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# Special identities with $(\alpha, \beta)$-derivations (**)

## 1 - Introduction

Throughout this paper, $R$ will be a prime ring with nonzero ideal $U$ and *symmetric Martindale* ring of quotients $Q_s = Q_s(R)$, *right Martindale* ring of quotients $Q_r = Q_r(R)$, *maximal right* ring of quotients $Q_{mr} = Q_{mr}(R)$ and *extended centroid* $C$. See K. I. Beidar, W. S. Martindale III and A. V. Mikhalev [1] and W. S. Martindale III [7] for the definitions and basic properties of $Q_s$, $Q_r$, $Q_{mr}$ and $C$. We denote by $\mathcal{U}(R)$ the group of invertible elements of the ring $R$.

An additive mapping $f$: $R \to R$ is called a *derivation* of the ring $R$ if we have $f(xy) = xf(y) + f(x)y$ for all $x, y \in R$. We shall denote by $\mathrm{Aut}\, Q_s$ the automorphism group of the ring $Q_s$ and set

$$A(R) = \{\alpha \in \mathrm{Aut}\, Q_s \mid \text{there exist nonzero ideals } I_1 \text{ and } I_2 \text{ of } R \text{ such that } I_2 \subseteq \alpha(I_1) \subseteq R\}.$$

We note that every automorphism of $R$ can be extended uniquely to an automorphism of $Q_s$ (Lemma 1, [6]), and we shall make no distinction in what follows between these two automorphisms for brevity.

Let $\alpha, \beta \in A(R)$. An additive mapping $f$: $R \to Q_s$ is called an *$(\alpha, \beta)$-derivation* of $R$ if $f(xy) = \alpha(x) f(y) + f(x) \beta(y)$ for all $x, y \in R$ and there exists a nonzero ideal $I$ of $R$ such that $f(I) \subseteq R$. For example, $\alpha - \beta$ is an $(\alpha, \beta)$- and a $(\beta, \alpha)$-derivation of $R$. Given a fixed element $a \in R$, the mapping $\mathrm{ad}_{\alpha, \beta}(a)$: $R \to R$ defined via $\mathrm{ad}_{\alpha, \beta}(a)(x) = \alpha(x) a - a\beta(x)$ for all $x \in R$ is also an $(\alpha, \beta)$-derivation of $R$.

For $t \in \mathcal{U}(Q_s)$ and $f$: $R \to Q_{mr}$, the mappings $ft$: $R \to Q_{mr}$ and $tf$: $R \to Q_{mr}$ are defined via $(ft)(x) = f(x) t$ and $(tf)(x) = tf(x)$ for all $x \in R$. The following statements can be verified easily and will be used without further references.

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Remark 1. Let $f$ be an $(\alpha, \beta)$-derivation and $t \in \mathcal{U}(Q_s)$. Denote by $\mathrm{Inn}(t)$ the inner automorphism of $Q_s$ induced by $t$. Set $\delta = \mathrm{Inn}(t)\alpha$ and $\gamma = \mathrm{Inn}(t^{-1})\beta$. Then:

1. $\alpha^{-1}f$ is an $(1, \alpha^{-1}\beta)$-derivation of $R$
2. $t^{-1}f$ is an $(\delta, \beta)$-derivation of $R$
3. $ft^{-1}$ is an $(\alpha, \gamma)$-derivation of $R$.

We remind the reader that for a nonzero $(\alpha, \beta)$-derivation $f$ and a nonzero $(\gamma, \delta)$-derivation $g$, the notation «$f = g$» means that $f$ and $g$ are equal as derivations, which asserts that, besides $f(x) = g(x)$ for all $x \in R$, $\alpha = \gamma$ and $\beta = \delta$ as well.

Lemma 1. (Lemma 3, [4]). *Let $f$ be a nonzero $(\alpha, \beta)$-derivation of $U$ into $R$ and $g$ an $(\gamma, \delta)$-derivation of $U$ into $R$. The following conditions are equivalent:*

1. *$f(x) = g(x)$ for all $x \in U$*
2. *either $f = g$ or there exists a $t \in \mathcal{U}(Q_s)$ such that $\delta = \mathrm{Inn}(t)\alpha$, $\beta = \mathrm{Inn}(t)\gamma$, $f(x) = (\alpha(x) - \gamma(x))t$ and $g(x) = t(\delta(x) - \beta(x))$.*

In 1993 M. Brešar (Lemma 2.3 [2]) studied an identity $f_1(x)f_2(y) = f_3(x)f_4(y)$ for all $x, y \in R$ where $f_i$'s are nonzero derivations of $R$, and showed that there exists a $\lambda \in C$ such that $f_3(x) = \lambda^{-1}f_1(x)$ and $f_4(x) = \lambda f_2(x)$ for all $x \in R$. Recently, J.-C. Chang (Lemma 1 [4]) considered a more general case where $f_2$ and $f_3$ are $(\alpha, \beta)$-derivations, $f_1$ is an $(\alpha, \alpha)$-derivation and $f_4$ is a $(\beta, \beta)$-derivation. He proved that there exists a $t \in \mathcal{U}(Q_s)$ such that $f_3(x) = f_1(x)t^{-1}$ and $f_4(x) = tf_2(x)$ for all $x \in R$. Our goal is to prove the following generalization of Brešar's and Chang's results.

