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ABDESLEM LYAGHFOURI (*)

A unified formulation for the dam problem (**)

1 - Statement of the problem

Let Ω be a bounded, locally Lipschitz, domain in $\mathbb{R}^n$ $(n \geq 2)$. Ω represents a porous medium. The boundary Γ of Ω is denoted by Γ. Assuming that the flow in Ω has reached a steady state, we are concerned with finding the pressure p of the fluid and the saturated region of the porous medium, i.e., the subset S of Ω where $p > 0$. Let us first describe the formulation of our problem.

The boundary of S that we denote by ∂S, is divided into four parts: an impervious part, Γ_1, a free boundary, Γ_2, a part covered by the fluid, Γ_3, and finally a seepage front, Γ_4, where the fluid flows outside Ω but does not remain there in a significant amount to modify the pressure (see Fig. 1).

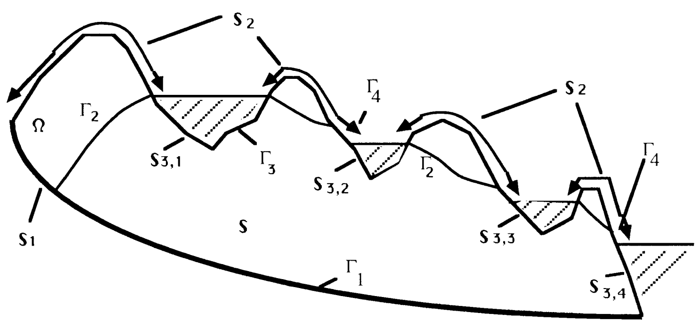

Fig. 1.

(*) Institut für Angewandte Mathematik der Universität Zürich, Winterthurerstr. 190, CH-8057, Zürich, Switzerland.

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In the saturated region, the fluid velocity $\vec{v}$ and its pressure p are related by the generalized Darcy law

$$\vec{v} = -\,\mathcal{A}(x,\nabla(p+x_n)) = -\,\mathcal{A}(x,\nabla u) \tag{1.1}$$

where $x=(x_1,\dots,x_n)$ denotes points in $\mathbb{R}^n$, $u=p+x_n$ is the hydrostatic head and $\mathcal{A}:\Omega\times\mathbb{R}^n\to\mathbb{R}^n$ is a mapping that satisfies the following assumptions with some constants $q>1$ and $\beta\geqslant\alpha>0$:

$$\begin{cases} \text{the function } x\mapsto\mathcal{A}(x,\xi) \text{ is measurable } \forall\xi\in\mathbb{R}^n, \text{ and} \\ \text{the function } \xi\mapsto\mathcal{A}(x,\xi) \text{ is continuous for a.e. } x\in\Omega, \end{cases} \tag{1.2}$$

for all $\xi\in\mathbb{R}^n$ and a.e. $x\in\Omega$

$$\mathcal{A}(x,\xi)\,.\,\xi\geqslant\alpha|\xi|^q, \tag{1.3}$$

$$|\mathcal{A}(x,\xi)|\leqslant\beta|\xi|^{q-1}, \tag{1.4}$$

for all $\xi,\zeta\in\mathbb{R}^n$ and a.e. $x\in\Omega$

$$(\mathcal{A}(x,\xi)-\mathcal{A}(x,\zeta))\,.\,(\xi-\zeta)\geqslant 0\,. \tag{1.5}$$

A typical example of nonlinear Darcy's laws for a homogeneous porous medium (see [**13**], [**16**], [**24**], [**28**]) corresponds to the q-Laplacian

$$\mathcal{A}(x,\xi)=|\xi|^{q-2}\xi\,.$$

Now we have the following strong formulation

$$\begin{cases} \vec{v}=-\mathcal{A}(x,\nabla u) & \text{in } S\,, \\ \operatorname{div}(\vec{v})=0 & \text{in } S\,, \\ u>x_n & \text{in } S \quad\text{and}\quad u=x_n \quad \text{in } S^c, \\ -\vec{v}\,.\,\nu\in\mathcal{B}(x,\psi-u) & \text{on } \Gamma\,, \end{cases}$$

where $\psi=\varphi+x_n$ and φ is a nonnegative Lipschitz continuous function in $\overline{\Omega}$, representing the exterior pressure on Γ. $\mathcal{B}$ is a multivalued monotone function and the goal of this modelisation is to give a unified formulation to the boundary conditions for the dam problem. In the classical formulation of this problem, we have $\Gamma=S_1\cup S_2\cup S_3$, where S_1 denotes the impervious part of Γ, S_2 is the part in contact with the air and $S_3=\bigcup_{1\leqslant i\leqslant N}S_{3,i}$ the part covered by fluid (see Fig. 1). So if $\mathcal{B}$

is given by:

$$\mathcal{B}(x, .) = \begin{cases} \mathbb{R} \times \{0\} & \text{for a.e.} \quad x \in S_1, \\ \{0\} \times \mathbb{R} & \text{for a.e.} \quad x \in S_2 \cup S_3. \end{cases} \tag{1.7}$$

$$\text{(resp.} \quad \mathcal{B}(x, .) = \begin{cases} \mathbb{R} \times \{0\} & \text{for a.e.} \quad x \in S_1, \\ \{0\} \times \mathbb{R} & \text{for a.e.} \quad x \in S_2, \\ \{(u, \beta(x, u))/u \in \mathbb{R}\} & \text{for a.e.} \quad x \in S_3, \end{cases} \tag{1.8}$$

where β: $S_3 \times \mathbb{R} \to \mathbb{R}$ is a continuous monotone function with respect to the second variable), we are in the case of Dirichlet boundary conditions i.e. $p = \varphi$ on $S_2 \cup S_3$ (see [**3, 4, 5, 6, 8, 11, 13, 27**]) (resp. leaky boundary conditions i.e. $p = \varphi$ on S_2 and $\vec{v}.\nu = -\beta(x, \varphi - p)$ on S_3 (see [**6, 12, 14, 15, 16, 17, 28, 30, 32**])).

For $\mathcal{B}$, we assume that

$$\text{for a.e.} \quad x \in \Gamma, \qquad s \mapsto \mathcal{B}(x, s) \text{ is a maximal monotone graph of } \mathbb{R}^2, \tag{1.9}$$

$$0 \in \mathcal{B}(x, 0) \text{ for a.e.} \quad x \in \Gamma, \tag{1.10}$$

$$\text{for a.e. } x \in \Gamma, \quad D(\mathcal{B}(x, .)) \text{ is closed}. \tag{1.11}$$

For a.e. $x \in \Gamma$, let $(a(x), b(x)) = \operatorname{Int}(D(\mathcal{B}(x, .)))$ where $-\infty \leqslant a(x) \leqslant 0 \leqslant b(x) \leqslant +\infty$. Assumptions (1.10) and (1.11) imply that for a.e. $x \in \Gamma$, there exists a unique pair $\mathcal{B}_1(x, .)$ and $\mathcal{B}_2(x, .)$ of maximal monotone graphs in $\mathbb{R}^2$ such that

$$\begin{cases} D(\mathcal{B}_2(x, .)) = \mathbb{R}, \\ \mathcal{B}_2(x, .) = \mathcal{B}(x, .), \qquad \text{in } (a(x), b(x)), \\ \mathcal{B}_2(x, a(x)) = \{a(x)\} \times [\mathcal{B}^0(x, a(x)), \mathcal{B}^0(x, a(x)+)], \\ \mathcal{B}_2(x, s) = \{(s, \mathcal{B}^0(x, a(x)))\}, \qquad \forall s < a(x), \\ \mathcal{B}_2(x, s) = \{(s, \mathcal{B}^0(x, b(x)))\}, \qquad \forall s > b(x), \\ \mathcal{B}_2(x, b(x)) = \{b(x)\} \times [\mathcal{B}^0(x, b(x)-), \mathcal{B}^0(x, b(x))], \\ D(\mathcal{B}_1(x, .)) = D(\mathcal{B}(x, .)), \\ \mathcal{B}_1(x, .) = 0 \quad \text{in } \operatorname{Int}(D(\mathcal{B}(x, .))), \\ \mathcal{B} = \mathcal{B}_1 + \mathcal{B}_2, \end{cases} \tag{1.12}$$

where $\mathcal{B}^0$ is the minimal section of $\mathcal{B}$ and for a.c. $x \in \Gamma$, $\mathcal{B}^0(x, s_-)$ (resp. $\mathcal{B}^0(x, s_+)$) is the left (resp. right) limit of $\mathcal{B}^0(x, \cdot)$ at s.

Moreover, we assume that

$$\text{(1.13)} \qquad \exists H > \max\left(\max\{x_n, (x', x_n) \in \overline{\Omega}\},\ \max\{\psi(x), x \in \Gamma\}\right) \text{ such that}$$

$$\int_\Gamma |\mathcal{B}_2^0(x, \psi(x) - H)|^{q'} d\sigma(x), \quad \int_\Gamma |\mathcal{B}_2^0(x, \varphi(x))|^{q'} d\sigma(x) < +\infty$$

$$\text{i.e.} \quad \mathcal{B}_2^0(., \psi - H), \quad \mathcal{B}_2^0(., \varphi) \in L^{q'}(\Gamma).$$

Note that (1.13) is satisfied for example if we have

$$\exists R_0 > 0, \qquad \forall R \geqslant R_0, \qquad \exists C_R \text{ such that } \mathcal{B}_2(x, s) \subset (-C_R, C_R),$$

$$\forall s \in (-R, R), \qquad \text{for a.e. } x \in \Gamma.$$

Then we have the following weak unified formulation of the Dam problem (see [**2**])

$$(P) \quad \begin{cases} \text{Find}\,(u, g, \gamma) \in W^{1,q}(\Omega) \times L^\infty(\Omega) \times L^{q'}(\Gamma), \text{ such that:} \\ \text{(i)} \quad \psi(x) - u(x) \in D(\mathcal{B}(x, .)) \text{ for a.e. } x \in \Gamma, \\ \text{(ii)} \quad u \geqslant x_n, \quad 0 \leqslant g \leqslant 1, \quad g.(u - x_n) = 0 \text{ a.e. } in\ \Omega, \\ \text{(iii)} \quad \gamma(x) \in \mathcal{B}_2(x, \psi(x) - u(x)) \text{ for a.e. } x \in \Gamma, \\ \qquad\qquad \text{and } \gamma(x) \leqslant 0 \text{ for a.e. } x \in \Gamma \text{ such that } \psi(x) = x_n, \\ \text{(iv)} \quad \displaystyle\int_\Omega (\mathfrak{A}(x, \nabla u) - g\mathfrak{A}(x, e)).\nabla(\xi - u)\, dx \geqslant \int_\Gamma \gamma.(\xi - u)\, d\sigma(x), \\ \forall \xi \in \mathbb{K} = \{\xi \in W^{1,q}(\Omega) / a(x) \leqslant \psi(x) - \xi(x) \leqslant c(x) \text{ for a.e. } x \in \Gamma\} \end{cases}$$

where q' is the conjuguate exponent of q and for a.e. $x \in \Gamma$: $c(x) = b(x)$ if $\varphi(x) > 0$ and $c(x) = +\infty$ if $\varphi(x) = 0$.

In the following paragraph, we establish an existence of a solution of (P). In the last section we consider the case of unbounded domains.

Remark 1.1. Throughout this paper we denote by $|.|_{r,\Gamma}$, (Resp. $|.|_r$) the usual L^r-norm on Γ, (Resp. Ω) and $|E|$ denotes the Lebesgue measure of the measurable set E. $|.|_{1,l}$ denotes the usual norm of the Sobolev space $W^{1,l}$.

2 - The case of bounded domains

2.1 - *Existence of a solution*

We have

Theorem 2.1. *Assume that φ is a nonnegative Lipschitz continuous function, $\mathcal{A}$ satisfies* (1.2)-(1.5) *and $\mathcal{B}$ satisfies* (1.9)-(1.13), *then there exists a solution* (u, g, γ) *to problem* (P).

For $\varepsilon > 0$, we introduce the following approximated problem

$$(P_\varepsilon)\quad \begin{cases} \text{Find } u_\varepsilon \in V \text{ such that} \\ \displaystyle\int_\Omega \varepsilon(|u_\varepsilon|^{q-2}u_\varepsilon - |x_n|^{q-2}x_n).\xi + (\mathcal{A}(x,\nabla u_\varepsilon) - G_\varepsilon(u_\varepsilon)\mathcal{A}(x,e)).\nabla\xi \,\mathrm{d}x \\ \qquad\qquad\qquad\qquad \displaystyle + \int_\Gamma \varepsilon(|u_\varepsilon|^{q'-2}u_\varepsilon - |x_n|^{q'-2}x_n).\xi \,\mathrm{d}\sigma(x) \\ \qquad\qquad\qquad\qquad \displaystyle = \int_\Gamma (\mathcal{B}_1^\varepsilon(x,\psi - u_\varepsilon) + \mathcal{B}_2^\varepsilon(x,\psi - u_\varepsilon)).\xi \,\mathrm{d}\sigma(x), \\ \forall \xi \in V = \{\xi \in W^{1,q}(\Omega)/\xi_{|\Gamma} \in L^{q'}(\Gamma)\}, \end{cases}$$

where $G_\varepsilon : L^q(\Omega) \to L^\infty(\Omega)$ is defined for a.e. $x \in \Omega$ by

$$G_\varepsilon(v(x)) = \begin{cases} 0 & \text{if} \quad v(x) - x_n \geq \varepsilon, \\ 1 - (v(x) - x_n)/\varepsilon & \text{if} \quad 0 \leq v(x) - x_n \leq \varepsilon, \\ 1 & \text{if} \quad v(x) - x_n \leq 0. \end{cases} \tag{2.1}$$

$\mathcal{B}_i^\varepsilon$ denotes the Yoshida approximation of $\mathcal{B}_i$ for $i = 1, 2$. Note that $\mathcal{B}_i^\varepsilon$ is a nondecreasing Lipschitz continuous function with respect to the second variable. The constant of Lipschitz is equal to $1/\varepsilon$ and by (1.10) and (1.12) we have

$$\mathcal{B}_i^\varepsilon(x, 0) = 0 \quad \text{for a.e.} \quad x \in \Gamma,$$

which leads by the monotonicity of $\mathcal{B}_i^\varepsilon(x, .)$ to

$$\mathcal{B}_i^\varepsilon(x, u).u \geq 0 \ \text{ for a.e. } \ x \in \Gamma, \quad \forall u \in \mathbb{R}. \tag{2.2}$$

The space V is equipped with the norm $\|u\| = |u|_{1,q} + |u|_{q',\Gamma}$ $\forall u \in V$. Then we have

Theorem 2.2. *Assume that φ is a nonnegative Lipschitz continuous function, that $\mathcal{A}$ satisfies* (1.2)-(1.5) *and $\mathcal{B}$ satisfies* (1.9)-(1.12). *Then, there exists a unique solution u_ε of* (P_ε).

