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On the growth of entire generalized axisymmetric potentials (**)

1 - Introduction

Solutions of the n-dimensional *Laplace equation*

$$\frac{\partial^2 H}{\partial x_1^2} + \frac{\partial^2 H}{\partial x_2^2} + \dots + \frac{\partial^2 H}{\partial x_n^2} = 0 \tag{1.1}$$

which depend only on the two variables $x = x_1$ and $y = (x_2^2 + x_3^2 + \dots + x_n^2)^{\frac{1}{2}}$ are called *axisymmetric potentials*. These satisfy the partial differential equation

$$\frac{\partial^2 H}{\partial x^2} + \frac{\partial^2 H}{\partial y^2} + \frac{2\mu}{y}\frac{\partial H}{\partial y} = 0 \tag{1.2}$$

where $2\mu = n - 2$. Solutions of (1.2) for $\mu > 0$ (2μ not necessarily an integer) were first investigated by Weinstein [8] and are called *generalized axisymmetric potentials*.

Let $H(x, y)$ be a generalized axisymmetric potential (GASP). If H is *entire*, then it can be expanded as

$$H(r, \theta) = \sum_{n=0}^{\infty} a_n r^n C_n^{\mu}(\cos\theta) \tag{1.3}$$

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where $x = r\cos\theta$, $y = r\sin\theta$ and $C_n^\mu(\cos\theta)$ are *Gegenbauer polynomials* of degree n. The series on right-hand side of (1.3) converges uniformly on compact sets. R. P. Gilbert ([4], p. 168-173) connected an entire GASP with its A_μ *associate* h, which is an entire function of one complex variable z. Thus we have

$$H(x, y) = A_\mu(h) = \alpha_\mu \int_L h(z)(\xi - \xi^{-1})^{2\mu}\xi^{-1}\,d\xi$$

where $\quad z = x + \frac{i}{2}y(\xi + \xi^{-1}) \qquad L = \{e^{i\theta} | 0 \leq \theta \leq \pi\}$

$$\alpha_\mu = [\int_L (\xi - \xi^{-1})^{2\mu - 1}\xi^{-1}\,d\xi]^{-1} + 4\Gamma(2\mu)(4i)^{-2\mu}\Gamma(\mu)^{-2}$$

and Γ denotes the classical gamma function.

The *inverse transformation* A_μ^{-1} is given by

$$h(z) = A_\mu^{-1}(H) = \int_0^\pi H(r, \theta)K(\frac{z}{r}, \theta)\,d\theta \qquad |z| < r$$

where $$K(\frac{z}{r}, \theta) = \frac{\mu\Gamma(2\mu)}{2^{2\mu-1}\Gamma(\mu + \frac{1}{2})^2} \frac{(\sin\theta)^{2\mu}(1 - \frac{z^2}{r^2})}{[1 - 2(\frac{z}{r}\cos\theta + \frac{z^2}{r^2}]^{\mu+1}}.$$

If $H(x, y)$ has series expansion (1.3) then its A_μ associate h has series expansion

$$h(z) = \sum_{n=0}^{\infty} \frac{\Gamma(n + 2\mu)}{\Gamma(2\mu)\Gamma(n + 1)} a_n z^n \qquad |z| = r. \tag{1.4}$$

A. Fryant [3] introduced the concepts of *order* and *type* for an entire GASP and obtained their characterizations in terms of coefficients a_n in the series (1.3). He also obtained necessary and sufficient conditions for entire GASP to be of *regular growth* (*perfectly regular*). However, important growth parameters such as *lower order*, *lower type* etc. were not considered by Fryant. In this paper, we introduce the concepts of lower order, lower type etc. and obtain the coefficient characterizations for these growth parameters. We also obtain alternative coefficient characterizations for order and type in terms of the ratio $|a_n a_{n+1}^{-1}|$.

We use well known results from entire function theory.

The paper forms a natural sequel to the paper [3].

2 - Order and lower order of an entire GASP

Let $H(x, y)$ be an *entire* GASP and $M(r, H) = \max_{\theta} |H(r, \theta)|$. Then the *order* ρ of H is defined in [3] as

$$\rho = \lim_{r \to \infty} \sup \frac{\log \log M(r, H)}{\log r} . \tag{2.1}$$

Similarly, we define the *lower order* λ of H by

$$\lambda = \lim_{r \to \infty} \inf \frac{\log \log M(r, H)}{\log r} . \tag{2.2}$$

We shall denote the order and the lower order of the corresponding A_μ associate h by ρ' and λ' respectively. It was proved by Fryant ([3], Th. 2.1) that $\rho = \rho'$.

