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Generalized Lucas polynomials and Fibonacci polynomials (**)

1 - Introduction

In some preceding articles (see e.g. [15], [3], [4]), the generalized Lucas polynomials of the second and first kind have been studied. These polynomials naturally arise in the study of the solutions of a bilateral homogeneous recurrence relation with constant (real or complex) coefficients u_k $(k = 1, 2, \dots, r)$, $(u_r \neq 0)$

$$X_n = u_1 X_{n-1} - u_2 X_{n-2} + \dots + (-1)^r u_r X_{n-r} \qquad n \in \boldsymbol{Z}. \tag{1.1}$$

A particular case of these multivariable polynomials is constituted by the multivariable Chebyshev polynomials, which have been studied by R. Lidl [11], M. Bruschi and P. E. Ricci [4]. More general definitions and generalizations of these multidimensional polynomials can be found in papers by R. Lidl et al. [6], [10], T. Koornwinder [8], [9], R. J. Beerends [1].

It is well known that the generalized Lucas polynomials of the second kind are closely related to the representation formulas for the power of a square matrix (see e.g. M. Bruschi and P. E. Ricci [3]). They have also been applied successfully in order to obtain representation formulas for the sum rules of the zeros of Orthogonal Polynomial Sets (see P. E. Ricci [16], P. Natalini [14]). The generalized Lucas polynomials of the first kind have been used in the problem of computing the moments of the density of zeros for Orthogonal Polynomial Sets (see e.g. B. Germano, P. Natalini and P. E. Ricci [7]).

The classical Chebyshev polynomials are also included as particular solutions of the recurrence (1.1). Nevertheless there is another class of polynomials in one variable which is worth to be considered, the Fibonacci polynomials. These have

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probably the same importance as the classical Chebyshev ones, but the literature in this field has not the same extension.

In this paper some properties of these polynomials are deduced from the general properties of the above mentioned Lucas polynomials.

I want to point out that the definition for the Fibonacci polynomials I consider here is different from the usual one considered e.g. by M. Bichnell and V. E. Hoggatt [2]. This difference leads to interesting properties for the zeros of the polynomials under consideration.

2 - Fibonacci polynomials

By letting $u_1 = x$, $u_2 = -1$, and $u_3 = \ldots = u_r = 0$ the recurrence relation (1.1) becomes

$$X_n = x X_{n-1} + X_{n-2} \qquad n \in \boldsymbol{Z}.$$

The corresponding *Lucas polynomials*

$$F_{1,n}(x, -1) = \Phi_n(x, -1) = \varphi_n(x) \tag{2.1}$$

are defined by the recurrence relation and initial conditions

$$\varphi_n(x) = x\varphi_{n-1}(x) + \varphi_{n-2}(x) \qquad \varphi_{-1}(x) = 0 \qquad \varphi_0(x) = 1. \tag{2.2}$$

The *Fibonacci polynomials* are defined by the same recurrence relation but different initial conditions, i.e.

$$F_n(x) = x F_{n-1}(x) + F_{n-2}(x) \qquad F_{-1}(x) = F_0(x) = 1. \tag{2.3}$$

Then we have

$$F_1(x) = x + 1$$

$$F_2(x) = x^2 + x + 1$$

$$F_3(x) = x^3 + x^2 + 2x + 1$$

$$F_4(x) = x^4 + x^3 + 3x^2 + 2x + 1$$

$$F_5(x) = x^5 + x^4 + 4x^3 + 3x^2 + 3x + 1$$

$$F_6(x) = x^6 + x^5 + 5x^4 + 4x^3 + 6x^2 + 3x + 1$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

Note that $F_k(1) = f_k$, $k \in N$, are the *Fibonacci numbers*.

By definition (2.3) a table for the coefficients of $F_n(x)$ can be obtained by using the following law of construction of squares, by means of which the subsequent table can be completely derived

(2.4)

a	b
a	$a+b$

c	d
c	$c+d$

1	x	x^2	x^3	x^4	x^5	x^6	x^7	x^8	x^9
1	0								
1	1	0							
1	1	1	0						
1	2	1	1	0					
1	2	3	1	1	0				
1	3	3	4	1	1	0			
1	3	6	4	5	1	1	0		
1	4	6	10	5	6	1	1	0	
1	4	10	10	15	6	7	1	1	0

....

Theorem 1. *For any $n \geqslant -1$ we can write*

$$F_n(x) = \varphi_n(x) + \varphi_{n-1}(x). \tag{2.5}$$

Proof. Use induction and verify initial conditions.

Formula (2.5) can be generalized in different ways.