Theorem 1. *Let $R$ be a prime ring with nonzero ideal $U$ and $f_i \neq 0$ an $(\alpha_i, \beta_i)$-derivation of $R$, $i = 1, 2, 3, 4$. Suppose that $f_1(x)f_2(y) = f_3(x)f_4(y)$ for all $x, y \in U$. Then there exists a $t \in \mathcal{U}(Q_s)$ such that $f_3(x) = f_1(x)t^{-1}$ and $f_4(x) = tf_2(x)$ for all $x \in R$.*

The above theorem rests on the following result, which has an independent interest.

Theorem 2. *Let $R$ be a prime ring with nonzero ideal $U$, $a, b \in Q_r \setminus \{0\}$, $f \neq 0$ an $(\alpha, \beta)$-derivation and $g \neq 0$ a $(\gamma, \delta)$-derivation of $R$. Suppose that*

*$af(x) - bg(x) = 0$ for all $x \in U$. Then there exists a $t \in \mathcal{U}(Q_s)$ such that $b = at^{-1}$, and one of the following statements holds:*

1. *$g(x) = tf(x)$ for all $x \in R$ and either $\gamma = \mathrm{Inn}(t^{-1})\alpha$, $\delta = \beta$ or there exists a $s \in \mathcal{U}(Q_s)$ such that $\beta = \mathrm{Inn}(s^{-1})\gamma$, $\delta = \mathrm{Inn}(st^{-1})\alpha$*

2. *there exists a $q \in Q_s$ such that $aq = 0$, $f = \mathrm{ad}_{\alpha,\beta}(q)$ and $g = t\,\mathrm{ad}_{\alpha,\delta}(q)$.*

The study of identities with derivations goes back to 1957 when E.C. Posner [9] proved that a prime ring $R$ with nonzero derivation $d$ is commutative if it satisfies the identity $[[d(x), x], y] = 0$ for all $x, y \in R$. Since then, more than 50 papers have been published on the related matters, and there are theories developed on *rings with generalized identities* [1], [6], on *Hopf algebras action on rings* [8], and on *commuting additive mappings* [2], [3] and Chapter 9 [1]. Besides the mentioned applications, Theorem 2 is motivated by results of I. N. Herstein [5], M. Brešar [2], J.-C. Chang [4] as well as some others.

In order to state an important result of V. K. Kharchenko and A. Z. Popov [6], which will be used in the proofs of Theorem 1 and 2, we need the following definitions.

First of all, we call a $(1, \beta)$-derivation of $R$ a *skew* derivation (connected with $\beta$). If there exists a nonzero ideal $I$ of $R$ such that $\beta(I) \subseteq R$, we consider the set $L_\beta(R) = \{f \mid f$ is a skew derivation of $Q_s$ connected with $\beta\}$.

Let $\beta \in A(R)$ and $q \in Q_s$. The skew derivation $\mathrm{ad}_{1,\beta}(q)$ is said to be *inner* (connected with $\beta$). Here, 1 denotes the identity mapping of $R$. The set of all inner skew derivations of $R$ connected with $\beta$, denoted by $\mathrm{Inn}\, L_\beta$. is a $C$-subspace of $L_\beta(R)$.

Two automorphisms $g, h \in A(R)$ are said to be *mutually outer* if $gh^{-1}$ is not an inner automorphism of $Q_s$.

Finally, a set of skew derivations $S = \{f_1, f_2, \dots, f_n\}$ is called *reduced* if the following conditions are satisfied:

1. distinct automorphisms connected with skew derivations in $S$ are mutually outer

2. skew derivations in $S$ which are connected with a fixed automorphism $\beta$ are linearly independent over $C$ modulo $\mathrm{Inn}\, L_\beta$.

Since $Q_r(U) = Q_r$, the following result is a special case of V. K. Kharchenko and A. Z. Popov (Proposition 9 [6]).

