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Aspects of the geometry of the Jacobi operator (**)

1 - Introduction

A central topic in Riemannian geometry is the study of curvature. One of the useful tools for these studies are *Jacobi vector fields*. Recall that a Jacobi vector field is a solution of the vector-valued Jacobi equation

$$Y'' + R(Y, \dot{\gamma})\dot{\gamma} = 0$$

along a geodesic γ in a Riemannian manifold (M, g), where R denotes the Riemannian curvature tensor of M with the convention $R(X, Y) = [\nabla_X, \nabla_Y] - \nabla_{[X, Y]}$. The symmetric tensor field

$$R_\gamma = R(\cdot, \dot{\gamma})\dot{\gamma}$$

is called the *Jacobi operator along* γ. The Jacobi operator plays also, via Jacobi fields, a central role in the study of the intrinsic and extrinsic geometry of geodesic spheres, tubes, and of reflections with respect to points, curves and submanifolds.

In general, the explicit determination of the Jacobi fields is a very difficult problem except for Riemannian manifolds with a simple curvature tensor. But several properties of a Riemannian manifold may be discovered via those of its Jacobi operators without knowing explicitly the Jacobi fields. To give an example, consider the class $\mathfrak{S}$ of all Riemannian locally symmetric spaces. In [4] the following well-known property was considered: a Riemannian manifold

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(M, g) belongs to $\mathfrak{S}$ if and only if for any geodesic γ in M the following two properties hold:

C. R_γ has constant spectrum

P. R_γ is diagonalizable by a parallel orthonormal frame field along γ.

This characterization of locally symmetric spaces suggests to study the two classes $\mathfrak{C}$ and $\mathfrak{P}$ consisting of all Riemannian manifolds for which just the respective condition **C** and **P** holds. More precisely, a Riemannian manifold (M, g) is called a *$\mathfrak{C}$-space* if for any geodesic γ in M the associated Jacobi operator R_γ has constant eigenvalues, and M is called a *$\mathfrak{P}$-space* if for any geodesic γ in M the Jacobi operator R_γ can be diagonalized by a parallel orthonormal frame field along γ.

There are also various characterizations of locally symmetric spaces in terms of geometric properties of small geodesic spheres. We concentrate here on two of them. Consider a non-stationary geodesic γ in a Riemannian manifold (M, g) parametrized by arc length and so that $m = \gamma(0)$ is defined.

First, for sufficiently small $r \in \boldsymbol{R}_+$, the geodesic spheres $G_p(r)$ and $G_q(r)$ centered at $p = \gamma(r)$ and $q = \gamma(-r)$, respectively, and with radius r are smooth hypersurfaces of M and tangent to each other at m. In **[51]** it was proved that M is locally symmetric if and only if for any such configuration of geodesic spheres the shape operators $S_p(m)$ and $S_q(m)$ of $G_p(r)$ and $G_q(r)$, respectively, at m coincide. The latter condition means that $S_p(m)$ and $S_q(m)$ have the same eigenvalues and are simultaneously diagonalizable. Splitting up these conditions leads to the two classes $\mathfrak{T}\mathfrak{C}$ and $\mathfrak{T}\mathfrak{P}$. More precisely, a Riemannian manifold (M, g) is said to be a *$\mathfrak{T}\mathfrak{C}$-space* if for any such configuration $G_p(r)$ and $G_q(r)$ have the same principal curvatures at m, and M is called a *$\mathfrak{T}\mathfrak{P}$-space* if for any such configuration $S_p(m)$ and $S_q(m)$ are simultaneously diagonalizable.

Secondly, for sufficiently small $r \in \boldsymbol{R}_+$, the geodesic sphere $G_m(r)$ centered at m and with radius r is a smooth hypersurface of M. Denote by s_m a local geodesic symmetry of M at m. In **[40]** it was shown that a Riemannian manifold (M, g) is locally symmetric if and only if

$$s_{m*} \circ S_m(p) = S_m(q) \circ s_{m*}$$

for any such configuration. As in the first case one can split up the latter condition into two, leading to the classes $\mathfrak{S}\mathfrak{C}$ and $\mathfrak{S}\mathfrak{P}$. More precisely, a Riemannian manifold (M, g) is called an *$\mathfrak{S}\mathfrak{C}$-space* if and only if for any small geodesic sphere in M the principal curvatures (counted with multiplicities) are the same at antipodal points, and M is called an $\mathfrak{S}\mathfrak{P}$-*space* if for any such configuration $S_m(p)$ and $s_{m*}^{-1} \circ S_m(q) \circ s_{m*}$ are simultaneously diagonalizable.