A first generalization can be obtained by induction, by writing

$$F_n(x) = f_k \varphi_{n-k}(x) + f_{k-1} \varphi_{n-k-1}(x). \tag{2.6}$$

A second generalization can be obtained by using the *isobaricity property* of the Lucas polynomials of the second kind $\Phi_n(tx, -t^2) = t^n \Phi_n(x, -1)$, $\forall t \neq 0$ (see e.g. [4]). Namely

$$F_n(x) = \frac{\Phi_n(tx, -t^2) + t\Phi_{n-1}(tx, -t^2)}{t^n} \qquad \forall t \neq 0. \tag{2.7}$$

By putting $t = 1$, formula (2.5) follows.

3 - Reflection properties

By using the *reflection property* of the Lucas polynomials of the second kind (see [4]), we have:

$$F_{-n}(x) = F_{1,-n}(x, -1) + F_{1,-n-1}(x, -1)$$

$$F_{1,-n}(x, -1) = F_{1,n-2}(-x, -1) \quad \text{i.e.} \quad \varphi_{-n}(x) = \varphi_{n-2}(-x) \quad \forall n > 1$$

and consequently

$$F_{-n}(x) = \varphi_{n-2}(-x) + \varphi_{n-1}(-x) = F_{n-1}(-x).$$

Theorem 2. *For any $n \in \boldsymbol{Z}$ we can write*

$$F_{-n}(x) = F_{n-1}(-x) \tag{3.1}$$

Remark. Note that by using formulas (3.1) the result of Theorem 1 can be extended to be true for all $n \in \boldsymbol{Z}$.

4 - Generating functions and integral representations

By using the generating function of the generalized Lucas polynomials of the second kind (see e.g. [4])

$$\sum_{n=0}^{\infty} \Phi_{n+r-2}(u_1, u_2, \dots, u_r)\, t^n = \frac{1}{1 - u_1 t + \dots + (-1)^r u_r t^r}$$

and the reflection properties of Section 3, we find

Theorem 3. *The generating functions of the Fibonacci polynomials are given by*

$$\sum_{n=0}^{\infty} F_n(x)\, t^n = \frac{1+t}{1 - xt - t^2} \tag{4.1}$$

$$\sum_{n=-1}^{-\infty} F_n(x)\, t^n = -\frac{1+t}{1 - xt - t^2}\,. \tag{4.2}$$

By the *integral representation* for the Lucas polynomials of the second kind (see [4]) we deduce

Theorem 4. *Denote by γ a circle (if $n \geqslant -1$) or an annulus (if $n < -1$), with center at the origin surrounding the zeros of $\lambda^2 - x\lambda - 1 = 0$. Then the following integral representation for the Fibonacci polynomials is true*

$$F_n(x) = \int_{+\gamma} \frac{\lambda^n (\lambda + 1)}{\lambda^2 - x\lambda - 1}\, d\lambda\,. \tag{4.3}$$

5 - Matrix representation and location of the zeros

Consider for any $n \geqslant 1$, the $n \times n$ matrix

$$\mathcal{A} = \begin{bmatrix} -1 & -i & 0 & \cdots & 0 & 0 & 0 \\ -i & 0 & -i & \cdots & 0 & 0 & 0 \\ 0 & -i & 0 & \cdots & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & -i & 0 & -i \\ 0 & 0 & 0 & \cdots & 0 & -i & 0 \end{bmatrix}. \tag{5.1}$$

Theorem 5. *The zeros of $F_n(\lambda)$ ($n \geqslant 1$) are the eigenvalues of $\mathcal{A}$.*

Proof. It is sufficient to note that the function

$$\det(x\mathfrak{I} - \mathfrak{A}) = \begin{bmatrix} x+1 & i & 0 & \cdots & 0 & 0 & 0 \\ i & x & i & \cdots & 0 & 0 & 0 \\ 0 & i & x & \cdots & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & i & x & i \\ 0 & 0 & 0 & \cdots & 0 & i & x \end{bmatrix} \tag{5.2}$$

verifies the same recurrence relation (2.3) of the Fibonacci polynomials. This can be checked developing the above determinant with respect to the last column and applying the same method to the resulting determinant which is multiplied by $-i$. Furthermore the first two determinants (for $n = 1$ and $n = 2$) coincide respectively with $F_1(x)$ and $F_2(x)$.

By using the Gershgorin's theorem a further result immediately follows

Theorem 6. *If* $F_n(x) = 0$, *then* $|x| < 2$.

Corollary. *All the zeros of the polynomials* $\mathcal{F}_n(x) = F_n(2x)$ *are inside the unit circle.*

6 - Analogues of Christoffel-Darboux formulas and related formulas

Theorem 7. *For any* $n \geqslant 1$ *the following identities hold true*

$$(x-y)\sum_{k=0}^{n}(-1)^{n-k}F_k(x)F_k(y) = F_{n+1}(x)F_n(y) - F_{n+1}(y)F_n(x)$$

$$\sum_{k=0}^{n}(-1)^{n-k}F_k^2(x) = F'_{n+1}(x)F_n(x) - F_{n+1}(x)F'_n(x)$$

$$F_n^2(x) = \sum_{k=0}^{n}F'_{k+1}(x)F_k(x) - F_{k+1}(x)F'_k(x).$$

Proof. The above identities can be obtained by applying the same technique used in the proof of the classical Christoffel-Darboux formulas.

