, v \in L$, *then L is central.*

Proof. Suppose L is not central. Since R has characteristic different from 2 then, by a classical result of Herstein in [**7**], there exists a non-zero two-sided ideal I of R such that $[I, I] \subseteq L$.

Therefore $[d([[r_1, r_2], [r_3, r_4]]_k), [[r_1, r_2], [r_3, r_4]]_k] = 0$, for any $r_1, r_2, r_3, r_4 \in I$ and Lemma 1.1 we obtain the contradiction that R is commutative. ■

2 - Commutators with invertible values

In this section we study the following situation: R is a 2-torsion free semiprime ring such that

$$[d([[r_1, r_2],[r_3, r_4]]_k),[[r_1, r_2],[r_3, r_4]]_k] \quad \text{is zero or invertible}$$

for any $r_1, r_2, r_3, r_4 \in R$. By the results in previous section, we may assume that there exist $a, b, c, d \in R$ such that

$$[d([[a, b],[c, d]]_k),[[a, b],[c, d]]_k] \neq 0 .$$

At first we observe that the only case is the one in which R is a simple ring, as the following lemma states:

Lemma 2.1. *R is simple.*

Proof. Suppose that there exists a two-sided ideal $0 \neq I \neq R$ of R. Since I does not contain any invertible element of R then

$$[d([[r_1, r_2],[r_3, r_4]]_k),[[r_1, r_2],[r_3, r_4]]_k] = 0$$

for any $r_1, r_2, r_3, r_4 \in I$. In this case, by [**12**], one has that

$$[d([[r_1, r_2],[r_3, r_4]]_k),[[r_1, r_2],[r_3, r_4]]_k]\, s = 0 ,$$

for any $r_1, r_2, r_3, r_4 \in R$ and $s \in I$. In particular

$$[d([[a, b],[c, d]]_k),[[a, b],[c, d]]_k] s = 0 ,$$

i.e. $s = 0$, since $[d([[a, b],[c, d]]_k),[[a, b],[c, d]]_k]$ is invertible. By the arbitrariety of $s \in I$ we have $I = 0$, which contradicts our assumption. ∎

In all that follows R will be a simple ring with 1.

Lemma 2.2. *Let $R = M_n(D)$, for D a division ring. If $d \neq 0$ then $n = 1$.*

Proof. Since $R = M_n(D)$, by [**14**] there exists a derivation $\delta: D \to D$ and a matrix $A \in M_n(D)$ such that $d = d_A + \overline{\delta}$, where d_A is the inner derivation induced by A, that is $d_A(x) = Ax - xA$, for all $x \in M_n(D)$, and $\overline{\delta}$: $M_n(D) \to M_n(D)$ is the derivation induced by δ, that is $\overline{\delta}(\sum_{i,j} r_{ij} e_{ij}) = \sum_{i,j} \delta(r_{ij})\, e_{ij}$. As in Lemma 1.1, here e_{ij} are the matrices unit, with 1 in (i, j)-entry and 0 elsewhere. Assume $n \geqslant 2$.

Now fix $i \neq j$ and choose $[[u_1, u_2],[u_3, u_4]]_k = [[e_{ij}, e_{jj}],[e_{ij}, e_{ji}]]_k = (-2)^k e_{ij}$.

Hence by our assumption

$$[d([[u_1, u_2],[u_3, u_4]]_k),[[u_1, u_2],[u_3, u_4]]_k] = (-2)^k[d(e_{ij}), e_{ij}]$$

is zero or invertible.

Since the set $\{[[x_1, x_2],[x_2, x_3]]_k / x_i \in M_n(D)\}$ is invariant under the action of all Z(D)-automorphisms of $M_n(D)$, then for all $s \neq t$ there exist $v_1, v_2, v_3, v_4 \in R$ such that $[[v_1, v_2],[v_3, v_4]] = e_{st} \neq 0$. Now we have

$$[d(e_{st}), e_{st}] = [Ae_{st} - e_{st}A, e_{st}]$$

$$= -e_{st}Ae_{st} - e_{st}Ae_{st} = -2e_{st}Ae_{st}$$

which is a matrix of rank ≤ 1, and so it is not invertible in $M_n(D)$. Then, by our hypotesis, $-2e_{st}Ae_{st} = 0$, and so $e_{st}Ae_{st} = 0$.