Proof. First let us define for $u \in V$

$Au \colon V \to \mathbb{R}$,

$$\xi \mapsto \langle Au, \xi \rangle = \int_{\Omega} \varepsilon |u|^{q-2} u.\xi + \mathcal{A}(x, \nabla u).\nabla \xi \, dx + \int_{\Gamma} \varepsilon |u|^{q'-2} u.\xi \, d\sigma(x)$$

$$- \int_{\Gamma} (\mathcal{B}_1^{\varepsilon}(x, \psi - u) + \mathcal{B}_2^{\varepsilon}(x, \psi - u)).\xi \, d\sigma(x).$$

Then it is clear that the operator A defined by $A \colon u \in V \mapsto Au$ is continuous from V into V'. Moreover as a consequence of the following lemma which is proved in [**18**] and the monotonicity of $\mathcal{A}$ and $\mathcal{B}_i^{\varepsilon}$, one can see easily that A is monotone,

Lemma 2.3. *Assume* $q > 1$. *There exists* $\mu > 0$ *such that for all* $(x, y) \in (\mathbb{R}^n)^2$ *we have*

i) *if* $q \geq 2$, $\quad \mu |x - y|^q \leq (|x|^{q-2} x - |y|^{q-2} y, x - y)$,

ii) *if* $1 < q < 2$, $\quad \mu |x - y|^2 \leq (|x| + |y|)^{2-q} (|x|^{q-2} x - |y|^{q-2} y, x - y)$.

Now, we have for $u \in V$

$$\langle Au, u \rangle = \int_{\Omega} \varepsilon |u|^q + \mathcal{A}(x, \nabla u).\nabla u \, dx + \varepsilon \int_{\Gamma} |u|^{q'} d\sigma(x) - \sum_{i=1}^{2} \int_{\Gamma} \mathcal{B}_i^{\varepsilon}(x, \psi - u).u \, d\sigma(x).$$

Note that by (2.2) and the Lipschitz continuity of $\mathcal{B}_i^{\varepsilon}(x, .)$, we have

$$\int_{\Gamma} \mathcal{B}_i^{\varepsilon}(x, \psi - u).u \, d\sigma(x) \leq \int_{\Gamma} \mathcal{B}_i^{\varepsilon}(x, \psi - u).\psi d\sigma(x) \tag{2.3}$$

$$\leq \frac{1}{\varepsilon} \int_{\Gamma} |\psi - u|.|\psi| \, d\sigma(x) \leq c_0 + c_1 |u|_{q', \Gamma}.$$

Using (1.3) and (2.3), one can check for some constants c_i

$$\langle Au, u \rangle \geq c_2 (|u|_{1,q}^q + |u|_{q',\Gamma}^{q'}) - 2c_1 |u|_{q',\Gamma} - 2c_0, \qquad \forall u \in V. \tag{2.4}$$

Thus, since $q, q' > 1$

$$\lim_{\|u\| \to +\infty} \frac{\langle Au, u\rangle}{\|u\|} = +\infty .$$

So A is coercive. ■

Next for $v \in L^q(\Omega)$, we consider the map:

$$f_v : V \to \mathbb{R},$$

$$\xi \mapsto \int_\Omega \varepsilon |x_n|^{q-2} x_n . \xi \, \mathrm{d}x + \int_\Omega G_\varepsilon(v) \mathfrak{A}(x, e) . \nabla \xi \, \mathrm{d}x + \int_\Gamma \varepsilon |x_n|^{q'-2} x_n . \xi \, \mathrm{d}\sigma(x).$$

It is clear that f_v is a continuous linear form on V. Since A is continuous, coercive, monotone, we deduce (see [**26**]) that for every $v \in L^q(\Omega)$ there exists a unique u_ε solution of the variational problem

$$\text{(2.5)} \qquad \begin{cases} u_\varepsilon \in V , \\ \langle Au_\varepsilon, w\rangle = \langle f_v, w\rangle , \qquad \forall w \in V. \end{cases}$$

Now, let us consider the map $\mathcal{T}_\varepsilon$ defined by: $\mathcal{T}_\varepsilon : L^q(\Omega) \to V$, $v \mapsto u_\varepsilon$.

Then one has

$$\text{i) } \exists R(\varepsilon) > 0 / \mathcal{T}_\varepsilon(\overline{B}(0, R(\varepsilon))) \subset \overline{B}(0, R(\varepsilon)),$$

$$\text{ii) } \mathcal{T}_\varepsilon : L^q(\Omega) \to L^q(\Omega) \quad \text{is continuous} ,$$

where $\overline{B}(0, R(\varepsilon))$ denotes the closed ball in $L^q(\Omega)$ of center 0 and radius $R(\varepsilon)$.

Indeed, note that u_ε is a suitable test function of (2.5), so:

$$\text{(2.6)} \qquad \langle Au_\varepsilon, u_\varepsilon\rangle = \langle f_v, u_\varepsilon\rangle .$$

Using (2.4) and (2.6), we deduce, for some constant $R(\varepsilon)$ depending on ε, that

$$\|u_\varepsilon\| \leq R(\varepsilon) .$$

So we have : $|u_\varepsilon|_{q,\Omega} \leq \|u_\varepsilon\| \leq R(\varepsilon)$ and $\mathcal{T}_\varepsilon(\overline{B}(0, R(\varepsilon))) \subset \overline{B}(0, R(\varepsilon))$. Moreover $\mathcal{T}_\varepsilon(\overline{B}(0, R(\varepsilon)))$ is bounded in $W^{1,q}(\Omega)$ since $|u_\varepsilon|_{1,q} \leq \|u_\varepsilon\| \leq R(\varepsilon)$ and thus it is relatively compact in $L^q(\Omega)$.

ii) Let $(v_k)_k$ be a sequence in $L^q(\Omega)$ which converges to v in $L^q(\Omega)$.

Set $u_\varepsilon^k = \mathcal{T}_\varepsilon(v_k)$ and $u_\varepsilon = \mathcal{T}_\varepsilon(v)$. Since $u_\varepsilon^k - u_\varepsilon$ is a suitable test function for

(2.5), one has by writting (2.5) for u_ε^k and u_ε and subtracting the equations

$$(2.7)\quad \int_\Omega \big(\mathcal{A}(x, \nabla u_\varepsilon^k) - \mathcal{A}(x, \nabla u_\varepsilon)\big).\nabla(u_\varepsilon^k - u_\varepsilon)$$

$$+\varepsilon(|u_\varepsilon^k|^{q-2}u_\varepsilon^k - |u_\varepsilon|^{q-2}u_\varepsilon).(u_\varepsilon^k - u_\varepsilon)\,\mathrm{d}x$$

$$+\int_\Gamma \varepsilon(|u_\varepsilon^k|^{q'-2}u_\varepsilon^k - |u_\varepsilon|^{q'-2}u_\varepsilon).(u_\varepsilon^k - u_\varepsilon)\,\mathrm{d}\sigma(x)$$

$$= \int_\Omega \big(G_\varepsilon(v_k) - G_\varepsilon(v)\big)\,\mathcal{A}(x, e).\nabla(u_\varepsilon^k - u_\varepsilon)\,\mathrm{d}x$$

$$+\sum_{i=1}^{i=2}\int_\Gamma \big(\mathcal{B}_i^\varepsilon(x, \psi - u_\varepsilon^k) - \mathcal{B}_i^\varepsilon(x, \psi - u_\varepsilon)\big).(u_\varepsilon^k - u_\varepsilon)\,\mathrm{d}\sigma(x).$$

Now we have

$$\Big|\int_\Omega \big(G_\varepsilon(v_k)-G_\varepsilon(v)\big)\,\mathcal{A}(x, e).\nabla(u_\varepsilon^k-u_\varepsilon)\,\mathrm{d}x\Big| \leqslant \int_\Omega \beta|G_\varepsilon(v_k)-G_\varepsilon(v)|.|\nabla(u_\varepsilon^k-u_\varepsilon)|\,\mathrm{d}x$$

$$\leqslant \beta\Big(\int_\Omega |G_\varepsilon(v_k) - G_\varepsilon(v)|^{q'}\,\mathrm{d}x\Big)^{1/q'}.\Big(\int_\Omega |\nabla(u_\varepsilon^k - u_\varepsilon)|^q\,\mathrm{d}x\Big)^{1/q}.$$

Moreover since $q' > 1$ and $0 \leqslant G_\varepsilon(v_k)$, $G_\varepsilon(v) \leqslant 1$ a.e. in Ω, one has

$$|G_\varepsilon(v_k) - G_\varepsilon(v)|^{q'}$$

$$= |G_\varepsilon(v_k) - G_\varepsilon(v)|^{q'-1}|G_\varepsilon(v_k) - G_\varepsilon(v)| \leqslant |G_\varepsilon(v_k) - G_\varepsilon(v)| \leqslant \frac{1}{\varepsilon}|v_k - v|$$

which leads by Hölder's inequality to

$$\Big(\int_\Omega |G_\varepsilon(v_k)-G_\varepsilon(v)|^{q'}\,\mathrm{d}x\Big)^{1/q'} \leqslant \frac{1}{\varepsilon^{1/q'}}\Big(\int_\Omega |v_k-v|\,\mathrm{d}x\Big)^{1/q'} \leqslant \frac{|\Omega|^{1/q'^2}}{\varepsilon^{1/q'}}\Big(\int_\Omega |v_k-v|^q\,\mathrm{d}x\Big)^{1/qq'}.$$

Hence we obtain

$$\left| \int_\Omega (G_\varepsilon(v_k) - G_\varepsilon(v))\,\mathcal{A}(x, e).\nabla(u_\varepsilon^k - u_\varepsilon)\,\mathrm{d}x \right| \leq \frac{\beta|\Omega|^{1/q'^2}}{\varepsilon^{1/q'}} \|u_\varepsilon^k - u_\varepsilon\| \left(\int_\Omega |v_k - v|^q\,\mathrm{d}x \right)^{1/qq'}. \tag{2.8}$$

Taking into account (2.8), the fact that $\|u_\varepsilon^k - u_\varepsilon\| \leq 2R(\varepsilon)$, the monotonicity of $\mathcal{B}_i^\varepsilon(x, .)$, $\mathcal{A}(x, .)$ and Lemma 2.3, we deduce from (2.7)

$$\int_\Omega \varepsilon(|u_\varepsilon^k|^{q-2}u_\varepsilon^k - |u_\varepsilon|^{q-2}u_\varepsilon).(u_\varepsilon^k - u_\varepsilon)\,\mathrm{d}x \leq c.|v_k - v|_q^{1/q'}$$

which leads to

$$\lim_{k \to +\infty} \int_\Omega (|u_\varepsilon^k|^{q-2}u_\varepsilon^k - |u_\varepsilon|^{q-2}u_\varepsilon).(u_\varepsilon^k - u_\varepsilon)\,\mathrm{d}x = 0\,. \tag{2.9}$$

Using (2.9) and Lemma 2.3, we deduce that if $q \geq 2$

$$u_\varepsilon^k \to u_\varepsilon \quad \text{ in } L^q(\Omega).$$

When $1 < q < 2$, we set $w_\varepsilon^k = |u_\varepsilon^k|^{q-2}u_\varepsilon^k$ and $w_\varepsilon = |u_\varepsilon|^{q-2}u_\varepsilon$, so that $u_\varepsilon^k = |w_\varepsilon^k|^{q'-2}w_\varepsilon^k$, $u_\varepsilon = |w_\varepsilon|^{q'-2}w_\varepsilon$ and (2.9) becomes

$$\lim_{k \to +\infty} \int_\Omega (|w_\varepsilon^k|^{q'-2}w_\varepsilon^k - |w_\varepsilon|^{q'-2}w_\varepsilon).(w_\varepsilon^k - w_\varepsilon)\,\mathrm{d}x = 0$$

which leads again by Lemma 2.3, since $q' > 2$ to $w_\varepsilon^k \to w_\varepsilon$ in $L^{q'}(\Omega)$. Now using the continuity of the operator: $L^{q'}(\Omega) \to L^q(\Omega)$, $w \mapsto |w|^{q'-2}w$, we get $u_\varepsilon^k \to u_\varepsilon$ in $L^q(\Omega)$. Hence the continuity of $\mathcal{T}_\varepsilon$ holds.

At this step, applying the Schauder fixed point theorem on $\overline{B}(0, R(\varepsilon))$ (see [**23**]), we derive that $\mathcal{T}_\varepsilon$ has a fixed point. Thus (P_ε) has at least one solution.

Let us now prove the uniqueness of the solution of (P_ε). Consider u_ε and u'_ε two solutions of (P_ε).