Our general project is to find examples, derive classifications and nice geometrical characteristic properties, study the geometry and the relations between these classes and other ones already discussed in the literature. In the following we provide a survey about our work related to these questions that has been done so far. We always suppose that a Riemannian manifold (M, g) is connected and smooth unless stated otherwise.

2 - The classes $\mathfrak{P}$, $\mathfrak{T}\mathfrak{P}$, and $\mathfrak{S}\mathfrak{P}$

We start with two useful characterizations of *real analytic* $\mathfrak{P}$-*spaces* [4]. Each of the following two conditions is necessary and sufficient in order that a real analytic Riemannian manifold M is a $\mathfrak{P}$-space:

- for any geodesic γ in M the associated Jacobi operator R_γ and its covariant derivative $R_\gamma' = (\nabla_{\dot\gamma} R)(\cdot, \dot\gamma)\dot\gamma$ commute
- all small geodesic spheres in M are curvature-adapted.

Note that a hypersurface of a Riemannian manifold is curvature-adapted if its shape operator commutes with the Jacobi operator with respect to the unit normal vectors. A detailed discussion of curvature-adapted submanifolds may be found in [5].

The fundamental result about these three classes is that

$$\mathfrak{P} = \mathfrak{T}\mathfrak{P} = \mathfrak{S}\mathfrak{P}$$

in the class of analytic manifolds [1], [6].

As regards examples, we first mention that any two-dimensional Riemannian manifold is a $\mathfrak{P}$-space. This is a trivial consequence of $R_\gamma \dot\gamma = 0$ and the symmetry of R_γ. The local classification of all three-dimensional analytic $\mathfrak{P}$-spaces is known [4]. Apart from the spaces of constant curvature certain warped products and triply orthogonal systems of surfaces appear in this classification. From the first characterization of $\mathfrak{P}$-spaces given above it follows also that the Riemannian product of two analytic $\mathfrak{P}$-spaces is again a $\mathfrak{P}$-space. Other sporadic examples of $\mathfrak{P}$-spaces are known in the class of semi-symmetric spaces [12], [16].

From our point of view the main open problems regarding $\mathfrak{P}$-spaces are to find further examples (perhaps there are some among the Stäckel manifolds), to continue the study of their geometry and to derive further (partial) classifications.

3 - The classes $\mathfrak{C}$, $\mathfrak{TC}$, and $\mathfrak{SC}$

The basic references for the results presented in this section are [4], [6], and [1].

The first important remark is that the Ricci tensor of any space belonging to $\mathfrak{C}$, $\mathfrak{TC}$ or $\mathfrak{SC}$ is invariant under the geodesic flow. Thus, any $\mathfrak{C}$-, $\mathfrak{TC}$- or $\mathfrak{SC}$-space has constant scalar curvature and is analytic in normal coordinates.

In dimensions less or equal than three the classes $\mathfrak{C}$, $\mathfrak{TC}$ and $\mathfrak{SC}$ coincide. A two-dimensional Riemannian manifold is a $\mathfrak{C}$-space if and only if it has constant curvature. Further, a three-dimensional Riemannian manifold is a $\mathfrak{C}$-space if and only if it is locally symmetric or locally isometric to a naturally reductive Riemannian homogeneous space. Note that the three-dimensional simply connected naturally reductive Riemannian homogeneous spaces are the symmetric spaces $\boldsymbol{R}^3, S^3, \boldsymbol{R}H^3, S^2 \times \boldsymbol{R}, \boldsymbol{R}H^2 \times \boldsymbol{R}$ and the Lie groups $SU(2)$, $\widetilde{SL(2, \boldsymbol{R})}$, H_3 (Heisenberg group) equipped with suitable left-invariant Riemannian metrics [32]. In [7] we gave some geometric realizations of the above naturally reductive Lie groups: the naturally reductive metrics on $SU(2) = S^3$ are the induced metrics on the geodesic spheres in $\boldsymbol{C}P^2$ and $\boldsymbol{C}H^2$; the naturally reductive metrics on $\widetilde{SL(2, \boldsymbol{R})}$ are realized by the Riemannian universal covering spaces of tubes about totally geodesic $\boldsymbol{C}H^1$ in $\boldsymbol{C}H^2$; and the naturally reductive metrics on H_3 are realized by the horospheres in $\boldsymbol{C}H^2$.

The most useful characterization of $\mathfrak{C}$-spaces says that a Riemannian manifold (M, g) is a $\mathfrak{C}$-space if and only if for any geodesic γ in M there exists a skew-symmetric tensor field T_γ along γ so that the associated Jacobi operator R_γ satisfies the *Lax-pair equation*

$$R'_\gamma = [R_\gamma, T_\gamma]$$

along γ. By means of this characterization we define the subclass $\mathfrak{C}_0$ of $\mathfrak{C}$ consisting of all $\mathfrak{C}$-spaces for which T_γ might always be chosen parallel along γ. It is possible to choose T_γ parallel if and only if the operators $R_\gamma(t)$ are related to $R_\gamma(t_0)$ for some fixed t_0 via conjugation with a one-parameter subgroup of the respective orthogonal group.