Now we prove

Theorem 1. *Let H be an entire* GASP *and let h be its A_μ associate. Then the lower orders of H and h are equal.*

Proof. The proof of Theorem 1 runs on the lines of Theorem 2.1 of Fryant ([3], p. 363). Let us set

$$M(r, h) = \max_{|z| = r} |h(z)| .$$

Then from the integral operator A_μ, we have $M(r, H) \leqslant M(r, h)$. Setting $z = pre^{it}$, $0 < p < 1$, in the reverse transform A_μ^{-1}, we have

$$h(pre^{it}) = \int_0^\pi H(r, \theta) K(pe^{it}, \theta) \, d\theta .$$

Hence

$$M(pr, h) = \max_t | \int_0^\pi H(r, \theta) K(pe^{it}, \theta) \, d\theta | \leqslant \max_t \max_\theta \pi |H(r, \theta) K(pe^{it}, \theta)|$$

$$\leqslant \pi M(r, h) \max_t |K(pe^{it}, \theta)| \leqslant \pi K(p) M(r, H)$$

where
$$K(p) = \frac{\pi \mu \Gamma(2\mu)}{2^{2\mu - 1} \Gamma(\mu + \frac{1}{2})^2} \max_{\theta, t} \left| \frac{(\sin \theta)^{2\mu} (1 - p^2 e^{2it})}{(1 - 2pe^{it} \cos \theta + p^2 e^{2it})^{\mu + 1}} \right| .$$

Hence
$$M(r, h) \leqslant K(p) M(p^{-1} r, H).$$

Combining two previous inequalities, we have

$$M(r, H) \leqslant M(r, h) \leqslant K(p) M(p^{-1} r, H). \tag{2.3}$$

We further note that $\lim_{p \to \infty} K(p) = \infty$.

Now, from the definition of lower order, we have from the left-hand inequality of (2.3) $\lambda \leqslant \lambda'$. Further

$$\liminf_{r \to \infty} \frac{\log \log M(r, h)}{\log r} \leqslant \liminf_{r \to \infty} \frac{\log \log [K(p) M(p^{-1} r, H)]}{\log r}$$

$$= \liminf_{r \to \infty} \frac{\log [\log K(p) + \log M(p^{-1} r, H)]}{\log (p^{-1} r) + \log p} = \liminf_{r \to \infty} \frac{\log \log M(p^{-1} r, H)}{\log (p^{-1} r)}$$

i.e. $\lambda' \leqslant \lambda$. This proves Theorem 1.

In the subsequent results, we assume that $2\mu > 1$. Now we prove

Theorem 2. *Let H be an entire* GASP *having series expansion* (1.2). *If* $\log |a_n a_{n+1}^{-1}|$ *is a non-decreasing function of n for all $n > n_0$, where n_0 is a fixed positive integer, then the lower order λ of H is given by*

$$\lambda = \liminf_{n \to \infty} \frac{n \log n}{\log |a_n|^{-1}}. \tag{2.4}$$

Proof. We consider the series expansion of h, the A_μ associate of H, given by

$$h(z) = \sum_{n=0}^{\infty} b_n z^n \qquad \text{where } b_n = a_n \frac{\Gamma(n + 2\mu)}{\Gamma(2\mu)\Gamma(n + 1)}. \tag{2.5}$$

Then $\left| \frac{b_n}{b_{n+1}} \right| = \left| \frac{a_n}{a_{n+1}} \right| \frac{n+1}{n + 2\mu}$, i.e. $\log \left| \frac{b_n}{b_{n+1}} \right| = \log \left| \frac{a_n}{a_{n+1}} \right| + \log \frac{n+1}{n + 2\mu}$.

Since $2\mu \geqslant 1$, $\log |b_n b_{n+1}^{-1}|$ forms a non-decreasing function of n, whenever $\log |a_n a_{n+1}^{-1}|$ satisfies this condition. We now refer to the coefficient formulae for lower order of an entire function obtained by various workers (see e.g. [2],

[6], etc.). In particular we use [6], Th. 2

$$\lambda' = \lim_{n\to\infty}\inf \frac{n\log n}{\log|b_n|^{-1}} = \lim_{n\to\infty}\inf \frac{n\log n}{\log\Gamma(2\mu)+\log[\Gamma(n+1)\Gamma(n+2\mu)^{-1}]+\log|a_n|^{-1}}.$$

Using the *Tricomi-Erdely asymptotic formula* for the ratio of gamma function ([1], p. 257), we obtain

$$\frac{\Gamma(n+1)}{\Gamma(n+2\mu)} \simeq n^{(1-2\mu)}.$$

Since $\lambda = \lambda'$, we obtain

$$\lambda = \lim_{n\to\infty}\inf \frac{n\log n}{\log\Gamma(2\mu)+(1-2\mu)\log n+\log|a_n|^{-1}} = \lim_{n\to\infty}\inf \frac{n\log n}{\log|a_n|^{-1}}.$$

This proves Theorem 2.