For the second kind Lucas polynomials (2.2) a similar argument leads to the following identities

$$\begin{aligned} &y\,\varphi_{2n-2}(y)\,\varphi_{2n-1}(x) - x\,\varphi_{2n-2}(x)\,\varphi_{2n-1}(y) \\ &= \varphi_{2n-1}(y)\,\varphi_{2n-3}(x) - \varphi_{2n-1}(x)\,\varphi_{2n-3}(y) \end{aligned} \tag{6.1}$$

$$
\begin{aligned}
&y\,\varphi_{2n}(y)\,\varphi_{2n-1}(x) - x\,\varphi_{2n}(x)\,\varphi_{2n-1}(y) \\
&= \varphi_{2n+1}(y)\,\varphi_{2n-1}(x) - \varphi_{2n+1}(x)\,\varphi_{2n-1}(y).
\end{aligned} \tag{6.2}
$$

7 - Representation of the even and the odd part of $F_m(x)$

By induction, and using the recurrence relation and initial conditions (2.2), it is easy to see that φ_{2n-1} is an odd function, and φ_{2n} is an even function. Then we can write

$$
\begin{aligned}
F_{2n-1}(x) &= \varphi_{2n-2}(x) + \varphi_{2n-1}(x) = \varphi_{2n-2}(x) + x\,\Big(\frac{\varphi_{2n-1}(x)}{x}\Big) \\
F_{2n}(x) &= \varphi_{2n}(x) + \varphi_{2n-1}(x) = \varphi_{2n}(x) + x\,\Big(\frac{\varphi_{2n-1}(x)}{x}\Big)
\end{aligned}
$$

i.e.:

$$
\begin{aligned}
\mathcal{E}(F_{2n-1}(x)) &= \varphi_{2n-2}(x) \qquad \mathcal{O}(F_{2n-1}(x)) = x\Big(\frac{\varphi_{2n-1}(x)}{x}\Big) \\
\mathcal{E}(F_{2n}(x)) &= \varphi_{2n}(x) \qquad \mathcal{O}(F_{2n}(x)) = x\Big(\frac{\varphi_{2n-1}(x)}{x}\Big)
\end{aligned}
$$

where $\mathcal{E}$, $\mathcal{O}$ denote the even-part, the odd-part, respectively.

8 - Stability of $F_n(x)$

We remember here a classical result for stability, which is due to A. Liénard and A. Chipart [12], and is equivalent to the Routh-Hurwitz conditions.

Let $f(x)$ be a polynomial. Put:

$$f(x) = \phi(x^2) + x\psi(x^2) = \mathcal{E}(f(x)) + \mathcal{O}(f(x)) \tag{8.1}$$

$$\mathcal{F}(x^2, y^2) = \frac{\phi(y^2)\,\psi(x^2) - \phi(x^2)\,\psi(y^2)}{y^2 - x^2} \tag{8.2}$$

$$x^2 y^2\, \mathcal{F}(x^2, y^2) = \sum c_{\alpha\beta}(x^2)^{\alpha}(y^2)^{\beta} \tag{8.3}$$

$$\Theta(\Delta_1, \Delta_2, \ldots, \Delta_m) = \sum c_{\alpha\beta}\Delta_a\Delta_b\,. \tag{8.4}$$

Then, a necessary and sufficient condition in order that the polynomial $f(x)$

is *stable* (i.e. all zeros of it have a negative real part) is given by the conditions:

I. the quadratic form Θ is positive definite

II. the polynomial $\phi(x^2)$ is complete and all of its coefficients have the same sign as the leading coefficient of f.

In order to apply this result to the Fibonacci polynomials $F_m(x)$, we must distinguish two cases:

Case 1. $m = 2n - 1$ (odd)

$$F_{2n-1}(x) = \varphi_{2n-2}(x) + \varphi_{2n-1}(x) = \varphi_{2n-2}(x) + x\left(\frac{\varphi_{2n-1}(x)}{x}\right).$$

By using formula (6.1) we can write:

$$x^2 y^2 \mathcal{F}(x^2, y^2) = \frac{xy}{y^2 - x^2}\left[y\varphi_{2n-2}(y)\,\varphi_{2n-1}(x) - x\varphi_{2n-2}(x)\,\varphi_{2n-1}(y)\right]$$

$$= \frac{xy}{y^2 - x^2}\left[\varphi_{2n-1}(y)\,\varphi_{2n-3}(x) - \varphi_{2n-1}(x)\,\varphi_{2n-3}(y)\right].$$