Using the preceding characterization and the theory of homogeneous structures [48] we immediately get that any *naturally reductive Riemannian homogeneous space* is a $\mathfrak{C}$-space. It is clear that if a geodesic γ is the orbit of a one-parameter group of isometries, the eigenvalues of R_γ must be constant. Thus any Riemannian g.o. space provides an example of a $\mathfrak{C}$-space. Note that any naturally reductive Riemannian homogeneous space is a Riemannian g.o. space and recall that a *Riemannian g.o.* space is a Riemannian manifold for which every geodesic is the orbit of a one-parameter subgroup of the group of isometries of the space. Riemannian g.o. spaces have been studied in [39], where also classifications up to dimension five and a partial classification for dimension six were achieved. The simply connected Riemannian g.o. spaces up to dimension five are precisely the naturally reductive Riemannian homogeneous spaces which have been classified in [32], [34], [38]. In dimension six there are examples of Riemannian g.o. spaces which are in no way naturally reductive. Note that we have the even stronger result that any Riemannian g.o. space is a $\mathfrak{C}_0$-space, but we will see later that the converse does not hold.

Using Killing tensors and geodesic flows it also follows that any commutative space is a $\mathfrak{C}$-space. A *commutative space M* is a Riemannian homogeneous space for which the algebra of all differential operators, which are invariant under the connected component of the full isometry group of M, is commutative. The simply connected commutative spaces up to dimension five are precisely the naturally reductive Riemannian homogeneous spaces [32], [36], [9]. All commutative spaces known to us are also Riemannian g.o. spaces, and we do not know whether this relation is true in general or not. But there are examples (for instance, a particular generalized Heisenberg group, see below) of Riemannian g.o. spaces which are not commutative.

By means of power series expansions for the shape operators of small geodesic spheres it can be proved that both classes $\mathfrak{TC}$ and $\mathfrak{SC}$ are contained in $\mathfrak{C}$. So we have the general inclusions:

two-point homogeneous spaces → symmetric spaces → naturally reductive spaces → Riemannian g.o. spaces → $\mathfrak{C}_0$-spaces → $\mathfrak{C}$-spaces

commutative spaces → $\mathfrak{C}$-spaces

$\mathfrak{TC}$-spaces → $\mathfrak{C}$-spaces $\mathfrak{SC}$-spaces → $\mathfrak{C}$-spaces.

We now describe some special examples (see [3] for the various references). The first ones are geodesic spheres and certain tubes in two-point homogeneous spaces. Geodesic spheres in $\boldsymbol{R}^n$, S^n, $\boldsymbol{RP}^n$, and $\boldsymbol{RH}^n$ are symmetric spaces, but not those in $\boldsymbol{CP}^n$, $\boldsymbol{CH}^n$, $\boldsymbol{HP}^n$, $\boldsymbol{HH}^n$, $\boldsymbol{Cay}\ P^2$, and $\boldsymbol{Cay}\ H^2$. Any geodesic sphere in a complex or quaternionic projective or hyperbolic space is a naturally reductive Riemannian homogeneous space, a property that does not hold for the geodesic spheres in Cayley projective or hyperbolic plane. It is still an open problem whether the latter ones are Riemannian g.o. spaces or not. We also remark that any geodesic sphere in a two-point homogeneous space is a commutative space (see also section 7). Horospheres in $\boldsymbol{CH}^n$ and $\boldsymbol{HH}^n$ are naturally reductive, and any horosphere in a non-compact two-point homogeneous space is commutative. Any tube around a totally geodesic $\boldsymbol{CP}^k$ in $\boldsymbol{CP}^n$, or $\boldsymbol{CH}^k$ in $\boldsymbol{CH}^n$, or $\boldsymbol{HP}^k$ in $\boldsymbol{HP}^n$ ($1 \leq k \leq n-2$), or $\boldsymbol{HH}^k$ in $\boldsymbol{HH}^n$ ($1 \leq k \leq n-1$) is naturally reductive and commutative. The same holds for the Riemannian universal covering space of any tube around a totally geodesic $\boldsymbol{CH}^{n-1}$ in $\boldsymbol{CH}^n$. Also, any tube around $\boldsymbol{Cay}\ H^1$ in $\boldsymbol{Cay}\ H^2$ is a commutative space. (Note that in the last four cases we mean commutativity with respect to the full isometry group.)