For $\delta > 0$, we define as in [**9**] a function $T_\delta : \mathbb{R} \to \mathbb{R}$ by

$$T_\delta(s) = \begin{cases} s & \text{if } \quad |s| \leq \delta\,, \\ \delta \dfrac{s}{|s|} & \text{if } \quad |s| > \delta\,. \end{cases}$$

Since $T_\delta \in C(\mathbb{R})$ and $T'_\delta \in L^\infty(\mathbb{R})$, we have $\forall u \in W^{1,q}(\Omega)$: $T_\delta o u \in W^{1,q}(\Omega)$ and

$$\nabla(T_\delta o u) = T'_\delta(u) . \nabla u = \chi([|u| \leqslant \delta]) \nabla u . \tag{2.10}$$

Set $v = u_\varepsilon - u'_\varepsilon$. Choose $T_\delta(v)$ as a test function for (P_ε) written for u_ε and u'_ε. Subtract the equations, so that

$$\begin{aligned}\int_{\Omega_\delta} (\mathcal{A}(x, \nabla u_\varepsilon) - \mathcal{A}(x, \nabla u'_\varepsilon)) . \nabla(u_\varepsilon - u'_\varepsilon)\, dx \\ + \int_\Omega \varepsilon(|u_\varepsilon|^{q-2} u_\varepsilon - |u'_\varepsilon|^{q-2} u'_\varepsilon) . T_\delta(v)\, dx \\ + \int_\Gamma \varepsilon(|u_\varepsilon|^{q'-2} u_\varepsilon - |u'_\varepsilon|^{q'-2} u'_\varepsilon) . T_\delta(v)\, d\sigma(x) \\ = \int_{\Omega_\delta} (G_\varepsilon(u_\varepsilon) - G_\varepsilon(u'_\varepsilon))\, \mathcal{A}(x, e) . \nabla(u_\varepsilon - u'_\varepsilon)\, dx \\ + \sum_{i=1}^{i=2} \int_\Gamma (\mathcal{B}_i^\varepsilon(x, \psi - u_\varepsilon) - \mathcal{B}_i^\varepsilon(x, \psi - u'_\varepsilon)) . T_\delta(v)\, d\sigma(x)\end{aligned} \tag{2.11}$$

where $\Omega_\delta = \{x \in \Omega / |v(x)| \leqslant \delta\}$.

Using the monotonicity of $\mathcal{A}(x, .)$, T_δ, $\mathcal{B}_i^\varepsilon(x, .)$ and Lemma 2.3, we get from (2.11)

$$\begin{aligned}\int_\Omega \varepsilon(|u_\varepsilon|^{q-2} u_\varepsilon - |u'_\varepsilon|^{q-2} u'_\varepsilon) . T_\delta(v)\, dx \\ \leqslant \int_{\Omega_\delta} (G_\varepsilon(u_\varepsilon) - G_\varepsilon(u'_\varepsilon))\, \mathcal{A}(x, e) . \nabla(u_\varepsilon - u'_\varepsilon)\, dx .\end{aligned} \tag{2.12}$$

But since $\Omega = \Omega_\delta \cup \Omega'_\delta$, where $\Omega'_\delta = \{x \in \Omega / |v(x)| > \delta\}$ and $\Omega_\delta \cap \Omega'_\delta = \emptyset$, we get by using the Lipschitz continuity of G_ε

$$\begin{aligned}\int_{\Omega'_\delta} \varepsilon\delta(|u_\varepsilon|^{q-2} u_\varepsilon - |u'_\varepsilon|^{q-2} u'_\varepsilon) . \frac{(u_\varepsilon - u'_\varepsilon)}{|u_\varepsilon - u'_\varepsilon|}\, dx \\ + \int_{\Omega_\delta} \varepsilon(|u_\varepsilon|^{q-2} u_\varepsilon - |u'_\varepsilon|^{q-2} u'_\varepsilon) . T_\delta(u_\varepsilon - u'_\varepsilon)\, dx \leqslant \frac{\beta\delta}{\varepsilon} \int_{\Omega_\delta} |\nabla(u_\varepsilon - u'_\varepsilon)|\, dx .\end{aligned}$$

Using the monotonicity of T_δ and $w \mapsto |w|^{q-2}w$, we get:

$$\int_{\Omega'_\delta} \big|\, |u_\varepsilon|^{q-2}u_\varepsilon - |u'_\varepsilon|^{q-2}u'_\varepsilon \big|\, dx \leqslant \frac{\beta}{\varepsilon^2} \int_{\Omega_\delta} |\nabla(u_\varepsilon - u'_\varepsilon)|\, dx \,. \tag{2.13}$$

Letting $\delta \to 0$ in (2.13), we get

$$\int_{\Omega} \big|\, |u_\varepsilon|^{q-2}u_\varepsilon - |u'_\varepsilon|^{q-2}u'_\varepsilon \big|\, dx = 0$$

which leads to $u_\varepsilon = u'_\varepsilon$ a.e. in Ω. ■

Let us now show that our sequence (u_ε) is uniformly bounded in $L^\infty(\Omega)$.

Lemma 2.4. *Let u_ε be a solution of (P_ε) and let $\varepsilon_0 > 0$ such that*

$$H \geqslant \max\left(\max\{\varepsilon_0 + x_n, (x', x_n) \in \overline{\Omega}\}, \max\{\psi(x), x \in \Gamma\}\right), \tag{2.14}$$

where H is the nonnegative constant given by (1.13). *Then we have for any $\varepsilon \in (0, \varepsilon_0)$*

$$x_n \leqslant u_\varepsilon \leqslant H \qquad \text{a.e. } in \ \Omega\,. \tag{2.15}$$

Proof. i) Since $(u_\varepsilon - H)^+$ is a suitable test function for (P_ε), we have

$$\begin{aligned} &\int_\Omega \mathcal{A}(x, \nabla u_\varepsilon).\nabla(u_\varepsilon - H)^+ + \varepsilon(|u_\varepsilon|^{q-2}u_\varepsilon - |x_n|^{q-2}x_n).(u_\varepsilon - H)^+\, dx \\ &+ \int_\Gamma \varepsilon(|u_\varepsilon|^{q'-2}u_\varepsilon - |x_n|^{q'-2}x_n).(u_\varepsilon - H)^+\, d\sigma(x) \\ &= \int_\Omega G_\varepsilon(u_\varepsilon)\, \mathcal{A}(x, e).\nabla(u_\varepsilon - H)^+\, dx + \sum_{i=1}^{i=2} \int_\Gamma \mathcal{B}_i^\varepsilon(x, \psi - u_\varepsilon)(u_\varepsilon - H)^+\, d\sigma(x)\,. \end{aligned} \tag{2.16}$$

Note that by (2.14), one has for $\varepsilon \in (0, \varepsilon_0)$ and $u_\varepsilon(x) \geqslant H$: $u_\varepsilon(x) \geqslant H \geqslant \varepsilon_0 + x_n \geqslant \varepsilon + x_n$ for a.e. $x \in \Omega$ and then by (2.1) $G_\varepsilon(u_\varepsilon(x)) = 0$. So

$$G_\varepsilon(u_\varepsilon)\, \mathcal{A}(x, e).\nabla(u_\varepsilon - H)^+ = 0 \qquad \text{a.e. in } \Omega\,. \tag{2.17}$$

Using (2.2), (2.14) and the monotonicity of $\mathcal{B}_i^\varepsilon(x, .)$, one has

$$\mathcal{B}_i^\varepsilon(x, \psi - u_\varepsilon)(u_\varepsilon - H)^+ \leqslant \mathcal{B}_i^\varepsilon(x, H - u_\varepsilon)(u_\varepsilon - H)^+ \leqslant 0 \qquad \text{for } a.e. \ \ x \in \Gamma. \tag{2.18}$$

Then we deduce from (1.3), (2.14)-(2.18) and Lemma 2.3

$$\int_{\Omega} \varepsilon(|u_\varepsilon|^{q-2}u_\varepsilon - |H|^{q-2}H).(u_\varepsilon - H)^+ + \alpha|\nabla(u_\varepsilon - H)^+|^q \, dx \leqslant 0$$

which leads to $(u_\varepsilon - H)^+ = 0$ and $u_\varepsilon \leqslant H$ a.e. in Ω.

ii) We denote by $(.)^-$ the negative part of a function. Then $\xi = (u_\varepsilon - x_n)^-$ is a test function for (P_ε) and one has

$$\text{(2.19)} \quad \int_{\Omega} \mathcal{A}(x, \nabla u_\varepsilon).\nabla(u_\varepsilon - x_n)^- + \varepsilon(|u_\varepsilon|^{q-2}u_\varepsilon - |x_n|^{q-2}x_n).(u_\varepsilon - x_n)^- \, dx$$

$$+ \int_{\Gamma} \varepsilon(|u_\varepsilon|^{q'-2}u_\varepsilon - |x_n|^{q'-2}x_n).(u_\varepsilon - x_n)^- \, d\sigma(x)$$

$$= \int_{\Omega} G_\varepsilon(u_\varepsilon)\, \mathcal{A}(x, e).\nabla(u_\varepsilon - x_n)^- \, dx + \sum_{i=1}^{i=2} \int_{\Gamma} \mathcal{B}_i^\varepsilon(x, \psi - u_\varepsilon)(u_\varepsilon - x_n)^- \, d\sigma(x).$$

Using (2.1), one has

$$\int_{\Omega} G_\varepsilon(u_\varepsilon)\, \mathcal{A}(x, e).\nabla(u_\varepsilon - x_n)^- \, dx = -\int_{\Omega \cap [u_\varepsilon \leqslant x_n]} G_\varepsilon(u_\varepsilon)\, \mathcal{A}(x, e).\nabla(u_\varepsilon - x_n) \, dx$$

$$= -\int_{\Omega \cap [u_\varepsilon \leqslant x_n]} \mathcal{A}(x, e).\nabla(u_\varepsilon - x_n) \, dx\,.$$

Moreover using (2.2) and the monotonicity of $\mathcal{B}_i^\varepsilon(x, .)$, one has also

$$\int_{\Gamma} \mathcal{B}_i^\varepsilon(x, \psi - u_\varepsilon)(u_\varepsilon - x_n)^- \, d\sigma(x) \geqslant \int_{\Gamma} \mathcal{B}_i^\varepsilon(x, x_n - u_\varepsilon)(u_\varepsilon - x_n)^- \, d\sigma(x)$$

$$= \int_{\Gamma \cap [u_\varepsilon \leqslant x_n]} \mathcal{B}_i^\varepsilon(x, x_n - u_\varepsilon)(x_n - u_\varepsilon) \, d\sigma(x) \geqslant 0\,.$$

Then we obtain from (2.19):

$$\int_{\Omega \cap [u_\varepsilon \leqslant x_n]} (\mathcal{A}(x, \nabla u_\varepsilon) - \mathcal{A}(x, \nabla x_n)).(\nabla u_\varepsilon - \nabla x_n)$$

$$+ \varepsilon(|u_\varepsilon|^{q-2}u_\varepsilon - |x_n|^{q-2}x_n).(u_\varepsilon - x_n) \, dx$$

$$+ \int_{\Gamma \cap [u_\varepsilon \leqslant x_n]} \varepsilon(|u_\varepsilon|^{q'-2}u_\varepsilon - |x_n|^{q'-2}x_n).(u_\varepsilon - x_n) \, d\sigma(x) \leqslant 0\,.$$

Using (1.5) and Lemma 2.3, we conclude from the last inequality that $u_\varepsilon \geqslant x_n$ a.e. in Ω. ■

Now we give an a priori estimate for ∇u_ε.

Lemma 2.5. *Under assumptions of Lemma* 2.4, *we have for any* $\varepsilon \in (0, \varepsilon_0)$

$$\int_\Omega |\nabla u_\varepsilon|^q \, dx \leqslant C, \tag{2.20}$$

where C *is a constant independent of* ε.