Proposition 1. *Let* $\{f_1, f_2, \ldots, f_n\}$ *be a reduced set of skew derivations and* $\{h_1, h_2, \ldots, h_m\}$ *a set of mutually outer automorphisms satisfying an identity of the type*

$$\sum_{j,k} a_j^{(k)} f_k(x)\, b_j^{(k)} + \sum_{i,j} d_{ij} h_i(x)\, e_{ij} = 0 \qquad x \in U$$

*where* $a_j^{(k)}$, $b_j^{(k)}$, $d_{ij}$ *and* $e_{ij}$ *are coefficients from* $Q_r$. *Then the following relations are fulfilled in the tensor product* $Q_r \otimes_C Q_r$:

$$\sum_j a_j^{(k)} \otimes b_j^{(k)} = 0 \qquad \sum_j d_{ij} \otimes e_{ij} = 0 \qquad 1 \leqslant i \leqslant m,\ q \leqslant k \leqslant n\,.$$

*In particular, the identities*

$$\sum_j a_j^{(k)} x b_j^{(k)} = 0 \qquad \sum_j d_{ij} x e_{ij} = 0 \qquad 1 \leqslant i \leqslant m,\ q \leqslant k \leqslant n$$

*are fulfilled in* $R$.

## 2 - Proofs of the main results

We shall proceed to the proofs of Theorems 1 and 2 with several Lemmas.

Lemma 2. *Let* $f$ *and* $g$ *be* $(\alpha, \beta)$*-derivations of* $R$ *and* $a$ *be an nonzero element in* $Q_{mr}$ *such that*

$$(1) \qquad af(x) = ag(x) \qquad \textit{for all } x \in U\,.$$

*Then* $f = g$.

Proof. Substituting $xy$ for $x$ in (1), we obtain

$$a\alpha(x)\, f(y) + af(x)\, \beta(y) = a\alpha(x)\, g(y) + ag(x)\, \beta(y)$$

for all $x \in U$ and $y \in R$. Using (1), we obtain $a\alpha(x)(f(y) - g(y)) = 0$ for all $x \in U$ and $y \in R$. Since $\alpha(U)$ contains a nonzero ideal of $R$, we get $f(x) = g(x)$ for all $x \in R$.

Lemma 3. *Let* $\alpha, \beta$ *be elements of* $A(R)$ *and let* $a$, $b$, $c$, *and* $d$ *be nonzero elements of* $Q_{mr}$ *such that* $a\alpha(x)\, b = c\beta(x)\, d$ *for all* $x \in U$. *Then there exists a* $t \in \mathfrak{U}(Q_s)$ *such that* $\alpha = \mathrm{Inn}(t^{-1})\beta$, $c = at$ *and* $d = t^{-1} b$. *In particular, if* $a\alpha(x) = b\beta(x)$ *for all* $x \in U$, *then* $a = b$ *and* $\alpha = \beta$.

Proof. Let $a_1 = \beta^{-1}(a)$, $b_1 = \beta^{-1}(b)$, $c_1 = \beta^{-1}(c)$, $d_1 = \beta^{-1}(d)$ and $\alpha_1 = \beta^{-1}\alpha$. Clearly, $a_1$, $b_1$, $c_1$ and $d_1$ are nonzero. Therefore, $a_1 \alpha_1(x)\, b_1 = c_1 x d_1$ for all $x \in U$.

By Proposition 1, there exists an $s \in \mathcal{U}(Q_s)$ such that $\alpha_1(x) = \mathrm{Inn}(s)x$ for all $x \in R$. Thus $a_1 s^{-1} x s b_1 = c_1 x d_1$. By Theorem 6.1.2 [1], there exists a $\lambda \in C$ such that $c_1 = \lambda a_1 s^{-1}$ and $d_1 = \lambda^{-1} s b_1$. Therefore $c = a\beta(\lambda s^{-1})$ and $d = \beta(\lambda^{-1}s)b$. Let $t = \beta(\lambda s^{-1})$. Then $c = at$, $d = t^{-1}b$ and

$$\alpha(x) = \beta(\mathrm{Inn}(s)x) = \beta(s^{-1})\beta(x)\beta(s) = \beta(\lambda s^{-1})\beta(x)\beta(\lambda^{-1}s) = \mathrm{Inn}(t^{-1})\beta(x).$$

In particular, if $a\alpha(x) = b\beta(x)$, for all $x \in U$ then $t = 1$, and so $a = b$ and $\alpha = \beta$.