Further special examples arise from the theory of contact geometry and flow geometry [10], [11], [23], [24], [25], [47], and [14] for further references. Any Sasakian space form is a $\mathfrak{T}\mathfrak{C}$-space and also an $\mathfrak{S}\mathfrak{C}$-space [1]. Sasakian space forms belong to the larger class of φ-symmetric spaces (Sasakian manifolds with complete characteristic field such that the reflections with respect to the integral curves of that field are global isometries), which itself are special examples of Killing-transversally symmetric spaces (Riemannian manifolds equipped with a complete unit Killing vector field such that the reflections with respect to the flow lines of that field are global isometries). The simply connected manifolds among the latter ones provide also examples of naturally reductive Riemannian homogeneous spaces. It is worthwhile to mention that all these spaces (Sasakian space forms, φ-symmetric spaces, Killing-transversally symmetric spaces) are reflection spaces in the sense of O. Loos [41].

A further consideration arises from the conjecture of Osserman [42] stating that every Riemannian manifold with globally constant eigenvalues for the Jacobi operators is locally isometric to a two-point homogeneous space. A manifold satisfying the hypothesis of this conjecture is called a *globally Osserman space* and is obviously a $\mathfrak{C}$-space. We refer to [22] for further results on globally Osserman spaces. In this paper the notion of *pointwise Osserman spaces* is introduced and related, for four-dimensional manifolds, to self-dual Einstein manifolds. In this context it is also worthwhile to mention that it is conjectured in [49] that Riemannian manifolds all of whose small geodesic spheres are isoparametric are locally isometric to a two-point homogeneous space. In [22] it is proved that the isoparametric condition implies that the manifold is a globally Osserman space. An intrinsic analogue concerning the Ricci tensor of small geodesic spheres is treated in [22].

All examples of $\mathfrak{C}$-spaces known so far are locally homogeneous manifolds and one of the basic problems is whether this is a general fact: Is any $\mathfrak{C}$-space locally homogeneous?

4 - Further motivating results

As mentioned above the classes $\mathfrak{T}\mathfrak{C}$ and $\mathfrak{S}\mathfrak{C}$ are both contained in $\mathfrak{C}$. By now we do not know whether these inclusions are strict or not. Also the relation between $\mathfrak{TC}$ and $\mathfrak{SC}$ is not clear, but a partial result was obtained in [1] and says that if M is a commutative space or a $\mathfrak{C}_0$-space, then M is a $\mathfrak{TC}$-space if and only if it is an $\mathfrak{SC}$-space.

An interesting class of Riemannian manifolds is formed by the *D'Atri spaces*, that is, Riemannian manifolds whose local geodesic symmetries are volume-preserving up to sign. These spaces were first introduced and studied in [19], [20], [21]. The following relations between D'Atri spaces and the former classes are known:

$$\text{Riemannian g.o. spaces} \to \text{D'Atri spaces [37]}$$

$$\mathfrak{C}_0\text{-spaces} \to \text{D'Atri spaces [1]} \qquad \text{commutative spaces} \to \text{D'Atri spaces [35]}$$

$$\mathfrak{TC}\text{-spaces} \to \text{D'Atri spaces [6]} \qquad \mathfrak{SC}\text{-spaces} \to \text{D'Atri spaces [1]}.$$

None of the above inclusions is strict. An obvious problem, on which we come back again later, is to study the relation between the $\mathfrak{C}$- and D'Atri spaces. Nice candidates for these studies are Riemannian harmonic spaces, generalized Heisenberg groups, and weakly symmetric spaces. All these space are indeed D'Atri spaces, but what can be said about their relation with the other classes discussed above? Many of these and related questions have been settled in [3] (see also [2] for a short summary), on which we report in the following two sections. The manifolds we are considering first are the generalized Heisenberg groups and the Damek-Ricci harmonic spaces. The first ones are two-step nilpotent Lie groups, the other ones are solvable Lie groups arising as one dimensional extensions of the nilpotent ones. These Lie groups have previously been studied in detail in harmonic analysis. In Riemannian geometry these spaces have been proved useful as they provide nice examples and counterexamples, for instance regarding the Lichnerowicz conjecture and related open problems on harmonic spaces [18], [49], in spectral geometry [26], [46], and in the study of the relation between the classes discussed in this paper [3]. We begin with the generalized Heisenberg groups.

5 - Generalized Heisenberg groups

We refer to [3] for details.