This means that, for all $s, t = 1, \ldots, n$, $s \neq t$, the (s, t)-entry of the matrix A is zero. Hence $A = \sum_i \alpha_i e_{ii}$, where $\alpha_i \in D$, that is A is a diagonal matrix. Moreover we remark that if φ is a Z(D)-automorphism of $M_n(D)$, then the derivation $d_\varphi = \varphi d \varphi^{-1}$ satisfies the same condition of d, that is

$$[d_\varphi([[r_1, r_2],[r_3, r_4]]_k),[[r_1, r_2],[r_3, r_4]]_k] \text{ is zero or invertible in } M_n(D).$$

For $n \geq t \geq 2$ and $b \in D$, let $\varphi = \varphi_{t,b}$ be the automorphism of $M_n(D)$ defined by $\varphi(x) = (1 - be_{1t})\, x(1 + be_{1t})$. Then $\varphi^{-1}(x) = (1 + be_{1t})\, x(1 - be_{1t})$ and

$$d_\varphi(x) = \varphi d_A(\varphi^{-1}(x)) + \varphi \overline{\delta}(\varphi^{-1}(x))$$

$$= \varphi(A\varphi^{-1}(x) - \varphi^{-1}(x)\, A)$$

$$+ \varphi(\delta(b)\, e_{1t}x(1 - be_{1t}) + (1 + be_{1t})\, \overline{\delta}(x)(1 - be_{1t}) + (1 + be_{1t})\, x(-\delta(b)\, e_{1t}))$$

$$= \varphi(A)\, x - x\varphi(A) + (1 - be_{1t})\, \delta(b)\, e_{1t}x + \overline{\delta}(x) + x(-\delta(b)\, e_{1t})(1 + be_{1t})$$

$$= \varphi(A)\, x - x\varphi(A) + \delta(b)\, e_{1t}x - x\delta(b)\, e_{1t} + \overline{\delta}(x)$$

$$= d_B(x) + \overline{\delta}(x), \quad \text{where } B = \varphi(A) + \delta(b)\, e_{1t}.$$

Therefore, as above, B must be a diagonal matrix. Since $A = \sum_i \alpha_i e_{ii}$, we obtain

$$(-b\alpha_t + \alpha_1 b + \delta(b))e_{1t} = 0, \quad \text{for any } b \in D \text{ and } n \geq t \geq 2\,.$$

In particular set $b = 1$, then $\alpha_1 = \alpha_t$ for any t, that is A is a scalar matrix, $A = \alpha I_n$, $(\alpha = \alpha_1)$. Therefore $\delta(b) = -(\alpha b - b\alpha)$, which inplies $\overline{\delta} = -d_{\alpha I_n}$ and $d = d_A + \overline{\delta} = d_{\alpha I_n} - d_{\alpha I_n} = 0$, a contradiction. Hence n must be 1 and the proof is complete. ∎

Remark. Notice that the proof of the previous Lemma can be easily deduced by the main Theorem's one in [5]. We have included it for sake of clearness.

Now we are ready to prove the following:

Theorem 2.1. *Let R a 2-torsion free semiprime ring, d a non-zero derivation of R and $a, b, c, d \in R$ such that $[d([[a, b],[c, d]]_k),[[a, b],[c, d]]_k] \neq 0$. If $[d([[r_1, r_2],[r_3, r_4]]_k),[[r_1, r_2],[r_3, r_4]]_k]$ is zero or invertible in R, for any $r_1, r_2, r_3, r_4 \in R$, then R is a division ring.*

Proof. Suppose at first that there exists a right ideal ϱ of R such that

$$[d([[x_1, x_2],[x_3, x_4]]_k),[[x_1, x_2],[x_3, x_4]]_k]$$

is an identity in ϱ.

