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A remark on Brody's theorem on homogeneous complex manifolds (**)

1 - Introduction

In [1], it has been given the proof that on compact complex manifolds, the Kobayashi's hyperbolicity can be characterized by the non existence of non constant complex line. The following is the *standard* example of a non hyperbolic manifold, that has no complex line:

$$M = \{(z_1, z_2) \in C^2 \colon |z_1 z_2| < 1 \text{ and } |z_2| < 1\} - \{(z_1, 0) \colon |z_1| \leq 1\}.$$

In [13], J. Winkelmann proves, among other things, that Brody's theorem holds on complex manifold of type G/H, where G is a real, solvable Lie group and H is a closed subgroup.

In this brief note, we want to point out that Brody's theorem holds also in homogeneous manifolds G/H, with G real semisimple Lie group of the first category, which have a G-invariant complex structure [8].

We don't have any example of homogeneous manifolds, where Brody's theorem fails and, as far as we know, there is no proof of a Brody's theorem on homogeneous manifolds.

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2 - The anticanonical, the Kobayashi and the hyperbolic reductions

Let G be a connected real Lie group acting almost effectively and transitively as a group of holomorphic transformations on the complex manifold $X = G/H$. Let N be, as usual, the normalizer of H°, let J be the subgroup of N given by

$$J = \{k \in N \mid R(k): G/H^\circ \to G/H^\circ,\ gH^\circ \to gkH^\circ \text{ is holomorphic}\}$$

where G/H° is endowed with the complex structure induced by the covering $G/H^\circ \to G/H$.

Then J/H° is a complex Lie group and $G/H^\circ \to G/J$ is a holomorphic J/H°-principal fiber bundle, the fibering $G/H \to G/J$ is locally holomorphically trivial and it is called the *anticanonical reduction.*

Moreover, if Z denotes the centralizer of H°, the identity connected component of H, then the group J contains Z so, if Z° does not coincide with H°, then G/H has some complex line. Moreover, let's remark that *when G is solvable, and X has no non constant functions, then $G = J$, so that G is a complex Lie group, H is discrete and the action of G on X is holomorphic.*

Note that, when G and H are complex Lie groups, then $J = N$, where N is the normalizer of H° (For the anticanonical reduction see [3], p. 61).

Let's briefly recall the definition of the *Kobayashi reduction.* Let's consider the Kobayashi pseudodistance d on G/H, and define the equivalence relation

$$x \approx y \quad \text{if and only if} \quad d(x, y) = 0\,.$$

The quotient space and the fiber are (not necessarily complex) homogeneous spaces and the fiber cannot be hyperbolic.

The *hyperbolic reduction* is defined in [13] by the equivalence relation

$$x \approx y \quad \text{if and only if} \quad d(f(x), f(y)) = 0$$

for any bounded holomorphic function f of G/H.

The main fact about the hyperbolic reduction is given by Theorem 1 of [13]; hence, we get a domain D' and a holomorphic mapping $p : G/L \to D'$ such that all holomorphic maps $f: G/L \to B$ to a hyperbolic manifold B, must fiber over D', that is there is $f': D' \to B$ such that $f' \circ p = f'$.

Note also, from Corollary 9 and Theorem 10 of [3] p. 78, that

if X has more than one end, then G/J is a complex homogeneous manifold and J/H must be positive dimensional. This fact implies that X has some complex line. So the problem of Brody should remain just for the one end case.

It is easy to prove that

if X has more than one end, then the hyperbolic and Kobayashi reductions must reduce to a point.

The following Theorem 1 is a generalisation of Theorem 1 of A. Kodama [7], where the same is proven under condition of hyperbolicity. Let's first describe, in some details, the situation: if $\mathfrak{H}$ denotes the Lie algebra of H, then the Lie algebra of J is given by

$$\mathfrak{J} = \{X \in \mathfrak{G} : J[X, Y] - [X, JY] \in \mathfrak{H} \text{ for all } Y \text{ in } \mathfrak{G}\}$$

$\mathfrak{J}$ is stable with respect to J and coincides with $\mathfrak{G} \cap \mathcal{N}(\mathfrak{q})$ (see [7], Lemma 1) where $\mathfrak{G}$ is the Lie algebra of G, $\mathfrak{q}$ denotes the complex subalgebra of the complexified algebra $\mathfrak{G}^c$ which defines the complex structure J and

$$\mathcal{N}(\mathfrak{q}) = \{X \in \mathfrak{G}^c, [X, \mathfrak{q}] \subset \mathfrak{q}\}.$$