Generalized Heisenberg groups were introduced by A. Kaplan [29] and are defined as follows. Let $\mathfrak{v}$ and $\mathfrak{z}$ be real vector spaces of dimensions n, $m \in N_+$, respectively, and β: $\mathfrak{v} \times \mathfrak{v} \to \mathfrak{z}$ a skew-symmetric bilinear map. We endow the direct sum $\mathfrak{n} = \mathfrak{v} \oplus \mathfrak{z}$ with an inner product $\langle , \rangle$ such that $\mathfrak{v}$ and $\mathfrak{z}$ are perpendicular and define an $\boldsymbol{R}$-algebra homomorphism

$$J\colon \mathfrak{z} \to \mathrm{End}(\mathfrak{v}) \qquad Z \mapsto J_Z$$

by $$\forall U, V \in \mathfrak{v} \qquad \forall Z \in \mathfrak{z} \qquad \langle J_Z U, V \rangle = \langle \beta(U, V), Z \rangle$$

and a Lie algebra structure on $\mathfrak{n}$ by

$$\forall U, V \in \mathfrak{v}, \qquad \forall X, Y \in \mathfrak{z} \qquad [U + X, V + Y] = \beta(U, V).$$

The Lie algebra $\mathfrak{n}$ is said to be a *generalized Heisenberg algebra* if

$$\forall Z \in \mathfrak{z} \qquad J_Z^2 = -\langle Z, Z \rangle id_{\mathfrak{v}}.$$

The attached simply connected Lie group N, endowed with the induced left-invariant Riemannian metric g, is called a *generalized Heisenberg group.*

The classification of generalized Heisenberg groups is known and related to the classification of representations of Clifford algebras of finite-dimensional vector spaces equipped with negative definite quadratic forms. We first recall some known properties of generalized Heisenberg groups.

– Every generalized Heisenberg group is a two-step nilpotent Lie group.

– Every generalized Heisenberg group is diffeomorphic to the Euclidean space of the respective dimension. For example, the Lie exponential map provides a suitable diffeomorphism.

– Every generalized Heisenberg group is a D'Atri space [**31**].

– A generalized Heisenberg group N is a Riemannian g.o. space if and only if

i. $\dim \mathfrak{z} \in \{1, 2, 3\}$, or

ii. $\dim \mathfrak{z} = 5$ and $\dim N = 13$, or

iii. $\dim \mathfrak{z} = 6$ and $\dim N = 14$, or

iv. $\dim \mathfrak{z} = 7$ and

(1) $\dim N = 15$, or

(2) $\dim N \in \{23, 31\}$ and $\mathfrak{v}$ is isotypic [**44**].

– A generalized Heisenberg group is a commutative space if and only if it is a Riemannian g.o. space, except when $\dim \mathfrak{z} = 7$ and $\dim N = 31$ [**43**].

– A generalized Heisenberg group is naturally reductive if and only if $\dim \mathfrak{z} \in \{1, 3\}$ [31], [48].

– A generalized Heisenberg group is said to satisfy the J^2-*condition* if for all X, $Y \in \mathfrak{z}$ with $\langle X, Y \rangle = 0$ and all non-zero $U \in \mathfrak{v}$ there exists a $Z \in \mathfrak{z}$ so that $J_X J_Y U = J_Z U$. A generalized Heisenberg group satisfies the J^2-condition if and only if it is isometric to a horosphere in $\boldsymbol{C}H^{n+1}$, $\boldsymbol{H}H^{n+1}$ or $\boldsymbol{Cay}\,H^2$ [17].

– Every geodesic in a generalized Heisenberg group lies in a suitable totally geodesically embedded three-dimensional Heisenberg group [30], [31].

We continue with some new properties of generalized Heisenberg groups. First we mention that it is possible to write down explicitly the Jacobi operators R_γ, its covariant derivatives R_γ', and the Jacobi vector fields (vanishing at a point). This detailed and complicated work leads to the following properties:

– Any generalized Heisenberg group is irreducible and non-symmetric, has non-parallel Ricci tensor, and hence is not a harmonic space.

– The Ricci tensor of any generalized Heisenberg group is a Killing tensor, or equivalently, is invariant under the geodesic flow.

– None of the generalized Heisenberg groups carries a Kähler structure which is compatible with its left-invariant Riemannian metric.

– Any generalized Heisenberg group is a $\mathfrak{C}$-space and hence, since it is non-symmetric, never a $\mathfrak{P}$-space. This result is obtained by an explicit calculation of the eigenvalues of the Jacobi operators R_γ.

– Any generalized Heisenberg group is a $\mathfrak{C}_0$-space. This is proved by an explicit calculation of a parallel skew-symmetric tensor field T_γ along any geodesic γ so that $R_\gamma' = [R_\gamma, T_\gamma]$, and this shows that the rotational behaviour of the eigenspaces of R_γ is described by a one-parameter subgroup of the respective orthogonal group.