Let $a \in \varrho - \{0\}$, then R satisfies the differential identity

$$f(x_1, \ldots, x_4, d(x_1), \ldots, d(x_4))$$

$$= [d([[ax_1, ax_2],[ax_3, ax_4]]_k),[[ax_1, ax_2],[ax_3, ax_4]]_k].$$

In other words, for any $r_1, r_2, r_3, r_4 \in R$

$$0 = \left[\left(\sum_h (-1)^h \binom{k}{h} d([ar_3, ar_4]^h [ar_1, ar_2][ar_3, ar_4]^{k-h}),[[ar_1, ar_2],[ar_3, ar_4]]_k\right)\right].$$

Now let

$$f_h = d([ar_3, ar_4]^h [ar_1, ar_2][ar_3, ar_4]^{k-h})$$

and say

$$f_h = f_h^1 + f_h^2 + f_h^3$$

where

$$f_h^1 = d([ar_3, ar_4]^h)[ar_1, ar_2][ar_3, ar_4]^{k-h}$$

$$= \Big(\sum_{s+t=h-1} [ar_3, ar_4]^s (d([ar_3, ar_4]))[ar_3, ar_4]^t\Big) [ar_1, ar_2][ar_3, ar_4]^{k-h}$$

$$f_h^2 = [ar_3, ar_4]^h d([ar_1, ar_2])[ar_3, ar_4]^{k-h}$$

$$f_h^3 = [ar_3, ar_4]^h [ar_1, ar_2]\, d([ar_3, ar_4]^{k-h})$$

$$= [ar_3, ar_4]^h [ar_1, ar_2] \Big(\sum_{s+t=k-h-1} [ar_3, ar_4]^s d([ar_3, ar_4])[ar_3, ar_4]^t\Big).$$

Consider the following generalized differential polynomials:

$$\Phi_h^1(x_1, x_2, x_3, x_4, d(x_3), d(x_4)) = d([ax_3, ax_4]^h)[ax_1, ax_2][ax_3, ax_4]^{k-h}$$

$$= \Big(\sum_{s+t=h-1} [ax_3, ax_4]^s d([ax_3, ax_4])[ax_3, ax_4]^t\Big) [ax_1, ax_2][ax_3, ax_4]^{k-h}$$

$$\Phi_h^2(x_1, x_2, x_3, x_4, d(x_1), d(x_2)) = [ax_3, ax_4]^h d([ax_1, ax_2])[ax_3, ax_4]^{k-h}$$

$$\Phi_h^3(x_1, x_2, x_3, x_4, d(x_3), d(x_4)) = [ax_3, ax_4]^h [ax_1, ax_2]\, d([ax_3, ax_4]^{k-h})$$

$$= [ax_3, ax_4]^h [ax_1, ax_2] \Bigg(\sum_{s+t=k-h-1} [ax_3, ax_4]^s d([ax_3, ax_4])[ax_3, ax_4]^t\Bigg).$$

Denote

$$\Phi_h(x_1, x_2, x_3, x_4, d(x_1), d(x_2), d(x_3), d(x_4)) = \Phi_h^1 + \Phi_h^2 + \Phi_h^3.$$

Therefore R satisfies the differential identitiy

$$\Phi(x_1, \ldots, x_4, d(x_1), \ldots, d(x_4))$$

$$= \left[\left(\sum_h (-1)^h \binom{k}{h} \Phi_h(x_1, \ldots, x_4, d(x_1), \ldots, d(x_4)), [[ax_1, ax_2], [ax_3, ax_4]]_k\right)\right].$$

If d is not inner then R satisfies the non-trivial generalized polynomial identity

$$\Phi(x_1, x_2, x_3, x_4, y_1, y_2, y_3, y_4).$$

In particular R satisfies

$$\Phi(x_1, x_2, x_3, x_4, y_1, 0, 0, 0)$$

$$= \left[\left(\sum_h (-1)^h \binom{k}{h} \Phi_h^2(x_1, x_2, x_3, x_4, y_1, 0), [[ax_1, ax_2], [ax_3, ax_4]]_k\right)\right]$$

$$= \left[\left(\sum_h (-1)^h \binom{k}{h} [[ay_1, ax_2], [ax_3, ax_4]]_k, [[ax_1, ax_2], [ax_3, ax_4]]_k\right)\right].$$

Moreover R is simple with 1, hence $R \cong M_n(D)$.