Proof. Note that $u_\varepsilon - \psi$ is a suitable test function for (P_ε). Then we get by (2.2)

$$\int_\Omega \varepsilon(|u_\varepsilon|^{q-2} u_\varepsilon - |x_n|^{q-2} x_n).(u_\varepsilon - \psi) + (\mathcal{a}(x, \nabla u_\varepsilon) - G_\varepsilon(u_\varepsilon)\mathcal{a}(x, e)).\nabla(u_\varepsilon - \psi)\, dx$$

$$+ \int_\Gamma \varepsilon(|u_\varepsilon|^{q'-2} u_\varepsilon - |x_n|^{q'-2} x_n).(u_\varepsilon - \psi)\, d\sigma(x)$$

$$= \int_\Gamma (\mathcal{B}_1^\varepsilon(x, \psi - u_\varepsilon) + \mathcal{B}_2^\varepsilon(x, \psi - u_\varepsilon)).(u_\varepsilon - \psi)\, d\sigma(x) \leqslant 0$$

which leads to

$$\int_\Omega \mathcal{a}(x,\nabla u_\varepsilon).\nabla u_\varepsilon dx \leqslant \int_\Omega \mathcal{a}(x,\nabla u_\varepsilon).\nabla \psi\, dx - \int_\Omega \varepsilon(|u_\varepsilon|^{q-2} u_\varepsilon - |x_n|^{q-2} x_n).(u_\varepsilon - \psi)\, dx \tag{2.21}$$

$$+ \int_\Omega G_\varepsilon(u_\varepsilon)\, \mathcal{a}(x, e).\nabla(u_\varepsilon - \psi)\, dx - \int_\Gamma \varepsilon(|u_\varepsilon|^{q'-2} u_\varepsilon - |x_n|^{q'-2} x_n).(u_\varepsilon - \psi)\, d\sigma(x).$$

By (1.4) and Hölder's inequality, we have:

$$\left| \int_\Omega \mathcal{a}(x, \nabla u_\varepsilon).\nabla\psi\, dx \right| \leqslant \beta \left(\int_\Omega |\nabla u_\varepsilon|^q dx \right)^{1/q'} . \left(\int_\Omega |\nabla \psi|^q dx \right)^{1/q}, \tag{2.22}$$

$$\left| \int_\Omega G_\varepsilon(u_\varepsilon)\, \mathcal{a}(x, e).\nabla(u_\varepsilon - \psi)\, dx \right| \leqslant \beta |\Omega|^{1/q'} \left(\int_\Omega |\nabla u_\varepsilon|^q dx \right)^{1/q} + \beta \int_\Omega |\nabla \psi|\, dx\,. \tag{2.23}$$

Using (1.3), (2.15), (2.21)-(2.23), we derive for some constant $c>0$

$$\int_\Omega |\nabla u_\varepsilon|^q \,\mathrm{d}x \leq \frac{c}{\alpha}\left(1+\left(\int_\Omega |\nabla u_\varepsilon|^q \,\mathrm{d}x\right)^{1/q}+\left(\int_\Omega |\nabla u_\varepsilon|^q \,\mathrm{d}x\right)^{1/q'}\right). \tag{2.24}$$

Hence we get (2.20) from (2.24) since $q, q'>1$. ■

Proof of Theorem 2.1. The proof will consist in passing to the limit, when ε goes to 0, in (P_ε). First remark that $G_\varepsilon(u_\varepsilon)$ is uniformly bounded ($0\leq G_\varepsilon(u_\varepsilon)\leq 1$, see (2.1)) and u_ε is bounded in $W^{1,q}(\Omega)$ by (2.15) and (2.20), thus one has for some constant C independent of ε

$$|G_\varepsilon(u_\varepsilon)|_{q'}\leq C, \qquad |u_\varepsilon|_{1,q}\leq C, \qquad |\mathcal{A}(x,\nabla u_\varepsilon)|_{q'}\leq C\,.$$

So, due to the Rellich theorem and the complete continuity of the trace operator, there exist a subsequence ε_k, $u\in W^{1,q}(\Omega)$, $g\in L^{q'}(\Omega)$ and $\mathcal{A}_0\in\mathbb{L}^{q'}(\Omega)$ such that

$$G_{\varepsilon_k}(u_{\varepsilon_k})\rightharpoonup g \quad \text{in} \quad L^{q'}(\Omega)\,, \tag{2.25}$$

$$u_{\varepsilon_k}\rightharpoonup u \quad \text{in} \quad W^{1,q}(\Omega), \qquad u_{\varepsilon_k}\to u \quad \text{in} \quad L^q(\Omega) \text{ and a.e. in } \Omega\,, \tag{2.26}$$

$$u_{\varepsilon_k}\to u \quad \text{in } L^q(\Gamma) \text{ and a.e. on } \Gamma\,. \tag{2.27}$$

$$\mathcal{A}(x,\nabla u_{\varepsilon_k})\rightharpoonup \mathcal{A}_0 \quad \text{in} \quad \mathbb{L}^{q'}(\Omega). \tag{2.28}$$

Moreover by (2.15), one has

$$\psi-H\leq\psi-u_{\varepsilon_k}\leq\psi-x_n=\varphi \quad \text{a.e. in } \Omega\,. \tag{2.29}$$

Using (2.29) and the monotonicity of $\mathcal{B}_2^{\varepsilon_k}(x,.)$, we get for a.e. $x\in\Gamma$ (see [7], Proposition 1.1 page 42)

$$|\mathcal{B}_2^{\varepsilon_k}(x,\psi-u_{\varepsilon_k})|\leq\max\left(|\mathcal{B}_2^0(x,\psi-H)|,\,|\mathcal{B}_2^0(x,\varphi)|\right)$$

from which we deduce by (1.13), for some constant C

$$|\mathcal{B}_2^{\varepsilon_k}(x,\psi-u_{\varepsilon_k})|_{q',\Gamma}\leq C\,.$$

Then we deduce that there exists $\gamma\in L^{q'}(\Gamma)$ such that

$$\mathcal{B}_2^{\varepsilon_k}(x,\psi-u_{\varepsilon_k})\rightharpoonup\gamma\in L^{q'}(\Gamma)\,. \tag{2.30}$$

We are going to show that (u,g,γ) is a solution of (P).

Since the set $K_1 = \{v \in W^{1,q}(\Omega)/v \geqslant x_n \text{ a.e. in } \Omega\}$ is weakly closed and $u_{\varepsilon_k} \in K_1$, u is in this set so that

$$u \geqslant x_n \quad \text{a.e. in } \Omega. \tag{2.31}$$

Next, the set $K_2 = \{v \in L^{q'}(\Omega)/0 \leqslant v \leqslant 1 \text{ a.e. in } \Omega\}$ is also weakly closed in $L^{q'}(\Omega)$, and thus

$$0 \leqslant g \leqslant 1 \quad \text{a.e. in } \Omega. \tag{2.32}$$

Since $G_{\varepsilon_k}(u_{\varepsilon_k}(x)) = 1 - H_{\varepsilon_k}(u_{\varepsilon_k}(x) - x_n)$ with $H_{\varepsilon_k}(s) = \min(s^+/\varepsilon_k, 1)$, one has

$$\int_\Omega G_{\varepsilon_k}(u_{\varepsilon_k}).(u_{\varepsilon_k} - x_n)\,\mathrm{d}x = \int_{\Omega \cap [0 \leqslant u_{\varepsilon_k}(x) - x_n \leqslant \varepsilon_k]} (1 - H_{\varepsilon_k}(u_{\varepsilon_k}(x) - x_n)).(u_{\varepsilon_k}(x) - x_n)\,\mathrm{d}x$$

and

$$0 \leqslant \int_\Omega G_{\varepsilon_k}(u_{\varepsilon_k}).(u_{\varepsilon_k} - x_n)\,\mathrm{d}x \leqslant \varepsilon_k\,|\Omega|,$$

which leads by (2.25)-(2.26) to

$$0 = \lim_{k \to +\infty} \int_\Omega G_{\varepsilon_k}(u_{\varepsilon_k}).(u_{\varepsilon_k} - x_n)\,\mathrm{d}x = \int_\Omega g.(u - x_n)\,\mathrm{d}x.$$

So by (2.31)-(2.32), we get

$$g.(u - x_n) = 0 \quad \text{a.e. in } \Omega. \tag{2.33}$$

Now since we have for a.e. $x \in \Gamma$ such that

$$\psi = x_n, \qquad \mathcal{B}_2^{\varepsilon_k}(x, \psi - u_{\varepsilon_k}) = \mathcal{B}_2^{\varepsilon_k}(x, x_n - u_{\varepsilon_k}) \leqslant 0,$$

we deduce that

$$\gamma(x) \leqslant 0 \qquad \text{for a.e. } x \in \Gamma \text{ such that } \psi = x_n. \tag{2.34}$$

Moreover one has $\psi - u_{\varepsilon_k} \to \psi - u$ in $L^q(\Gamma)$ and $\mathcal{B}_2^{\varepsilon_k}(x, \psi - u_{\varepsilon_k}) \rightharpoonup \gamma$ in $L^{q'}(\Gamma)$. Then (see [**7**], Lemma 1.3, page 42) we have

$$\gamma(x) \in \mathcal{B}_2(x, \psi(x) - u(x)) \qquad \text{for a.e. } x \in \Gamma. \tag{2.35}$$

Note that since $\psi - u_{\varepsilon_k}$ is a test function for (P_{ε_k}), $\mathcal{B}_1^{\varepsilon_k}(x, .)$ is nondecreasing and due to (2.15) and (2.20), we derive for some nonnegative constant C

$$0 \leqslant \int_{\Gamma} \mathcal{B}_1^{\varepsilon_k}(x, \psi - u_{\varepsilon_k}).(\psi - u_{\varepsilon_k})\, d\sigma(x) \leqslant C$$

which can be written since for a.e. $x \in \Gamma$, $\forall u \in D(\mathcal{B})$: $\mathcal{B}_1^{\varepsilon_k}(x, u) = (1/\varepsilon_k)((u-b)^+ - (a-u)^+)$, with the convention that if $a = -\infty$ (resp. $b = +\infty$), one has $(a-u)^+ = 0$ (resp. $(u-b)^+ = 0$)

$$0 \leqslant \int_{\Gamma} (\min(\psi - u_{\varepsilon_k} - a, 0) + \max(\psi - u_{\varepsilon_k} - b, 0)).(\psi - u_{\varepsilon_k})\, d\sigma(x) \leqslant \varepsilon_k C$$

and then by letting $k \to +\infty$, we obtain by (2.27), (2.29) and the Lebesgue theorem

$$\int_{\Gamma} (\min(\psi - u - a, 0) + \max(\psi - u - b, 0)).(\psi - u)\, d\sigma(x) = 0\,.$$

Since $a \leqslant 0 \leqslant b$ a.e. in Γ, one has

$$\min(\psi - u - a, 0)(\psi - u) \geqslant 0 \quad \text{and} \quad \max(\psi - u - b, 0).(\psi - u) \geqslant 0 \quad \text{a.e. on } \Gamma$$

which leads to

$$a \leqslant \psi - u \leqslant b \qquad \text{a.e. on } \Gamma.$$

Hence we get

$$(\psi - u)(x) \in D(\mathcal{B}(x, .)) \qquad \text{for a.e. } x \in \Gamma.$$

Thus (P) i), ii) and iii) follow. Let us prove (P) iv). First note that any element of $W^{1,q}(\Omega)$ is a test function of (P_{ε_k}). Let $\xi \in \mathbb{K}$ and note that

$$\mathcal{B}_1^{\varepsilon_k}(x, \psi - u_{\varepsilon_k}).(\xi - u_{\varepsilon_k}) \geqslant 0 \qquad \text{a.e. in } \Gamma. \tag{2.36}$$

Indeed one has

$$\mathcal{B}_1^{\varepsilon_k}(x, \psi - u_{\varepsilon_k}).(\xi - u_{\varepsilon_k})$$

$$= \frac{1}{\varepsilon_k}(\psi - u_{\varepsilon_k} - b)^+ .(\xi - u_{\varepsilon_k}) - \frac{1}{\varepsilon_k}(a - (\psi - u_{\varepsilon_k}))^+ .(\xi - u_{\varepsilon_k})\,.$$

Since $\psi - \xi \geqslant a$ a.e. in Γ, then $\xi - u_{\varepsilon_k} \leqslant \psi - u_{\varepsilon_k} - a$ and $-(a - (\psi - u_{\varepsilon_k}))^+ .(\xi - u_{\varepsilon_k}) \geqslant (a - (\psi - u_{\varepsilon_k}))^+ .(a - (\psi - u_{\varepsilon_k})) \geqslant 0$ a.e. in Γ.

Let us distinguish two cases

– $\underline{\psi > x_n}$: By definition $c = b$ and $\psi - \xi \leqslant b$. So $\xi \geqslant \psi - b$ which leads to

$$(\psi - u_{\varepsilon_k} - b)^+ .(\xi - u_{\varepsilon_k}) \geqslant (\psi - u_{\varepsilon_k} - b)^+ .(\psi - u_{\varepsilon_k} - b) \geqslant 0 .$$

– $\underline{\psi = x_n}$: In this case $(\psi - u_{\varepsilon_k} - b)^+ = (x_n - u_{\varepsilon_k} - b)^+ = 0$ since $u_{\varepsilon_k} - x_n \geqslant 0$ and $b \geqslant 0$.

It follows then from (2.36)

$$\begin{aligned}(2.37)\qquad & \int_\Omega \big(\mathcal{A}(x, \nabla u_{\varepsilon_k}) - G_{\varepsilon_k}(u_{\varepsilon_k})\, \mathcal{A}(x, e)\big) . \nabla(\xi - u_{\varepsilon_k}) \\ & \qquad + \varepsilon_k (|u_{\varepsilon_k}|^{q-2} u_{\varepsilon_k} - |x_n|^{q-2} x_n) . (\xi - u_{\varepsilon_k})\, dx \\ & \qquad + \int_\Gamma \varepsilon_k (|u_{\varepsilon_k}|^{q'-2} u_{\varepsilon_k} - |x_n|^{q'-2} x_n) . (\xi - u_{\varepsilon_k})\, d\sigma(x) \\ & \qquad \geqslant \int_\Gamma \mathcal{B}_2^{\varepsilon_k}(x, \psi - u_{\varepsilon_k}) . (\xi - u_{\varepsilon_k})\, d\sigma(x) .\end{aligned}$$

To pass to the limit, we will need the following lemma

Lemma 2.6. *We have*

$$(2.38)\qquad \int_\Omega \mathcal{A}(x, \nabla u) . \nabla \xi\, dx = \int_\Omega \mathcal{A}_0(x) . \nabla \xi\, dx , \qquad \forall \xi \in W^{1,q}(\Omega) .$$

Proof. Since $u \in \mathbb{K}$, we deduce from (2.37) by taking $\xi = u$

$$\begin{aligned}(2.39)\qquad \int_\Omega \mathcal{A}(x, \nabla u_{\varepsilon_k}) . \nabla u_{\varepsilon_k}\, dx & \leqslant \int_\Omega \mathcal{A}(x, \nabla u_{\varepsilon_k}) . \nabla u\, dx \\ & \quad - \int_\Omega \varepsilon_k (|u_{\varepsilon_k}|^{q-2} u_{\varepsilon_k} - |x_n|^{q-2} x_n) . (u_{\varepsilon_k} - u)\, dx \\ & \quad - \int_\Gamma \varepsilon_k (|u_{\varepsilon_k}|^{q'-2} u_{\varepsilon_k} - |x_n|^{q'-2} x_n) . (u_{\varepsilon_k} - u)\, d\sigma(x) \\ & \quad + \int_\Omega G_{\varepsilon_k}(u_{\varepsilon_k})\, \mathcal{A}(x, e) . \nabla(u_{\varepsilon_k} - u)\, dx \\ & \quad + \int_\Gamma \mathcal{B}_2^{\varepsilon_k}(x, \psi - u_{\varepsilon_k}) . (u_{\varepsilon_k} - u)\, d\sigma(x) .\end{aligned}$$