– Any generalized Heisenberg group is a $\mathfrak{T}\mathfrak{C}$-space. Here we use the explicit expressions of the Jacobi vector fields to determine the shape operators of small geodesic spheres. As any generalized Heisenberg group is in $\mathfrak{C}_0$, it also follows that any generalized Heisenberg group is an $\mathfrak{S}\mathfrak{C}$-space.

– From the explicit expressions of the Jacobi vector fields it is possible to determine all conjugate points in generalized Heisenberg groups.

– The scalar curvature of any geodesic sphere in a generalized Heisenberg group is the same at antipodal points.

– On any generalized Heisenberg group the eigenvalues of the metric tensor with respect to normal coordinates are the same at antipodal points (with respect to the center of the normal coordinates).

– The left-invariant distribution $\mathfrak{v}$ is not integrable, whereas the left-invariant distribution $\mathfrak{z}$ is. Its maximal leaves are totally geodesic and isometric to $\boldsymbol{R}^m$ with its standard Euclidean metric.

These properties also give alternative proofs of some known results stated earlier. Moreover, it follows that:

– there exist $\mathfrak{C}_0$-spaces which are not Riemannian g.o. spaces and not commutative spaces

– the property stating that the eigenvalues of the metric with respect to normal coordinates are the same at antipodal points , does not characterize commutative spaces, Riemannian g.o. spaces or naturally reductive Riemannian homogeneous spaces.

6 - DR-spaces

We refer to [3] for details.

The basic idea for the construction of the DR-spaces from generalized Heisenberg groups is to imitate the construction of the non-compact rank-one symmetric spaces as solvable Lie groups by means of the Iwasawa decomposition of their isometry groups.

Let $\mathfrak{n}$ be a generalized Heisenberg algebra, $\mathfrak{a}$ a one-dimensional real vector space and A a non-zero vector in $\mathfrak{a}$. We denote the inner product and the Lie bracket on $\mathfrak{n}$ by $\langle\,,\rangle_{\mathfrak{n}}$ and $[\,,]_{\mathfrak{n}}$, respectively, and define a new vector space

$$\mathfrak{s} = \mathfrak{n} \oplus \mathfrak{a}$$

as the direct sum of $\mathfrak{n}$ and $\mathfrak{a}$. Each vector in $\mathfrak{s}$ can be written in a unique way in the form $V + Y + sA$ with some $V \in \mathfrak{v}$, $Y \in \mathfrak{z}$ and $s \in \boldsymbol{R}$. We now define an inner product $\langle\,,\rangle$ and a Lie bracket $[\,,]$ on $\mathfrak{s}$ by

$$\langle U + X + rA,\ V + Y + sA\rangle = \langle U + X,\ V + Y\rangle_{\mathfrak{n}} + rs$$

$$[U + X + rA,\ V + Y + sA] = [U,\ V]_{\mathfrak{n}} + \frac{1}{2}rV - \frac{1}{2}sU + rY - sX\,.$$

In this way $\mathfrak{s}$ becomes a Lie algebra with an inner product. The attached simply connected Lie group, equipped with the induced left-invariant metric, is denoted by S and is called a *Damek-Ricci space*, or briefly, a DR-*space*.

Also here, we start with some known properties:

– Any DR-space is a solvable Lie group diffeomorphic to the Euclidean space of the respective dimension. Suitable diffeomorphisms are provided by the Lie exponential map and by the exponential map at a single point.

– Any DR-space is harmonic and hence an irreducible Einstein manifold [18].

– Any DR-space is a D'Atri space which is probabilistic commutative [33].

– Any DR-space is a homogeneous Hadamard manifold.

– A DR-space is a Riemannian symmetric space if and only if the attached generalized Heisenberg group satisfies the J^2-condition. Note that the symmetric DR-spaces are $\boldsymbol{C}H^{n+1}$, $\boldsymbol{H}H^{n+1}$ and $\boldsymbol{Cay}\,H^2$ [17].

– The associated generalized Heisenberg group of a DR-space S is isometric to any maximal leaf of the Riemannian horosphere foliation in S centered at the point at infinity corresponding to A. Each leaf of this foliation is an isoparametric hypersurface of S with constant principal curvatures $\frac{1}{2}$ and 1 and respective eigenspaces $\mathfrak{v}$ and $\mathfrak{z}$.

– Every geodesic in a DR-space lies in a suitable totally geodesically embedded $\boldsymbol{C}H^2$ with constant holomorphic sectional curvature -1 [17].

We then have the following new results:

– A DR-space S admits a nearly Kähler structure which is invariant under the group of left translations on S if and only if S is isometric to a complex hyperbolic space.