Let now d the inner derivation induced by $A \in Q$, that is

$$[A, [[r_1, r_2], [r_3, r_4]]_k]_2 = 0 \qquad \text{for any} \qquad r_1, r_2, r_3, r_4 \in \varrho\,.$$

Fix $u \in \varrho$. Let $\alpha \in Z(R)$ such that $(A - \alpha)u = 0$. Thus, for any $r_1, r_2, r_3, r_4 \in R$, $(A - \alpha)[[ur_1, ur_2], [ur_3, ur_4]]_k = 0$, and

$$0 = [A - \alpha, [[ur_1, ur_2], [ur_3, ur_4]]_k]_2$$

$$= ([[ur_1, ur_2], [ur_3, ur_4]]_k)^2 (A - \alpha)$$

i.e. R satisfies the generalized polynomial identity

$$([[ux_1, ux_2], [ux_3, ux_4]]_k)^2 (A - \alpha)\,.$$

As above $R \cong M_n(D)$.

Now we may assume that, for any $\alpha \in Z(R)$, $(A - \alpha)u \neq 0$, which means that Au and u are linearly independent over $Z(R)$.

Also in this case, by [**4**], $[A, [[ux_1, ux_2], [ux_3, ux_4]]_k]_2$ is a non-trivial generalized polynomial identity in R and this implies again $R \cong M_n(D)$.

In any case the conclusion follows by the previous lemma.

Now we supose that for any ϱ right ideal of R,

$$[d([[x_1, x_2], [x_3, x_4]]_k), [[x_1, x_2], [x_3, x_4]]_k]$$

is not an identity in ϱ.

For any $r_1, r_2, r_3, r_4 \in \varrho$, $[d([[r_1, r_2], [r_3, r_4]]_k),\ [[r_1, r_2],\ [r_3, r_4]]_k] \in d(\varrho)\,\varrho + \varrho \subseteq d(\varrho) + \varrho$, since $d(\varrho) + \varrho$ is a right ideal of R.

Therefore $d(\varrho) + \varrho$ contains an invertible element of R, that is $d(\varrho) + \varrho = R$, for any right ideal ϱ.

Let $\varrho_1 \subseteq \varrho_2$ be right ideals of R. Thus $d(\varrho_1) + \varrho_1 = R = d(\varrho_2) + \varrho_2$. Fix $c \in \varrho_2 - \varrho_1$, $c = a + d(b)$, where $a, b \in \varrho_1$, $d(b) \neq 0$ and $d(b) \in \varrho_2$, since $c \notin \varrho_1$. In particular bR is a right ideal of R, and so $d(bR) + bR = R$, and also $d(bR) = d(b)R + bd(R) \subseteq \varrho_2$. Therefore $R = d(bR) + bR \subseteq \varrho_2$, i.e. $R = \varrho_2$. Also in this case R is a division ring. ■

We conclude this note with an easy application to Lie ideal in prime rings:

Corollary 2.1. *Let R be a prime ring with characteristic different from 2, d a non-zero derivation of R, L a non-central Lie ideal of R such that* $[d([u_1, u_2]_k)), [u_1, u_2]_k]$ *is zero or invertible, for any $u_1, u_2 \in L$. Then R is a division ring.*

Proof. As in Corollary 1.1, there exists a non-zero two-sided ideal I of R such that $[I, I] \subseteq L$. Thus $[d([[x, y], [z, t]]_k), [[x, y], [z, t]]_k]$ is zero or invertible, for any $x, y, z, t \in I$.

In the case $[d([[x, y], [z, t]]_k), [[x, y], [z, t]]_k] = 0$ is a differential identity in I, by Lemma 1.1, we obtain the contradiction that R is commutative.

If $[d([[a, b], [c, d]]_k), [[a, b], [c, d]]_k] \neq 0$, for suitable $a, b, c, d \in I$, then I contains an invertible element of R and $R = I$. In this case we conclude, by previous theorem, that R is a division ring. ■

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Abstract

Let R be a 2-torsion free semiprime ring, d a non-zero derivation of R and $a, b, c, d \in R$ such that $[d([[a, b],[c, d]]_k),[[a, b],[c, d]]_k] \neq 0$. We prove that if $[d([[r_1, r_2],[r_3, r_4]]_k),[[r_1, r_2],[r_3, r_4]]_k]$ is zero or invertible, for any $r_1, r_2, r_3, r_4 \in R$, then R is a division ring.

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