In the next theorem we obtain coefficient characterization for ρ and λ in terms of the *ratio* $|a_n a_{n+1}^{-1}|$. We thus prove

THEOREM 3. *Let the entire* GASP H *be of order* ρ *and lower order* λ. *If* $\log|a_n a_{n+1}^{-1}|$ *forms a non-decreasing function of* n *for* $n > n_0$, *then*

$$\begin{matrix}\rho\\ \lambda\end{matrix} = \lim_{n\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log n}{\log|a_n a_{n+1}^{-1}|}. \tag{2.6}$$

PROOF. As shown earlier, the coefficients b_n in the series expansion of h satisfy the condition that $\log|b_n b_{n+1}^{-1}|$ is a non-decreasing function of n for $n > n_0$. Hence for entire function h, we have ([6], [2] Th. 2)

$$\begin{matrix}\rho'\\ \lambda'\end{matrix} = \lim_{n\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log n}{\log|b_n b_{n+1}^{-1}|}.$$

Now $$\log\left|\frac{b_n}{b_{n+1}}\right| = \log\left|\frac{a_n}{a_{n+1}}\right| - \log\frac{n+2\mu}{n+1} = \log\left|\frac{a_n}{a_{n+1}}\right| - o(1).$$

Since $\rho = \rho'$ and $\lambda = \lambda'$, we finally get (2.6).

3 - Type and lower type of entire GASP

Let H be an entire GASP of order ρ, $0 < \rho < \infty$. Then the *type* of H is defined in [3] as

$$T = \lim_{r\to\infty} \sup \frac{\log M(r, H)}{r^\rho} \qquad 0 \leqslant T \leqslant \infty . \tag{3.1}$$

Similarly, we now define the *lower type* τ as

$$\tau = \lim_{r\to\infty} \inf \frac{\log M(r, H)}{r^\rho} . \tag{3.2}$$

We shall denote by T' and τ', the type and lower type of the corresponding A_μ associate h. We now prove

Theorem 4. *Let H be an entire* GASP *of order ρ, $0 < \rho < \infty$. Then lower types of H and its A_μ associate h are equal.*

Proof. We already know that $\rho = \rho'$. From the left-hand inequality of (2.3) we have

$$\lim_{r\to\infty} \inf \frac{\log M(r, H)}{r^\rho} \leqslant \lim_{r\to\infty} \inf \frac{\log M(r, h)}{r^{\rho'}}$$

or $\tau \leqslant \tau'$. From the right-hand inequality of (2.3) we have

$$\frac{\log M(r, h)}{r^{\rho'}} \leqslant \frac{\log K(p) + \log M(p^{-1} r, H)}{r^\rho} = \frac{\log K(p) + \log M(p^{-1} r, H)]}{(p^{-1} r)^\rho\, p^\rho} .$$

Hence $$\lim_{r\to\infty} \inf \frac{\log M(r, h)}{r^{\rho'}} \leqslant p^{-\rho} \lim_{r\to\infty} \inf \frac{\log K(p) + \log M(p^{-1} r, H)}{(p^{-1} r)^\rho}$$

or $\tau' \leqslant p^{-\rho} \tau$. Since p $(0 < p < 1)$ is arbitrary, we get $\tau' \leqslant \tau$. Combining the two inequalities, we thus obtain $\tau = \tau'$.

Next we have

Theorem 5. *Let H be an entire* GASP *of order ρ, $0 < \rho < \infty$, and lower type τ. Then*

$$e\rho\tau = \lim_{n\to\infty} \inf n\, |a_n|^{\frac{\rho}{n}} \tag{3.3}$$

provided that $\log |a_n a_{n+1}^{-1}|$ *forms a non-decreasing function of n for $n > n_0$.*

Proof. By assumption, $\log|b_n b_{n+1}^{-1}|$ is also a non-decreasing function of n for $n > n_0$. Hence by known formulae for lower type of an entire function ([7], [2] Theorem 6), we have for h,

$$e\rho'\tau' = \lim_{n\to\infty} \inf n|b_n|^{\frac{\rho'}{n}}.$$

Now $|b_n|^{\frac{1}{n}} = |a_n|^{\frac{1}{n}} \left(\frac{\Gamma(n+1)}{\Gamma(2\mu)\Gamma(n+2\mu)}\right)^{\frac{1}{n}} \simeq |a_n|^{\frac{1}{n}} (n^{1-2\mu}\Gamma(2\mu)^{-1})^{\frac{1}{n}}.$

Thus $\lim_{n\to\infty} \inf n|b_n|^{\frac{\rho'}{n}} = \lim_{n\to\infty} \inf n|a_n|^{\frac{\rho'}{n}}$. Since $\rho = \rho'$ and $\tau = \tau'$, we finally get (3.3).