By (2.28) we have

$$\lim_{k\to+\infty}\int_\Omega \mathcal{A}(x,\nabla u_{\varepsilon_k}).\nabla u\,dx=\int_\Omega \mathcal{A}_0.\nabla u\,dx. \tag{2.40}$$

According to (2.26)-(2.27) and since u_{ε_k} is bounded uniformly, the second and third terms in the right hand side of (2.39) converge to 0 when $k\to+\infty$. From (2.27) and (2.30), we have

$$\lim_{k\to+\infty}\int_\Gamma \mathcal{B}_2^{\varepsilon_k}(x,\psi-u_{\varepsilon_k}).(u_{\varepsilon_k}-u)\,d\sigma(x)=0\,. \tag{2.41}$$

Now one can write

$$\int_\Omega G_{\varepsilon_k}(u_{\varepsilon_k})\,\mathcal{A}(x,e).\nabla(u_{\varepsilon_k}-u)\,dx \tag{2.42}$$

$$=\int_\Omega G_{\varepsilon_k}(u_{\varepsilon_k})\,\mathcal{A}(x,e).\nabla(u_{\varepsilon_k}-x_n)\,dx-\int_\Omega G_{\varepsilon_k}(u_{\varepsilon_k})\,\mathcal{A}(x,e).\nabla(u-x_n)\,dx\,.$$

By (2.25) and (2.33), the second term in the right hand side of (2.42) converges to 0 when $k\to+\infty$. For the first term, note that

$$\int_\Omega G_{\varepsilon_k}(u_{\varepsilon_k})\,\mathcal{A}(x,e).\nabla(u_{\varepsilon_k}-x_n)\,dx=\int_\Omega \mathcal{A}(x,e).\nabla v_k\,dx \tag{2.43}$$

with $v_k=\int_0^{u_{\varepsilon_k}-x_n}(1-H_{\varepsilon_k}(s))\,ds$. Moreover since $|v_k(x)|\leqslant\varepsilon_k$ for a.e. $x\in\Omega$ and $|v_k|_{1,q}\leqslant C$ for some constant $C>0$, it is not difficult to see that $v_k\rightharpoonup 0$ weakly in $W^{1,q}(\Omega)$ and then we obtain from (2.42)-(2.43)

$$\lim_{k\to+\infty}\int_\Omega G_{\varepsilon_k}(u_{\varepsilon_k})\,\mathcal{A}(x,e).\nabla(u_{\varepsilon_k}-u)\,dx=0\,. \tag{2.44}$$

Combining (2.39)-(2.41) and (2.44), we conclude that

$$\overline{\lim}\int_\Omega \mathcal{A}(x,\nabla u_{\varepsilon_k}).\nabla u_{\varepsilon_k}\,dx\leqslant\int_\Omega \mathcal{A}_0(x).\nabla u\,dx\,. \tag{2.45}$$

Let now $v \in W^{1,q}(\Omega)$. By (1.5), we have

$$\int_\Omega (\mathcal{A}(x, \nabla u_{\varepsilon_k}) - \mathcal{A}(x, \nabla v)) . \nabla(u_{\varepsilon_k} - v) \, dx \geq 0 , \qquad \forall k \in \mathbb{N}$$

and

$$\int_\Omega \mathcal{A}(x, \nabla u_{\varepsilon_k}) . \nabla u_{\varepsilon_k} dx - \int_\Omega \mathcal{A}(x, \nabla u_{\varepsilon_k}) . \nabla v \, dx - \int_\Omega \mathcal{A}(x, \nabla v) . \nabla(u_{\varepsilon_k} - v) \, dx \geq 0 , \qquad \forall k \in \mathbb{N} . \tag{2.46}$$

Passing to the limsup in (2.46) and taking into account (2.26), (2.28) and (2.45), we get

$$\int_\Omega \mathcal{A}_0(x) . \nabla u dx - \int_\Omega \mathcal{A}_0(x) . \nabla v \, dx - \int_\Omega \mathcal{A}(x, \nabla v) . \nabla(u - v) \, dx \geq 0$$

or

$$\int_\Omega (\mathcal{A}_0(x) - \mathcal{A}(x, \nabla v)) . \nabla(u - v) \, dx \geq 0. \tag{2.47}$$

If we choose $v = u \pm \lambda\xi$ with $\xi \in W^{1,q}(\Omega)$ and $\lambda \in [0, 1]$, in (2.47), we obtain by letting λ go to 0 and using (1.2), (1.4) and the Lebesgue theorem

$$\int_\Omega (\mathcal{A}_0(x) - \mathcal{A}(x, \nabla u)) . \nabla \xi \, dx = 0 .$$

Thus we have proved (2.38). ■

Let us now finish the proof of Theorem 2.1. Consider $\xi = u$ in (2.38), we get

$$\int_\Omega \mathcal{A}(x, \nabla u) . \nabla u \, dx = \int_\Omega \mathcal{A}_0(x) . \nabla u \, dx . \tag{2.48}$$

Using (2.45) and (2.48), we obtain

$$\overline{\lim} \int_\Omega \mathcal{A}(x, \nabla u_{\varepsilon_k}) . \nabla u_{\varepsilon_k} dx \leq \int_\Omega \mathcal{A}(x, \nabla u) . \nabla u dx. \tag{2.49}$$

Now we have

$$\int_\Omega \big(\mathcal{A}(x, \nabla u_{\varepsilon_k}) - \mathcal{A}(x, \nabla u)\big).\nabla(u_{\varepsilon_k} - u)\,\mathrm{d}x = \int_\Omega \mathcal{A}(x, \nabla u_{\varepsilon_k}).\nabla u_{\varepsilon_k}\,\mathrm{d}x - \int_\Omega \mathcal{A}(x, \nabla u).\nabla u_{\varepsilon_k}\,\mathrm{d}x - \int_\Omega \big(\mathcal{A}(x, \nabla u_{\varepsilon_k}) - \mathcal{A}(x, \nabla u)\big).\nabla u\,\mathrm{d}x\,. \tag{2.50}$$

Combining (2.26), (2.28), (2.48)-(2.50) and the monotonicity of $\mathcal{A}(x, .)$, we get

$$\lim_{k\to+\infty} \int_\Omega \mathcal{A}(x, \nabla u_{\varepsilon_k}).\nabla u_{\varepsilon_k}\,\mathrm{d}x = \int_\Omega \mathcal{A}(x, \nabla u).\nabla u\,\mathrm{d}x\,. \tag{2.51}$$

Letting k go to $+\infty$ in (2.37) and using (2.25)-(2.28), (2.30), (2.38), (2.44), (2.51) and the fact that u_{ε_k} is uniformly bounded, we get

$$\int_\Omega \big(\mathcal{A}(x, \nabla u) - g\mathcal{A}(x, e)\big).\nabla(\xi - u)\,\mathrm{d}x \geqslant \int_\Gamma \gamma.(\xi - u)\,\mathrm{d}\sigma(x)\,.$$

This achieves the proof of Theorem 2.1. ■

2.2 - *Some properties of the solutions*

Proposition 2.7. *Let (u, g, γ) be a solution of (P). Then we have*

$$0 \leqslant u - x_n \leqslant c + h - x_n \qquad \text{a.e. in } \Omega \tag{2.52}$$

where c is some nonnegative constant and h is such that

$$h > \max\big(\max\{x_n, (x', x_n) \in \overline{\Omega}\}, \max\{\psi(x), x \in \Gamma\}\big)\,. \tag{2.53}$$

Proof. i) Since $u - (u-h)^+$ is a suitable test function for (P), we have

$$\int_\Omega \big(\mathcal{A}(x, \nabla u) - g\mathcal{A}(x, e)\big).\nabla(u-h)^+\,\mathrm{d}x \leqslant \int_\Gamma \gamma(u-h)^+\,\mathrm{d}\sigma(x)\,. \tag{2.54}$$

Note that by (P) ii) and (2.53) one has $g\mathcal{A}(x, e).\nabla(u-h)^+ = 0$ a.e. in Ω and by the monotonicity of $\mathcal{B}_2(x, .)$ and (2.53) $\gamma(u-h)^+ \leqslant 0$ a.e. in Γ. Then we deduce from (1.3) and (2.54)

$$\int_\Omega \alpha |\nabla(u-h)^+|^q\,\mathrm{d}x \leqslant 0$$

which leads to $\nabla(u-h)^+ = 0$ a.e. in Ω. Thus $(u-h)^+ = c$ for some nonnegative constant c and the result follows. ■

Proposition 2.8. *Let* (u, g, γ) *be a solution of* (P). *Then we have in the distributional sense*

$$\operatorname{div}\big(\mathcal{A}(x, \nabla u) - g\mathcal{A}(x, e)\big) = 0 . \tag{2.55}$$

Proof. Let $\xi \in \mathcal{D}(\Omega)$. Taking $\pm\xi + u$ as a test function for (P), we get (2.55). ■

Remark 2.9. i) We deduce from (2.55) (see [20, 31]) that $u \in C^{0,\alpha}_{\mathrm{loc}}(\Omega)$ for some $\alpha \in (0, 1)$ and then $[u > x_n]$ is an open set.

ii) We also deduce from (2.55) and (P) ii) that $\operatorname{div}\big(\mathcal{A}(x, \nabla u)\big) = 0$ in $\mathcal{D}'([u > x_n])$ i.e. u is $\mathcal{A}$-harmonic in $[u > x_n]$. So if $\mathcal{A}$ is sufficiently smooth (for example if $\mathcal{A}(x, \zeta) = |\zeta|^{q-2}\zeta$ with $q > 1$), then (see [19, 25]) $u \in C^{1,\gamma}_{\mathrm{loc}}([u > x_n])$ for some $\gamma \in (0, 1)$.

3 - The case of unbounded domains

Let Ω be an unbounded domain in $\mathbb{R}^n$ $(n \geq 2)$ whose boundary Γ is locally Lipschitz. Ω represents an unbounded porous medium. Assuming that the flow in Ω has reached a steady state, we look for the pressure p of the fluid and the saturated region S of Ω. We suppose that $\Omega \subset \mathbb{R}^{n-1} \times (-\infty, H)$, $H \in \mathbb{R}$.

As in the bounded case and with the same notations, we have the following strong formulation

$$\begin{cases} \vec{v} = -\mathcal{A}(x, \nabla u) & \text{in } S , \\ \operatorname{div}(\vec{v}) = 0 & \text{in } S , \\ u > x_n \quad \text{in } S \quad \text{and} \quad u = x_n & \text{in } S^c, \\ -\vec{v}.\nu \in \mathcal{B}(x, \psi - u) & \text{on } \Gamma , \end{cases} \tag{3.1}$$

where $\psi = \varphi + x_n$ and φ is a nonnegative Lipschitz continuous function in $\overline{\Omega}$ representing the pressure on Γ. We assume that we have

$$\exists H' \in \mathbb{R} \quad \text{such that} \quad \max_{x \in \overline{\Omega}} \psi(x) \leq H' , \tag{3.2}$$

$\mathcal{A}\colon \Omega \times \mathbb{R}^n \to \mathbb{R}^n$ is a mapping that satisfies (1.2)-(1.5). $\mathcal{B}$ is a multivalued function that satisfies (1.9)-(1.12). Moreover, we assume that

$$(3.3) \qquad \exists H_1 > H, H' \quad \text{such that} \quad \mathcal{B}_2^0(.\,, \psi - H_1),\ \mathcal{B}_2^0(.\,, \varphi) \in L^{q'}_{\mathrm{loc}}(\Gamma).$$

Note that (3.3) is satisfied for example if one has

$$\exists R_0 > 0, \qquad \forall R \geq R_0, \qquad \exists C_R \text{ such that } \mathcal{B}_2(x, s) \subset (-C_R, C_R),$$

$$\forall s \in (-R, R), \quad \text{for a.e. } x \in \Gamma.$$

Now set for a.e. $x \in \Gamma$: $c(x) = b(x)$ if $\varphi(x) > 0$ and $c(x) = +\infty$ if $\varphi(x) = 0$. Then the weak unified formulation is the following:

$$(P_\infty)\quad \begin{cases} \text{Find } (u, g, \gamma) \in W^{1,q}_{\mathrm{loc}}(\Omega) \times L^\infty(\Omega) \times L^{q'}_{\mathrm{loc}}(\Gamma), \text{ such that} \\ \text{(i)} \quad \psi(x) - u(x) \in D(\mathcal{B}(x, .)) \quad \text{for a.e. } x \in \Gamma, \\ \text{(ii)} \quad u \geq x_n, \quad 0 \leq g \leq 1, \quad g.(u - x_n) = 0 \quad \text{a.e. in } \Omega, \\ \text{(iii)} \quad \gamma(x) \in \mathcal{B}_2(x, \psi(x) - u(x)) \quad \text{for a.e. } x \in \Gamma \\ \qquad\qquad \text{and } \gamma(x) \leq 0 \text{ for a.e. } x \in \Gamma \text{ such that } \psi(x) = x_n, \\ \text{(iv)} \quad \displaystyle\int_\Omega (\mathcal{A}(x, \nabla u) - g\mathcal{A}(x, e)).\nabla(\xi - u)\, dx \geq \int_\Gamma \gamma.(\xi - u)\, d\sigma(x), \\ \forall \xi \in \mathbb{K}(u) = \{\xi \in W^{1,q}_{\mathrm{loc}}(\Omega)/\mathrm{supp}(\xi - u) \text{ is bounded} \\ \qquad\qquad \text{and } a(x) \leq \psi(x) - \xi(x) \leq c(x) \text{ for a.e. } x \in \Gamma\}, \end{cases}$$

where supp ξ denotes the support of the function ξ.