– The left-invariant distributions $\mathfrak{v}$ and $\mathfrak{v} \oplus \mathfrak{a}$ are not integrable. The left-invariant distribution $\mathfrak{z}$ is integrable and the induced foliation is Riemannian; each maximal leaf is a spherical submanifold (extrinsic sphere) of S with mean curvature vector A and isometric to $\boldsymbol{R}^m$ with its standard Euclidean metric. The left-invariant distribution $\mathfrak{a}$ is integrable and the induced foliation of S is not Riemannian; each maximal leaf is a totally geodesic submanifold of S isometric to $\boldsymbol{R}$. The left-invariant distribution $\mathfrak{z} \oplus \mathfrak{a}$ is integrable and the induced foliation of S is not Riemannian; each maximal leaf is a totally geodesic submanifold of S isometric to the real hyperbolic space $\boldsymbol{R}H^{m+1}$ of constant curvature -1.

The explicit calculation of the eigenvalues and eigenspaces of the Jacobi operators R_Y leads to the following further results:

– The sectional curvatures $K(\sigma)$ of a DR-space S satisfy $-1 \leqslant K(\sigma) \leqslant 0$ and -1 is attained. If S is symmetric, then $-1 \leqslant K(\sigma) \leqslant -\frac{1}{4}$ and $-\frac{1}{4}$ is attained. S attains zero as a value of the sectional curvature if and only if there exist a unit vector $V + Y \in \mathfrak{n}$ with $|V|^2 = \frac{2}{3}$ and a non-zero vector $X \in Y^{\perp}$ so that $J_X J_Y V$ is orthogonal to $J_3 V$. If S is odd-dimensional, then zero is attained as a value of $K(\sigma)$. (Note that Boggino's proof [**13**] for the result stating that zero is always attained as a value of $K(\sigma)$ in the non-symmetric case is not correct. For the even-dimensional DR-spaces this statement is therefore still an open problem.)

– A DR-space is a $\mathfrak{C}$-space if and only if it is a symmetric space.

– A DR-space is a $\mathfrak{P}$-space if and only if it is a symmetric space.

The result on $\mathfrak{C}$-spaces yields the following corollaries:

– The following statements are equivalent for a DR-space S:

i. S is a symmetric space
ii. S is a naturally reductive Riemannian homogeneous space
iii. S is a Riemannian g.o. space
iv. S is a weakly symmetric space (see below)
v. S is a commutative space
vi. S is a $\mathfrak{C}_0$-space
vii. S is a $\mathfrak{T}\mathfrak{C}$-space
viii. S is an $\mathfrak{S}\mathfrak{C}$-space
ix. S is a globally Osserman space
x. S admits a quotient of finite volume
xi. S is semi-symmetric
xii. all geodesic spheres in S are curvature-homogeneous
xiii. all geodesic spheres in S are isoparametric.

– There are D'Atri spaces which are not $\mathfrak{C}$-spaces. We do not know whether any $\mathfrak{C}$-space is a D'Atri space or not.

– There are probabilistic commutative spaces, which are not commutative spaces (see [**33**] for the notion of probabilistic commutative spaces).

– The DR-spaces provide counterexamples for various conjectures about k-harmonic spaces [**49**].

7 - Weakly symmetric spaces

The basic reference for this section is [8].

Weakly symmetric spaces were introduced in 1956 by A. Selberg [45] in the framework of his generalization of the Poisson summation formula to what is now known as the Selberg trace formula. Selberg's original definition is as follows. A Riemannian manifold M is called a *weakly symmetric space* if there exists a subgroup G of the isometry group $I(M)$ of M and an isometry f of M with $f^2 \in G$ and $fGf^{-1} = G$ so that for any two points $p, q \in M$ there exists an isometry $g \in G$ with $g(p) = f(q)$ and $g(q) = f(p)$. Note that such a group G acts necessarily transitive on M, so that a weakly symmetric space is always homogeneous. Taking $G = I(M)$ and $f = \mathrm{id}_M$, one sees readily that any Riemannian symmetric space is weakly symmetric. This definition is rather abstract and Selberg gave only one example of a non-symmetric weakly symmetric space, namely $SL(2, \boldsymbol{R})$ with a special left-invariant Riemannian metric.

An equivalent definition for weakly symmetric spaces is provided by the *ray symmetric spaces* introduced by Z. Szabó [46]. A Riemannian manifold M is weakly symmetric if and only if for every maximal geodesic γ in M and any $m \in \gamma$ there exists an isometry of M which is a non-trivial involution on γ with m as fixed point.

Finally, we mention an appealing geometrical characterization [8]. A Riemannian manifold M is weakly symmetric if and only if for any two points $p, q \in M$ there exists an isometry of M mapping p to q and q to p.

The last two characterizations lead us to a variety of new examples of weakly symmetric spaces which we shall discuss below.