Lemma 4. *Let $\alpha$, $\beta$ and $\delta$ be elements of $A(R)$ and let $f$ be a nonzero $(\alpha, \beta)$-derivation of $R$. If $a$ and $b$ are nonzero elements of $Q_{mr}$ such that*

$$af(x) = b(\beta(x) - \delta(x)) \qquad \text{for all } x \in U, \tag{2}$$

*then there exists a $t \in \mathcal{U}(Q_s)$ such that $\delta = \mathrm{Inn}(t^{-1})\alpha$, $bt = a$ and $f = t^{-1}(\beta - \delta)$.*

Proof. Substituting $xy$ for $x$ in (2), we obtain

$$a\alpha(x)f(y) + af(x)\beta(y) = af(xy) = b(\beta(xy) - \delta(xy)).$$

It follows from (2) that

$$a\alpha(x)f(y) + b(\beta(x) - \delta(x))\beta(y) = b(\beta(xy) - \delta(xy)) \qquad x \in U, y \in R.$$

Now, substitute $x\alpha^{-1}(a)$ for $x$ to get

$$a\alpha(x)b(\beta(y) - \delta(y)) + b[\beta(x\alpha^{-1}(a)) - \delta(x\alpha^{-1}(a))]\beta(y) = b(\beta(x\alpha^{-1}(a)y) - \delta(x\alpha^{-1}(a)y)).$$

Therefore, we can write

$$b\delta(x)\delta(\alpha^{-1}(a))(\beta(y) - \delta(y)) = a\alpha(x)b(\beta(y) - \delta(y)) \qquad x \in U, y \in R. \tag{3}$$

Since $a \neq 0$, if $\delta(\alpha^{-1}(a))(\beta(y) - \delta(y)) = 0$ for all $y \in R$, then $\beta = \delta$ by Lemma 3. But then $f = 0$, a contradiction. Hence there exists some $y' \in R$ such that $\delta(\alpha^{-1}(a))(\beta(y') - \delta(y')) \neq 0$. Then by Lemma 3, there exists a $t \in \mathcal{U}(Q_s)$ such that $\delta = \mathrm{Inn}(t^{-1})\alpha$, $bt = a$ and

$$t^{-1}(\delta(\alpha^{-1}(a))(\beta(y') - \delta(y'))) = b(\beta(y') - \delta(y')).$$

Since $\delta = \mathrm{Inn}(t^{-1})\alpha$, $t^{-1}(\beta - \delta)$ is an $(\alpha, \beta)$-derivation of $R$ by Remark 1. With $bt = a$, (2) becomes

$$af(x) = at^{-1}(\beta(x) - \delta(x)) \qquad \text{for all } x \in U.$$

By Lemma 2, we have $f = t^{-1}(\beta - \delta)$ as desired.

Lemma 5. *Let* $a \in Q_r \setminus \{0\}$, *and let* $f$ *and* $h$ *be nonzero skew derivations connected with* $\beta$ *and* $\delta$, *respectively, such that*

$$af(x) - ah(x) = 0 \qquad \text{for all } x \in U. \tag{4}$$

*Then one of the following statements holds:*

**i.** $f = h$

**ii.** *there exists a* $q \in Q_s$ *such that* $f = \mathrm{ad}_{1,\beta}(q)$, $h = \mathrm{ad}_{1,\delta}(q)$ *and* $aq = 0$.

Proof. If $\{f, h\}$ is a reduced set, then $a \otimes 1 = 0$ by Proposition 1, which is impossible. Therefore, $\{f, h\}$ is not reduced, and we have to consider three cases.

Case **I**. $f \in \mathrm{Inn}\, L_\beta$. In this case, there exists a $p \in Q_s$ such that $f = \mathrm{ad}_{1,\beta}(p)$. Then (4) implies $axp - ap\beta(x) - ah(x) = 0$. If $h$ is reduced, then $a \otimes 1 = 0$ by Proposition 1, which is impossible. Therefore, $h \in \mathrm{Inn}\, L_\delta$, i.e., there exists a $p' \in Q_s$ such that $h = \mathrm{ad}_{1,\delta}(p')$. Suppose that $p = p'$. Then we obtain

$$ap(\delta(x) - \beta(x)) = axp - ap\beta(x) - (axp - ap\delta(x)) = af(x) - ah(x) = 0$$

for all $x \in U$. If $ap \neq 0$, then $\beta = \delta$ by Lemma 3, and so $f = h$. Hence **i** holds. If $ap = 0$, then **ii** holds with $q = p$.