In [**22**] G. Gilardi and D. Kröner considered the problem of an unbounded dam with Linear Darcy's law and Dirichlet boundary conditions on the bottoms of the reservoirs. They obtained a result of existence of a solution by regularization. In [**14**] we imposed a leaky boundary condition and we obtained a solution as a monotone limit of a sequence of solutions for bounded subdomains. Here we establish an existence of a solution as a limit of a sequence of solutions for bounded subdomains.

3.1 - *Existence of a solution of* (P_r)

In this paragraph, we introduce an auxilliary problem (P_r) on a truncated domain and we establish an existence of a solution of this problem.

Let $r > 1$ and let $\zeta_r \in \mathcal{D}(\mathbb{R}^n)$ be such that for some positive constant m

$$0 \leqslant \zeta_r \leqslant 1, \qquad |\nabla \zeta_r| \leqslant m, \qquad \zeta_r = 1 \text{ in } B(0, r-1),$$

$$\zeta_r = 0 \text{ in } \mathbb{R}^n \setminus B(0, r), \qquad \zeta_r \leqslant \zeta_{r'} \qquad \forall r \leqslant r'.$$

Set $\Omega_r = \Omega \cap B(0, r)$, $\Gamma_r = \partial\Omega_r \cap \partial\Omega$, $\Gamma'_r = \partial\Omega_r \cap \Omega$, $\varphi_r = \varphi\zeta_r$ and $\psi_r = \varphi_r + x_n$, where $B(0, r)$ is the ball of $\mathbb{R}^n$ of center 0 and radius r.

Let us define $\overline{\mathcal{B}}$ for $r > 0$, by

$$\overline{\mathcal{B}}(x, .) = \mathcal{B}(x, .) \quad \text{for a.e. } x \in \Gamma_r,$$
$$\overline{\mathcal{B}}(x, .) = \{0\} \times \mathbb{R} \quad \text{for a.e. } x \in \Gamma'_r,$$

then it is clear that $\overline{\mathcal{B}}$ satisfies (1.9)-(1.12) on $\partial\Omega_r$. In particular for a.e. $x \in \Gamma'_r$

$$\overline{\mathcal{B}}_1(x, .) = \overline{\mathcal{B}}(x, .) = \{0\} \times \mathbb{R} \quad \text{and} \quad \overline{\mathcal{B}}_2(x, .) = \mathbb{R} \times \{0\}.$$

Thus by (3.3), $\overline{\mathcal{B}}$ satisfies also (1.13). For a.e. $x \in \partial\Omega_r$, we set:

$$c_r(x) = b(x) \quad \text{if } \varphi_r(x) > 0 \quad \text{and} \quad c_r(x) = +\infty \quad \text{if } \varphi_r(x) = 0.$$

Using the results of the previous section, we know that there exists a solution (u_r, g_r, γ_r) for the following problem

$$(\overline{P}_r) \quad \begin{cases} \text{Find } (u_r, g_r, \gamma_r) \in W^{1,q}(\Omega_r) \times L^\infty(\Omega_r) \times L^{q'}(\partial\Omega_r), \text{ such that} \\ \text{(i)} \quad \psi_r(x) - u_r(x) \in D(\overline{\mathcal{B}}(x, .)) \quad \text{for a.e. } x \in \partial\Omega_r, \\ \text{(ii)} \quad u_r \geqslant x_n, \quad 0 \leqslant g_r \leqslant 1, \quad g_r.(u_r - x_n) = 0 \quad \text{a.e. in } \Omega_r, \\ \text{(iii)} \quad \gamma_r(x) \in \overline{\mathcal{B}}_2(x, \psi_r(x) - u_r(x)) \quad \text{for a.e. } x \in \partial\Omega_r, \\ \qquad\qquad \text{and } \gamma_r(x) \leqslant 0 \text{ for a.e. } x \in \partial\Omega_r \text{ such that } \psi_r(x) = x_n, \\ \text{(iv)} \quad \displaystyle\int_{\Omega_r} (\mathcal{A}(x, \nabla u_r) - g_r \mathcal{A}(x, e)).\nabla(\xi - u_r)\, dx \geqslant \int_{\partial\Omega_r} \gamma_r.(\xi - u_r)\, d\sigma(x), \\ \forall \xi \in \mathbb{K}_r = \{\xi \in W^{1,q}(\Omega_r) / a(x) \leqslant \psi_r(x) - \xi(x) \leqslant c_r(x) \text{ for a.e. } x \in \partial\Omega_r\}, \end{cases}$$

as a limit when $\varepsilon \to 0$ of the solution of the following approximated problem

$$(\overline{P}_{\varepsilon,r}) \quad \begin{cases} \text{Find } u_{\varepsilon,r} \in V_r = \{v \in W^{1,q}(\Omega_r)/v_{|\partial\Omega_r} \in L^{q'}(\partial\Omega_r)\} \text{ such that} \\ \displaystyle\int_{\Omega_r} \varepsilon(|u_{\varepsilon,r}|^{q-2}u_{\varepsilon,r} - |x_n|^{q-2}x_n).\xi + (\mathcal{A}(x, \nabla u_{\varepsilon,r}) - G_\varepsilon(u_{\varepsilon,r})\mathcal{A}(x,e)).\nabla\xi\, dx \\ \qquad + \displaystyle\int_{\partial\Omega_r} \varepsilon(|u_{\varepsilon,r}|^{q'-2}u_{\varepsilon,r} - |x_n|^{q'-2}x_n).\xi\, d\sigma(x) \\ \qquad = \displaystyle\int_{\partial\Omega_r} (\overline{\mathcal{B}}_1^\varepsilon(x, \psi_r - u_{\varepsilon,r}) + \overline{\mathcal{B}}_2^\varepsilon(x, \psi_r - u_{\varepsilon,r})).\xi\, d\sigma(x), \qquad \forall \xi \in V_r \end{cases}$$

which satisfies for ε small enough

$$x_n \leqslant u_{\varepsilon,r} \leqslant H_1 \qquad \text{a.e. in } \Omega_r. \tag{3.4}$$

In particular

$$\overline{\mathcal{B}}_i^\varepsilon(x,.) = \mathcal{B}_i^\varepsilon(x,.) \qquad \text{for a.e. } x \in \Gamma_r, \qquad i = 1, 2,$$

$$\overline{\mathcal{B}}_1^\varepsilon(x,u) = \frac{u}{\varepsilon}, \qquad \overline{\mathcal{B}}_2^\varepsilon(x,u) = 0 \qquad \text{for a.e. } x \in \Gamma'_r, \quad \forall u \in \mathbb{R}.$$

Remark 3.1. i) Since for a.e. $x \in \Gamma'_r$, we have $D(\overline{\mathcal{B}}(x,.)) = \{0\}$, we deduce from $(\overline{P}_r)$ i) that $u_r = \psi_r = x_n$ on Γ'_r.

ii) Since $\varphi_r = 0$, $\psi_r = x_n$ and $c_r = +\infty$ on Γ'_r, the condition $a(x) \leqslant \psi_r(x) - \xi(x) \leqslant c_r(x)$ for a.e. $x \in \Gamma'_r$ is equivalent to $0 \leqslant x_n - \xi(x)$ for a.e. $x \in \Gamma'_r$ or $\xi \leqslant x_n$ on Γ'_r.

iii) For any $\xi \in \mathbb{K}_r$, we have

$$\int_{\Gamma'_r} \gamma_r.(\xi - u_r)\, d\sigma(x) = \int_{\Gamma'_r} \gamma_r.(\xi - x_n)\, d\sigma(x) \geqslant 0$$

since on Γ'_r, $\xi \leqslant x_n$ by ii) and $\gamma_r \leqslant 0$ by $(\overline{P}_r)$ iii).

iv) It follows then from i), ii) and iii) that any solution (u_r, g_r, γ_r) of $(\overline{P}_r)$ is such that $(u_r, g_r, \gamma_{r|\Gamma_r})$ is a solution of the following problem

$$(P_r)\quad\begin{cases} \text{Find } (u_r, g_r, \gamma_r) \in W^{1,q}(\Omega_r) \times L^\infty(\Omega_r) \times L^{q'}(\Gamma_r), \text{ such that} \\ \text{i)}\quad u_r = x_n \text{ on } \Gamma'_r \text{ and } \psi_r(x) - u_r(x) \in D(\mathcal{B}(x, .)) \text{ for a.e. } x \in \Gamma_r, \\ \text{ii)}\quad u_r \geq x_n,\ 0 \leq g_r \leq 1,\ g_r.(u_r - x_n) = 0 \text{ a.e. in } \Omega_r, \\ \text{iii)}\quad \gamma_r(x) \in \mathcal{B}_2(x, \psi_r(x) - u_r(x)) \text{ for a.e. } x \in \Gamma_r \\ \text{and } \gamma_r(x) \leq 0 \text{ for a.e. } x \in \Gamma_r \text{ such that } \psi_r(x) = x_n, \\ \text{iv)} \displaystyle\int_{\Omega_r} (\mathcal{A}(x, \nabla u_r) - g_r \mathcal{A}(x, e)).\nabla(\xi - u_r)\, dx \geq \int_{\Gamma_r} \gamma_r.(\xi - u_r)\, d\sigma(x), \\ \forall \xi \in \mathbb{K}_r = \{\xi \in W^{1,q}(\Omega_r)/a(x) \leq \psi_r(x) - \xi(x) \leq c_r(x) \\ \qquad\qquad \text{for a.e. } x \in \Gamma_r \text{ and } \xi(x) \leq x_n \text{ for a.e. } x \in \Gamma'_r\}. \end{cases}$$

Remark 3.2. i) In the remainder of this paper, we only consider solutions (u_r, g_r, γ_r) of (P_r) obtained as a limit when $\varepsilon \to 0$, of

$$(u_{\varepsilon, r}, G_\varepsilon(u_{\varepsilon, r}), \mathcal{B}_2^\varepsilon(x, \psi_r - u_{\varepsilon, r})).$$

ii) For any solution (u_r, g_r, γ_r) of (P_r), we shall extend respectively u_r and g_r into $\Omega \backslash \Omega_r$ by x_n and 1. We also extend γ_r into $\Gamma \backslash \Gamma_r$ by $\mathcal{B}_2^0(. , \varphi_r)$ and still denotes by u_r, g_r and γ_r these extensions.

Then we deduce from (3.4) that

$$x_n \leq u_r \leq H_1 \qquad \text{a.e. in } \Omega, \tag{3.5}$$

$$|\gamma_r(x)| \leq \max(|\mathcal{B}_2^0(x, \varphi_r)|, |\mathcal{B}_2^0(x, \psi_r - H_1)|) \qquad \text{for a.e. } x \in \Gamma. \tag{3.6}$$

3.2 - *Existence of a solution of* (P_∞)

We are now able to state our existence result.

Theorem 3.1. *Assume that* $\mathcal{A}$ *satisfies* (1.2)-(1.5), $\mathcal{B}$ *satisfies* (1.9)-(1.12) *and* (3.3). *Then there exists a solution to* (P_∞).

Proof. Let $\varrho > 0$ and let us first prove that for $r \gg \varrho$ we have $|u_r|_{1, q, \Omega_\varrho} \leq c(\varrho)$, where $c(\varrho)$ is a constant depending on ϱ only. Consider $\zeta_{\varrho+1}$ which we denote by ζ for simplicity. Since $\zeta^q(u_{\varepsilon, r} - \psi_r)$ is a suitable test function for the problem $(\overline{P}_{\varepsilon, r})$, we have by (2.2)

$$\int_{\Omega_r} \varepsilon(|u_{\varepsilon, r}|^{q-2} u_{\varepsilon, r} - |x_n|^{q-2} x_n) . \zeta^q (u_{\varepsilon, r} - \psi_r) \, dx$$

$$+ \int_{\Omega_r} (\mathcal{A}(x, \nabla u_{\varepsilon, r}) - G_\varepsilon(u_{\varepsilon, r}) \mathcal{A}(x, e)) . \nabla(\zeta^q (u_{\varepsilon, r} - \psi_r)) \, dx$$

$$+ \int_{\partial\Omega_r} \varepsilon(|u_{\varepsilon, r}|^{q'-2} u_{\varepsilon, r} - |x_n|^{q'-2} x_n) . \zeta^q (u_{\varepsilon, r} - \psi_r) \, d\sigma(x)$$

$$= \int_{\partial\Omega_r} (\overline{\mathcal{B}}_1^\varepsilon(x, \psi_r - u_{\varepsilon, r}) + \overline{\mathcal{B}}_2^\varepsilon(x, \psi_r - u_{\varepsilon, r})) . \zeta^q (u_{\varepsilon, r} - \psi_r) \, d\sigma(x) \leqslant 0 .$$

Taking into account the fact that $\zeta = 0$ on $\Gamma'_r = \partial\Omega_r \cap \Omega \subset \partial B(0, r)$, $\overline{\mathcal{B}}_i^\varepsilon = \mathcal{B}_i^\varepsilon$ on Γ_r, supp $\zeta \subset B(0, \varrho + 1)$, $\psi_r = \psi$ on $B(0, \varrho + 1)$ for $r \gg \varrho$, the above inequality becomes

$$\int_{\Omega_{\varrho+1}} \varepsilon(|u_{\varepsilon, r}|^{q-2} u_{\varepsilon, r} - |x_n|^{q-2} x_n) . \zeta^q (u_{\varepsilon, r} - \psi) \, dx \tag{3.7}$$

$$+ \int_{\Omega_{\varrho+1}} (\mathcal{A}(x, \nabla u_{\varepsilon, r}) - G_\varepsilon(u_{\varepsilon, r}) \mathcal{A}(x, e)) . \nabla(\zeta^q (u_{\varepsilon, r} - \psi)) \, dx$$

$$+ \int_{\Gamma_{\varrho+1}} \varepsilon(|u_{\varepsilon, r}|^{q'-2} u_{\varepsilon, r} - |x_n|^{q'-2} x_n) . \zeta^q (u_{\varepsilon, r} - \psi) \, d\sigma(x) \leqslant 0 .$$