We first continue with some basic properties of weakly symmetric spaces:

– Any weakly symmetric space is a commutative space (with respect to the full isometry group). This fundamental property was already proved by Selberg who also stated the still unsolved question whether the converse holds.

– The Riemannian product of two weakly symmetric spaces is a weakly symmetric space and conversely, each factor of a reducible weakly symmetric space is also weakly symmetric.

– The Riemannian universal covering space of a weakly symmetric space is a weakly symmetric space.

– Any weakly symmetric space is both a $\mathfrak{TC}$-space and an $\mathfrak{SC}$-space.

The new examples of weakly symmetric spaces were obtained via reflection theory (see [50] for details about reflections and for further references). First, suppose that B is a submanifold of a symmetric space. Then the reflection of the symmetric space in B is an isometry if and only if at each point of B both the tangent space and the normal space are curvature-invariant [15]. In order to find the submanifolds B in symmetric spaces such that the reflections in B are global isometries we need to determine the subspaces of the tangent spaces which are curvature-invariant and such that their orthogonal complements are also curvature-invariant. This can be carried out easily in two-point homogeneous spaces since we know the complete totally geodesic submanifolds and we have just to select those which have a curvature-invariant normal space at each point. This gives the following list:

ambient space	submanifold
$\boldsymbol{E}^n$	$\boldsymbol{E}^k$
S^n	S^k
$\boldsymbol{RP}^n$	$\boldsymbol{RP}^k$
$\boldsymbol{CP}^n$	$\boldsymbol{RP}^n$, $\boldsymbol{CP}^k$
$\boldsymbol{HP}^n$	$\boldsymbol{CP}^n$, $\boldsymbol{HP}^k$
$\boldsymbol{Cay}\,P^2$	$\boldsymbol{HP}^2$, $\boldsymbol{Cay}\,P^1$, $\{p\}$
$\boldsymbol{RH}^n$	$\boldsymbol{RH}^k$
$\boldsymbol{CH}^n$	$\boldsymbol{RH}^n$, $\boldsymbol{CH}^k$
$\boldsymbol{HH}^n$	$\boldsymbol{CH}^n$, $\boldsymbol{HH}^k$
$\boldsymbol{Cay}\,H^2$	$\boldsymbol{HH}^2$, $\boldsymbol{Cay}\,H^1$, $\{p\}$.

From this we may obtain the following weakly symmetric spaces:

ambient space	hypersurface
$\boldsymbol{CP}^n$	tube around $\{p\}$, $\boldsymbol{CP}^1$, ..., or $\boldsymbol{CP}^{n-1}$
$\boldsymbol{HP}^n$	tube around $\{p\}$, $\boldsymbol{HP}^1$, ..., or $\boldsymbol{HP}^{n-1}$
$\boldsymbol{Cay}\,P^2$	tube around $\{p\}$, or $\boldsymbol{Cay}\,P^1$
$\boldsymbol{CH}^n$	horosphere; tube around $\{p\}$, $\boldsymbol{CH}^1$, ..., or $\boldsymbol{CH}^{n-1}$
$\boldsymbol{HH}^n$	horosphere; tube around $\{p\}$, $\boldsymbol{HH}^1$, ..., or $\boldsymbol{HH}^{n-1}$
$\boldsymbol{Cay}\,H^2$	horosphere; tube around $\{p\}$, or $\boldsymbol{Cay}\,H^1$.

All these hypersurfaces are non-symmetric (which is not obvious and needs a proof). In the compact cases the radius of the tube has to be less than the distance to the focal set of the respective submanifold. Note that in $\boldsymbol{R}^n$, S^n, $\boldsymbol{RP}^n$ and $\boldsymbol{RH}^n$ any tube around a complete totally geodesic submanifold is a Riemannian symmetric space.

We continue with some conclusions:

– Any geodesic sphere in a two-point homogeneous space is weakly symmetric and hence a commutative space.

– If all small geodesic spheres in a Riemannian manifold M ($\dim M > 2$) are weakly symmetric, then M is locally isometric to a two-point homogeneous space.

– Any horosphere in a non-compact two-point homogeneous space is weakly symmetric. These horospheres are generalized Heisenberg groups (except for $\boldsymbol{R}H^n$ where the horosphere is isometric to $\boldsymbol{R}^{n-1}$) and it is still an open problem to determine the weakly symmetric spaces among the generalized Heisenberg groups. But note that any weakly symmetric generalized Heisenberg group is commutative and a g.o. space, which reduces the possible list considerably.

We mention that the simply connected weakly symmetric spaces are classified for dimensions $\leqslant 4$. In these dimensions the weakly symmetric spaces coincide with the naturally reductive Riemannian homogeneous spaces.

Finally we mention that Sasakian space forms are weakly symmetric. This fact may be shown easily by using reflection theory or the results above.

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