Next, we assume that $p \neq p'$. Then (4) implies

$$axp - ap\beta(x) - axp' + ap'\delta(x) = 0 \qquad x \in U. \tag{5}$$

Suppose that $\beta$ and $\delta$ are mutually outer. Then either $\beta$ or $\delta$ is outer, and so either $ap \otimes 1 = 0$ or $ap' \otimes 1 = 0$ by Proposition 1. Thus, either $ap = 0$ or $ap' = 0$. We consider here only $ap = 0$, and an analogous argument can be used for $ap' = 0$. Now, (5) implies

$$ax(p - p') + ap'\delta(x) = 0 \qquad x \in U.$$

Since $p \neq p'$, we have $ap' \neq 0$. It follows from Lemma 3 that $\delta$ is inner, i.e., $\delta = \mathrm{Inn}(s)$ for some $s \in \mathcal{U}(Q_s)$. Now, we can rewrite (5) as

$$ax(p - p') + ap's^{-1}xs = 0 \qquad x \in U.$$

By Theorem 6.1.2 [**1**], there exists a $\lambda \in C$ such that $p - p' = \lambda s$. Therefore

$$h(x) = \mathrm{ad}_{1,\delta}(p')(x) = xp' - p'\delta(x) = x(p - \lambda s) - (p - \lambda s)s^{-1}xs$$

$$= xp - ps^{-1}xs = xp - p\delta(x) = \mathrm{ad}_{1,\delta}(p)(x) \qquad x \in R.$$

Therefore **ii** holds.

The situation when $\beta$ and $\delta$ are not mutually outer will be discussed in Case **III**.

Case **II.** $h \in \operatorname{Inn} L_\delta$. This case is similar to Case **I**, and we omit the proof.

Case **III.** $\beta = \operatorname{Inn}(s)\delta$ for some $s \in \mathcal{U}(Q_s)$. In this case, set $g = hs$. Then by Remark 1, $g$ is a $(1,\beta)$-derivation of $R$. Now, (4) implies

$$af(x)s = ah(x)s = ag(x) \qquad x \in U.$$

Hence $f \equiv g(\operatorname{mod} \operatorname{Inn} L_\beta)$. Otherwise, $a \otimes s = 0$ by Proposition 1, which is impossible since $a \neq 0$. Therefore, there exist $c \in C$ and $b \in Q_s$ such that $g = cf + \operatorname{ad}_{1,\beta}(b)$, and so $af(x)s = ag(x) = caf(x) + axb - ab\beta(x)$ for all $x \in U$. It follows that $af(x)(s-c) - axb + ab\beta(x) = 0$ for all $x \in U$.

If $f \notin \operatorname{Inn} L_\beta$, then $a \otimes (s-c) = 0$ by Proposition 1, and so $s = c$. Therefore $\beta = \delta$ and $f = h$ by Lemma 2, and **i** holds.

Assume $f \in \operatorname{Inn} L_\beta$. Then $g \in \operatorname{Inn} L_\beta$, also. Hence, there exist $p, p' \in Q_s$ such that $f = \operatorname{ad}_{1,\beta}(p)$ and $g = \operatorname{ad}_{1,\beta}(p')$. Since $af(x)s = ag(x)$, we have $axps - ap\beta(x)s = axp' - ap'\beta(x)$; thus

$$(6) \qquad ax(p' - ps) + ap\beta(x)s - ap'\beta(x) = 0 \qquad x \in U.$$

Suppose that $\beta$ is outer. Then $a \otimes (p' - ps) = 0$ and $ap \otimes s - ap' \otimes 1 = 0$ by Proposition 1. It follows that $p' = ps$ and either $s \in C$ or $ap = 0 = ap'$. If $s \in C$, then $\beta = \delta$ and $f = h$ by Lemma 2, i.e. **i** holds. Next, assume that $ap = 0 = ap'$. Then we have

$$h(x) = g(x)s^{-1} = xp's^{-1} - p'\beta(x)s^{-1} = xp - ps\beta(x)s^{-1} = xp - p\delta(x)$$

for all $x \in U$, and so $h = \operatorname{ad}_{1,\delta}(p)$. It follows that **ii** holds.

Now assume that $\beta$ is inner, i.e., $\beta = \operatorname{Inn}(t)$ for some $t \in \mathcal{U}(Q_s)$. Then (6) becomes $ax(p' - ps) + apt^{-1}xts - ap't^{-1}xt = 0$ for all $x \in U$. If $p' = ps$, then $apt^{-1}xts = ap't^{-1}xt$ for all $x \in U$, and so either $ap = 0 = ap'$ or $s \in C$. Both cases have been considered above. So, we assume that $p' - ps \neq 0$.

Since $a$, $apt^{-1}$, $ts$, $ap't^{-1}$ and $t$ are nonzero elements of $Q_r$, by Lemma 6.1.2 [1] we have that $\{p' - ps, ts, t\}$ is $C$-dependent. If $ts$ and $t$ are $C$-dependent, then $s \in C$, which has just been discussed. Therefore we can assume that $t$ and $ts$ are $C$-independent, and so there exist $\lambda, \mu \in C$ such that $p' - ps = \lambda ts + \mu t$.