Using (3.5), it is clear that the first and third integrals of (3.7) are bounded by a constant depending only on ϱ. This leads to

$$\int_{\Omega_{\varrho+1}} (\mathcal{A}(x, \nabla u_{\varepsilon, r}) - G_\varepsilon(u_{\varepsilon, r}) \mathcal{A}(x, e)) . \nabla(\zeta^q (u_{\varepsilon, r} - \psi)) \, dx \leqslant c_1(\varrho)$$

which can be written

$$\int_{\Omega_{\varrho+1}} \zeta^q \mathcal{A}(x, \nabla u_{\varepsilon, r}) . \nabla u_{\varepsilon, r} \, dx \leqslant c_1(\varrho) - \int_{\Omega_{\varrho+1}} q\zeta^{q-1} u_{\varepsilon, r} \mathcal{A}(x, \nabla u_{\varepsilon, r}) . \nabla\zeta \, dx \tag{3.8}$$

$$+ \int_{\Omega_{\varrho+1}} \zeta^q \mathcal{A}(x, \nabla u_{\varepsilon, r}) . \nabla\psi \, dx + \int_{\Omega_{\varrho+1}} q\zeta^{q-1} \psi \mathcal{A}(x, \nabla u_{\varepsilon, r}) . \nabla\zeta \, dx$$

$$+ \int_{\Omega_{\varrho+1}} \zeta^q G_\varepsilon(u_{\varepsilon,r})\, \mathcal{A}(x, e) . \nabla u_{\varepsilon,r} \, dx$$

$$+ \int_{\Omega_{\varrho+1}} q\zeta^{q-1}(u_{\varepsilon,r} - \psi) G_\varepsilon(u_{\varepsilon,r}) \mathcal{A}(x, e) . \nabla \zeta \, dx - \int_{\Omega_{\varrho+1}} \zeta^q G_\varepsilon(u_{\varepsilon,r})\, \mathcal{A}(x, e) . \nabla \psi \, dx .$$

Using (1.4), (3.5) and the Hölder inequality, we derive easily for some constants $c_i(\varrho)$

$$\cdot \ \left| \int_{\Omega_{\varrho+1}} q\zeta^{q-1} u_{\varepsilon,r} \mathcal{A}(x, \nabla u_{\varepsilon,r}) . \nabla \zeta \, dx \right| \leqslant q\beta \int_{\Omega_{\varrho+1}} |u_{\varepsilon,r}|\, |\zeta \nabla u_{\varepsilon,r}|^{q-1} . |\nabla \zeta| \, dx$$

$$\leqslant qm\beta \left(\int_{\Omega_{\varrho+1}} |u_{\varepsilon,r}|^q \, dx \right)^{1/q} \left(\int_{\Omega_{\varrho+1}} |\zeta \nabla u_{\varepsilon,r}|^q \, dx \right)^{1/q'} \leqslant c_2(\varrho) \left(\int_{\Omega_{\varrho+1}} |\zeta \nabla u_{\varepsilon,r}|^q \, dx \right)^{1/q'},$$

$$\cdot \ \left| \int_{\Omega_{\varrho+1}} \zeta^q \mathcal{A}(x, \nabla u_{\varepsilon,r}) . \nabla \psi \, dx \right| \leqslant \beta \int_{\Omega_{\varrho+1}} |\zeta \nabla u_{\varepsilon,r}|^{q-1} . |\nabla \psi| \, dx$$

$$\leqslant c_3(\varrho) \left(\int_{\Omega_{\varrho+1}} |\zeta \nabla u_{\varepsilon,r}|^q \, dx \right)^{1/q'},$$

$$\cdot \ \left| \int_{\Omega_{\varrho+1}} q\zeta^{q-1} \psi \mathcal{A}(x, \nabla u_{\varepsilon,r}) . \nabla \zeta \, dx \right| \leqslant qm\beta \int_{\Omega_{\varrho+1}} |\zeta \nabla u_{\varepsilon,r}|^{q-1} . |\psi| \, dx$$

$$\leqslant c_4(\varrho) \left(\int_{\Omega_{\varrho+1}} |\zeta \nabla u_{\varepsilon,r}|^q \, dx \right)^{1/q'},$$

$$\cdot \ \left| \int_{\Omega_{\varrho+1}} q\zeta^{q-1}(u_{\varepsilon,r} - \psi)\, G_\varepsilon(u_{\varepsilon,r})\, \mathcal{A}(x, e) . \nabla \zeta \, dx \right| \leqslant qm\beta \int_{\Omega_{\varrho+1}} (|u_{\varepsilon,r}| + |\psi|) \, dx \leqslant c_5(\varrho),$$

$$\cdot \ \left| \int_{\Omega_{\varrho+1}} \zeta^q G_\varepsilon(u_{\varepsilon,r})\, \mathcal{A}(x, e) . \nabla u_{\varepsilon,r} \, dx \right| \leqslant \beta \int_{\Omega_{\varrho+1}} |\zeta \nabla u_{\varepsilon,r}| \, dx \leqslant c_6(\varrho) \left(\int_{\Omega_{\varrho+1}} |\zeta \nabla u_{\varepsilon,r}|^q \, dx \right)^{1/q}$$

$$\cdot \ \left| \int_{\Omega_{\varrho+1}} \zeta^q G_\varepsilon(u_{\varepsilon,r})\, \mathcal{A}(x, e) . \nabla \psi \, dx \right| \leqslant \beta \int_{\Omega_{\varrho+1}} |\nabla \psi| \, dx \leqslant c_7(\varrho),$$

Moreover by (1.3), we have

$$\int_{\Omega_{\varrho+1}} |\zeta \nabla u_{\varepsilon,r}|^q \, dx = \int_{\Omega_{\varrho+1}} \zeta^q\, |\nabla u_{\varepsilon,r}|^q \, dx \leqslant \frac{1}{\alpha} \int_{\Omega_{\varrho+1}} \zeta^q \mathcal{A}(x, \nabla u_{\varepsilon,r}) . \nabla u_{\varepsilon,r} \, dx .$$

So we deduce from (3.8) and the above inequalities, for some constant c_ϱ

$$\int_{\Omega_{\varrho+1}} |\zeta \nabla u_{\varepsilon,r}|^q \,dx \leqslant c_\varrho\left(\left(\int_{\Omega_{\varrho+1}} |\zeta \nabla u_{\varepsilon,r}|^q \,dx\right)^{1/q'} + \left(\int_{\Omega_{\varrho+1}} |\zeta \nabla u_{\varepsilon,r}|^q \,dx\right)^{1/q} + 1\right).$$

From this follows, since $q, q' > 1$

$$\int_{\Omega_{\varrho+1}} |\zeta \nabla u_{\varepsilon,r}|^q \,dx \leqslant c_\varrho.$$

Hence for some constant also denoted by c_ϱ

$$\int_{\Omega_\varrho} |\nabla u_{\varepsilon,r}|^q \,dx = \int_{\Omega_\varrho} |\zeta \nabla u_{\varepsilon,r}|^q \,dx \leqslant \int_{\Omega_{\varrho+1}} |\zeta \nabla u_{\varepsilon,r}|^q \,dx \leqslant c_\varrho.$$

Now, since one has when $\varepsilon \to 0$

$$\nabla u_{\varepsilon,r} \rightharpoonup \nabla u_r \quad \text{in} \quad L^q(\Omega_\varrho),$$

we have

$$\left(\int_{\Omega_\varrho} |\nabla u_r|^q \,dx\right)^{1/q} \leqslant \liminf_{\varepsilon \to 0}\left(\int_{\Omega_\varrho} |\nabla u_{\varepsilon,r}|^q \,dx\right)^{1/q}$$

and

$$\int_{\Omega_\varrho} |\nabla u_r|^q \,dx \leqslant c_\varrho.$$

Since u_r is locally uniformly bounded, this implies:

$$|u_r|_{1,q,\Omega_\varrho} \leqslant c(\varrho). \tag{3.9}$$

Now, by the reflexivity of $W^{1,q}(\Omega_\varrho)$, $L^{q'}(\Omega_\varrho)$, $L^{q'}(\Gamma_\varrho)$ and Rellich's theorem, there exists a subsequence $(u^\varrho_{r_k}, g^\varrho_{r_k}, \gamma^\varrho_{r_k})$ such that:

$$u^\varrho_{r_k} \rightharpoonup u^\varrho \quad \text{weakly in } W^{1,q}(\Omega_\varrho), \qquad u^\varrho_{r_k} \to u^\varrho \text{ strongly in } L^q(\Omega_\varrho),$$

$$u^\varrho_{r_k} \to u^\varrho \quad \text{a.e. in } \Omega_\varrho,$$

$$\mathcal{a}(.\,, \nabla u^\varrho_{r_k}) \rightharpoonup \mathcal{a}^\varrho_0 \text{ weakly in } L^{q'}(\Omega_\varrho),$$

$$g^\varrho_{r_k} \rightharpoonup g^\varrho \quad \text{weakly in } L^{q'}(\Omega_\varrho),$$

$$u^{\varrho}_{r_k} \to u^{\varrho} \quad \text{strongly in } L^q(\Gamma_\varrho), \quad u^{\varrho}_{r_k} \to u^{\varrho} \text{ a.e. in } \Gamma_\varrho,$$

$$\gamma^{\varrho}_{r_k} \rightharpoonup \gamma^{\varrho} \quad \text{weakly in } L^{q'}(\Gamma_\varrho),$$

where $(u^\varrho, \mathcal{A}^\varrho_0, g^\varrho, \gamma^\varrho) \in W^{1,q}(\Omega_\varrho) \times \mathbb{L}^{q'}(\Omega_\varrho) \times L^{q'}(\Omega_\varrho) \times L^{q'}(\Gamma_\varrho)$.

By a diagonal process there exist a subsequence also denoted by $(u_{r_k}, g_{r_k}, \gamma_{r_k})$ and $(u, \mathcal{A}_0, g, \gamma) \in W^{1,q}_{\text{loc}}(\Omega) \times \mathbb{L}^{q'}_{\text{loc}}(\Omega) \times L^{q'}_{\text{loc}}(\Omega) \times L^{q'}_{\text{loc}}(\Gamma)$ such that:

$$(3.10) \quad \begin{cases} u_{r_k} \rightharpoonup u \quad \text{weakly in } W^{1,q}_{\text{loc}}(\Omega), \quad u_{r_k} \to u \text{ strongly in } L^q_{\text{loc}}(\Omega), \\ u_{r_k} \to u \quad \text{a.e. in } \Omega, \\ \mathcal{A}(.,\nabla u_{r_k}) \rightharpoonup \mathcal{A}_0 \quad \text{weakly in } \mathbb{L}^{q'}_{\text{loc}}(\Omega), \\ g_{r_k} \to g \quad \text{strongly in } L^{q'}_{\text{loc}}(\Omega), \\ g_{r_k} \rightharpoonup g \quad \text{weakly in } L^{q'}_{\text{loc}}(\Omega), \\ u_{r_k} \to u \quad \text{strongly in } L^q_{\text{loc}}(\Gamma), \quad u_{r_k} \to u \text{ a.e. in } \Gamma, \\ \gamma_{r_k} \rightharpoonup \gamma \quad \text{weakly in } L^{q'}_{\text{loc}}(\Gamma). \end{cases}$$

From (P_{r_k}) we have for $\varrho > 0$ fixed and $r_k \gg \varrho$ by taking into account the fact that $\psi_{r_k} = \psi$ in $B(0, \varrho)$

$$u_{r_k} \geq x_n, \quad g_{r_k}.(u_{r_k} - x_n) = 0, \quad 0 \leq g_{r_k} \leq 1 \text{ a.e. in } \Omega_\varrho,$$

$$\psi(x) - u_{r_k}(x) \in D(\mathcal{B}(x,.)) \quad \text{for a.e. } x \in \Gamma_\varrho,$$

$$\gamma_{r_k}(x) \in \mathcal{B}_2(x, \psi(x) - u_{r_k}(x)) \quad \text{for a.e. } x \in \Gamma_\varrho,$$

$$\gamma_{r_k}(x) \leq 0 \quad \text{for a.e. } x \in \Gamma_\varrho \text{ such that } \psi(x) = x_n,$$

from which we derive by (3.10) for any $\varrho > 0$

$$u \geq x_n, \quad g.(u - x_n) = 0, \quad 0 \leq g \leq 1 \text{ a.e. in } \Omega_\varrho,$$

$$\psi(x) - u(x) \in D(\mathcal{B}(x,.)) \quad \text{for a.e. } x \in \Gamma_\varrho,$$

$$\gamma(x) \in \mathcal{B}_2(x, \psi(x) - u(x)) \quad \text{for a.e. } x \in \Gamma_\varrho,$$

$$\gamma(x) \leq 0 \quad \text{for a.e. } x \in \Gamma_\varrho \text{ such that } \psi(x) = x_n,$$

and thus

$$
(3.11)\qquad \begin{cases} u \geqslant x_n, \quad g.(u - x_n) = 0, \qquad 0 \leqslant g \leqslant 1 \text{ a.e. in } \Omega\,, \\ \psi(x) - u(x) \in D(\mathcal{B}(x,\,.)) \qquad \text{for a.e. } \; x \in \Gamma, \\ \gamma(x) \in \mathcal{B}_2(x,\,\psi(x) - u(x)) \qquad \text{for a.e. } \; x \in \Gamma\,, \\ \gamma(x) \leqslant 0 \qquad \text{for a.e. } \; x \in \Gamma \text{ such that } \psi(x) = x_n. \end{cases}
$$

To pass to the limit, we shall need the following lemma:

Lemma 3.2. *We have*

$$
(3.12)\qquad \forall \varrho > 0, \int_{\Omega_\varrho} \mathcal{A}(x,\,\nabla u)\,.\nabla \xi \,\mathrm{d}x
$$

$$
= \int_{\Omega_\varrho} \mathcal{A}_0(x)\,.\nabla \xi \,\mathrm{d}x \;\; \forall \xi \in W^{1,\,q}(\Omega) \;\; \textit{such that} \;\; \operatorname{supp}(\xi) \subset \Omega_\varrho,
$$

$$
(3.13)\qquad \lim_{k \to +\infty} \int_\Omega \theta \mathcal{A}(x,\,\nabla u_{r_k})\,.\nabla u_{r_k} \,\mathrm{d}x = \int_\Omega \theta \mathcal{A}(x,\,\nabla u)\,.\nabla u \,\mathrm{d}x\,, \qquad \forall \theta \in \mathcal{D}(\mathbb{R}^n)\,, \qquad \theta \geqslant 0.
$$

Proof. Let $\varrho > 0$ and let $\theta \in \mathcal{D}(\mathbb{R}^n)$ such that $\theta = 0$ in $\mathbb{R}^n \setminus B(0,\,\varrho + 1)$ and $\theta \geqslant 0$. Without loss of generality, we can assume that $0 \leqslant \theta \leqslant 1$.