Now, Kodama proved that, when X is hyperbolic, $\mathfrak{J}$ coincides with $\mathfrak{H}$, and we get $\mathcal{N}(\mathfrak{q}) = \mathfrak{q}$. In the proof of Kodama, the hyperbolicity has been used only in order to get that in $\mathfrak{G}^c$ there are no elements X which commute with JX, but this fact is implied already by the assumption of non existence of non constant complex line since, if $[X, JX] = 0$ then $\exp(aX + bJX)$, for $z = a + ib$, would be a non constant complex line.

Moreover it holds the following: the group G acts on the Grassmann manifold of all complex subspaces of dimension m in $\mathfrak{G}^c$, where m is the complex dimension of $\mathfrak{q}$, via its adjoint representation Ad.

Furthermore, if G^c is the complex subgroup of $GL(\mathfrak{G}^c, \boldsymbol{C})$ corresponding to the subalgebra ad $\mathfrak{G}^c$ of $\mathfrak{GL}(\mathfrak{G}^c, \boldsymbol{C})$, then one gets that G/H is G-equivariantly immersed as an open complex submanifold in the complex homogeneous space G^c/Q, where Q is the isotropy group of G^c at $\mathfrak{q}$, the Lie algebra of Q being ad $\mathfrak{q}$.

Theorem 1. *Let $X = G/H$ be a complex manifold without any non constant complex line. Then $\mathcal{N}(\mathfrak{q}) = \mathfrak{q}$. Moreover, the group G acts on the Grassmann manifold* $\mathrm{Gr}(\mathfrak{G}^c, m)$ *via its adjoint representation* Ad; *since H leaves $\mathfrak{q}$ invariant (that is $\mathrm{Ad}\,k(\mathfrak{q}) = \mathfrak{q}$, $\forall k \in H$), then one can define a map $gH \to \mathrm{Ad}\,g(\mathfrak{q})$, which is a G-equivariant immersion in the Grassmannian and is a covering of X over an open G-orbit in G^c/Q.*

3 - The first category

Let G be a semisimple Lie group of the first category. A Lie algebra is of the *first category* if the involutive automorphism generated by a corresponding Cartan decomposition is an inner automorphism (of this type are all the real forms of simple complex Lie algebras with the exception of $\mathcal{SL}(n+1, \boldsymbol{R})$ $(n \geqslant 2)$, $\mathcal{SU}^*(2n)$ $(n \geqslant 2)$, $\mathcal{SO}(p, 2n-p)$ (p odd, $n \geqslant 4$), $E\ I$, $E\ IV$).

In [8], F. M. Malysev proved that if G is a semisimple Lie group of the first category and L is a connected closed subgroup of G, then G/L has an invariant complex structure iff G/L has even dimension, L is reductive and its semisimple part coincides with the semisimple part of the centralizer of a torus in G.

In terms of Lie algebras, he proved that the subalgebra $\mathfrak{q}$ of $\mathfrak{g}^c$, which defines the complex structure, must be contained in a parabolic subalgebra $\mathfrak{p}$; moreover it holds that $\mathfrak{p} = \mathcal{N}(\mathfrak{q})$.

Theorem 2. *If G is a semisimple Lie group of first category and $X = G/L$ is a complex homogeneous manifold, then X is hyperbolic if and only if every complex line from* $\boldsymbol{C}$ to X is constant.

Proof. Since the hyperbolicity as the existence of non constant complex lines are invariant under holomorphic covering map, we can suppose that L is connected.

Thus, we can apply the result of Malysev and, if $X = G/L$ has no non constant complex lines, Theorem 1, to get that $\mathfrak{p}$ must coincide with $\mathfrak{q}$ since $\mathfrak{p} = \mathcal{N}(\mathfrak{q}) = \mathfrak{q}$, and G/L is holomorphically and equivariantly immersed, via $gH \to \mathrm{Ad}\, g(\mathfrak{q})$ as an open G-orbit in the flag manifold G^c/Q. Since open G-orbits in flag manifolds are simply connected (Theorem 5.4 of [14], p. 1146) G/L is realized as such an orbit.