Rewrite (6) as $\lambda axts + \mu axt + apt^{-1}xts - ap't^{-1}xt = 0$ and obtain $(\lambda a + apt^{-1})xts = (ap't^{-1} - \mu a)xt$. It follows from the $C$-independency of $t$ and $ts$ that $\lambda a + apt^{-1} = 0$ and $ap't^{-1} - \mu a = 0$, and so $a(p + \lambda t) = 0$. Hence

$$f(x) = xp - p\beta(x) = xp - p\beta(x) - \lambda t\beta(x) + \lambda t(t^{-1}xt)$$

$$= x(p + \lambda t) - (p + \lambda t)\beta(x) = \operatorname{ad}_{1,\beta}(p + \lambda t)(x)$$

and

$h(x) = g(x)s^{-1} = xp's^{-1} - p'\beta(x)s^{-1} = x(ps + \lambda ts + \mu t)s^{-1} - (ps + \lambda ts + \mu t)\beta(x)s^{-1}$

$= x(p + \lambda t) - (p + \lambda t)(s\beta(x)s^{-1}) = x(p + \lambda t) - (p + \lambda t)\delta(x) = \mathrm{ad}_{1,\delta}(p + \lambda t)(x)$.

Now **ii** holds with $q = p + \lambda t$. This completes the proof.

We can now prove Theorems 1 and 2. First we prove Theorem 2.

Proof of Theorem 2. Let $x \in U$ and $y \in R$. From $af(xy) - bg(xy) = 0$ we have $a\alpha(x)f(y) + af(x)\beta(y) - b\gamma(x)g(y) - bg(x)\delta(y) = 0$. Since $bg(x) = af(x)$, this equality becomes

$$(7) \qquad a\alpha(x)f(y) + af(x)\beta(y) - b\gamma(x)g(y) - af(x)\delta(y) = 0 \quad x \in U,\ y \in R.$$

Substituting $x\gamma^{-1}(b)$ for $x$ in (7), we have

$$a\alpha(x\gamma^{-1}(b))f(y) + af(x\gamma^{-1}(b))(\beta(y) - \delta(y)) = b\gamma(x)bg(y) = b\gamma(x)af(y)$$

with $x \in U$, $y \in R$. Thus

$$(b\gamma(x)a - a\alpha(x\gamma^{-1}(b)))f(y) = af(x\gamma^{-1}(b))(\beta(y) - \delta(y)) \quad x \in U,\ y \in R.$$

Case **I**. Suppose that $b\gamma(x)a = a\alpha(x)\alpha(\gamma^{-1}(b))$ for all $x \in U$. By Lemma 3 there exists a $t \in \mathcal{U}(Q_s)$ such that $\gamma = \mathrm{Inn}(t^{-1})\alpha$, $at^{-1} = b$ and $t^{-1}a = \alpha(\gamma^{-1}(b))$. Set $h = t^{-1}g$. Then $h$ is an $(\alpha, \delta)$-derivation of $R$, and so $af(x) = bg(x) = at^{-1}g(x) = ah(x)$ for all $x \in U$.

Putting $a_1 = \alpha^{-1}(a)$, $f_1 = \alpha^{-1}f$, $h_1 = \alpha^{-1}h$, $\beta_1 = \alpha^{-1}\beta$ and $\delta_1 = \alpha^{-1}\delta$, we have $a_1 f_1(x) - a_1 h_1(x) = 0$ for all $x \in U$. We note that $f_1$ is a skew derivation connected with $\beta_1$ and $h_1$ is a skew derivation connected with $\delta_1$. According to Lemma 5 there are two possibilities:

**a.** $f_1 = h_1$. Then $f = h$; therefore $at^{-1} = b$, $\beta = \delta$ and $f = t^{-1}g$.

**b.** $f_1 = \mathrm{ad}_{1,\beta_1}(p_1)$ $h_1 = \mathrm{ad}_{1,\delta_1}(p_1)$ and $a_1 p_1 = 0$ for some $p_1 \in Q_s$. Let $p = \alpha(p_1) \in Q_s$. Then $ap = 0$, $f = \mathrm{ad}_{\alpha,\beta}(p)$ and $h = \mathrm{ad}_{\alpha,\delta}(p)$, and so $g = t\,\mathrm{ad}_{\alpha,\delta}(p)$.

In both cases, the theorem holds.

Case **II**. Suppose that there exists an $x \in U$ with $b\gamma(x)a - a\alpha(x\gamma^{-1}(b)) \neq 0$. By Lemma 4 there exists some $t \in \mathcal{U}(Q_s)$ such that $\delta = \mathrm{Inn}(t^{-1})\alpha$ and $f = t^{-1}(\beta - \delta)$. By symmetry, we have $\beta = \mathrm{Inn}(s^{-1})\gamma$ and $g = s^{-1}(\delta - \beta)$ for some $s \in \mathcal{U}(Q_s)$. It follows that $f(x) = t'^{-1}g(x)$ for $t' = -t^{-1}s \in \mathcal{U}(Q_s)$ and for all $x \in R$. Thus $at'^{-1}g(x) = af(x) = bg(x)$ for all $x \in U$ and by Lemma 2 $b = at'^{-1}$. The proof is now complete.