Lastly we prove

Theorem 6. *Let H be an entire* GASP *of order ρ, $0 < \rho < \infty$, type T and lower type τ. Then*

$$\lim_{n\to\infty} \inf n\left|\frac{a_{n+1}}{a_n}\right|^{\rho} \leqq \rho\tau \leqq \rho T \leqq \lim_{n\to\infty} \sup n\left|\frac{a_{n+1}}{a_n}\right|^{\rho}. \tag{3.4}$$

Further, if $\log\left|\frac{a_n}{a_{n+1}}\right|$ *forms a non-decreasing function of n for $n > n_0$ then*

$$\lim_{n\to\infty} \sup n\left|\frac{a_{n+1}}{a_n}\right|^{\rho} \leqq e\rho T. \tag{3.5}$$

Proof. Let h be the A_μ associate of H. Then $\rho' = \rho$, $T' = T$ and $\tau' = \tau$. Using [5] Theorem 1 and [2] Theorem 8, we have

$$\lim_{n\to\infty} \inf n\left|\frac{b_{n+1}}{b_n}\right|^{\rho'} \leqq \rho'\tau' \leqq \rho' T' \leqq \lim_{n\to\infty} \sup n\left|\frac{b_{n+1}}{b_n}\right|^{\rho'}.$$

Now $\frac{b_{n+1}}{b_n} = \frac{n+2\mu}{n+1}\frac{a_{n+1}}{a_n}$. Hence we obtain

$$\lim_{n\to\infty} \inf n\left|\frac{a_{n+1}}{a_n}\right|^{\rho} \leqq \rho\tau \leqq \rho T \leqq \lim_{n\to\infty} \sup n\left|\frac{a_{n+1}}{a_n}\right|^{\rho}.$$

If $\log|b_n b_{n+1}^{-1}|$ forms a non-decreasing function of n for $n > n_0$ then ([5] Theorem 3 and [2] Theorem 9) we have

$$\lim_{n\to\infty} \sup n\left|\frac{b_{n+1}}{b_n}\right|^{\rho'} \leqq e\rho' T' \quad \text{i.e.} \quad \lim_{n\to\infty} \sup n\left|\frac{a_{n+1}}{a_n}\right|^{\rho} \leqq e\rho T.$$

This proves Theorem 6.

Corollary. *Let H be an entire* GASP *of order* ρ $(0 < \rho < \infty)$ *and type T. Then H is of perfectly regular growth, i.e.* $T = \tau$ *if* $\lim_{n \to \infty} n \left| \frac{a_{n+1}}{a_n} \right|^{\rho}$ *exists.*

References

[1] M. Abramowitz and I. A. Stegun, *Handbook of mathematical functions*, Appl. Math. Series, National Bureau of Standards, Washington 1964.

[2] S. K. Bajpai, G. P. Kapoor and O. P. Juneja, *On entire functions of fast growth*, Trans. Amer. Math. Soc. **203** (1975), 275-296.

[3] A. J. Fryant, *Growth and complete sequences of generalized axisymmetric potentials*, J. Approx. Theory **19** (1977), 361-370.

[4] R. P. Gilbert, *Function theoretic methods in partial differential equations*, Academic Press, New York 1969.

[5] O. P. Juneja, *On the coefficients of an entire series of finite order*, Arch. Math. **21** (1970), 374-378.

[6] S. M. Shah, *On the lower order of integral functions*, Bull. Amer. Math. Soc. **52** (1946), 1046-1052.

[7] S. M. Shah, *On the coefficients of an entire function of finite order*, J. London Math. Soc. **26** (1951), 45-46.

[8] A. Weinstein, *Generalized axisymmetric potential theory*, Bull. Amer. Math. Soc. **59** (1953), 20-38.

Sommario

Sia $H(x, y)$ *una trascendente intera, soluzione assisimmetrica generalizzata dell'equazione* (1.2) *con* $\mu > 0$. *A. J. Fryant ha definito per una tale funzione l'ordine e il tipo ed ha ottenuto caratterizzazioni dei coefficienti in funzione dei coefficienti della serie* $H(\rho, \theta) = \sum a_n r^n C_n^{\mu}(\cos \theta)$ *dove i* $C_n^{\mu}(\cos \theta)$ *sono i polinomi di Gegenbauer.*

In questo lavoro si introducono le nozioni di ordine inferiore e di tipo inferiore di H e si ottengono le relative espressioni in funzione degli a_n.

Si indicano anche nuove caratterizzazioni dell'ordine e del tipo mediante i coefficienti.

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