Moreover, by Lemma 6.2 and Theorem 6.3 at p. 1150 of [14] we obtain also that G/L has a G-invariant volume element and equivalently, has a G-invariant, possibly indefinite, Kähler metric.

Now, at p. 1147 of [14] there is a detailed exposition of the decomposition of a flag manifold S/P ($S = G^c$, $P = Q$) as a product of flag manifolds of type S_i/P_i, where S_i is simple (it comes from the complexification of the simple factors of G) and P_i is parabolic (it is the intersection of P with S_i). In relation with such a decomposition, there is a decomposition of any G-orbit at any point p as product of orbits in the flag manifolds S_i/P_i. An orbit is *open* iff each orbit in this decomposition is open.

Now, we can use the fact that in a product manifold the property of being hyperbolic is satisfied iff each factor is hyperbolic and in order to have a non constant complex line is necessary and sufficient that one factor has it. So we can restrict our attention to the case when we have:

a simple real Lie group G
a reductive connected subgroup H
G/H is realized as an open orbit of G in a flag manifold S/P (and it is simply connected).

Thus, following [13], there exists a maximal compact Lie subgroup K of G such that the K-orbit in X is a complex space. Now, if X does not contain any non constant complex line, then this K-orbit must reduce to a point, that is $K = K \cap H$. But H is also the maximal connected subgroup of G (since G is simple), so we get $K = H$.

Since H is compact, in X there is an invariant volume element v and then we can construct in a canonical way a Hermitian form h (see [6], p. 374-375, Theorems 4 and 5; or, for detailed proofs [7], [2]). Moreover, by Koszul [7], see also Theorem 4, p. 374 of [6], since the number of the negative squares in h is equal to the difference between the dimension of L and the dimension of a maximal compact subgroup, then h is positive definite and G/L is a Hermitian symmetric space of non compact type, so it is hyperbolic.

The following theorem holds for manifolds of semisimple Lie group without restriction on the category.

Theorem. (Brody's theorem for semisimple Lie group). *Let G be a semisimple Lie group, endowed with a left invariant complex structure. Then, G has some non constant complex line.*

Proof. The same of Proposition 5.3 of [9], keeping in mind the hypothesis that there are no complex lines.

4 - Kobayashi and hyperbolic reductions in the first category

We want to look now to the Kobayashi reduction of G/L, where G is semisimple of the first category, L is connected and G/L has a G-invariant complex structure. Then, G/J has the following property: if $\mathfrak{q}$ denotes, as usual, the subalgebra of $\mathfrak{g}^c$ which defines the complex structure of G/L, then $\mathfrak{p} = \mathcal{N}(\mathfrak{q})$ is a parabolic subalgebra.

The complex structure of G/L is given by $\mathfrak{p}$ itself and, since $\mathfrak{p} = \mathcal{N}(\mathfrak{p})$, being $\mathfrak{p}$ parabolic, we have again Theorem 1 of Kodama without any condition of hyperbolicity. So G/J can be realized as a simply connected open orbit of a flag manifold. In particular, the fibres of the anticanonical reduction are connected and we can apply the theorem of Illarionov [4] to the locally trivial holomorphic fibration $\pi : G/L \to G/J$ that is $d_{G/L} = \pi^* d_{G/J}$. Moreover, G/J is the product, see [13], of a bounded homogeneous domain D and a totally degenerate (with respect to the Kobayashi distance) flag domain F. Thus, since $d_{G/L}$ degenerates along $\pi^{-1}(F)$, then the Kobayashi reduction of G/L is simply the projection h over D.

Furthermore, since the fiber of h is totally degenerate, then all holomorphic mapping $f: G/L \to B$ must be constant along the fibres of h, when B is hyperbolic, so also the hyperbolic reduction coincides with h.

Theorem 3. *Let G be a real semisimple Lie group of the first category and let G/L have a G-invariant complex structure. Then the basis of the Kobayashi reduction of G/L is a bounded homogeneous domain of $\mathbf{C}^n$, and the fiber F^* is a Kobayashi totally degenerate complex manifold. Moreover it coincides with the hyperbolic reduction.*

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Sommario

Si prova il teorema di Brody per varietà omogenee sotto l'azione di un gruppo di Lie semisemplice di prima categoria.

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