Proof of Theorem 1. Fix an $x_0 \in U$ such that $f_1(x_0) \neq 0$. Then we have $f_1(x_0) f_2(y) = f_3(x_0) f_4(y)$ for all $y \in U$. Again, according to Theorem 2, there are two cases to consider.

Case I. $f_2(y) = t^{-1} f_4(y)$ for all $y \in R$ and $f_1(x_0) t^{-1} = f_3(x_0)$ for some $t \in \mathcal{U}(Q_s)$. Thus

$$(f_1(x) t^{-1} - f_3(x)) f_4(y) = f_1(x) f_2(y) - f_3(x) f_4(y) = 0 \qquad x, y \in U.$$

It follows from Lemma 2 and $f_4 \neq 0$ that $f_1(x) t^{-1} = f_3(x)$ for all $x \in R$.

Case II. $f_2 = \mathrm{ad}_{\alpha_2, \beta_2}(p)$, $f_4 = t\,\mathrm{ad}_{\alpha_2, \beta_4}(p)$ and $f_1(x_0) p = 0$ for some $p = p(x_0) \in Q_s$ and $t = t(x_0) \in \mathcal{U}(Q_s)$. In particular, $\alpha_4 = \mathrm{Inn}(t^{-1}) \alpha_2$ and $f_4 = \mathrm{ad}_{\alpha_4 \alpha_4}(tp)$.

If $p(x_0) = p(x_1)$ for all $x_0$, $x_1 \in U$ such that $f_1(x_0) \neq 0$ and $f_1(x_1) \neq 0$, then $f_1(x) p(x_0) = 0$ for all $x \in U$. But then $p(x_0) = 0$ and $f_2 = f_4 = 0$, a contradiction. Therefore, there are $x_0$, $x_1 \in U$ with $f_1(x_0) \neq 0$ and $f_1(x_1) \neq 0$ such that $p(x_0) \neq p(x_1)$. Let $p_i = p(x_i)$ and $t_i = t(x_i)$, $i = 0, 1$. Since

$$f_2 = \mathrm{ad}_{\alpha_2, \beta_2}(p_0) = \mathrm{ad}_{\alpha_2, \beta_2}(p_1)$$

we have $(p_0 - p_1) \beta_2(y) - \alpha_2(y)(p_0 - p_1) = 0$ for all $y \in U$. Since $p_0 \neq p_1$, it follows from Lemma 3 that there exists some $s \in \mathcal{U}(Q_s)$ such that $\beta_2 = \mathrm{Inn}(s^{-1}) \alpha_2$. Similarly,

$$f_4 = \mathrm{ad}_{\alpha_4, \beta_4}(t_0 p_0) = \mathrm{ad}_{\alpha_4, \beta_4}(t_1 p_1)$$

which implies the existence of some $r \in \mathcal{U}(Q_s)$ such that $\beta_4 = \mathrm{Inn}(r^{-1}) \alpha_2$. Therefore:

$$(8) \qquad f_2 = \mathrm{ad}_{\alpha_2, \mathrm{Inn}(s^{-1}) \alpha_2}(p) \qquad f_4 = t\,\mathrm{ad}_{\alpha_2, \mathrm{Inn}(r^{-1}) \alpha_2}(p).$$

Since $f_1(x) f_2(y) = f_3(x) f_4(y)$, we have

$$f_1(x)\,\mathrm{ad}_{\alpha_2, \mathrm{Inn}(s^{-1}) \alpha_2}(p)(y) = f_3(x)\, t\,\mathrm{ad}_{\alpha_2, \mathrm{Inn}(r^{-1}) \alpha_2}(p)(y).$$

Then

$$(9) \quad f_1(x) p s \alpha_2(y) s^{-1} - f_3(x) t p r \alpha_2(y) r^{-1} - (f_1(x) - f_3(x) t) \alpha_2(y) p = 0 \qquad x, y \in U.$$

Let $f_1' = \alpha_2^{-1} f_1$, $f_3' = \alpha_2^{-1} f_3$, $p' = \alpha_2^{-1}(p)$, $s' = \alpha_2^{-1}(s)$, $t' = \alpha_2^{-1}(t)$ and $r' = \alpha_2^{-1}(r)$. We have

$$(10) \quad f_1'(x) p' s' y s'^{-1} - f_3'(x) t' p' r' y r'^{-1} - (f_1'(x) - f_3'(x) t') y p' = 0 \qquad x, y \in U.$$

Suppose that $f_1'(x) = f_3'(x) t'$ for all $x \in U$. Then

$$f_1(x) = \alpha_2 f_1'(x) = \alpha_2 f_3'(x) \alpha_2(t') = f_3(x) t \qquad x \in R.$$

Therefore

$$f_3(x)(tf_2(y)-f_4(y))=f_1(x)f_2(y)-f_3(x)f_4(y)=0 \qquad x, y \in U$$

and so Lemma 2 implies that $tf_2(x)=f_4(x)$ for all $x \in R$.