Case 2. $m = 2n$ (even)

$$F_{2n}(x) = \varphi_{2n}(x) + \varphi_{2n-1}(x) = \varphi_{2n}(x) + x\left(\frac{\varphi_{2n-1}(x)}{x}\right).$$

By using formula (6.2) we can write

$$x^2 y^2 \mathcal{F}(x^2, y^2) = \frac{xy}{y^2 - x^2}\left[y\varphi_{2n}(y)\,\varphi_{2n-1}(x) - x\varphi_{2n}(x)\,\varphi_{2n-1}(y)\right]$$

$$= \frac{xy}{y^2 - x^2}\left[\varphi_{2n+1}(y)\,\varphi_{2n-1}(x) - \varphi_{2n+1}(x)\,\varphi_{2n-1}(y)\right].$$

Note that by Sturm's theorem $F_n(x)$ can not have *positive real roots*. The absence of complex roots with positive real part can be proved by using the condition of A. Liénard and A. Chipart, or by direct inspection for the first values of the index n.

The stability condition has been checked for the consecutive values of n from 1 to 12.

By using Bendixon theorem it can be easily seen that the zeros of the Fibonacci polynomials always belong to the rectangle of the complex plane with vertexes at the points $(0, 2i)$, $(-1, 2i)$, $(-1, -2i)$, $(0, -2i)$.

This leads to the

Conjecture. *The Fibonacci polynomials $F_n(x)$ verify the stability condition for any $n \in N$.*

The problem is to prove that the zeros of the considered Fibonacci polynomials can never be purely imaginary.

References

[1] R. J. BEERENDS, *Chebyshev polynomials in several variables and the radial part of the Laplace-Beltrami operator*, Trans. Am. Math. Soc. **328** (1991), 779-814.

[2] M. BICKNELL and V. E. HOGGATT JR., *Roots of Fibonacci Polynomials*, Fibonacci Quarterly **11** (1973), 271-274.

[3] M. BRUSCHI and P. E. RICCI, *An explicit formula for f(A) and the generating function of the generalized Lucas polynomials*, Siam J. Math. Anal. **13** (1982), 162-165.

[4] M. BRUSCHI e P. E. RICCI, *I polinomi di Lucas e di Tchebycheff in più variabili*, Rend. Mat. Appl. **13** (1980), 507-530.

[5] A. DI CAVE e P. E. RICCI, *Sui polinomi di Bell ed i numeri di Fibonacci e di Bernoulli*, Le Matematiche **35** (1980), 84-95.

[6] K. B. DUNN and R. LIDL, *Multi-dimensional generalizations of the Chebyshev polynomials I-II*, Proc. Japan Acad. **56** (1980), 154-165.

[7] B. GERMANO, P. NATALINI and P. E. RICCI, *Computing the moments of the density of zeros for orthogonal polynomials*, Computers Math. Applic. **30** (1995), 69-81.

[8] T. H. KOORNWINDER, *Orthogonal polynomials in two variables which are eigenfunctions of two algebraically independent partial differential operators I-II*, Kon. Ned. Akad. Wet. **77**, 46-66.

[9] T. H. KOORNWINDER, *Orthogonal polynomials in two variables which are eigenfunctions of two algebrically independent partial differential operators*, III-IV, Indag. Math. **36** (1974), 357-381.

[10] R. LIDL and C. WELLS, *Chebyshev polynomials in several variables*, J. Reine Angew. Math. **255** (1972), 104-111.

[11] R. LIDL, *Tschebyscheffpolynome in mehreren variabeln*, J. Reine Angew. Math. **273** (1975), 178-198.

[12] A. LIÉNARD et A. CHIPART, *Sur le signe de la partie réelle des racines d'une équation algébrique*, Journ. de Math. Pures Appl. **10** (1914).

[13] É. Lucas, *Théorie des nombres*, Gauthier-Villars, Paris 1891.
[14] P. Natalini, *Sul calcolo dei momenti della densità degli zeri dei polinomi ortogonali classici e semi-classici*, Calcolo, to appear.
[15] I. V. V. Raghavacharyulu and A. R. Tekumalla, *Solution of the difference equations of generalized Lucas polynomials*, J. Math. Phys. **13** (1972), 321-324.
[16] P. E. Ricci, *Sum rules for zeros of polynomials and generalized Lucas Polynomials*, J. Math. Phys. **34** (1993), 4884-4891.

Sommario

Partendo dai polinomi di Lucas generalizzati, vengono dedotte alcune proprietà di una particolare classe di polinomi di Fibonacci, i cui valori, nell'origine, generano la classica successione dei numeri di Fibonacci.

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