Note that for $r_k \gg \varrho$, $\zeta = \theta u + (1 - \theta)\, u_{r_k} = u_{r_k} - \theta(u_{r_k} - u)$ is a test function for (P_{r_k}). Indeed

$$
\begin{aligned}
\psi_{r_k} - \zeta &= \psi_{r_k} - u_{r_k} + \theta u_{r_k} - \theta u \\
&= \theta(\psi_{r_k} - u) + (1 - \theta)(\psi_{r_k} - u_{r_k}) \\
&= \theta(\psi - u) + (1 - \theta)(\psi_{r_k} - u_{r_k}) + \theta(\psi_{r_k} - \psi) \\
&= \theta(\psi - u) + (1 - \theta)(\psi_{r_k} - u_{r_k})
\end{aligned}
$$

since $\theta = 0$ in $\mathbb{R}^n \setminus B(0,\,\varrho + 1)$ and $\psi_{r_k} - \psi = 0$ in $B(0,\,\varrho + 1)$. Moreover for a.e. $x \in \Gamma_{r_k}$, we have

$$
\theta(x)\; a(x) \leqslant \theta(x)(\psi - u)(x) \leqslant \theta(x)\; c(x) = \theta(x)\; c_{r_k}(x)
$$

and

$$
(1 - \theta)(x)\; a(x) \leqslant (1 - \theta)(x)(\psi_{r_k} - u_{r_k})(x) \leqslant (1 - \theta)(x)\; c_{r_k}(x)
$$

so

$$a(x) \leqslant (\psi_{r_k} - \zeta)(x) \leqslant c_{r_k}(x)\,.$$

Then one has

$$\int_{\Omega_{\varrho+1}} (\mathcal{A}(x, \nabla u_{r_k}) - g_{r_k}\mathcal{A}(x, e)).\nabla(\theta(u_{r_k} - u))\,dx \leqslant \int_{\Gamma_{\varrho+1}} \gamma_{r_k}\theta(u_{r_k} - u)\,d\sigma(x)$$

which can be written

$$(3.14)\qquad \int_{\Omega_{\varrho+1}} \theta\mathcal{A}(x, \nabla u_{r_k}).\nabla u_{r_k}\,dx \leqslant \int_{\Omega_{\varrho+1}} \theta\mathcal{A}(x, \nabla u_{r_k}).\nabla u\,dx$$

$$- \int_{\Omega_{\varrho+1}} (u_{r_k} - u)\,\mathcal{A}(x, \nabla u_{r_k}).\nabla\theta\,dx + \int_{\Omega_{\varrho+1}} g_{r_k}\mathcal{A}(x, e).\nabla(\theta(u_{r_k} - u))\,dx$$

$$+ \int_{\Gamma_{\varrho+1}} \gamma_{r_k}\theta(u_{r_k} - u)\,d\sigma(x).$$

Note that

$$\int_{\Omega_{\varrho+1}} g_{r_k}\mathcal{A}(x, e).\nabla(\theta(u_{r_k} - u))\,dx = \int_{\Omega_{\varrho+1}} g_{r_k}\mathcal{A}(x, e).\nabla(\theta(u_{r_k} - x_n))\,dx$$

$$- \int_{\Omega_{\varrho+1}} g_{r_k}\mathcal{A}(x, e).\nabla(\theta(u - x_n))\,dx = - \int_{\Omega_{\varrho+1}} g_{r_k}\mathcal{A}(x, e).\nabla(\theta(u - x_n))\,dx$$

since $g_{r_k}(u_{r_k} - x_n) = 0$ a.e. in Ω_{r_k}. By (3.10), we have

$$\lim_{k\to+\infty} \int_{\Omega_{\varrho+1}} g_{r_k}\mathcal{A}(x, e).\nabla(\theta(u - x_n))\,dx = \int_{\Omega_{\varrho+1}} g\mathcal{A}(x, e).\nabla(\theta(u - x_n))\,dx = 0$$

since by (3.11) $g(u - x_n) = 0$ a.e. in Ω. So

$$(3.15)\qquad \lim_{k\to+\infty} \int_{\Omega_{\varrho+1}} g_{r_k}\mathcal{A}(x, e).\nabla(\theta(u_{r_k} - x_n))\,dx = 0\,.$$

Also by (3.10) and (3.14)-(3.15), we conclude that

$$(3.16)\qquad \overline{\lim} \int_{\Omega_{\varrho+1}} \theta\mathcal{A}(x, \nabla u_{r_k}).\nabla u_{r_k}\,dx \leqslant \int_{\Omega_{\varrho+1}} \theta\mathcal{A}_0(x).\nabla u\,dx\,.$$

Let now $v \in W^{1,q}_{\mathrm{loc}}(\Omega)$. By (1.5), we have

$$\int_{\Omega_{\varrho+1}} \theta\big(\mathcal{A}(x, \nabla u_{r_k}) - \mathcal{A}(x, \nabla v)\big).\nabla(u_{r_k} - v)\,dx \geq 0, \qquad \forall k \in \mathbb{N}$$

and

$$\int_{\Omega_{\varrho+1}} \theta\mathcal{A}(x, \nabla u_{r_k}).\nabla u_{r_k}\,dx - \int_{\Omega_{\varrho+1}} \theta\mathcal{A}(x, \nabla u_{r_k}).\nabla v\,dx - \int_{\Omega_{\varrho+1}} \theta\mathcal{A}(x, \nabla v).\nabla(u_{r_k} - v)\,dx \geq 0, \qquad \forall k \in \mathbb{N}. \tag{3.17}$$

Passing to the limsup in (3.17) and taking into account (3.10) and (3.16), we get

$$\int_{\Omega_{\varrho+1}} \theta\mathcal{A}_0(x).\nabla u\,dx - \int_{\Omega_{\varrho+1}} \theta\mathcal{A}_0(x).\nabla v\,dx - \int_{\Omega_{\varrho+1}} \theta\mathcal{A}(x, \nabla v).\nabla(u - v)\,dx \geq 0$$

or

$$\int_{\Omega_{\varrho+1}} \theta\big(\mathcal{A}_0(x) - \mathcal{A}(x, \nabla v)\big).\nabla(u - v)\,dx \geq 0. \tag{3.18}$$

If we choose $v = u \pm \lambda\xi$ with $\xi \in W^{1,q}_{\mathrm{loc}}(\Omega)$ and $\lambda \in [0, 1]$ in (3.18), we obtain by letting λ go to 0 and using (1.2), (1.4) and the Lebesgue theorem

$$\int_{\Omega_{\varrho+1}} \theta\big(\mathcal{A}_0(x) - \mathcal{A}(x, \nabla u)\big).\nabla\xi\,dx = 0. \tag{3.19}$$

If moreover, one takes θ such that $\theta = 1$ in $B(0, \varrho)$ and ξ as in Lemma 3.2, we obtain

$$\int_{\Omega_\varrho} \big(\mathcal{A}_0(x) - \mathcal{A}(x, \nabla u)\big).\nabla\xi\,dx = 0$$

which is (3.12).

Let us now prove (3.13). Take $\xi = u$ in (3.19), we get

$$\int_\Omega \theta\mathcal{A}(x, \nabla u).\nabla u\,dx = \int_\Omega \theta\mathcal{A}_0(x).\nabla u\,dx. \tag{3.20}$$

Using (3.16) and (3.20), we obtain

(3.21) $$\overline{\lim} \int_\Omega \theta \mathcal{A}(x, \nabla u_{r_k}) . \nabla u_{r_k} \, dx \leq \int_\Omega \theta \mathcal{A}(x, \nabla u) . \nabla u \, dx .$$

Now we have

(3.22) $$\int_\Omega \theta \mathcal{A}(x, \nabla u_{r_k}) . \nabla u_{r_k} \, dx = \int_\Omega \theta (\mathcal{A}(x, \nabla u_{r_k}) - \mathcal{A}(x, \nabla u)) . \nabla (u_{r_k} - u) \, dx$$
$$+ \int_\Omega \theta \mathcal{A}(x, \nabla u) . \nabla u_{r_k} \, dx + \int_\Omega \theta (\mathcal{A}(x, \nabla u_{r_k}) - \mathcal{A}(x, \nabla u)) . \nabla u \, dx .$$

Combining (3.10), (3.20)-(3.22) and the monotonicity of $\mathcal{A}(x, .)$, we get

$$\lim_{k \to +\infty} \int_\Omega \theta \mathcal{A}(x, \nabla u_{r_k}) . \nabla u_{r_k} \, dx = \int_\Omega \theta \mathcal{A}(x, \nabla u) . \nabla u \, dx . \qquad \blacksquare$$

Let us now complete the proof of Theorem 3.1. Choose $\xi \in \mathbb{K}(u)$. Then for some $\varrho > 0$ and for r_k large enough we have: $\operatorname{supp}(\xi - u) \subset \Omega_\varrho \subset \Omega_{r_k}$. Let $\theta \in \mathcal{D}(\mathbb{R}^n)$ such that

$$0 \leq \theta \leq 1, \qquad \theta = 1 \text{ in } B(0, \varrho), \quad \theta = 0 \text{ in } \mathbb{R}^n \setminus B(0, \varrho + 1).$$

Set $\zeta = \theta \xi + (1 - \theta) u_{r_k} = u_{r_k} + \theta(\xi - u_{r_k})$ and let us verify that $\zeta \in \mathbb{K}_{r_k}$. We have

$$\begin{aligned} \psi_{r_k} - \zeta &= \psi_{r_k} - u_{r_k} - \theta \xi + \theta u_{r_k} \\ &= \theta(\psi - \xi) + (1 - \theta)(\psi_{r_k} - u_{r_k}) + \theta(\psi_{r_k} - \psi) \\ &= \theta(\psi - \xi) + (1 - \theta)(\psi_{r_k} - u_{r_k}) \end{aligned}$$

since $\theta = 0$ in $\mathbb{R}^n \setminus B(0, \varrho + 1)$ and $\psi_{r_k} - \psi = 0$ in $B(0, \varrho + 1)$. Then we have

$$\int_{\Omega_{\varrho + 1}} (\mathcal{A}(x, \nabla u_{r_k}) - g_{r_k} \mathcal{A}(x, e)) . \nabla (\theta(\xi - u_{r_k})) \, dx \geq \int_{\Gamma_{\varrho + 1}} \gamma_{r_k} . \theta(\xi - u_{r_k}) \, d\sigma(x),$$

letting $k \to +\infty$, we get by (3.10) and Lemma 3.2

$$\int_{\Omega_{\varrho + 1}} (\mathcal{A}(x, \nabla u) - g \mathcal{A}(x, e)) . \nabla (\theta(\xi - u)) \, dx \geq \int_{\Gamma_{\varrho + 1}} \gamma . \theta(\xi - u) \, d\sigma(x).$$

But since $\theta = 1$ in Ω_ϱ and $\mathrm{supp}\,(\xi - u) \subset \Omega_\varrho$, we obtain

$$\int_{\Omega_\varrho} (\mathcal{a}(x, \nabla u) - g\mathcal{a}(x, e)).\nabla(\xi - u)\,\mathrm{d}x \geq \int_{\Gamma_\varrho} \gamma.(\xi - u)\,\mathrm{d}\sigma(x)$$

which is

$$\int_{\Omega} (\mathcal{a}(x, \nabla u) - g\mathcal{a}(x, e)).\nabla(\xi - u)\,\mathrm{d}x \geq \int_{\Gamma} \gamma.(\xi - u)\,\mathrm{d}\sigma(x)$$

and (u, g, γ) is a solution of (P_∞). ■

Remark 3.5. Since for any $\xi \in \mathcal{D}(\Omega)$, $u \pm \xi$ is a well test function for (P_∞), it is clear that the results of Proposition 2.8 and Remark 2.9 remain true. Hence $u \in C^{0,\alpha}_{\mathrm{loc}}(\Omega)$ for some $\alpha \in (0, 1)$ and the set $[u > x_n]$ is open. Moreover u is $\mathcal{a}$-harmonic in $[u > x_n]$ and if $\mathcal{a}$ is sufficiently smooth, we would have $u \in C^{1,\gamma}_{\mathrm{loc}}([u > x_n])$ for some $\gamma \in (0, 1)$.

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Abstract

In this paper we propose a unified formulation for the stationary dam problem that includes the cases of linear or nonlinear Darcy's laws and Dirichlet or leaky boundary conditions via the theory of maximal monotone graphs. We prove an existence of a solution both for bounded or unbounded domains.

* * *