Now, assume that there exists an $x \in U$ such that $f_1'(x)-f_3'(x)t' \neq 0$. Since $p'$ is a nonzero elements of $Q_r$, Lemma 6.1.2, **[1]** implies that $p' \in Cs'^{-1}+Cr'^{-1}$.

If $s'^{-1}$ and $r'^{-1}$ are $C$-dependent, then $s$ and $r$ are also $C$-dependent. It follows that $\mathrm{Inn}(s^{-1})=\mathrm{Inn}(r^{-1})$, and by (8), $tf_2=f_4$; consequently, $f_1(x)=f_3(x)t$ for all $x \in R$, and we are done. If $s'^{-1}$ and $r'^{-1}$ are $C$-independent, then there exist unique $\lambda$ and $\mu \in C$ such that $p'=\mu s'^{-1}+\lambda r'^{-1}$, and (10) becomes

$$f_1'(x)p's'ys'^{-1}-f_3'(x)t'p'r'yr'^{-1}-(f_1'(x)-f_3'(x)t')y(\mu s'^{-1}+\lambda r'^{-1})=0\,.$$

Therefore,

$$[f_1'(x)p's'-(f_1'(x)-f_3'(x)t')\mu]ys'^{-1}-[f_3'(x)t'p'r'+(f_1'(x)-f_3'(x)t')\lambda]yr'^{-1}=0$$

for all $x, y \in U$.

For any fixed $x \in U$, the above equality and Lemma 6.1.2 **[1]** implies that $f_1'(x)p's'=(f_1'(x)-f_3'(x)t')\mu$ and $f_3'(x)t'p'r'=(f_1'(x)-f_3'(x)t')\lambda$.

If $\mu=0$, then $f_1'(x)p's'=0$ for all $x \in U$ and if $\lambda=0$, then $f_3'(x)t'p'r'=0$ for all $x \in U$. In both cases, $p=0$, and so $f_2=f_4=0$ by (8), which is impossible. Therefore $\lambda$ and $\mu$ are both nonzero.

Since $p'=\mu s'^{-1}+\lambda r'^{-1}$, we have

$$(f_1'(x)-f_3'(x)t')\mu=f_3'(x)p's'=f_1'(x)(\mu s'^{-1}+\lambda r'^{-1})s'=f_1'(x)\mu+f_1'(x)\lambda r'^{-1}s'$$

for all $x \in U$. Therefore

$$f_1'(x)\lambda r'^{-1}s'=-f_3'(x)t'\mu \quad \text{and} \quad f_1'(x)=-f_3'(x)t'\mu s'^{-1}r'\lambda^{-1} \qquad x \in U.$$

Clearly, $q=-t\alpha(\mu)s^{-1}r\alpha(\lambda)^{-1} \in \mathcal{U}(Q_s)$, and so $f_1(x)=f_3(x)q$ for all $x \in U$. It follows from Lemma 2 that $f_1(x)=f_3(x)q$ and so $qf_2(x)=f_4(x)$ for all $x \in R$. The proof is now complete.

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## Sommario

*Sia R un anello primo. In un lavoro del 1993 M. Brešar ha studiato l'identità* $f_1(x) f_2(y) = f_3(x) f_4(y)$ *per ogni* $x, y \in R$ *dove le* $f_i$ *sono derivazioni in R. Recentemente J.-C. Chang ha considerato il caso più generale in cui* $f_2$ *ed* $f_3$ *sono* $(\alpha, \beta)$*-derivazioni,* $f_1$ *è una* $(\alpha, \alpha)$*-derivazione ed* $f_4$ *una* $(\beta, \beta)$*-derivazione. In questo lavoro viene considerato il caso generale in cui le* $f_i$ *sono* $(\alpha_i, \beta_i)$*-derivazioni.*

*Si dimostra che nell'anello simmetrico di Martindale dei quozienti di R esiste un elemento invertibile t tale che* $f_1(x) = f_3(x)t$ *e* $f_4(x) = tf_2(x)$ *per ogni x di R.*

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